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ON THE DYNAMICS OF MAGNETIC WAVE IN FERRITES: INFLUENCE OF DAMPING AND INHOMOGENEOUS EXCHANGE EFFECTS

THESIS

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On the Dynamics of Magnetic Wave in Ferrites: Influence of Damping and Inhomogeneous Exchange Effects

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Dedication

To **God** almighty,

my late father **TCHUDJO Pierre** and my mother **POUOKAM**

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Abstract

Throughout this dissertation, we investigate in detail the contributions of the damping effects and inhomogeneous exchange on the propagation of electromagnetic waves in ferrimagnetic insulator. Restricting our interests to the ultra-fast process assumption, we discuss the Kraenkel-Manna-Merle (KMM) system, derived from the Maxwell's equations in which damping effects and inhomogeneity set in from the Landau-Lifshitz-Gilbert equation. We construct analytical expression of a solution that well approximates the aforementioned global system. We proceed to some analytical analysis of the contribution of damping effects on the solutions to the system of our interest and thereafter, the influence of the inhomogeneous exchange effects on the dynamics of waves and finally, the contribution of the two effects simultaneously. It appears as a result that contribution of damping reduces the amplitude of waves during its motion. We unearth two types of inhomogeneities that effects act on the waves in different manner. When the wave moves with constant amplitude, inhomogeneity acts principally on the width of the wave, otherwise inhomogeneity acts on all the parameters of the wave. For ferrites made of poly-crystals it may coexist different types of inhomogeneities appearing periodically. Wave moving therein will breathe as it moves and there must be a case where two or more types of inhomogeneities can be combined in such a way to reduce their effects on the dynamics of the wave. We complete our analysis by proceeding to some numerical simulations. Such a study is helpful to characterize ferrite.

Keywords: Ferrites; Damping effects; Inhomogeneous exchange; Numerical simulations.

Résumé

Dans ce travail, nous étudions la contribution des effets de la dissipation et des échanges inhomogènes sur la propagation des ondes électromagnétiques dans les ferrites. En nous interressant uniquement aux processus ultrat-rapides, nous étudions le système de Kraenkel-Manna-Merle (KMM) dérivé des équations de Maxwell dans lesquelles les termes traduisant la dissipation et l'inhomogénéité proviennent de l'équation de Landau-Lifshitz-Gilbert. Nous construisons la solution analytique de l'équation sus-mentionnée. Nous procédons à l'analyse de l'effet de la dissipation sur la dynamique de notre système, puis celui des effets inhomogènes et enfin l'influence des deux effets combinés au moyen des méthodes mathématiques telles que la méthode de diffusion inverse, la métode d'expension des fonctions élliptiques de Jacobi ainsi que la méthode de différence finie. Il apparaît comme résultats que la contribution de la dissipation a pour effet de réduire l'amplitude de l'onde durant sa propagation. Nous retrouvons deux types d'inhomogénéités qui agissent sur l'onde de différentes manières. Lorsque l'onde se déplace avec une amplitude constante, l'inhomogénéité agit principalement sur sa largeur, dans le cas contraire, l'inhomogénéité impacte sur tous les paramètres de l'onde. Pour les ferrites polycristallins, il peut exister différents types d'inhomogénéités apparaissant de manière périodique. L'onde qui s'y propage respire au cours de son déplacement et il doit exister des cas où deux ou plusieurs types d'inhomogénéités peuvent se combiner de manière à réduire leurs effets sur la dynamique de l'onde. Nous complétons notre analyse en procédant à quelques simulations numériques. Une telle étude est utile pour caractériser les ferrites.

Mots clés: Ferrite; Dissipation; Échange Inhomogène; Simulation Numérique.

General Introduction

In recent years, it has been developed significant interest in the study of nonlinear waves in media that are governed by nonlinear evolution systems. These waves are able to move in various media including fluids [1], stretched ropes [2], optical fibers [3–6] magnetic materials [7,8] just to name a few. Propagation of waves in such media aroused great interests through explanation of some huge phenomena that occur in our daily life such as rogue waves in oceans [9, 10], or because of their wide prospect of industrial, technological or theoretical applications. Indeed, the investigations carried out on the dynamics of waves in particular medium, often enable to unearth nonlinear equations that modeled the dynamics of wave moving therein. It then appears in the literature a wide number of evolution equations that describe propagation of waves in various media. As example, one may enumerate the Korteweg-de Vries (KdV) equation [11] that models the dynamics of water waves, the vector counterpart of the KdV-equation that describes evolution of waves in multi-layer fluids [12, 13], the nonlinear Schrodinger equation [14] that models dynamics of waves in optical fibers , just to name a few. The real need that come out from these equations is their solutions that are more expressive than the equations themselves. Therefore, the question of integrability is posed along with explicit expressions of solutions. To carry out such a task, there are techniques available in the literature that can be handled namely: The inverse scattering transform (IST) method [14, 19] that is useful when Lax pairs to nonlinear evolution equation is provided. Beside the IST method, the Hirota's bilinearization [20, 22] which stands as a more direct method to investigate solutions to nonlinear evolution equations when the bilinear equation associated to nonlinear systems are provided. To find such bilinear equation associated to nonlinear systems, one often make use of the Painleve analysis [23, 24] that is also a technique that help verifying if a system is integrable or not before providing analytical solutions. Having solutions at hand, understanding of the behavior of waves in considered media is possible and, some investigations may also be carried out theoretically before going back in laboratories to test their efficiency. In the case where, exact solution is difficult to be constructed, one may use the aforementioned methods to find solutions that approximates very well the solution of the system under investigation and numerical methods can be used to confirm results obtained analytically. As far as we are concerned with a nonlinear system describing the propagation of short waves in magnetic insulators, Kraenkel, Manna and Merle [25] have derived a system known as the KMM-system, that describes propagation of ultra-short light pulses in these media. Integrability of such a system has been fully investigated from prolongations structure analysis, Hirota's bilinearization [26], auxiliary equation method [27, 28] and inverse scattering transform method [29]. Pursuing in the same analysis as Kraenkel-Manna-Merle, the investigation of the dynamics of waves in ferrites, Nguepjouo and coworkers [30] have derived a system that takes into account the damping effects from the Landau-Lifshitz-Gilbert equation [31, 32] which can be derived from the Heisenberg equation of spin, which is related to the energy conservation of magnetic free energy. Since this system has not been proven integrable, phase portraits analysis has been carried out to discuss the influence of such damping effects on the wave. Going further, Kuetche and co-authors [33] have proposed a nonlinear system, that takes into account inhomogeneous exchanges along with damping effects. They have investigated phase portraits analysis, to predict the effects of the combined effects on the dynamics of waves in ferrites. For the case of system with the term standing for inhomogeneous exchange process, loop wave has been constructed from Hirota's bilinear method [34] which is a direct technic of finding exact solutions to a large number of nonlinear evolution systems among which the Kraenkel, Manna and Merle equation. His procedure is to replace the dependent variables by a ratio of functions which satisfy some coupled bilinear differential system. Recently, the magnetic field has been constructed versus external magnetization nevertheless the case of magnetic field versus position has not been investigated. This contribution to the resolution of KMM-System does not take into account the contribution of damping term. The comparison with the solutions to the KMM-system has not been investigated explicitly to know precisely the contribution of damping and inhomogeneous exchange of the magnetic field in ferrites. A question can arise as follows: what are the influences of the combined damping and inhomogeneous effects on the wave's propagations? This question constitute the main aim of our investigations. Thus, the present work is organized as follows:

We present in chapter one the generality on the magnetic materials and on the ferrites in particular. We dwell too long on ferrites and in particular on yttrium-iron-garnet (YIG), by giving its characteristics as well as its technological, industrial and economic importance. We also present the non-linear integrable differential equation of KMM without damping and inhomogeneous exchange and as well as the KMM-system including the damping and inhomogeneous exchange effects which is the model which we study in the remainder of our work.

In Chapter two, we present the Wahlquist and Estabrook formalism while investigating the prolongation structure of $(1+1)$-dimensional evolution equation which must lead to the Lax Pair. We illustrate this method by applying it to the Schäfer Wayne system. We pursue in the same chapter by exposing the inverse scattering method transform and also Jacobi elliptic function expansion method. We also present numerical method according to the finite difference scheme.

In Chapter three, we firstly, investigate analytically the influence of the damping effects on the dynamics of short waves in ferrites and to confirm analytical results while proceeding to some numerical simulations. Indeed, to carry out our goal, the mathematical toll described by Konno is used to deal with the system under current interest. The Lax-pairs of this system is unknown until now, we provide a system of equations that is Lax integrable and that approximates in some conditions which will be specify, the damped-KMM system. Under these condition, we provide analytical expression of the solutions to the damped-KMM system. We investigate, numerical simulations to confirm the results obtained analytically

We secondly, construct new traveling wave solutions to the inhomogeneous system of our interest using Jacobi elliptic expansion method. We discuss the influence of the inhomogeneous exchange effects on the traveling waves and we address some physical implications.

We thirdly, present the solution of the KMM-system that is free of damping and inhomogeneous exchange effects. We provide an approximate solution to the system taking into account only damping effects, and also in this section we discuss solution to the system in which involve only inhomogeneous exchange effects. We then deduce and approximate analytical solution of the system taking into account the combined effects of damping and inhomogeneous exchange. We proceed to some numerical simulations to complete the analytical analysis.

Chapter 1

Literature Review

Introduction

Magnetism is a phenomenon that comes from moving charges. Three sources are at the origin of the magnetic moment in an atom: the quantum state of spin of electrons, the orbital motion around the nucleus of electrons and the quantum state of spin of the nucleus. In a magnetic atom, the predominant contribution at its total moment comes from the spin and the orbital moment of the electrons [35]. Magnetization $\mathbf{M}$ is the macroscopic magnitude that describes the magnetic character of a material. It is defined as the total magnetic moment per unit volume [36]. A fundamental characteristic of magnetic materials is their response to a magnetic field $\mathbf{H}$ applied to. Magnetic susceptibility χ is the quantity that describes this response [35, 36]. In a general way, $\overline{\chi}$ is a tensor, but if the material is isotropic, the susceptibility becomes a scalar. The objective of this chapter is first to present the magnetism concept and main characteristic of magnetic material. We also present the mathematical model equation which describe the magnetic wave propagation in ferrite.

1.1 Generality on magnetic materials

The different types of magnetic materials are usually classified under the basis of their susceptibility or permeability. Therefore we must define these related properties before describing the differences between ferromagnetic, paramagnetic, diamagnetic antiferromagnetic and ferrimagnetic materials.

1.1.1 Magnetic Susceptibility

The most common way of classifying magnetic properties of materials is by their response to an applied magnetic field. Materials that are magnetized to a certain extent by a magnetic field are called magnetic [37]. In particular, it is the quantity termed magnetic susceptibility that characterizes the magnetic response through the relationship $\mathbf{M} = \chi\mathbf{H}$, where $\mathbf{M}$ is the magnetization, also known as the magnetic moment per unit volume, and $\mathbf{H}$ is the applied magnetic field intensity [38]. Magnetic susceptibility is usually a tensor and a function of both field $\mathbf{H}$ and magnetization $\mathbf{M}$. For a magnetically isotropic material, $\mathbf{M}$ is parallel to $\mathbf{H}$, and magnetic susceptibility is reduced to a scalar quantity. The unit for the permeability of vacuum is the same as for magnetic susceptibility [38]. Hence, it is possible to measure susceptibility in units of permeability of vacuum. Generally the behavior of susceptibility χ leads to various types of magnetism [38] which is approached in the following section.

1.1.2 Classification of magnetic materials

Ferromagnetism: Ferromagnetic materials contain spontaneously magnetized magnetic domains where an individual domain magnetization is oriented differently with respect to the magnetization of neighboring domains [37]. The spontaneous domain magnetization is a result of unpaired electron spins from partially filled shells, spins aligned parallel to each other due to a strong exchange interaction. The arrangement of spins depends on temperature and so does the spontaneous domain magnetization [37]. When the total resultant magnetization for all magnetic domains is zero, the ferromagnetic material is said to be demagnetized. However, an applied magnetic field changes the total resultant magnetization from zero to a saturation value [37]. When the magnetic field is decreased and reverses in sign, the magnetization of a ferromagnetic material does not retrace its original path of values, the material exhibiting so-called hysteresis [37].

Diamagnetism: Diamagnetism is a behavior of the materials which leads them, when

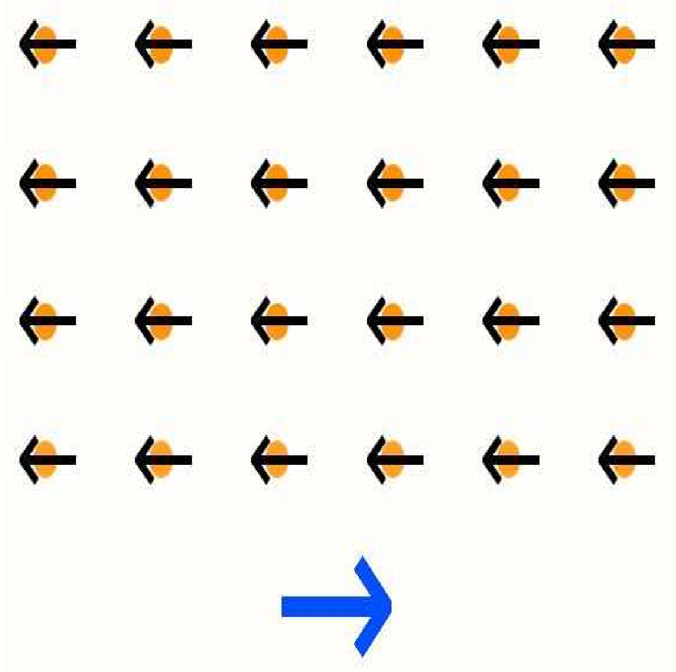

Figure 1: The magnetic moments of each atom is opposed to the external magnetic field here in blue.

subjected to a magnetic field, to create a very weak magnetization opposite to the external field, and thus to generate a magnetic field opposite to the external field [37] as seen in Figure (1). Diamagnetic materials exhibit an antiparallel magnetization with respect to the direction of the applied magnetic field, opposing the latter according to Lenz's law. Thus, the magnetization of a diamagnetic material is proportional to the applied magnetic field . An isotropic diamagnetic material is therefore characterized by a negative magnetic susceptibility. Diamagnets have a negative and very weak relative susceptibility, of the order 10^{-5} [38].

Paramagnetism: Paramagnetic material is characterized by a positive susceptibility [40]. It meets in the substances whose atoms have one permanent magnetic moment, when these moments are not coupled the ones with the others. When an external magnetic field is applied, a weak induced magnetization is produced parallel to the field as presented in Figure (2). However, the polarization which results remain very weak, because the effect of the thermal agitation which directs the magnetic moments in a random way remains dominant [41]. Are paramagnetic the majority of gases, certain metals, in

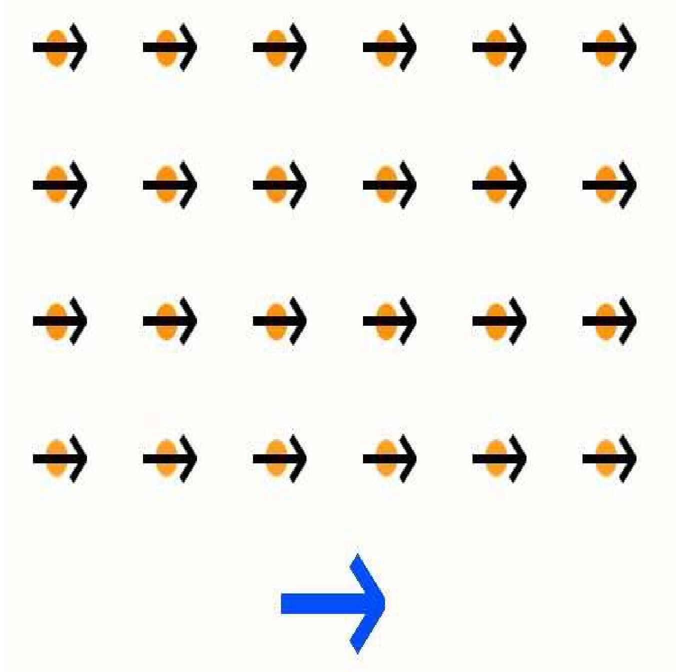

Figure 2: The magnetic moments of each atom are aligned with the external magnetic field in blue.

particular the alkaline materials, some salts, the ferromagnetic and ferrimagnetic materials when they are heated above their Curie temperature which is the transition temperature from ferromagnetic to paramagnetic behaviour. At this temperature the permeability of the material drops suddenly and both coercivity and remanence become zero [39]. **Antiferromagnetism:** Analogous to paramagnetism, antiferromagnetism also exhibits a small positive relative susceptibility that varies with temperature [11]. However, this dependence differs significantly in the fact that in an antiferromagnetic material it displays a change at the so-called [11] Néel temperature. Néel's temperature is the transition temperature from anti-ferromagnetic to paramagnetic behaviour. Below this temperature, the electron spins are arranged antiparallel so that they cancel each other and an external magnetic field is faced with a strong opposition due to the interaction between these spins. one finds that the total magnetization of an anti-ferromagnetic materials is essentially zero Figure (3)

Ferrimagnetism: Ferrimagnetism is a magnetic property of certain solid bodies. In a ferrimagnetic material, the magnetic moments are anti-parallel but of different ampli-

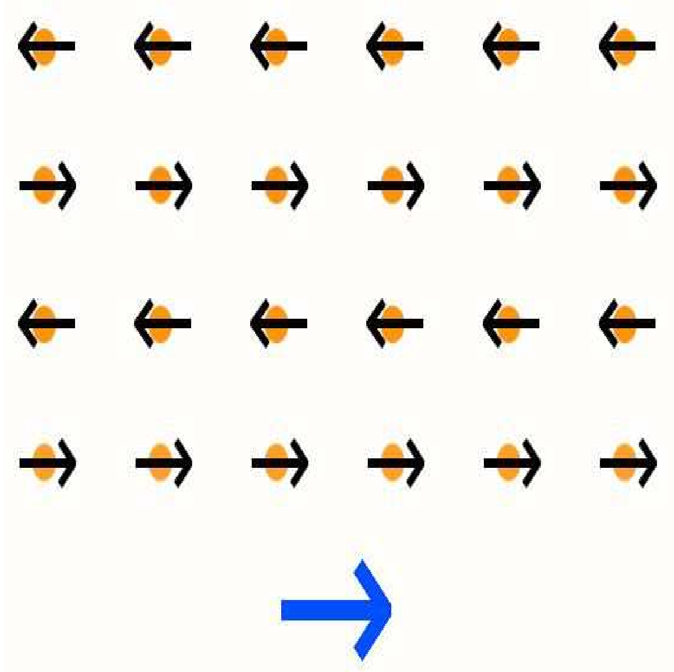

Figure 3: The magnetic moments of an atom are reversed with that of the neighboring atom.

tude as in Figure (4). This results in a spontaneous magnetization of the material. It is thus distinguished both from antiferromagnetism, for which the resulting magnetic moment is zero, and from ferromagnetism, for which spontaneous magnetization results at the microscopic level from a parallel arrangement of magnetic moments [38]. Nevertheless, an increase in temperature brings about a disturbance in the spin arrangement that culminates in completely random orientation of spins at the Curie temperature. At this temperature, the ferrimagnet loses its spontaneous magnetization and becomes paramagnetic [39].

1.1.3 Ferrites and applications

Magnetic materials which have combined electrical and magnetic properties are known as ferrites. The main constituents of the ferrites are iron oxide and metal oxides [43]. The importance of ferrite material has been known to mankind for many centuries. Ferrites may be defined as magnetic materials composed of oxides containing ferric ions as the main constituent and classified as magnetic materials because they exhibit ferrimagnetic

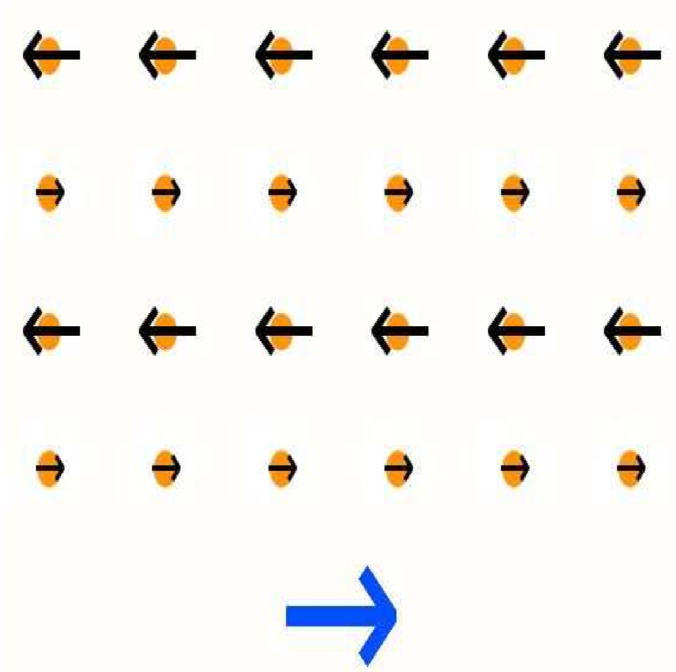

Figure 4: The magnetic moments in a direction do not have the same intensity as those in the other direction.

behavior [43]. Ferrites materials are insulating magnetic oxides and possess high electrical resistivity, low eddy current and dielectric losses, high saturation magnetization and high permeability, what makes ferrites the unique magnetic materials which find applications in almost all fields [44–46]. Ferrites are magnetic oxide materials with semiconducting nature which are of great technological importance by virtue of their interesting electrical and magnetic properties. Exhibiting dielectric properties means that even though electromagnetic waves can pass through ferrites, they do not readily conduct electricity [47]. This also gives them an advantage over iron, nickel and other transition metals that have magnetic properties in many applications because these metals conduct electricity. The ferrites in nanocrystalline form find applications in new fields like magnetically guided drug delivery, magnetic resonance imaging, catalyst, humidity and gas sensors, magnetic fluids just to name a few. [47–49]. The low loss polycrystalline ferrites should be used in a high frequency range. For the good performance in application and classified by the relative permeability, for the low and high frequency applications, the most important technological properties are saturation magnetization M_s, coercive force H_c relative permeability μ_i and losses. It is not possible generally to obtain the best combination of these properties for any specific application. By varying the compositions or adding additives or by varying the preparation technique, one can, to a large extent, control most parameters for any particular applications. Ferrites are generally classified into three major families that are spinels, garnets and hexaferrites [50]. The main advantage of ferrites comes from their dielectric character. A dielectric is an electrical insulator that can be polarized by an applied electric field. The study of dielectric properties concerns storage and dissipation of electric and magnetic energy in materials. Indeed, this characteristic makes it possible to avoid losses related to the skin effect. Among all the ferrites, Yttrium Iron Garnet (YIG) is the most found in microwave devices. Yttrium Iron Garnet (YIG) is a magnetic material whose chemical formula is given by $Y_3F_{e_2}(F_eO_4)_3$. This material is often used under sphere shape because in this geometry, the resonance frequency only depends of the intensity of applied field. When the (YIG) is used as a film, it is chemical vapor deposition on a support [50].

1.1.4 Magnetization

The magnetization of the usual ferromagnetic materials can be explain by the theory of magnetic domains which was developed by Pierre Weiss [42,43]. This theory states that the material consists of domains where the internal magnetization is aligned in a single direction. This magnetization inside the domains is called spontaneous magnetization. The domains are separated by walls called Bloch walls and represent regions where the orientation of the magnetization varies rapidly [46]. The global state of magnetization of material is given by the relative orientation of the direction of the magnetization in the domains. The provision of the domains is such as the magnetic sum of the moments is null and resulting global magnetization is null. In a magnetized state with saturation, the magnetization of all the magnetic domains is aligned according to a single direction [35]. The movement of the walls of the magnetic domains, main mechanism of variation of magnetization to the low fields, is disturb by the presence of defects which arise in the form of dislocations in the crystal lattice of material [42]. These defects involve a reduction of the permeability and an increase of the magnetic losses. The mechanism of magnetization comprises three main phenomena which occur successively according to the intensity of the magnetic field applied [42]: The first mechanism of magnetization is the elastic movement of the walls of domains, which occurs with the low magnetic fields and which represents a spontaneously reversible magnetization of material. The system can turn over naturally at the initial state if the field applied is canceled. The second mechanism of magnetization is the irreversible movement of the walls of domains, during the withdrawal of the magnetic field applied, one does not find the same distribution in fields. This magnetization is described as remnant. The third mechanism of magnetization occurs when the material comprises only fields having a component aligned with the directions of easy magnetization of the crystals closest to the direction of the field. The increase in the magnetic field applied then causes the orientation of magnetization inside these domains to align itself with the magnetic field applied. The magnetization of material presents a hysteretic character which is due to irreversible displacement of the walls of the domains, known as Bloch walls [43].

1.1.5 Inhomogeneities in magnetic materials

Inhomogeneities such as defined by Balakrishnan [52] in the Heisenberg spin chain system is a strength interaction of localization of nearest neighbour. Otherwise known as deformations in a system may either be due to external fields or to the presence of defects, voids and gaps in the material. In the case where inhomogeneities are considered as deformation, they arise when a ferromagnetic medium lying in the plane is magnetized either in the longitudinal or transverse direction. When saturated along the longitudinal axis, a medium exhibits a homogeneous effective field but when magnetized in the transverse axis, the induced effective field is inhomogeneous [53]. These types of deformations are called field deformations. In the other case, a site-dependent function is introduced into the Hamiltonian to model the lattice defects such that the corresponding exchange integral is bond dependent [54]. These lattice defects introduce a lattice distortion thereby leading the material to a deformed one. We can have the case where the distance between neighboring atoms varies along the chain, hence altering the overlap of electronic wave-functions assumed to be identical at all sites, or the case where the wave-function itself varies from site to site, even for equally spaced atoms. In the case where the deformation is assumed identical at all sites, the inhomogeneity function acting as a coefficient of exchange interactions is a set of random variables [54]. In such organometallic insulators [54] inhomogeneity function randomly alternates between two values along the chain. A chain system is natural for modeling inhomogeneities due to defects, although it may still be applicable in the case of weakly disordered systems with peaked wave-functions such that a small change in the lattice constant causes a relatively large change in the atom overlap [55, 56]. These inhomogeneities can be modeled in the effective Hamiltonian for a one-dimensional magnetic insulator placed in a weak, static, inhomogeneous electric field or by the introduction of imperfections (impurities or organic complexes) in the vicinity of a bond to alter the electronic wave-functions without causing appreciable lattice distortions. By gradually changing the concentration of impurities along the chain, it is possible for engineers to control inhomogeneity function [55].

1.2 Mathematical model

1.2.1 Landau-Lifshitz equation

A phenomenological theory of ferromagnetism has been establish by Landau-Lifshitz and has been corroborated by a considerable amount of experimental data [31]. The Heisenberg's equation describing the motion of spin is given by the following equation

$$\frac{dS_i}{dt} = \frac{1}{i\hbar}[S_i, \mathbb{H}] = \frac{g\mu_B}{i\hbar}[S_i, S_j]H_j = \frac{g\mu_B}{\hbar}\varepsilon_{ijk}S_kH_j, \tag{1.1}$$

where the commutator of the spin is $[S_i, S_j] = \imath\varepsilon_{ijk}S_k$ in an ferromagnetic material. The atomic moment in a cell can be written as $\mathbf{M} = -ng\mu_B\mathbf{S}$, where n is the atomic density. The atomic magnetic moment $\overrightarrow{\mu_a} = -g\mu_B\mathbf{S}$, where g is the g-factor, μ_B is the Bohr magnet, $\mathbf{S}$ is the atomic spin magnetic moment. If we omit energy dissipation, the eigenvalue of Hamiltonian $\mathbb{H} = -\mu_a.H$, is conserved, then we can use the Heisenberg's equations to describe the motion of spin $\mathbf{S} = \mathbf{e}_iS_i$. Based on the preceding equation (1.1), if the energy is conserved [31], the equation of motion of magnetic moment is in the form

$$\frac{d\mathbf{M}}{dt} = -ng\mu_B\frac{d\mathbf{S}}{dt} = -g\frac{e}{2mc}(\mathbf{M}\times H) = -\gamma_0(\mathbf{M}\times H), \tag{1.2}$$

where the constant $\gamma_0 = g\frac{e}{2mc}$ is the gyromagnetic ratio which is generally as being the relationship between the magnetic moment and the kinetic moment for an atomic spin.

The preceding equation is free of damping effects. When the dissipation of magnetic free energy is included, the atomic spins have a non-equilibrium statistics, and its motion becomes nonlinear. Based on the Hamiltonian $\mathbb{H}$, the dissipation of magnetic energy means the moment $\mathbf{M}$ rotate to the direction of local magnetic field $\mathbf{H}$, which equals the effective magnetic field $\mathbf{H}_{eff}$ [57]. Therefore Landau and Lifshitz have added a damping term in the equation of motion of spin, which results in the famous LL equation:

$$\frac{d\mathbf{M}}{dt} = -\gamma(\mathbf{M} \times \mathbf{H}_{eff}) - \gamma\frac{\alpha}{M}\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{eff}), \tag{1.3}$$

where the dimensionless constant α is called the Landau damping constant, and describes the dissipation effect in the ferromagnetic material. Similar to the frictional coefficient, the damping constant α is also phenomenological. Usually in such metals or alloys, α is less than 0.1, sometimes even below 0.01. However, in ferromagnetic oxides or ferrites, the dissipation process is much slower, thus α can be one or two orders lower than the value in ferromagnetic metals [57]. A common way of introducing a damping term of this kind into classical equations of motion for a physical system is to use a Lagrangian formulation of the equations of motion and add a velocity-dependent term derived from a quadratic function of the time derivatives of the dynamical variables called Rayleigh's dissipation function [58]. In micromagnetics, the equation of motion for magnetic moments is always called the Landau-Lifshitz-Gilbert equation (LLG)-equation, because Thomas Gilbert explained the damping term of (LL)-equation by the dissipative Lagrange equation with a Rayleigh's dissipation function in [31]:

$$\frac{d}{dt}\frac{\delta\mathcal{L}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} - \frac{\delta\mathcal{L}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} + \frac{\delta\mathcal{R}[\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} = 0, \tag{1.4}$$

where Gilbert assumed that the Lagrange equation itself will result in the equation (1.2) of motion for the magnetic moment under the constraint of energy conservation. In the Lagrange $\mathcal{L}[\mathbf{M},\dot{\mathbf{M}}] = \mathcal{T} - \mathcal{U}$, the role of magnetization $\mathbf{M}$ is the same as the position r in $\mathcal{L}[\mathbf{r},\dot{\mathbf{r}}]$ of classical mechanics. The Rayleigh dissipation functional is also constructed analogous to that of the frictional force in mechanics:

$$\mathcal{R}[\dot{r}] = \frac{\eta}{2}\int\int\int d^3r\dot{r}^2(r,t) \quad \Rightarrow \quad \mathcal{R}[\dot{M}] = \frac{\eta}{2}\int\int\int d^3M\dot{M}^2(r,t), \tag{1.5}$$

where η is a damping parameter that is characteristic of the material. Then the (LLG)-equation can be obtained by inserting the Rayleigh dissipation functional into equation (1.4), where the effective field appears as $\delta\mathcal{U}/\delta M = -H_{eff}$

$$\frac{d}{dt}\frac{\delta\mathcal{T}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} - \frac{\delta\mathcal{T}[\mathbf{M},\dot{\mathbf{M}}]}{\delta\dot{\mathbf{M}}} + (-\mathbf{H}_{eff} + \eta\dot{\mathbf{M}}) = 0, \tag{1.6}$$

implies

$$\frac{\partial\mathbf{M}}{\partial t} = -\gamma_0\mathbf{M}\times(\mathbf{H}_{eff} - \eta\dot{\mathbf{M}}) = -\gamma_0\mathbf{M}\times\mathbf{H}_{eff} + \frac{\alpha}{M}\mathbf{M}\times\frac{\partial\mathbf{M}}{\partial t}, \tag{1.7}$$

where the damping $\alpha = \gamma_0\eta M$. This equation must be equivalent to equation (1.2) when the friction coefficient $\eta = 0$ [31]. The (LLG)-equation and (LL)-equation are totally equivalent to one another, except that the gyromagnetic ratio γ in the two equations has a small difference related to the damping. If we insert the right-hand side of the (LLG)-equation in equation (1.7) into the last term $\partial M/\partial t$, it is easy to prove that

$$(1+\alpha^2)\frac{\partial\mathbf{M}}{\partial t} = -\gamma_0\mathbf{M}\times\ \mathbf{H}_{eff} - \gamma_0\frac{\alpha}{M}(\mathbf{M}\times\frac{\partial\mathbf{M}}{\partial t}), \tag{1.8}$$

Comparing this equation (1.8) with the LL equation (1.3), it is clear that $\gamma = \gamma_0/(1+\alpha^2)$. Therefore, the gyromagnetic ratio γ in the LL equation should be smaller than the gyromagnetic ratio γ_0 in the (LLG)-equation, by a small factor related to the damping coefficient α.

1.2.2 Maxwell equation

Maxwell's equations summarize several important findings in electromagnetism. They describe in a mathematical way how are bound and how interact electric charges, electrical currents, electric fields and magnetic fields. For simply statement, they describe the electric, magnetic and luminous phenomena quantitatively. These equations are very significant in physics.

Maxwell-Gauss equation: This equation can be express as follows and show that the divergence of the electric field is proportional to the electric charge distribution.

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{1.9}$$

A particle or an electrically charged body, constitutes a concentration of electric charges of the same sign. This means that the electrical field is diverging since the source of electric charges is proportional to the distribution of these charges.

Maxwell- Flux equation: The divergence of the magnetic field is nought and is given as follows

$$\nabla \cdot \mathbf{B} = 0. \tag{1.10}$$

There is no divergence of the magnetic field, so the magnetic field lines do not point towards infinity. This law reflects the simple fact that there is no magnetic monopole. A magnet monopole does not exist as there are electric monopoles such as electron and proton. If we break a magnet, we get several magnets with north and south pole. Mathematically, the law can also be read as the lines of outgoing magnetic field of one of the poles of a magnet return in the other pole. This formulation explains better the fact that the sum of all lines of fields is equal to nought. What leaves one side returns to the other and final one does not lose nor creates thing.

Maxwell-Faraday equation: The curl of electric field is proportional to the variation of the magnetic field over time.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.11}$$

If we take a varying magnetic field in a conductor, then there appears a rotating electric field around the magnet. In the Maxwell-Faraday equation, the curl of electric field is proportional to the variation of magnetic field. Indeed, it is the variation of the magnetic field that generates an electric field and not the magnetic field alone. If we place a magnet in a coil, nothing happens. On the other hand, if you move the magnet, an electric field is created around, which itself will generate an electric current in the wire.

Maxwell-Ampere equation: The curl of magnetic field is the sum of his time dependence variation of electric field and electric current

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.12}$$

This equation shows that the magnetic field is produced by the variation of electric field during the time. The term $\mu_0 \mathbf{J}$ shows also that the magnetic field as well depends on electric current in the case of conductor.

We follow the classical manipulation of the Maxwell's equations namely, take the curl of the Ampere law and eliminate $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{B}$ by making use of the constitutive relation and the Faraday law, and obtain

$$-\nabla(\nabla \cdot \mathbf{H}) + \nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}(\mathbf{H}+\mathbf{M}) + \nabla \times \mathbf{J}, \tag{1.13}$$

where $c^2 = \frac{1}{\varepsilon_0 \mu_0}$

1.2.3 System without damping and inhomogeneous effects

Within the framework of their investigation relating to the wave propagation in ferrimagnetic media, Kraenkel et al. [25] have used the Maxwell's equations in the absence of current and charges in a medium of scalar permittivity, which are supplemented with a relation between the magnetization and the magnetic field in the materials called Landau-Lifshitz equation given as follows

$$
\begin{aligned}
-\nabla(\nabla\cdot\mathbf{H}) + \nabla^2\mathbf{H} &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{H}+\mathbf{M}), && (1.14a)\\
\frac{\partial \mathbf{M}}{\partial t} &= -\mu_0\gamma\mathbf{M}\times\mathbf{H}, && (1.14b)
\end{aligned}
$$

where vectors $\mathbf{H}$ and $\mathbf{M}$ stand for the magnetic induction and magnetization density, respectively. γ being the gyromagnetic ratio, the first term on the right-hand side describes the precession of the magnetization around the micro-magnetic field and μ_0 the magnetic permeability of the vacuum field. From a practical viewpoint, the above coupled equations are actually fundamental for investigation of the data loading processes in reversal magnetic memory devices in the ferrites. These equations are fundamentally nonlinear and describe the propagation of magnetic waves in ferrite in the absence of current and charges when losses are negligible.

Linear regime: propagating plane wave.

In order to clear the notation, let us rescale the variables as

$$
\mathbf{M} \longrightarrow \frac{\mu_0\gamma}{c}\mathbf{M}, \quad \mathbf{H} \longrightarrow \frac{\mu_0\gamma}{c}\mathbf{H}, \quad t \longrightarrow \frac{t}{c}. \tag{1.15}
$$

For better understanding the propagation of waves in ferrites we proceed firstly to the analysis of linear limit regime considering the small perturbations of solution with steady state given as follows

$$
\mathbf{M}_0 = (0, M_0\sin(\varphi), 0), \quad \mathbf{H}_0 = \alpha\mathbf{M}_0, \tag{1.16}
$$

where α M_0 and H_0 are positives constant respectively. In order to linearize the previews system around this state, we assume for the perturbations a plane wave solution propagating along the x-direction so that

$$\mathbf{M} = \mathbf{M}_0 + \epsilon \mathbf{M}_1 \exp[\imath(kx - \omega t)], \tag{1.17a}$$

$$\mathbf{H} = \mathbf{H}_0 + \epsilon \mathbf{H}_1 \exp[\imath(kx - \omega t)], \tag{1.17b}$$

where the parameter ϵ represents the small perturbation related to the wavelength of the short-wave perturbation along the ferrite. ω and k are, respectively, the frequency and wave number of the wave. It is important in the continuation of our analysis to establish the dispersion relation of the preceding system (1.14) by expressing beforehand the terms that constitute it. We start by expressing terms of Maxwell's equation (1.14a) as follows

$$\nabla(\nabla\cdot\mathbf{H}) = (H^i_x + H^j_y + H^k_z)_x\mathbf{e}_x + (H^i_x + H^j_y + H^k_z)_y\mathbf{e}_y + (H^i_x + H^j_y + H^k_z)_z\mathbf{e}_z.$$

The terms bearing i, j and k indices correspond to the components of the vectors along the x, y and z directions respectively.

$$\begin{aligned}\nabla^2\mathbf{H} &= (H^i_{xx} + H^i_{yy} + H^i_{zz})\mathbf{e}_x + (H^j_{xx} + H^j_{yy} + H^j_{zz})\mathbf{e}_y + (H^k_{xx} + H^k_{yy} + H^k_{zz})\mathbf{e}_z,\\ -\nabla(\nabla\cdot\mathbf{H}) + \nabla^2\mathbf{H} &= (H^i_{yy} + H^i_{zz} - H^j_{xy} - H^k_{zx})\mathbf{e}_x + (H^j_{xx} + H^j_{zz} - H^i_{xy} - H^k_{yz})\mathbf{e}_y\\ &\quad + (H^k_{xx} + H^k_{yy} - H^i_{xz} - H^j_{yz})\mathbf{e}_z,\end{aligned}$$

$$\frac{\partial^2}{\partial t^2}(\mathbf{H}+\mathbf{M}) = \frac{\partial^2}{\partial t^2}(H^i + M^i)\mathbf{e}_x + \frac{\partial^2}{\partial t^2}(H^j + M^j)\mathbf{e}_y + \frac{\partial^2}{\partial t^2}(H^k + M^k)\mathbf{e}_z,$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(H^i + M^i) &= 0,\\ \frac{\partial^2}{\partial t^2}(H^j + M^j) &= -\epsilon k^2 H^j_1 \exp[\imath(kx - \omega t)],\\ \frac{\partial^2}{\partial t^2}(H^k + M^k) &= -\epsilon k^2 H^k_1 \exp[\imath(kx - \omega t)].\end{aligned}$$

The Maxwell equation (1.14a) takes the following form in view of the terms express above according to directions x, y and z respectively

$$\begin{cases} H^i_1 = -M^i_1,\\ H^j_1 = -\frac{\omega^2}{\omega^2 - k^2} M^j_1,\\ H^k_1 = -\frac{\omega^2}{\omega^2 - k^2} M^k_1.\end{cases} \tag{1.18}$$

Handing Landau-lifshizt equation (1.14b) according to the perturbations plane wave solution propagating along the x-direction, we obtain the following relations

$$\begin{cases} \frac{\partial M^i}{\partial t} \equiv -\imath\omega\epsilon M_1^i \exp[\imath(kx - wt)] = (M^j H^k - M^k H^j), \\ \frac{\partial M^j}{\partial t} \equiv -\imath\omega\epsilon M_1^j \exp[\imath(kx - wt)] = (M^k H^i - M^i H^k), \\ \frac{\partial M^k}{\partial t} \equiv -\imath\omega\epsilon M_1^k \exp[\imath(kx - wt)] = (M^i H^j - M^j H^i), \end{cases}$$

$$\mathbf{M} \times \mathbf{H} = (M^j H^k - M^k H^j)\overrightarrow{e_x} + (M^k H^i - M^i H^k)\overrightarrow{e_y} + (M^i H^j - M^j H^i)\overrightarrow{e_k}.$$

Landau-lifshizt equation (1.14b) become according to x, y and z direction, respectively

$$\imath\omega M_1^i = H_1^k M_0 \sin\varphi - M_1^k H_0 \sin\varphi, \tag{1.19a}$$

$$\imath\omega M_1^j = M_1^k H_0 \cos\varphi - H_1^k M_0 \cos\varphi, \tag{1.19b}$$

$$\imath\omega M_1^k = H_1^j M_0 \cos\varphi + M_1^i H_0 \sin\varphi - H_1^i M_0 \sin\varphi - M_1^j H_0 \cos\varphi. \tag{1.19c}$$

Introducing equation (1.18) into equation (1.19) we have the following relation in $\mathbf{M}_1$

$$\imath(\omega^2 - k^2)M_1^i + M_0 \sin(\varphi)[\omega^2(1 + \alpha - \alpha k^2)]M_1^k = 0, \tag{1.20a}$$

$$\imath(\omega^2 - k^2)M_1^j - M_0 \cos(\varphi)[\omega^2(1 + \alpha) + \omega^2]M_1^k = 0, \tag{1.20b}$$

$$\begin{aligned} \imath(\omega^2 - k^2)M_1^k + M_0 \cos(\varphi)[\omega^2(1 + \alpha) - \alpha k^2]M_1^j \\ -M_0 \sin(\varphi)[(\omega^2 - k^2)(1 + \alpha)]M_1^i = 0. \end{aligned} \tag{1.20c}$$

This equation (1.20) can be written in the following matrix form

$$\begin{pmatrix} \imath\omega(\omega^2 - k^2) & 0 & M_0 A \sin(\varphi) \\ 0 & \imath\omega(\omega^2 - k^2) & -M_0 A \cos(\varphi) \\ -M_0 \sin(\varphi) & M_0 A \cos(\varphi) & \imath\omega(\omega^2 - k^2) \end{pmatrix} \begin{pmatrix} M_1^i \\ M_1^j \\ M_1^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.21}$$

with $A = \omega^2(1 + \alpha) - \alpha k^2$.

Equation (1.20) admit a nontrivial solution if the determinant of this matrix is equal to zero from where comes the following equation

$$M_0^2[\omega^2(1 + \alpha) - \alpha k^2][\omega^2(1 + \alpha) - k^2(\alpha + \sin^2\varphi] - \omega^2(\omega^2 - k^2)^2 = 0, \tag{1.22}$$

which is the dispersion relation. In the condition of studying short-waves, one has to consider wave vector tending toward infinity [25]. For that purpose, let us suppose $k \sim \epsilon^{-1}$

with $\epsilon \lll 1$. Consequently, the frequency is expanded accordingly as

$$\omega = \omega_{-1}\epsilon^{-1} + \omega\epsilon + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \cdots \tag{1.23}$$

Our hypothesis consider above corresponds just to the requirement that short-waves exist in the linear limit with the phase velocity and the group velocity always bounded. Now, replacing Eq.(1.23) into the dispersion relation above, we obtain a set of equations

At order ϵ^{-6} : $-w_{-1}^6 + 2w_{-1}^4 k_0^2 - w_{-1}^2 k_0^4$, we have

$$w_{-1} = \pm k_0. \tag{1.24}$$

At order ϵ^{-4} : $M_0^2(1+\alpha)(\alpha+\sin^2(\varphi)w_{-1}^2k_0^2 + M_0^2(1+\alpha)^2w_{-1}^4 - 6w_0^5w_1 + 8w_{-1}^3w_1k_0^2 - M_0^2\alpha(1+\alpha)w_{-1}^2k_0^2 - 2w_{-1}w_1k_0^4 + M_0^2\alpha(\alpha+\sin^2(\varphi)k_0^4 = 0$, we have

$$\sin^2\varphi = 1. \tag{1.25}$$

This express the fact that the short-waves can propagate only for one value of $\varphi = \pi/2$. At order ϵ^{-2} : $-15w_{-1}^4w_1^2 - w_1^2k_0^4 - 2M_0^2(1+\alpha)(\alpha+\sin^2(\varphi))w_{-1}w_1k_0^2 + 4M_0^2(1+\alpha)^2w_{-1}^3w_1 - 2M_0^2\alpha(1+\alpha)w_{-1}w_1k_0^2 + 12w_{-1}^2w_1^2k_0^2$, we have

$$w_1 = M_0^2\frac{(1+\alpha)}{2k_0}, \tag{1.26}$$

w_{-1}, w_1 and k_0 are characteristics of short-wave in the linear limit. Characteristics of the short waves thus to found in the linear limit, we now turn to the nonlinear aspect.

Nonlinear regime

We resort to the multiple-scale method, adapted to short-wave asymptotic. Thus, we introduce rescaled, space and time variables, such that

$$\frac{\partial}{\partial t} = -\frac{1}{\epsilon}\frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial \tau}, \tag{1.27a}$$

$$\frac{\partial}{\partial x} = \epsilon\frac{\partial}{\partial \tau}. \tag{1.27b}$$

We now consider general expansion in the following form

$$\mathbf{M} = \mathbf{M}_0 + \epsilon\mathbf{M}_1 + \epsilon^2\mathbf{M}_2 + \cdots \tag{1.28a}$$

$$\mathbf{H} = \mathbf{H}_0 + \epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2 + \cdots \tag{1.28b}$$

where the expansion coefficients depend on the variables ξ and τ. Prior to the direct substitution into the governing equations, namely the Maxwell's and Landau-Libshizt equations (1.14), the Maxwell equations provide the following relations

$$\epsilon^{-2}(H^i_{\xi\xi} + M^i_{\xi\xi}) + \epsilon^2(H^i_{\tau\tau} + M^i_{\tau\tau}) - 2(H^i_{\xi\tau} + M^i_{\xi\tau}) = 0, \quad (1.29a)$$

$$\epsilon^{-2}(H^j_{\xi\xi} + M^j_{\xi\xi}) + \epsilon^2(H^j_{\tau\tau} + M^j_{\tau\tau}) - 2(H^j_{\xi\tau} + M^j_{\xi\tau}) = 0, \quad (1.29b)$$

$$\epsilon^{-2}(H^k_{\xi\xi} + M^k_{\xi\xi}) + \epsilon^2(H^k_{\tau\tau} + M^k_{\tau\tau}) - 2(H^k_{\xi\tau} + M^k_{\xi\tau}) = 0. \quad (1.29c)$$

The Landau-Libshizt equation (1.14) provide the following relations

$$-\epsilon^{-1}M^i_\xi + \epsilon M^i_\tau = M^jM^k - M^kM^j, \quad (1.30a)$$

$$-\epsilon^{-1}M^j_\xi + \epsilon M^j_\tau = M^kM^i - M^iM^k, \quad (1.30b)$$

$$-\epsilon^{-1}M^k_\xi + \epsilon M^k_\tau = M^iM^j - M^jM^i. \quad (1.30c)$$

At order ϵ^{-2} we have, according to equation (1.29) and introducing general expansion relation (1.28), the following relation

$$H^i_{0\xi\xi} + M^i_{0\xi\xi} = 0, \quad (1.31a)$$

$$H^j_{0\xi\xi} + M^j_{0\xi\xi} = H^j_{0\xi\xi}, \quad (1.31b)$$

$$H^k_{0\xi\xi} + M^k_{0\xi\xi} = H^k_{0\xi\xi}. \quad (1.31c)$$

We seek the components of the vectors $\mathbf{H}_0$ and $\mathbf{M}_0$ respectively by using the initial conditions on the vectors and their derivative and we have $H^i_{0\xi\xi} + M^i_{0\xi\xi} = 0 \quad \Rightarrow (H^i_{0\xi} + M^i_{0\xi}) = c_1 \quad yet \quad \xi \longrightarrow -\infty, \quad H^i_{0\xi} \quad and \quad M^i_{0\xi} \longrightarrow 0 \quad then$
$c_1 = 0 \Rightarrow (H^i_0 + M^i_0) = c_2 \quad as \quad \xi \longrightarrow -\infty \quad H^i_0 \quad and \quad M^i_0 \to 0$

$H^j_{0\xi} + M^i_{0\xi} = H^j_{0\xi} + c_3 \Rightarrow \xi \longrightarrow \infty \quad H^j_{0\xi} \quad and \quad M^j_{0\xi} \longrightarrow 0 \quad then, \quad c_3 = 0$

$H^j_0 + M^i_0 = H^j_0 + c_4 \quad \xi \longrightarrow -\infty \quad H^j_0 = H_0, \quad M^j_0 = M_0, \quad c_4 = M_0$

we have according to the preview expressions

$$\mathbf{H}_0 = H^j_0\mathbf{e}_y + H^k_0\mathbf{e}_z, \quad (1.32a)$$

$$\mathbf{M}_0 = M^j_0\mathbf{e}_y, \quad (1.32b)$$

with $M^j_0 = M_0$ and $H^j_0 = H_0$.

At the order ϵ^{-1} we have, according to equation (1.29) and equation (1.30) and introducing general expansion relation (1.28), the following relation

$$(H_1^i + M_1^i)_{\xi\xi} = 0, \tag{1.33a}$$
$$(H_1^j + M_1^j)_{\xi\xi} = H_{\xi\xi}^j, \tag{1.33b}$$
$$(H_1^k + M_1^k)_{\xi\xi} = H_{\xi\xi}^k, \tag{1.33c}$$
$$-M_{0\xi}^i = 0, \tag{1.33d}$$
$$-M_{0\xi}^j = 0, \tag{1.33e}$$
$$-M_{0\xi}^k = 0, \tag{1.33f}$$

is constructed from which we have the following relation by using the initial conditions on the vectors and their derivatives

$$\begin{cases} H_1^i = -M_1^i, \\ M_1^j = 0, \\ M_1^k = 0, \\ M_0^i = 0, \\ M_0^j = m, \\ M_0^k = 0. \end{cases} \tag{1.34}$$

At the order ϵ^0 we have, according to equation (1.29) and equation (1.30) and introducing general expansion relation (1.28), the following relation

$$(H_2^i + M_2^i)_{\xi\xi} - 2(H_0^i + M_0^i)_{\xi\tau} = 0, \tag{1.35a}$$
$$(H_2^j + M_2^j)_{\xi\xi} - 2(H_0^j + M_0^j)_{\xi\tau} = H_{2\xi\xi}^j, \tag{1.35b}$$
$$(H_2^k + M_2^k)_{\xi\xi} - 2(H_0^k + M_0^k)_{\xi\tau} = H_{2\xi\xi}^k, \tag{1.35c}$$
$$-M_{1\xi}^i = H_0^i M_0^k - H_0^k M_0^j, \tag{1.35d}$$
$$-M_{1\xi}^j = H_0^k M_0^i - H_0^i M_0^k, \tag{1.35e}$$
$$-M_{1\xi}^k = H_0^i M_0^j - H_0^j M_0^i, \tag{1.35f}$$

is derived. We integrate equation (1.35) according to the limiting conditions on vectors and their derivatives and obtain the following relation

$$H_2^i = -M_2^i, M_{2\xi}^j = 2H_{0\tau}^j, M_{2\xi}^k = 2H_{0\tau}^k, \tag{1.36a}$$
$$M_{1\xi}^i = H_0^k M_0, M_1^j = 0, M_1^k = 0. \tag{1.36b}$$

At the order ϵ^1 we have, according to equation (1.29) and equation (1.30) and introducing general expansion relation (1.28), the following relation

$$\begin{aligned}
(H_3^i + M_3^i)_{\xi\xi} - 2(H_1^i + M_1^i)_{\xi\tau} &= 0 && (1.37a)\\
(H_3^j + M_3^j)_{\xi\xi} - 2(H_1^j + M_1^j)_{\xi\tau} &= H_{3\xi\xi}^j && (1.37b)\\
(H_3^k + M_3^k)_{\xi\xi} - 2(H_3^k + M_3^k)_{\xi\tau} &= H_{3\xi\xi}^k && (1.37c)\\
-M_{2\xi}^i &= M_0^i H_1^k + H_0^i M_1^k - H_0^k M_1^j - H_1^k M_0^j && (1.37d)\\
-M_{2\xi}^j &= H_0^k M_1^i + H_1^k M_0^i - H_0^i M_1^k - H_1^i M_0^k && (1.37e)\\
-M_{1\xi}^k &= H_0^i M_1^j + H_1^i M_0^j - H_0^j M_1^i - H_1^j M_0^i && (1.37f)
\end{aligned}$$

We integrate equation (1.37) according to the limiting conditions on vectors and their derivatives, and obtain the following relations

$$H_3^i = -M_3^i, \quad M_{3\xi}^j = 2(H_1^j + M_1^j)_\xi, \quad M_{3\xi}^k = 2(H_1^k + M_1^k)_\xi, \tag{1.38a}$$

$$M_{2\xi}^i = M_0 H_1^k, \quad M_{2\xi}^j = -H_0 H^k M_1^i, \quad M_{2\xi}^k = -M_0 H_1^i + H_0^j M_1^i. \tag{1.38b}$$

equation (1.36) enable us to write the following relations

$$\begin{aligned}
M_{2\xi}^j &= -\frac{1}{M_0} M_{1\xi}^i,\\
H_{0\tau}^j &= -\frac{1}{M_0} M_1^i M_{1\xi}^i,
\end{aligned}$$

then we have

$$\frac{\partial H_0^j}{\partial \tau} = -\frac{1}{M_0} M_1^i M_{1\xi}^i. \tag{1.39a}$$

equation (1.36) and equation (1.38) enable us to write the following relation

$$M_{2\xi}^k = \frac{2}{M_0} M_{1\xi\tau}^i = -M_0 H_1^i + H_0^j M_i^i,$$

then we have

$$\frac{\partial^2 M_1^i}{\partial \xi \partial \tau} = \frac{M_0}{2}(M_0 + H_0^j) M_1^i. \tag{1.40}$$

It is significant to note that equations (1.39) and (1.40) is a nonlinear system describing the evolution of the first nontrivial terms in the expansion of **H** and **M** which involve effects at order zero and one on ϵ respectively. Set $M_1^i = \beta B$ and $H_0^j = -M_0 + lC_\xi$, β and l are determined constants toward the following system

$$\beta B_{\xi\tau} = \frac{M_0}{2} l C_\xi \beta B, \tag{1.41a}$$

$$l C_{\xi\tau} = -\frac{\beta^2}{2M_0} B B_\xi. \tag{1.41b}$$

Then we have

$$\begin{cases} B_{\xi\tau} = BC_\xi \\ C_{\xi\tau} = -BB_\xi \end{cases} \tag{1.42}$$

with $l = \frac{2}{M_0}$ and $\beta^2 = 4$ Equation (1.42) is a well-known system called the Kraenkel-Manna-Merle system [25], describing the propagation of magnetic waves in ferrites. It is an integrable system which displays exact solutions.

1.2.4 Damping-Inhomogeneous magnetic system

We use a coupling between the classical Landau-Lifshitz-Gilbert model for the magnetization density function $\mathbf{M}(x,t) = (M^i, M^j, M^k)$ and Maxwell's equations for the propagation of electromagnetic waves in ferromagnetic chains [59]. This results permit to point up Maxwell-Landau-Gilbert model taking into account the Heisenberg exchange couplings and damping effect, given by

$$-\nabla(\nabla\cdot\mathbf{H}) + \nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{H}+\mathbf{M}), \tag{1.43a}$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\mu_0\gamma \mathbf{M}\wedge\mathbf{H}_{eff} + \sigma\frac{\mathbf{M}}{M_s}\wedge\frac{\partial \mathbf{M}}{\partial t}, \tag{1.43b}$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light, μ_0 being the magnetic permeability of the vacuum, and the vectors $\mathbf{H}$ and $\mathbf{B}$ stand for the magnetic induction and the magnetization density, respectively. The first term on the right hand side of Eq.(1.43) describe the precession of the magnetization around the micromagnetic effective field $\mathbf{H}_{eff}$ and the second term which is the Gilbert-damping term with the damping parameter σ drives the magnetization toward the direction of $\mathbf{H}_{eff}$ whereby angular momentum is transferred to magnetic degrees of freedom. The quantity M_s represents the saturation magnetization. The expression of $\mathbf{H}_{eff}$ is given by

$$\mathbf{H}_{eff} = \mathbf{H} - \beta\mathbf{n}(\mathbf{n}\cdot\mathbf{M}) + \alpha\triangle\mathbf{M}, \tag{1.44}$$

where α and β are the constants of the inhomogeneous exchange and the magnet anisotropy, respectively and $\mathbf{n}$ is the unit vector directed along the anisotropy axis and we assume $\mathbf{n} = \overrightarrow{e}_z$. In order to clear the notation, let us rescale the variables as follows

$$\mathbf{M} \to \frac{c\mathbf{M}}{\mu_0\gamma}, \quad \mathbf{H} \to \frac{c\mathbf{H}}{\mu_0\gamma}, \quad t \to \frac{t}{c}. \tag{1.45}$$

In order to well understand the inhomogeneous exchange in ferrites, we start at first by studying the linear limit regime looking at small perturbations of a given solution.

Linear regime: propagating of plane wave

We assume for the perturbations a plane wave solution propagating along the x direction, so that

$$\mathbf{M} = \mathbf{M}_0 + \epsilon\mathbf{M}_1 \exp[\imath(kx - wt)], \tag{1.46a}$$

$$\mathbf{H} = \mathbf{H}_0 + \epsilon\mathbf{H}_1 \exp[\imath(kx - wt)], \tag{1.46b}$$

$\mathbf{M}$ and $\mathbf{H}$ are vectors, where k and w are, respectively, the wave number and the frequency of the wave.

Maxwell equation (1.43) takes the new following form after substituting perturbations plane wave solution propagating along the x direction

$$\frac{\partial^2}{\partial t^2}(H^i + M^i) = (H^i_{yy} + H^i_{zz} - H^j_{xy} - H^k_{zx}), \tag{1.47a}$$

$$\frac{\partial^2}{\partial t^2}(H^j + M^j) = (H^j_{xx} + H^j_{zz} - H^i_{xy} - H^k_{yz}), \tag{1.47b}$$

$$\frac{\partial^2}{\partial t^2}(H^k + M^k) = (H^k_{xx} + H^k_{yy} - H^i_{xz} - H^j_{yz}). \tag{1.47c}$$

Landau-Lifshizt-Gilbert equation takes the new following forms

$$\frac{\partial M^i}{\partial t} = M^k H^j - M^j H^k + \beta M^k M^j + \alpha(M^k M^j_{xx} - M^j M^k_{xx}) + \frac{\sigma}{m}(M^j \frac{\partial M^k}{\partial t} - M^k \frac{\partial M^j}{\partial t}), \tag{1.48a}$$

$$\frac{\partial M^j}{\partial t} = M^i H^k - M^k H^i - \beta M^k M^i + \alpha(M^i M^k_{xx} - M^k M^i_{xx}) + \frac{\sigma}{m}(M^k \frac{\partial M^i}{\partial t} - M^i \frac{\partial M^k}{\partial t}), \tag{1.48b}$$

$$\frac{\partial M^k}{\partial t} = M^j H^i - M^i H^j + \alpha(M^i M^j_{xx} - M^j M^i_{xx}) + \frac{\sigma}{m}(M^i \frac{\partial M^j}{\partial t} - M^j \frac{\partial M^i}{\partial t}). \tag{1.48c}$$

Introducing equation (1.46) in equation (1.47), we obtain the following relations

$$H^i = -M^i, \tag{1.49a}$$
$$H_1^j = -\frac{w^2}{w^2-k^2}M_1^j, \tag{1.49b}$$
$$H_1^k = -\frac{w^2}{w^2-k^2}M_1^k. \tag{1.49c}$$

Introducing equation (1.46) and system (1.49) in equation (1.48), we obtain the following relations

$$\imath w(w^2-k^2)M_1^i + m[(w^2-k^2)(\tilde{\alpha}+\beta_0)+w^2]\sin\theta M_1^k = 0, \tag{1.50a}$$
$$\imath w(w^2-k^2)M_1^j + m[(w^2-k^2)(\tilde{\alpha}+\beta_0)+w^2]\cos\theta M_1^k = 0, \tag{1.50b}$$
$$-m(w^2-k^2)(1+\tilde{\alpha})\sin\theta M_1^i + m[(w^2-k^2)\tilde{\alpha}+w^2]\cos\theta M_1^j + \imath w(w^2-k^2)M_1^k = 0, \tag{1.50c}$$

where $\tilde{\alpha} = \mu + \alpha_0 k^2 - \imath w \frac{\sigma_0}{m}$.

This equation (1.50) can take the following matrix form in $\mathbf{M}_1$

$$\begin{pmatrix} \imath\omega(\omega^2-k^2) & 0 & D\sin\theta \\ 0 & \imath\omega(\omega^2-k^2) & D\cos\theta \\ E\sin\theta & m[(w^2-k^2)\tilde{\alpha}+w^2]\cos\theta & \imath\omega(\omega^2-k^2) \end{pmatrix} \begin{pmatrix} M_1^i \\ M_1^j \\ M_1^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1.51}$$

with $D = m(w^2-k^2)(\tilde{\alpha}+\beta_0)+w^2$ and $E = -m(w^2-k^2)(1+\tilde{\alpha})$.

This equation (1.51) admits nontrivial solution only if the determinant of the matrix is nil and that leads to the following dispersion relation

$$\omega^2(\omega^2-k^2)^2 + m^2\{(w^2-k^2)(\tilde{\alpha}+\beta_0)+w^2\}\{k^2\sin^2(\theta) - [\tilde{\alpha}(\omega^2-k^2)+\omega^2]\} = 0. \tag{1.52}$$

We are interested in studying the short-wave limit when wave number tend to infinity. This is possible if we consider $k \sim \epsilon^{-1}$ with $\epsilon \ll 1$. Consequently, the frequency is expanded accordingly as

$\omega = \omega_{-1}\epsilon^{-1} + \omega\epsilon + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \cdots$

Now, replacing the previous frequency into the dispersion relation above, we obtain a set of equations

At order ϵ^{-8}: $m^2\left({w_{-1}}^2 - {k_0}^2\right)^2 {\alpha_0}^2 {k_0}^4 = 0$ this relation permit to have

$$\omega_{-1} = \pm k_0. \tag{1.53}$$

At order ϵ^{-4}:

$$
\begin{aligned}
4w_{-1}^{3}\left(w_{-1}^{2}-k_{0}^{2}\right)w_{1}+2w_{-1}w_{1}(w_{-1}^{2}-k_{0}^{2})^{2}+m^{2}((w_{-1}^{2}-k_{0}^{2})(\mu+\beta_{0})+2w_{-1}w_{1}\alpha_{0}k_{0}^{2}\\
+w_{-1}^{2})(k_{0}^{2}sin^{2}(\theta)-(w_{-1}^{2}-k_{0}^{2})(\mu+\beta_{0})-2w_{-1}w_{1}\alpha_{0}k_{0}^{2}-w_{-1}^{2})-m^{2}(2w_{-1}w_{1}\\
+2w_{-1}w_{1}(\mu+\beta_{0})+w_{1}^{2}\alpha_{0}k_{0}^{2})(w_{-1}^{2}-k_{0}^{2})\alpha_{0}k_{0}^{2}+m^{2}(w_{-1}^{2}-k_{0}^{2})\\
\alpha_{0}k_{0}^{2}(-2w_{-1}w_{1}-2w_{-1}w_{1}(\mu+\beta_{0})-w_{1}^{2}\alpha_{0}k_{0}^{2})+m(\frac{\imath(w_{-1}^{2}-k_{0}^{2})w_{1}\sigma_{0}}{m}-\frac{2\imath w_{-1}^{2}w_{1}\sigma_{0}}{m})\\
(w_{-1}^{2}-k_{0}^{2})\imath w_{-1}\sigma_{0}-m(w_{-1}^{2}-k_{0}^{2})\imath w_{-1}\sigma_{0}(\frac{(w_{-1}^{2}-k_{0}^{2})^{2}\imath w_{1}\sigma_{0}}{m}+\frac{2w_{-1}^{2}w_{1}\imath\sigma_{0}}{m})=0.
\end{aligned}
$$

This relation express the following condition

$$
Either \quad 1+2w_{-1}w_{1}\alpha_{0}=0 \quad or \quad 1+2w_{-1}w_{1}\alpha_{0}=\sin^{2}(\theta). \tag{1.54}
$$

At order ϵ^{-3}:

$$
\begin{aligned}
m(2w_{-1}w_{1}+2w_{-1}w_{1}(\mu+\beta_{0})+w_{1}^{2}\alpha_{0}k_{0}^{2})(w_{-1}^{2}-k_{0}^{2})\imath w_{-1}\sigma_{0}\\
+6mw_{-1}w_{1}^{2}\imath\sigma_{0}(w_{-1}^{2}-k_{0}^{2})\alpha_{0}k_{0}^{2}+m^{2}((w_{-1}^{2}-k_{0}^{2})(\mu+\beta_{0})+2w_{-1}w_{1}\alpha_{0}k_{0}^{2}\\
+w_{-1}^{2})(\frac{\imath(w_{-1}^{2}-k_{0}^{2})w_{1}\sigma_{0}}{m}+\frac{2\imath w_{-1}^{2}w_{1}\sigma_{0}}{m})+m^{2}(\frac{-\imath(w_{-1}^{2}-k_{0}^{2})w_{1}\sigma_{0}}{m}-\frac{2\imath w_{-1}^{2}w_{1}\sigma_{0}}{m})\\
(k_{0}^{2}sin^{2}(\theta)-(w_{-1}^{2}-k_{0}^{2})(\mu+\beta_{0})-2w_{-1}w_{1}\alpha_{0}k_{0}^{2}-w_{-1}^{2})\\
-m(w_{-1}^{2}-k_{0}^{2})\imath w_{-1}\sigma_{0}(-2w_{-1}w_{1}-2w_{-1}w_{1}(\mu+\beta_{0})-w_{1}^{2}\alpha_{0}k_{0}^{2} &= 0.
\end{aligned}
$$

we obtain the following relation

$$
Either \quad \sigma_{0}=0 \quad or \quad 2(1+2w_{-1}w_{1}\alpha_{0})=\sin^{2}(\theta). \tag{1.55}
$$

At order ϵ^{-2}:

$$
\begin{aligned}
w_{-1}^{2}(2(w_{-1}^{2}-k_{0}^{2})w_{1}^{2}+4w_{-1}^{2}w_{1}^{2})+8w_{-1}^{2}w_{1}^{2}(w_{-1}^{2}-k_{0}^{2})\\
+w_{1}^{2}(w_{-1}^{2}-k_{0}^{2})^{2}+6w_{-1}^{2}w_{1}^{2}\sigma_{0}^{2}(w_{-1}^{2}-k_{0}^{2})-m^{2}(w_{1}^{2}(\mu+\beta_{0})+w_{1}^{2})(w_{-1}^{2}-k_{0}^{2})\alpha k_{0}^{2}\\
+m^{2}(\frac{-(w_{-1}^{2}-k_{0}^{2})\imath w_{1}\sigma_{0}}{m}-\frac{2w_{-1}^{2}w_{1}\imath\sigma_{0}}{m})(\frac{(w_{-1}^{2}-k_{0}^{2})\imath w_{1}\sigma_{0}}{m}+\frac{2w_{-1}^{2}w_{1}\imath\sigma_{0}}{m})+m^{2}((w_{-1}^{2}-k_{0}^{2})\\
(\mu+\beta_{0}+2w_{-1}w_{1}\alpha_{0}k_{0}^{2}+w_{-1}^{2})(-2w_{-1}w_{1}-2w_{-1}w_{1}(\mu+\beta_{0})-w_{1}^{2}\alpha_{0}k_{0}^{2})\\
+m^{2}(2w_{-1}w_{1}+2w_{-1}w_{1}(\mu+\beta_{0})+w_{1}^{2}\alpha_{0}k_{0}^{2})(k_{0}^{2}sin^{2}(\theta)-(w_{-1}^{2}-k_{0}^{2})\\
(\mu+\beta_{0})-2w_{-1}w_{1}\alpha_{0}k_{0}^{2}-w_{-1}^{2})+m^{2}(w_{-1}^{2}-k_{0}^{2})\alpha_{0}k_{0}^{2}(-w_{-1}^{2}(\mu+\beta_{0})-w_{1}^{2})=0.
\end{aligned}
$$

Handling now the previous expression in order ϵ^{-2} with $w_{-1} = k_0$, we obtain the following equation

$$\begin{aligned} 4k_0^2 w_1^2 + m^2\{-(1+2w_{-1}w_1\alpha_0)[2w_{-1}w_1(1+\mu+\beta_0) + \alpha_0 k_0^2(w_1^2+2w_{-1}w_3)] \\ +4k_0^2 w_1\sigma_0^2/m^2 + (\sin^2(\theta - 1 - 2w_{-1}w_1\alpha_0)[2w_{-1}w_1(1+\mu+\beta_0) \\ +\alpha_0 k_0^2(w_1^2+2w_{-1}w_3)]\} = 0. \qquad (1.56) \end{aligned}$$

It is also appropriate to find the frequency of propagation of wave at order one in ϵ, for that we assume $\sigma_0 = 0$ in equation (1.56) an w_1 is given by

$$w_1 = m^2\frac{(1+\mu)}{2k_0}. \qquad (1.57)$$

Nonlinear regime

To completely solve equation (1.43), we use the approach of multiscale analysis to reduce the coupled vector equations to a nonlinear equation for a scalar field. The multiscale analysis or the reductive perturbation method is a generalized asymptotic analysis method for solving the Maxwell-Landau-Lifshizt-Gilbert model by reducing it to soliton equations [60]. We first introduce the following space and time variables, depending on a small parameter ϵ

$$\zeta = (x-t)\epsilon^{-1}, \qquad (1.58a)$$

$$\tau = \epsilon t. \qquad (1.58b)$$

The slow space variable ζ describes the shape of the wave propagating at speed unity and the time variable t is the time evolution of this wave during the propagation. The parameter ϵ measures the weakness of the perturbation effect. As in the theory of ferromagnetic resonance experiment, we assume the model of sample immersed in a high intensity external magnetic field such that the sample is magnetized to its saturation limit. This assumption avoids the creation of domain walls which would require more care in the derivation of the reduced equation and the sample is free from the effects due to its geometry such as shape anisotropies. Under this assumption, we seek a reduction of the coupled system (1.43) through the multi-scale perturbation from the expansion

$$\begin{aligned}
\mathbf{M} &= \mathbf{M}_0 + \epsilon^1\mathbf{M}_1 + \epsilon^2\mathbf{M}_2 + \epsilon^3\mathbf{M}_3 + \cdots && (1.59a)\\
\mathbf{H} &= \mathbf{H}_0 + \epsilon^1\mathbf{H}_1 + \epsilon^2\mathbf{H}_2 + \epsilon^3\mathbf{H}_3 + \cdots && (1.59b)\\
\alpha &= \alpha_0 + \epsilon^1\alpha^1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + \cdots && (1.59c)\\
\beta &= \beta_0 + \epsilon^1\beta^1 + \epsilon^2\beta_2 + \epsilon^3\beta_3 + \cdots && (1.59d)
\end{aligned}$$

The terms $\mathbf{H}_0$ is called the external field despite the fact that it differs from the applied external field. $\mathbf{M}_0$ is called the magnetization density lined up with field $\mathbf{H}_0$. The expansion of equation (1.59) is thus considered for perturbations around these fields $(\mathbf{M}_0, \mathbf{H}_0)$. We combine relation (1.68) with the perturbation expansion (1.59) for the components of $\mathbf{M}$ and $\mathbf{H}$ in the aim of describing the nonlinear asymptotic behavior of the original system. By substituting the perturbed fields (1.59) into the coupled system (1.43), we obtain at the leading order of the perturbation

$$\begin{aligned}
\epsilon^{-2}(H^i_{\zeta\zeta} + M^i_{\zeta\zeta}) + \epsilon^2(H^i_{\tau\tau} + M^i_{\tau\tau}) - 2(H^i_{\zeta\tau} + M^i_{\zeta\tau}) &= 0, && (1.60a)\\
\epsilon^{-2}(H^j_{\zeta\zeta} + M^j_{\zeta\zeta}) + \epsilon^2(H^j_{\tau\tau} + M^j_{\tau\tau}) - 2(H^j_{\zeta\tau} + M^j_{\zeta\tau}) &= \epsilon^{-2}H^j_{\zeta\zeta}, && (1.60b)\\
\epsilon^{-2}(H^k_{\zeta\zeta} + M^k_{\zeta\zeta}) + \epsilon^2(H^k_{\tau\tau} + M^k_{\tau\tau}) - 2(H^k_{\zeta\tau} + M^k_{\zeta\tau}) &= \epsilon^{-2}H^k_{\zeta\zeta}, && (1.60c)
\end{aligned}$$

$$\begin{aligned}
-\epsilon^{-1}M^i_\zeta + \epsilon M^i_\tau &= M^kH^j - M^jH^k + \beta M^kM^j + \epsilon^{-2}\alpha(M^kM^j_{\zeta\zeta} - M^jM^k_{\zeta\zeta})\\
&\quad + \frac{\sigma}{m}[M^j(-\epsilon^{-1}M^k_\zeta + \epsilon M^k_\tau) + M^k(\epsilon^{-1}M^j_\zeta - \epsilon M^j_\tau)], && (1.61a)\\
-\epsilon^{-1}M^j_\zeta + \epsilon M^j_\tau &= M^iH^k - M^kH^i - \beta M^kM^i + \epsilon^{-2}\alpha(M^iM^k_{\zeta\zeta} - M^kM^i_{\zeta\zeta})\\
&\quad + \frac{\sigma}{m}[M^k(-\epsilon^{-1}M^i_\zeta + \epsilon M^i_\tau) + M^i(\epsilon^{-1}M^k_\zeta - \epsilon M^k_\tau)], && (1.61b)\\
-\epsilon^{-1}M^k_\zeta + \epsilon M^k_\tau &= M^jH^i - M^iH^j + \epsilon^{-2}\alpha(M^jM^i_{\zeta\zeta} - M^iM^j_{\zeta\zeta})\\
&\quad + \frac{\sigma}{m}[M^i(-\epsilon^{-1}M^j_\zeta + \epsilon M^j_\tau) + M^j(\epsilon^{-1}M^i_\zeta - \epsilon M^i_\tau)]. && (1.61c)
\end{aligned}$$

Handling the system (1.60) and (1.61) at the order ϵ^{-2}, by substituting the perturbed fields (1.59) into the coupled system, we obtain at the order of perturbation, the following relations, taking into account the boundary conditions on all derivatives of $\mathbf{M}$ and $\mathbf{H}$ and steady state $\mathbf{M}_0$ and $\mathbf{H}_0$

$$\begin{aligned}
H^i_{0\zeta\zeta} + M^i_{0\zeta\zeta} = 0 \Rightarrow H^i_{0\zeta} + M^i_{0\zeta} = C_1,\\
\lim_{\zeta\to-\infty} M^i_{0\zeta} = \lim_{\zeta\to-\infty} H^i_{0\zeta} = 0 \Rightarrow C_1 = 0 H^i_0 + M^i_0 = C_2,\\
as \quad \lim_{\zeta\to-\infty} M^i_0 = \lim_{\zeta\to-\infty} H^i_0 = 0, with M^i_0 = 0 \quad then \quad H^i_0 = 0, \qquad (1.62a)\\
H^j_{0\zeta\zeta} + M^j_{0\zeta\zeta} = H^j_{0\zeta\zeta} \Rightarrow H^j_{0\zeta} + M^j_{0\zeta} = H^j_{0\zeta} + C_3,\\
\lim_{\zeta\to-\infty} M^j_{0\zeta} = \lim_{\zeta\to-\infty} H^j_{0\zeta} = 0 \Rightarrow C_3 = 0, H^j_0 + M^j_0 = H^j_0 + C_4,\\
as \quad \lim_{\zeta\to-\infty} M^j_0 = M_0 and \lim_{\zeta\to-\infty} H^j_0 = H_0 \quad then \quad C_4 = M_0, \qquad (1.62b)\\
H^k_{0\zeta\zeta} + M^k_{0\zeta\zeta} = H^k_{0\zeta\zeta} \Rightarrow H^k_{0\zeta} + M^k_{0\zeta} = H^k_{0\zeta} + C_5,\\
\lim_{\zeta\to-\infty} M^j_{0\zeta} = \lim_{\zeta\to-\infty} H^j_{0\zeta} = 0 \Rightarrow C_5 = 0, H^k_0 + M^k_0 = H^k_0 + C_6,\\
as \quad \lim_{\zeta\to-\infty} M^j_0 = \lim_{\zeta\to-\infty} H^j_0 = 0 \quad then \quad C_6 = 0 \quad and \quad M^k_0 = 0, \qquad (1.62c)\\
\alpha_0 \mathbf{M}_0 \wedge \mathbf{M}_{0\zeta\zeta} = 0 \Rightarrow \alpha_0 = 0, \qquad (1.62d)
\end{aligned}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are constants of integration. In short, equations (1.62) permit to have $\mathbf{H}_0 = H^j_0 \mathbf{e}_y + H^k_0 \mathbf{e}_k$, $\mathbf{M}_0 = M^j_0 \mathbf{e}_y$ and $\alpha_0 = 0$

At the order ϵ^{-1} we have

$$(H^i_1 + M^i_0)_{\zeta\zeta} = 0 \Rightarrow H^i_1 = -M^i_1, \qquad (1.63a)$$

$$(H^j_1 + M^j_0)_{\zeta\zeta} = H^j_{1\zeta\zeta} \Rightarrow M^j_1 = 0, \qquad (1.63b)$$

$$(H^k_1 + M^k_0)_{\zeta\zeta} = H^k_{1\zeta\zeta} \Rightarrow M^k_1 = 0, \qquad (1.63c)$$

$$\begin{aligned}
-M^i_{0\zeta} &= \alpha_0 M^k_0 M^j_{1\zeta\zeta} + \alpha_1 M^k_0 M^j_{0\zeta\zeta} + \alpha_0 M^k_1 M^j_{0\zeta\zeta} - \alpha_0 M^j_0 M^k_{1\zeta\zeta}\\
&\quad -\alpha_1 M^j_0 M^k_{0\zeta\zeta} - \alpha_0 M^j_1 M^k_{0\zeta\zeta} - \frac{\sigma_0}{m} M^j_0 M^k_{0\zeta} + \frac{\sigma_0}{m} M^k_0 M^j_{0\zeta}, \qquad (1.63d)\\
-M^j_{0\zeta} &= \alpha_0 M^i_0 M^k_{1\zeta\zeta} + \alpha_1 M^i_0 M^k_{0\zeta\zeta} + \alpha_0 M^i_1 M^k_{0\zeta\zeta} - \alpha_0 M^k_0 M^i_{1\zeta\zeta}\\
&\quad -\alpha_1 M^k_0 M^i_{0\zeta\zeta} - \alpha_0 M^k_1 M^i_{0\zeta\zeta} - \frac{\sigma_0}{m} M^k_0 M^i_{0\zeta} + \frac{\sigma_0}{m} M^i_0 M^k_{0\zeta}, \qquad (1.63e)\\
-M^k_{0\zeta} &= \alpha_0 M^j_0 M^i_{1\zeta\zeta} + \alpha_1 M^j_0 M^i_{0\zeta\zeta} + \alpha_0 M^j_1 M^i_{0\zeta\zeta} - \alpha_0 M^i_0 M^j_{1\zeta\zeta}\\
&\quad -\alpha_1 M^i_0 M^j_{0\zeta\zeta} - \alpha_0 M^i_1 M^j_{0\zeta\zeta} - \frac{\sigma_0}{m} M^i_0 M^j_{0\zeta} + \frac{\sigma_0}{m} M^j_0 M^i_{0\zeta}, \qquad (1.63f)
\end{aligned}$$

we obtain at the order zero in ϵ from equation (1.63)d, (1.63)e and (1.63)f $M^i_0 = 0$, $M^j_0 = m$ and $M^k_0 = 0$

At the order ϵ^0 we have

$$(H_2^i + M_2^i)_{\zeta\zeta} - 2(H_0^i + M_0^i)\zeta\tau = 0 \Rightarrow H_2^i = -M_2^i, \tag{1.64a}$$

$$(H_2^j + M_2^j)_{\zeta\zeta} - 2(H_0^j + M_0^j)\zeta\tau = H^j_{2\zeta\zeta} \Rightarrow M^j_{2\zeta} = 2H^j_{0\tau}, \tag{1.64b}$$

$$(H_2^k + M_2^k)_{\zeta\zeta} - 2(H_0^k + M_0^k)\zeta\tau = H^k_{2\zeta\zeta} \Rightarrow M^k_{2\zeta} = 2H^k_{0\tau}, \tag{1.64c}$$

$$\begin{aligned} -M^i_{1\zeta} = {} & M_0^k H_0^j - M_0^j H_0^k + \beta_0 M_0^k M_0^j + \alpha_0(M_0^k M^j_{2\zeta\zeta} + M_2^k M^j_{0\zeta\zeta} - M_0^j M^k_{2\zeta\zeta} \\ & -M_1^j M^k_{1\zeta\zeta} + M_2^j M^k_{0\zeta\zeta}) + \alpha_1(M_1^k M^j_{0\zeta\zeta} + M_0^k M^j_{1\zeta\zeta} \\ & -M_1^j M^k_{0\zeta\zeta} - M_0^j M^k_{1\zeta\zeta}) + \alpha_2(M_0^k M^j_{0\zeta\zeta} - M_0^j M^k_{0\zeta\zeta}) \\ & -\frac{\sigma_0}{m}(M_0^j M^k_{1\zeta} + M_1^j M^k_{0\zeta} - M_0^k M^j_{1\zeta} - M_1^k M^j_{0\zeta}) - \frac{\sigma_1}{m}(M_0^j M^k_{0\zeta} - M_0^k M^j_{0\zeta}), \end{aligned} \tag{1.64d}$$

$$\begin{aligned} -M^j_{1\zeta} = {} & M_0^i H_0^k - M_0^k H_0^i + \beta_0 M_0^k M_0^i + \alpha_0(M_0^i M^k_{2\zeta\zeta} + M_2^i M^k_{0\zeta\zeta} - M_0^k M^i_{2\zeta\zeta} \\ & -M_1^k M^i_{1\zeta\zeta} + M_2^k M^i_{0\zeta\zeta}) + \alpha_1(M_1^i M^k_{0\zeta\zeta} + M_0^i M^k_{1\zeta\zeta} \\ & -M_1^k M^i_{0\zeta\zeta} - M_0^k M^i_{1\zeta\zeta}) + \alpha_2(M_0^i M^k_{0\zeta\zeta} - M_0^k M^i_{0\zeta\zeta}) \\ & -\frac{\sigma_0}{m}(M_0^k M^i_{1\zeta} + M_1^k M^i_{0\zeta} - M_0^i M^k_{1\zeta} - M_1^i M^k_{0\zeta}) - \frac{\sigma_1}{m}(M_0^k M^i_{0\zeta} - M_0^i M^k_{0\zeta}), \end{aligned} \tag{1.64e}$$

$$\begin{aligned} -M^k_{1\zeta} = {} & M_0^j H_0^i - M_0^i H_0^j + \alpha_0(M_0^j M_{2\zeta\zeta} + M_1^j M^i_{1\zeta\zeta} - M_2^j M^i_{0\zeta\zeta} \\ & -M_0^i M^j_{2\zeta\zeta} + M_1^i M^j_{1\zeta\zeta}) + \alpha_1(M_1^i M^k_{0\zeta\zeta} + M_0^i M^k_{1\zeta\zeta} \\ & -M_1^k M^i_{0\zeta\zeta} - M_0^k M^i_{1\zeta\zeta}) + \alpha_2(M_0^i M^k_{0\zeta\zeta} - M_0^k M^i_{0\zeta\zeta}) \\ & -\frac{\sigma_0}{m}(M_0^i M^j_{1\zeta} + M_1^i M^j_{0\zeta} - M_0^i M^i_{1\zeta} - M_1^j M^i_{0\zeta}) - \frac{\sigma_1}{m}(M_0^i M^j_{0\zeta} - M_0^j M^i_{0\zeta}), \end{aligned} \tag{1.64f}$$

we obtain at the order zero in ϵ from equation (1.64), (1.64) and (1.64) $M^i_{1\zeta} = M_0^j H_0^k$, $M_1^j = 0$ and $M_1^k = 0$.

At the order ϵ^1 we have

$$(H_3^i + M_3^i)_{\zeta\zeta} - 2(H_1^i + M_1^i)_{\zeta\tau} = 0 \Rightarrow H_3^i = -M_3^i, \quad (1.65a)$$

$$(H_3^j + M_3^j)_{\zeta\zeta} - 2(H_1^j + M_1^j)_{\zeta\tau} = H^j_{3\zeta\zeta} \Rightarrow M^j_{3\zeta} = 2(H_1^j + M_1^j)_{\tau}, \quad (1.65b)$$

$$(H_3^k + M_3^k)_{\zeta\zeta} - 2(H_1^k + M_1^k)_{\zeta\tau} = H^k_{3\zeta\zeta} \Rightarrow M^k_{3\zeta} = 2(H_1^k + M_1^k)_{\tau}, \quad (1.65c)$$

$$\begin{aligned}
-M^i_{2\zeta} + M^i_{0\tau} &= M_0^k H_1^j + M_1^k H_0^j - M_0^j H_1^k - M_1^j H_0^k + \beta_0(M_0^k M_1^j + M_1^k M_0^j) \\
&+\beta_1 M_0^k M_0^j + \alpha_0(M_1^k M^j_{2\zeta\zeta} + M_2^k M^j_{1\zeta\zeta} + M_3^k M^j_{0\zeta\zeta} + M_0^k M^j_{3\zeta\zeta} - M_1^j M^k_{2\zeta\zeta} \\
&M_2^j M^k_{1\zeta\zeta} - M_3^j M^k_{0\zeta\zeta} - M_0^j M^k_{3\zeta\zeta}) + \alpha_1(M_2^k M^j_{0\zeta\zeta} + M_1^k M^j_{1\zeta\zeta} \\
&+M_0^k M^j_{2\zeta\zeta} - M_2^j M^k_{0\zeta\zeta} - M_1^j M^k_{1\zeta\zeta} - M_0^j M^k_{0\zeta\zeta}) + \alpha_2(M_0^k M^j_{1\zeta\zeta} \\
&+M_1^k M^j_{0\zeta\zeta} - M_0^j M^k_{1\zeta\zeta} - M_2^j M^k_{0\zeta\zeta}) + \alpha_3(M_0^k M^j_{0\zeta\zeta} - M_0^j M^k_{0\zeta\zeta} \\
&-\frac{\sigma_0}{m}(M_0^j M^k_{2\zeta} + M_1^j M^k_{1\zeta} + M_2^j M^k_{0\zeta} - M_0^k M^j_{2\zeta} - M_1^k M^j_{1\zeta} - M_2^k M^j_{0\zeta}) \\
&-\frac{\sigma_1}{m}(M_1^j M^k_{0\zeta} + M_0^j M^k_{1\zeta} - M_1^k M^j_{0\zeta} - M_0^k M^j_{1\zeta}) - \frac{\alpha_2}{m}(M_0^j M^k_{0\zeta} - M_0^k M^j_{0\zeta}), \quad (1.65d) \\
-M^j_{2\zeta} + M^j_{0\tau} &= M_0^i H_1^k + M_1^i H_0^k - M_0^k H_1^i - M_1^k H_0^i - \beta_0(M_0^k M_1^i + M_1^k M_0^j) \\
&-\beta_1 M_0^k M_0^i + \alpha_0(M_1^i M^k_{2\zeta\zeta} + M_2^i M^k_{1\zeta\zeta} + M_3^i M^k_{0\zeta\zeta} + M_0^i M^k_{3\zeta\zeta} - M_1^k M^i_{2\zeta\zeta} \\
&-M_2^k M^i_{1\zeta\zeta} - M_3^k M^i_{0\zeta\zeta} - M_0^k M^i_{3\zeta\zeta}) + \alpha_1(M_2^i M^k_{0\zeta\zeta} + M_1^i M^k_{1\zeta\zeta}) \\
&+M_0^i M^k_{2\zeta\zeta} - M_2^k M^i M^i_{0\zeta\zeta} - M_1^k M^i_{1\zeta\zeta} + M_0^k M^i_{2\zeta\zeta}) + \alpha_2(M_0^i M^k_{1\zeta\zeta} \\
&+M_1^i M^k_{0\zeta\zeta} - M_0^k M^i_{1\zeta\zeta} - M_1^k M^i_{0\zeta\zeta}) + \alpha_3(M_0^i M^k_{0\zeta\zeta} - M_0^k M^i_{0\zeta\zeta} - M_2^i M^k_{0\zeta} + M_0^i M^k_{0\tau}) \\
&-\frac{\sigma_0}{m}(M_0^k M^i_{2\zeta} + M_1^k M^i_{1\zeta} + M_2^k M^i_{0\zeta} - M_0^k M^i_{0\tau} - M_0^i M^k_{2\zeta} - M_1^i M^k_{1\zeta} \\
&-\frac{\sigma_1}{m}(M_1^k M^i_{0\zeta} + M_0^k M^i_{1\zeta} + M_1^i M^k_{0\zeta} - M_0^i M^k_{0\zeta}) - \frac{\sigma_2}{m}(M_0^k M^i_{0\zeta} - M_0^i M^k_{0\zeta}), \quad (1.65e) \\
-M^k_{2\zeta} + M^k_{0\tau} &= M_0^j H_1^i + M_1^j H_0^i - M_0^i H_1^j - M_1^i H_0^j + \alpha_0(M_1^j M^i_{2\zeta\zeta} \\
&+M_2^j M^i_{1\zeta\zeta} + M_3^j M^i_{0\zeta\zeta} + M_0^j M^i_{3\zeta\zeta} - M_1^i M^j_{2\zeta\zeta} - M_2^i M^j_{1\zeta\zeta} - M_3^i M^j_{0\zeta\zeta} - M_0^i M^j_{3\zeta\zeta}) \\
&+\alpha_1(M_2^j M^i_{0\zeta\zeta} + M_0^j M^i_{2\zeta\zeta} - M_2^i M^j_{0\zeta\zeta} - M_0^i M^j_{2\zeta\zeta}) + \alpha_2(M_0^j M^i_{1\zeta\zeta} \\
&+M_1^j M^i_{0\zeta\zeta} - M_0^i M^j_{1\zeta\zeta} - M_1^i M^j_{0\zeta\zeta}) + \alpha_3(M_0^j M^i_{0\zeta\zeta} - M_0^i M^j_{0\zeta\zeta}) \\
&-\frac{\sigma_0}{m}(M_0^i M^j_{2\zeta} + M_1^i M^j_{1\zeta} + M_2^i M^j_{0\zeta} - M_0^j M^k_{0\tau} - M_0^j M^i_{2\zeta} - M_1^j M^i_{1\zeta} \\
&-M_2^j M^i_{0\zeta} + M_0^k M^i_{0\tau}) - \frac{\sigma_1}{m}(M_1^i M^j_{0\zeta} + M_0^i M^j_{1\zeta} - M_1^j M^i_{0\zeta} \\
&-M_0^j M^i_{1\zeta}) - \frac{\sigma_2}{m}(M_0^i M^j_{0\zeta} - M_0^j M^i_{0\zeta}). \quad (1.65f)
\end{aligned}$$

However, it must be noted that from the linear relations in (1.14) we get $M_0^i = 0$ and then $H_0^i = 0$. By collecting terms at the next order ϵ^1, we obtain

$$M^i_{2\zeta} = mH^k_1, \tag{1.66a}$$
$$M^j_{2\zeta} = -M^i_1 H^k_0, \tag{1.66b}$$
$$M^k_{2\zeta} = M^i_1 H^j_0 - m(H^i_1 + \alpha_2 M^i_{1\zeta\zeta}) + \sigma_1 M^i_{1\zeta}. \tag{1.66c}$$

Equation (1.64) enable us to have $M^i_{1\zeta} = mH^k_0$ from which we have $H^k_0 = \frac{1}{m}M^i_{1\zeta}$. In the same time equation (1.66) enable us to have $M^j_{2\zeta} = -M^i_1 H^k_0$ what leads to $M^j_{2\zeta} = -\frac{1}{m}M^i_{1\zeta}M^i_1$ which become, considering equation (1.64b) the following relations

$$-\frac{(M^i_1 M^i_{1\zeta})_\zeta}{m} = 2H^j_{0\zeta\tau}, \tag{1.67a}$$
$$\frac{2M^i_{1\zeta\tau}}{m} = (H^j_0 + m)M^i_1 - \alpha_2 m M^i_{1\zeta\zeta} + \sigma_1 M^i_{1\zeta}. \tag{1.67b}$$

This system is a nonlinear one describing the evolution of the first nontrivial terms in expansion of $\mathbf{M}$ and $\mathbf{H}$, no matter the second term of expansion of the magnetization $\mathbf{M}$ has been used. For deeply investigating equation (1.67), let us consider the new independent variables X and T such as

$$\frac{\partial}{\partial \zeta} = -\frac{m}{2}\frac{\partial}{\partial X}, \tag{1.68a}$$
$$\frac{\partial}{\partial \tau} = \frac{m}{2}\frac{\partial}{\partial T}, \tag{1.68b}$$

then we have

$$\begin{cases} \frac{-(M^i_1 M^i_{1X})_X}{4m^2} = \frac{H^j_{0XT}}{m}, \\ \frac{M^i_{1XT}}{2m} = -(1 + \frac{H^j_0}{m})\frac{M^i_1}{2m} + \frac{\alpha_2 m^2}{4}\frac{M^i_{1XX}}{2m} - \frac{\sigma_1}{2}\frac{M^i_{1X}}{2m}. \end{cases} \tag{1.69}$$

If we consider $B = \frac{M^i_1}{2m}$, $A = -(1 + \frac{H^j_0}{m}) = C_X$, $\varrho = \frac{\alpha_2 m^2}{4}$, $s = -\frac{\sigma_1}{2}$, equation (1.69) becomes

$$\begin{cases} B_{XT} = BC_X + \varrho B_{XX} - sB_X, \\ C_{XT} = -BB_X, \end{cases} \tag{1.70}$$

where $B = B(X,T)$ and $C = C(X,T)$ stand for the magnetization and the external magnetic fields , respectively. X and T represent the space-like and time-like variable, respectively, while the parameters s and ϱ are denoting the damping and inhomogeneous effects respectively. Kraenkel et al. investigated the propagation of short-wave in saturated ferromagnetic materials from Maxwell equations with the Landau-Lifshitz equation

(1.14), and derived the nonlinear system (1.42) [25]. The nonlinear system (1.70) was derided very recently, constructed by Nguepjouo et al.. The nonlinear wave dynamics described by this model have attracted special attention because it possesses interesting exact solutions of soliton-type.

Conclusion

We have presented in this chapter the generality on the magnetic materials and on the ferrites in particular. We have classified the materials according to their magnetic state in presence of applied magnetic field and their intrinsic characteristic such a susceptibility. We dwell too long on ferrites, by giving its characteristics as well as its technological, industrial and economic importance. We have also presented the non-linear integrable differential equation of KMM without damping and inhomogeneous exchange describing the propagation of wave in ferrites which already was the subject of many work and as well the KMM-system including the damping and inhomogeneous exchange effects which is the model whose we study in the remainder of our work. Our main goal being to show the damping and inhomogeneous exchange effects on the wave propagating in ferrites, it is important to propose some methods allowing to achieve this goal which will be covered in the next chapter.

Chapter 2

Methodology of Investigations

Introduction

Propagation of wave in nonlinear media appears to be one of the most famous topic that has attracted attention of number of researchers [61–65]. Such a topic has been applied in a wide variety of media including water [66–69], optical fibers [16, 18, 70–72], magnetic insulators [25, 26, 29, 30], just to name these ones. Investigations of propagation of wave in the previously mentioned media often concern experimental observations and theoretical descriptions. Concerning theoretical descriptions, it appears as an important part of investigations that often come after an experimental observation and, when theoretical explanation is well carried out to generate nonlinear evolution system to model the phenomenon observed, it is possible to predict theoretically the behavior of the wave in the medium of interest under some new hypothesis before returning in laboratories to verify if the theoretical hypothesis are confirmed experimentally. Such an inter-dependence between experiment and theoretical investigation make it clear the importance of both in looking for more innovative technological devices. As far as we are concerned throughout this thesis with the propagation of short wave in ferrites, the nonlinear system, derived in Ref. [25] has been investigated intensively under some tractable methods including Hirota's bilinearization [30, 73], inverse scattering transform method [18, 29, 65] and some expansion functions [74–77]. Beside this system, known in the literature as the KMM system [27], a new nonlinear system has been derived in Ref. [30] that takes into account damping effects. Influence of this damping effect has been investigated under the initial value problem and, as a result, it appeared that for non-zero damping, energy of the wave decreases. Going further in investigating propagation of short waves in real ferrites, the previously mentioned system that takes into account damping, has been subjected to amelioration while taking into account not only the damping effects, but also the inhomogeneous exchange effects in Ref. [33]. At fixed value of the inhomogeneous term, initial value problem of the new system has been investigated and, as a result, it appeared that energy decreases as wave evolves. In Ref. [34] soliton solutions of the inhomogeneous system have been constructed along with associated energy densities. By fixing the value of the parameter characterizing inhomogeneous exchange effects, it influence on the parameters of the propagating wave in ferrites subjected to this phenomenon is not clear. Such a question then constitutes the main aim of our investigations. In this chapter, we present some important method used within the investigation of damping and inhomogeneous exchange effects on the wave propagation in ferrites. Therefore we present the Wahlquist and Estabrook formalism while investigating the prolongation structure of $(1 + 1)$-dimensional evolution equation which must lead to the Lax Pair. We illustrate this method by applying it to the Schäfer Wayne system. We pursue by exposing the inverse scattering transform method and also Jacobi elliptic function expansion method. In order to complete our analysis, we perform a numerical simulation using finite difference scheme.

2.1 Prolongation structure analysis

The determination of symmetries of nonlinear evolution systems is actually an important and challenging problem to tackle owing to their miscellaneous physical implications [78]. In fact, there are many ways and approaches for finding symmetries of differential equations. For instance, Harrison and Estabrook [78] found a method using differential forms and Cartan formulation of differential systems while investigating the Maxwell's equations. It is however true that a fundamental understanding of the interrelationships between the various sorts of symmetries associated to evolution systems is still in the process of being developed. Among the very important investigations of the subject using many tractable methods to search for solutions to nonlinear evolution equations, and their properties, Estabrook and Wahlquist [1] carried out and important tool in the determination of symmetries using algebras known as "prolongation structure".

Nowadays, many researchers have developed the prolongation structure methodologies of Wahlquist and Estabrook [1]. As examples, Hermann [49] interpreted prolongation structure as a connection, Dodd and Fordy [80] studied prolongation structure of quasi-polynomial and complex quasi-polynomial evolution equations and provided a method for obtaining the matrix representations of the Lie algebra, Morris [81, 82] applied prolongation structure to nonlinear evolution equations in $(2 + 1)$ and $(3 + 1)$-dimensions for the first time, Guo, Ke Wu, Hsiang and Wang [83, 84] proposed a covariant geometry theory for prolongation structure method, Ji-Peng et al. [85] proposed a fermionic covariant prolongation structure theory for supernonlinear evolution equation, Deconinck [86] applied prolongation structure method to semi-discrete systems for the first time. They applied this method to series of nonlinear differential equations [87,88] and obtained their Lax-pairs, which undoubtedly plays an important role in the construction of Bäcklund transformation, soliton solutions related to the nonlinear differential equations. In this work, we assume that a system of equations is said to be integrable if it admits Lax-pair.

In this section, we describe the Wahlquist-Estabrook approach, originally formulated in the framework of differential forms and Cartan's exterior differential system which attempts to construct a linear spectral problem associated with the nonlinear evolution equation under consideration. We investigate this method firstly by giving some important definitions.

Let $\mathfrak{M}$ be a manifold. A collection of differential forms on $\mathfrak{M}$ is said to be an ideal, denoted by $\mathbf{I}$, if the following conditions are satisfied:

For α_1 and $\alpha_2 \in \mathbf{I}$, $\phi_1\alpha_1 + \phi_2\alpha_2 \in \mathbf{I}$, where ϕ_1 and ϕ_2 are functions on $\mathfrak{M}$.

For $\alpha \in \mathbf{I}$ and η any arbitrary form, $\alpha \wedge \eta \in \mathbf{I}$, where $\wedge$ denotes the exterior product (antisymmetric tensor product).

For any $\alpha \in \mathbf{I}$, $\mathrm{d}\alpha \in \mathbf{I}$. This is the closure condition and is necessary for closed ideals. In the literature, closed ideals are called differential ideals. An exterior differential system is a pair $(\mathfrak{M}, \mathbf{I})$, where $\mathfrak{M}$ is a manifold and $\mathbf{I}$ is a differential ideal. Usually it is a differential P-forms Ω, then the pair $(\mathbf{I}, \Omega)$ is called an exterior differential system with an independence condition. Here P denotes number of independent variables.

An integral manifold of the system $\mathbf{I}$ is a pair $(\mathcal{M}, f)$, where $\mathcal{M}$ is a sub manifold of $\mathfrak{M}$ and $f\colon \mathcal{M} \longrightarrow \mathfrak{M}$ is a differential mapping such that $f^*(\alpha) = 0$, for all $\alpha \in \mathbf{I}$, for which f^* is the pull-back map. The terms integral sub manifold or solution manifold are also use for integral manifolds. An integral manifold of $\mathbf{I}$ with the addition condition $f^*(\Omega) \neq 0$, satisfied. Note that, while for an exterior differential system, we do not specify any independent variables, we choose independent variables for exterior differential systems with an independence condition.

In order to represent a partial differential equation as a system of forms, firstly set new variable, which can be used to write the differential equation as a set of first order differential equation. Next, $\mathfrak{M}$ is described to be a manifold, the local coordinates of which consist of all independent and dependent variables in the partial differential equation and auxiliary variables introduced in the previous step. By taking the exterior derivative of these coordinates, the basis forms can be adopted in order to define the forms α_i on $\mathfrak{M}$. These forms generate a differential ideal. Then it is required that any integral manifold $\mathcal{M}$, which consists of independent variables, annuls this set of forms and restriction of initial set of forms α_i on $\mathfrak{M}$ to $\mathcal{M}$ gives us the partial differential equation from which we started, that is, the solution manifold gives our partial differential equation in the form of exterior differential equations. It is seen that, if $f^*(\alpha_i) = 0$, then for any η_i on $\mathfrak{M}$, we have

$$f^*(\sum_i \eta_i \wedge \alpha_i) = 0 \quad and \quad f^*(\mathrm{d}\alpha_i) = 0. \tag{2.1}$$

Hence, any form in $\mathbf{I}$ generate by α_i vanishes when restricted to the solution manifold $\mathcal{M}$. Therefore, it is the whole ideal $\mathbf{I}$ generated by α_i that represents the partial differential equation [89].

It is possible to construct different exterior systems which generate the same ideal. Two exterior system, α_i and $\alpha_i^{'}$, can be considered as algebraically equivalent if they generate the same ideal [89]. Obviously, algebraically equivalent systems have the same solutions since they represent the same partial differential equation.

To assert complete equivalence between the forms and the differential equation, we require that the set of forms must be closed. This requirement is necessary since on a solution manifold, if we have $f^*(\alpha_i) = 0$ which gives back the differential equation, we must also have $\mathrm{d}(f^*(\alpha_i)) = f^*(\mathrm{d}\alpha_i) = 0$ (since the pull-back map and the exterior derivative commute) and the equation $f^*(\mathrm{d}\alpha_i) = 0$ cannot impose any additional conditions. Hence, we need to have $\mathrm{d}\alpha_i \in \mathbf{I}$ which implies that the exterior derivatives for all the forms must be contained in the ring of forms generated by the set

$$d\alpha_i = \sum_{j=1} \eta_{ij} \wedge \alpha_j, \tag{2.2}$$

where the summation runs from 1 to the number of forms in the initial set and η_{ij} is some set of 1-forms. This condition also says that, all the integrability conditions of the first-order partial differential equations, which we found at the beginning by defining new variables, are satisfied.

Another idea in Wahlquist-Estabrook prolongation structure method is to search for the existence of 1-forms. The motivation comes from the existence of conservation laws, which describe quantities that remain invariant during the evolution of partial differential equation. It is well known that, integrable nonlinear evolution equations possess an infinite set of conservation laws [90, 91]. Thus it is natural to expect the existence various 1-forms which lead to conservation laws. Any differential equation in the form precisely those equations of evolution type which contain two independent variables x and t can be written as

$$U^\gamma_t = U^\gamma_{nx} - K^\gamma(U^\beta, U^\beta_x, \ldots, U^\beta_{(n-1)x}), \tag{2.3}$$

where K is an arbitrary function depending on U^β and x-derivatives of U^β. Clearly, the KdV equation is in this form. Equation (2.3) has local conservation laws of the form:

$$\frac{\partial f^k}{\partial t} + \frac{\partial g^k}{\partial x} = 0, \quad k = 1, 2, \ldots, \tag{2.4}$$

where f^k and g^k are local expressions depending on the jet variables. In fact, they depend on the independent variables, dependent variables and the x-derivatives of the dependent variables since the t-derivatives of the dependent variables can be eliminated by using equation (2.3).

One of the important properties of these conservation laws is that, they lead to conserved quantities with appropriate boundary conditions. For example integrating equation (2.4) with respect to x we get

$$\frac{d}{dt}\int_{X_A}^{X_B} f^k dx + [g^k]^{X_B}_{X_A} = 0. \tag{2.5}$$

Under asymptotic boundary conditions in which the dependent variables are rapidly vanishing as $x \longrightarrow \pm\infty$, the term $[g^k]_{X_A}^{X_B}$ vanishes. Hence, equation (2.4) gives rise to the constant of motion $\int_{X_A}^{X_B} f^k dx$. In fact, these conserved quantities can only be defined up to exact x-derivatives since equation (refEq2.4) is invariant under the following transformation

$$f^k \longrightarrow f^k + \Lambda_x^k, \quad g^k \longrightarrow g^k + \Lambda_t^k. \tag{2.6}$$

Another property of conservation laws is their close relation with potential function. A function ϕ of dependent and independent variables is called a potential function if it is a constant when the differential equation is satisfied. Thus cross derivative integrability condition implies that every conservation law of the form gives by equation (2.4) defines a potential function ϕ^k, where

$$\frac{\partial \phi^k}{\partial t} = g^k, \quad \frac{\partial \phi^k}{\partial x} = -f^k, \tag{2.7}$$

such as

$$d\phi^k = -f^k dx + g^k dt, \tag{2.8}$$

is an exact differential equation. In the language of differential forms, the above discussion about conservation laws corresponds to the existence of 2-forms contained in the ring of the α_i that generate the closed ideals. Therefore, we can find 2-forms

$$\beta^k = \sum_{i=1} h_i^k \alpha_i, \tag{2.9}$$

satisfying $d\beta^k = 0$, the condition for exactness. In equation (2.9), h_i^k are arbitrary functions of jet variables and the summation runs from 1 to the number of forms in the initial set that generates the ideal. In fact, $d\beta^k = 0$ can be considered as the integrability condition for the existence of 1-forms, say w^k, such that

$$\beta^k = dw^k. \tag{2.10}$$

Since the double exterior derivative of any differential form vanishes, equation (2.10) conversely implies that $d\beta^k = 0$. For each exact 2- form dw^k, the associated conservation law is obtained by using Stoke's theorem

$$\oint_{\partial M_1} w^k = \int_{M_1} dw^k, \tag{2.11}$$

where $\mathcal{M}_1$ is any two-dimensional manifold and $\partial\mathcal{M}_1$ is its closed one-dimensional boundary. If $\mathcal{M}_1$ is chosen to be the solution manifold, on which the restricted exact 2-form dw^k are annulled, then, equation (2.11) gives the conservation laws for w^k. In general the 1-forms w^k can be taken in the form of equation (2.8)

$$w^k = F^k dx + G^k dt. \tag{2.12}$$

Consider the Korteweg-de Vries equation given by:

$$u_t + u_{xxx} + 12uu_x = 0. \tag{2.13}$$

It has an infinite set of local conservation laws [1] all of which are in the form of equation (2.4). Actually, the KdV equation itself can be written as one of those conservation laws,

$$u_t + (p + 6u^2)_x = 0, \tag{2.14}$$

from which we can deduce its simple potential,

$$\phi_x = -u, \quad \phi_t = 6u^2 + p. \tag{2.15}$$

Now, in accordance with equation (2.14), take the 1-form w as

$$w \;=\; udx - (p + 6u^2)dt. \tag{2.16}$$

The exterior derivative of w,

$$dw \;=\; \beta = -\alpha_3 - 12\alpha_1, \tag{2.17}$$

shows that the 2-form β is in the ring of original set of form α_i and the exterior derivative of β,

$$d\beta \;=\; -d\alpha_3 - 12du \wedge \alpha_1 - 12ud\alpha_1, \tag{2.18}$$

vanishes identically.

As we are motivated by the conservation laws, the core idea in the method is the prolongation of the ideal **I** generated by α_i. The prolongation process can be summarized as follows.

Consider a differential ideal I on a manifold $\mathfrak{M}$ and a differential ideal $\mathbf{I}^{'}$ on a manifold $\mathfrak{M}^{'}$ with a projection map,

$$\pi : \mathfrak{M}^{'} \longrightarrow \mathfrak{M}, \tag{2.19}$$

such that $\mathbf{I}^{'}$ is constructed by lifting the generator α_i of $\mathbf{I}$ to $\mathfrak{M}^{'}$ and by adding new generator w^k. In fact, this is equivalent to saying that $\mathbf{I}^{'}$ is generated by the set $(\pi^*\alpha_i, w^k)$. Since $\mathbf{I}$ is constructed from a differential equation with two independent variables, the set of forms w^k in this case is a set of 1-form in the form of equation (2.12). In order to have all the integrability conditions satisfying $\mathbf{I}^{'}$, $(\pi^*\alpha_i, w^k)$ is also required to be closed [1]. However, since $\mathbf{I}^{'}$ is constructed from $\mathbf{I}$ which is known to be closed, it is enough to require that dw^k be in $\mathbf{I}^{'}$. Furthermore, we have the freedom to add any exact set of 1-forms to w^k, say dy^k, where y^k are arbitrary scalar function since they do not change the closure relation. Thus, the manifold $\mathfrak{M}$ is enlarged to a fibre bundle $\mathfrak{M}^{'} = \mathfrak{M} \times Y$ where $Y \subset \mathbb{R}^m$, by adding the coordinates y^k, $k = 1, \ldots, m$. Hence, prolongation structure of an ideal $\mathbf{I}$ can be done in two different ways [93]. In the first way requiring the equation:

$$w^k \;\; = \;\; dy^k + F^k dx + G^k dt. \tag{2.20}$$

To be in the ring of the initial set of forms, that is using

$$dw^k \;\; = \;\; \sum_{i=1}^{n} h_i^k \alpha_i, \tag{2.21}$$

where h_i^k are arbitrary functions and n is the number of forms α_i, one may search w and a prolongation variable y. If there exists one, it is possible to add that w to α_i and prolong the ideal I. Then one may think that F^k and G^k depend on y as well as the other variables upon which they depend before, and search for new y. However, this time the closure relation becomes

$$dw^k \;\; = \;\; \sum_{i=1}^{n} h_i^k \alpha_i + \eta^k \wedge w, \tag{2.22}$$

where η^k is some 1-form. This is the closure relation for the first prolonged ideal. If one is successful in finding new w, it may be possible to continue to prolong the ideal further. On the other hand, in the second way, it is possible to think from the beginning that F^k and G^k in equation (2.20) depend on all the prolongation variables y^k as well as the other variables upon which they depend before. In this case, to search for new w, one has to modify the closure relation equation (2.21) to be

$$dw^k - \sum_{i=1}^{n} h_i^k \alpha_i - \sum_{i=1}^{m} \eta_i^k \wedge w_i \;=\; 0, \tag{2.23}$$

where m is the number of prolongation variables and η_i^k is some set of 1-forms. Then, this equation can be used, just as equation (2.21), to find some set of partial differential equations for F^k and G^k, but this time there exist some nonlinear terms in these equations such as

$$\sum_i (G^i \frac{\partial F^k}{\partial y^i} - F^i \frac{\partial G^k}{\partial y^i}) dx \wedge dt. \tag{2.24}$$

We note an important consequence of the existence of two different ways for prolonging the ideal. In equation (2.15) and Eq.(2.16) what we call potential is indeed the prolongation variable y. Actually, the prolongation variable y^k appearing in the first way of prolonging the ideal are similar to the potential defined in equation (2.15) since the function F^k and G^k do not depend on the newest prolongation variable y^k. Thus it is also natural to refer to these variables as potentials. In fact, the term potential denotes an integral variable which can be defined by quadrature. However, this is not always the case. We see that in the second way of prolonging the ideal we may have 1-forms w^k in which F^k and G^k depend on all of the prolongation variables. In this case, we call those prolongation variables pseudo potentials. They denote integral variables which cannot be defined by quadrature, but those that can only be defined by an integrable set of first-order differential equations. The nonlinear terms in Eq.(2.24) are clearly essential for these variables and pseudo potentials cannot be found by using the first way in which we have only the linear equations.

Till now we have discussed the basic ideas in order to construct the Wahlquist-Estabrook prolongation structure method. Indeed most of the discussion above constitute some of the basic steps in the method. Now, we give a brief description of the method in the form of recipe.

Suppose we are given a nonlinear differential equation. The first step in the Wahlquist-Estabrook prolongation method may be thought of as the representation of this differential equation as a differential ideal. For this purpose, the original differential equation is written as a system of first-order equations by defining new variables. Then, this set of first order equations is represented as differential forms. Even in this first step of the representation of the differential equation as differential forms for an arbitrary equation, there may be some problems, but since we are dealing with only equations in the forms of Eq.(2.3), no such problem occur.

The second step is the extension of this ideal adding a system of 1-forms w^k, which depend upon the jet variables used in the construction of the differential forms and some new variables y^k, termed prolongation variables. The prolongation of the ideal is done by using the second way discussed above. The requirement that the prolonged ideal must be close under exterior differential gives some set of differential equation for F^k and G^k.We have some nonlinear terms in these equations. Fortunately, all of these nonlinear terms have always some commutator like structure in Eq.(2.24) and are like always solvable. The major part of the job in this step is to integrate these equations to find the dependence of F^k and G^k on the jet variables and y^k. Since the set of equations for F^k and G^k are over determined, some constraints on the constants of integration naturally arise. Due to the fact that the nonlinear terms are always in a commutator like structure, these constraints are in the form of commutation relations between a set of vectors, X_i, depending only on the prolongation variables y^k. It can be said that the integration constants, X_i, generate a free Lie algebra with constraints. Actually, for the equations in the forms of Eq.(2.3), the constants of integration of F^k generate the free Lie algebra and the constants of integration of G^k are elements of this free Lie algebra [92, 94]. Since the constraints are not strong enough for obtaining all the commutation relations, in general this Lie algebra is incomplete and possibly infinite-dimension in the fact implies the existence of infinite number of possible prolongation variables and associated conservation laws. Thus the next step in the Wahlquist-Estabrook prolongation method is to close this algebra that is to find the finite-dimensional algebra consistent with the commutation relation. Unfortunately, there is no general rule to attain this process. However, a strategy can be formed by help of the general theory of Lie algebras.

The last step can be considered as finding a representation for the resulting finite-dimensional Lie algebra. In order to obtain a linear scattering problem a linear representation must be found.

On a solution manifold $\mathcal{M}$, all of the generators of the prolonged ideal $\mathbf{I}'$ are annihilated. Thus the new w^k in Eq.(2.20), which are also generators of $\mathbf{I}'$, are annihilated to give

$$dy^k = -F^k dx - G^k dt. \tag{2.25}$$

Hence

$$y^k_x = -F^k \quad and \quad y^k_t = -G^k. \tag{2.26}$$

Now, without loss of generality, F^k and G^k can be assumed to be linear in y^k. Thus, when we write $y = (y^1, y^2, \ldots)^T$ and F and G for the matrices F^k_j and G^k_j where

$$F^k = F^k_j y^j \quad and \quad G^k = G^k_j y^j, \tag{2.27}$$

the linear scattering problem is achieved

$$y_x = -Fy, \quad y_t = -Gy. \tag{2.28}$$

To make these points clear, we give some example.

2.2 Prolongation structure of Schäfer-Wayne equation

Very recently, Schäfer and Wayne [5] derived an equation, an alternative model equation to the nonlinear Schrödinger equation which have the advantage to better approximate the solution of Maxwell's equation than the nonlinear Schrödinger equation as shown by Chung et al. [95]. This equation is often named "Schäfer-Wayne short-pulse equation". In the literature, many names are attributed to this equation such as "few cycle equation", "video-pulse equation", "extremely short-pulse equation", just to name a few [96]. It is important to remind that this equation appeared earlier as a Rabelo's system [97] possessing a zero curvature representation with a parameter. Schäfer and Wayne [5] studied the propagation of linearly polarized light in one-dimensional medium satisfying the Maxwell's equation. During their investigation, they state that since the medium response to an electric field is not instantaneous, the polarization is a non-local function of the electric field, and should be splitted into linear and nonlinear parts respectively. By considering the propagation of light in infrared range, they found for the

Fourier transform of the linear susceptibility a good approximation. Then, they proposed the Fourier transform of the linear part of the Maxwell's equation. For the nonlinear part of the Maxwell equation, they made an assumption that the contribution of instantaneous nonlinear term is dominant and they set therefore that the nonlinear susceptibility is a constant value. Combining the nonlinear counterpart of the polarization with the previously mentioned linear part of the Maxwell's equation, they obtained another equation by means of multiple scale method. And, introducing some suitable variable changes, they found

$$P_{xt} \quad = \quad P + (P^3)_{xx}/6, \tag{2.29}$$

where subscripts denote partial derivatives. It is on this equation that we will pay particular attention in this section. Other form of equation (2.29) is given as follows which describe the propagation of waves in anharmonic system

$$p_{\xi\tau} - \frac{3}{2}\beta(p^2 p_\xi)\xi - \gamma p \quad = \quad 0. \tag{2.30}$$

in differential forms, we define the variable $u = p_\xi$ where upon equation (2.30) can be written as the first order equation

$$u_\tau - (3\beta p^2 u_\xi)/2 - (3\beta u^2 + \gamma)p \quad = \quad 0, \tag{2.31}$$

where subscripts denote partial derivatives. In the four-dimensional space of all these dependent and independent variables $\{\xi, \tau, p, u\}$, we adopt the basic forms $\{\mathrm{d}\xi, \mathrm{d}\tau, \mathrm{d}p, \mathrm{d}u\}$. Following the previous consideration, the relating second-rank differential forms associated to equation (2.30) are expressed by

$$\alpha_1 \quad = \quad \mathrm{d}p \wedge \mathrm{d}\tau - u\mathrm{d}\xi \wedge \mathrm{d}\tau, \tag{2.32a}$$

$$\alpha_2 \quad = \quad \mathrm{d}u \wedge \mathrm{d}\xi + (3\beta/2)p^2\mathrm{d}u \wedge \mathrm{d}t + (3\beta u^2 + \gamma)p\mathrm{d}\xi \wedge \mathrm{d}\tau, \tag{2.32b}$$

where letter "d" denotes the exterior derivatives and "$\wedge$" denotes the exterior product [22, 98]. Any regular two-dimensional solution in the space $\mathrm{S}_2 = \{\xi, \tau, p, u\}$, satisfying equation (2.29) will annul this set of forms, that is sectioning the forms into the solution manifold [99–101]. Also it is easy to show that

$$\mathrm{d}\alpha_1 \quad = \quad \mathrm{d}\tau \wedge \alpha_2, \tag{2.33a}$$

$$\mathrm{d}\alpha_2 \quad = \quad -[3\beta p\mathrm{d}u + (3\beta u^2 + \gamma)\mathrm{d}\xi] \wedge \alpha_1 + 3\beta up\mathrm{d}\tau \wedge \alpha_2, \tag{2.33b}$$

which implies that the exterior derivatives of the above forms are contained in the ring of forms generated by the set $\{\alpha_1, \alpha_2\}$. Thus, equation (2.32) constitutes a closed ideal of differential forms and Cartan's theory [99–101] of such system can be applied.

Let $\mathbf{I}$ be the differential ideal generated by α_1 and α_2 in such a way that any two dimensional integral manifold of $\mathbf{I}$ on which $\mathrm{d}\xi \wedge \mathrm{d}\tau$ is nonzero determine a solution to equation (1.30). By introducing a one-dimensional space denoted Y and parameterized by a single variable ξ, we define a vector bundle $\mathrm{E} = S_2 \times Y$ over S_2, where Y is a fiber on which a symmetry group acts [102]. A connection [102] for this fiber bundle is then determined by a one-differential form Ω on E expressed as follows

$$\Omega = \mathrm{d}\zeta + \mathrm{F}(\zeta, \tau, p, u)\mathrm{d}\xi + \mathrm{G}(\zeta, \tau, p, u)\mathrm{d}\tau, \tag{2.34}$$

where the quantities F and G stand for arbitrary functions defined on F.

According to the Wahlquist-Estabrook method [1], the connection Ω being a pseudo-conservation law for the system (2.29), it is required that $\mathrm{d}\Omega$ lies in the exterior ideal generated by $\mathcal{I}$ and Ω. Therefore, F and G should satisfy

$$\mathrm{F}_P = 0, \tag{2.35a}$$

$$[\mathrm{F}, \mathrm{G}] = \mathrm{uG_p} - \mathrm{F_u}(3\beta \mathrm{u}^2 + \gamma)p. \tag{2.35b}$$

Solving the system (2.35), yields

$$F = X_1 + uX_2, \tag{2.36a}$$

$$G = \frac{3}{2}\beta p^2 u X_2 + \frac{1}{2} X_4 p^2 + X_3 p + X_5, \tag{2.36b}$$

where the functions X_i $(i = 1, ..., 5)$ depending on the single variable ξ satisfy

$$[X_1, X_4] = 0, \quad [X_2, X_3] = X_4, \tag{2.37a}$$

$$[X_1, X_5] = 0, \quad [X_2, X_4] = 3\beta \left[X_2, X_1\right], \tag{2.37b}$$

$$[X_1, X_2] = -\frac{1}{3\beta}\left[X_2, X_4\right], \quad [X_2, X_5] = X_3, \tag{2.37c}$$

$$[X_1, X_3] = -\gamma X_2. \tag{2.37d}$$

As it is observed, the above algebra is not completed. Nonetheless, by trying arbitrarily the choice $X_4 = X_1$ and $X_5 = \nu X_1$, a simple completeness of the previous Lie-algebra is given by

$$[X_1, X_2] = -X_3/\nu, \tag{2.38a}$$

$$[X_1, X_3] = -\gamma X_2, \tag{2.38b}$$

$$[X_2, X_3] = 3\beta X_1, \tag{2.38c}$$

ν being a nonzero arbitrary parameter. The system (2.38) constitutes a well-defined Lie-algebra. Now, let us discuss the different solutions to equation (2.38) from the viewpoint of some structural symmetries.

2.3 Hidden structural symmetries

2.3.1 The special linear symmetry $\mathrm{SL}(2, \mathbb{R})$

We know that the general representation of $\mathrm{SL}(2, \mathbb{R})$-symmetry obeys the following Lie-bracket

$$[Y_0, Y_1] = Y_1, \tag{2.39a}$$

$$[Y_0, Y_{-1}] = -Y_{-1}, \tag{2.39b}$$

$$[Y_1, Y_1] = -2Y_0, \tag{2.39c}$$

which exhibit a one-dimensional representation

$$Y_0 = Y\frac{\partial}{\partial Y}, \tag{2.40a}$$

$$Y_1 = Y^2\frac{\partial}{\partial Y}, \tag{2.40b}$$

$$Y_{-1} = -\frac{\partial}{\partial Y}. \tag{2.40c}$$

Using the complete Lie-algebra system (2.38), we can define the $\mathrm{SL}(2, \mathbb{R})$-symmetry which has the following representation

$$X_1 = d\zeta b, \tag{2.41a}$$

$$X_2 = e(\zeta^2 + 3\beta d^2/4\gamma e^2)b, \tag{2.41b}$$

$$X_3 = -e\gamma(\zeta^2 + 3\beta d^2/4\gamma e^2)b/d, \tag{2.41c}$$

with the vector field $b = \partial_\xi$. We note that d, e, γ and β are constants, the relations (2.35) become

$$F = 3\beta d^2 u/4\gamma e + ue\zeta^2, \tag{2.42a}$$

$$G = 3\beta d(3\beta d/2\gamma - 1)/4e + (3\beta p^2 u/2 - \gamma p/d)e\zeta^2 + (pd/2 + \nu d). \tag{2.42b}$$

The one form Ω is then written as follows

$$\begin{aligned}\Omega &= \mathrm{d}\zeta + (3\beta d^2 u/4\gamma e + ue\zeta^2)\mathrm{d}\xi \\ &\quad + [3\beta d^2(3\beta d/2\gamma - 1)/4e + (3\beta p^2 u/2 - \gamma p/d)e\zeta^2 + (p/2 + \nu)d\zeta]\mathrm{d}\tau, \end{aligned}\tag{2.43}$$

By restricting equation (2.35) on the solution manifold, we obtain

$$\zeta_\xi = -(3\beta d^2 u/4\gamma e + ue\zeta^2)\zeta, \tag{2.44a}$$

$$\zeta_\tau = -[3\beta d^2(3\beta d/2\gamma - 1)/4e + (3\beta p^2 u/2 - \gamma p/d)e\zeta^2 + (p/2 + \nu)d\zeta]\zeta, \tag{2.44b}$$

which constitutes the Lax-representation of equation (2.30). When the vector field $b = \partial_\xi$ and $\gamma = \nu d^2$, we obtain a $\mathrm{SL}(2, \mathbb{R})$-symmetry with two-dimensional representation of X_i given by

$$X_1 = -d\sigma_3/2, \tag{2.45a}$$

$$X_2 = -e(1 + 3\beta d^2/4\gamma e^2)\sigma_1/2 + \imath e(1 - 3\beta d^2/4\gamma e^2)\sigma_2/2, \tag{2.45b}$$

$$X_3 = \gamma e(1 - 3\beta d^2/4\gamma e^2)\sigma_1/2d - \imath e\gamma(1 + 3\beta d^2/4\gamma e^2)\sigma_2/2d. \tag{2.45c}$$

We introduce the above new generators in equation (2.35). Considering $X_4 = X_1$ and $X_5 = \nu X_1$, we obtain the matrix representation below

$$F = -eu(1 + 3\beta d^2/4\gamma e^2)\sigma_1/2 + \imath eu(1 - 3\beta d^2/4\gamma e^2)\sigma_2/2 - d\sigma_3/2, \tag{2.46a}$$

$$\begin{aligned} G &= \{-3\beta p^2 eu(1 + 3\beta d^2/4\gamma e^2)/4 + e\gamma p(1 - 3\beta d^2/4\gamma e^2)/2d\}\sigma_1 \\ &\quad + \imath\{3\beta p^2 eu(1 - 3\beta d^2/4\gamma e^2)/4 - e\gamma p(1 + 3\beta d^2/4\gamma e^2)/2d\}\sigma_2 \\ &\quad - (\gamma/d^2 + 3\beta p^2/2)d\sigma_3/2. \end{aligned}\tag{2.46b}$$

We note that the symbols σ_i(i=1, 2, 3) stand for Pauli matrices. Besides, we assume that $d = -2i\lambda$ and $e = \lambda$ in the case $1 + 3\beta d^2/4\gamma e^2 = 0$, with $3\beta/\gamma = 1$. We obtain the following Lax-representation of the previous system

$$\zeta_\xi = \imath\lambda(u\sigma_2 + \sigma_3)\zeta, \tag{2.47a}$$

$$\zeta_\tau = \imath\gamma[p\sigma_1 + \lambda p^2 u\sigma_2 + (\lambda p^2 - 1/2\lambda)\sigma_3]\zeta/2. \tag{2.47b}$$

This relations denote that our system (2.30) has multi-valued soliton solution

2.3.2 The special unitary symmetry SU(2)

It is known that, the general representation of SU(2)-symmetry are the following

$$Y_1 = -\sigma_1/2, \tag{2.48a}$$

$$Y_2 = -\sigma_2/2, \tag{2.48b}$$

$$Y_3 = -\sigma_3/2, \tag{2.48c}$$

satisfying the commutation relation

$$[Y_1, Y_2] = Y_3, \tag{2.49a}$$

$$[Y_2, Y_3] = Y_1, \tag{2.49b}$$

$$[Y_3, Y_1] = Y_2. \tag{2.49c}$$

Besides, by selecting

$$X_1 = -\imath\mu_1\sigma_3/2, \tag{2.50a}$$

$$X_2 = \imath\mu_1\sqrt{3\beta/\gamma}\sigma_2/2, \tag{2.50b}$$

$$X_3 = -\imath\sqrt{3\beta\gamma}\sigma_1, \tag{2.50c}$$

the generator X_i i=(1, 2, 3) satisfies the commutation relation of SU(2)-symmetry where $\nu\mu_1^2 = -\gamma$ and μ_1 is and arbitrary parameter. We use the above generators to write the matrix representation below

$$F = -\imath\mu_1\sigma_3/2 + \imath\mu_1 u\sqrt{3\beta/\gamma}\sigma_2/2, \tag{2.51a}$$

$$G = -\imath p\sqrt{3\beta\gamma}\sigma_1/2 + \imath 3\beta\mu_1 up^2\sqrt{3\beta/\gamma}\sigma_2/4 + \imath\mu_1(\gamma/\mu^2 - 3\beta p^2/2)\sigma_3/2, \tag{2.51b}$$

which is the general form of Wadati-Konno-Ichikawa eigenvalue problem [103]. We assume that $-\mu_1/2 = \lambda$ and $3\beta/\gamma = 1$. Then, the expression of matrix representation becomes

$$F = -\imath\lambda u\sigma_2 + \imath\lambda\sigma_3, \tag{2.52a}$$

$$G = -\imath\gamma[p\sigma_1 + \lambda up^2\sigma_2 + (1/2\lambda - \lambda p^2)\sigma_2]/2, \tag{2.52b}$$

from which we derive the Lax-representation

$$
\begin{aligned}
\zeta_\xi &= \imath\lambda(u\sigma_2+\sigma_3)\zeta, &(2.53a)\\
\zeta_\tau &= \imath\gamma[p\sigma_1+\lambda p^2u\sigma_2+(\lambda p^2-1/2\lambda)\sigma_3]\zeta/2, &(2.53b)
\end{aligned}
$$

which constitutes a variant form of Wadati-Konno-Ichikawa eigenvalue problem corresponding to the SU(2)-symmetry.

2.3.3 The special unitary symmetry $\mathrm{SU}(1,1)$

Now, we find a solution to equation (2.29) in the form

$$
\begin{aligned}
X_1 &= \delta\sigma_3/2, &(2.54a)\\
X_2 &= \delta\sqrt{3\beta/\gamma}(\sigma_1\cosh\varphi-\imath\sigma_2\sinh\varphi)/2, &(2.54b)\\
X_3 &= \sqrt{3\beta/\gamma}(\sigma_1\sinh\varphi-\imath\sigma_2\cosh\varphi)/2, &(2.54c)
\end{aligned}
$$

where $\gamma=\nu\delta^2$ and φ are the arbitrary nonzero parameters. The pseudo-conservation laws to equation (2.29) can then be written as

$$
\begin{aligned}
\Omega = {} & \mathrm{d}\zeta+[-((\sqrt{3\beta\gamma}/2)\cosh\varphi)u\sigma_1+\imath(\sqrt{3\beta\gamma}\cosh\varphi)u\sigma_2/2+\delta\sigma_3]\mathrm{d}\xi\\
&+[-((3\beta p^2\delta\tau\sqrt{3\beta/\gamma}/4)\cosh\varphi+((p\sqrt{3\beta\gamma})/2)\sinh\varphi)\sigma_1\\
&+\imath((3\beta\gamma\sqrt{3\beta/\gamma}p^2u\sinh\varphi)/2+p\sqrt{3\beta\gamma}\cosh\varphi)\sigma_2/2\\
&+(3\beta/2+\gamma/\delta^2)\delta\sigma_3/2]\mathrm{d}\tau. \qquad (2.55)
\end{aligned}
$$

Sectioning the form Ω into a solution manifold of the original, we have the following Lax-pairs

$$
\begin{aligned}
\zeta_\xi &= [(\sqrt{3\beta\gamma}u\sigma_1/2)\cosh\varphi-\imath(\sqrt{3\beta\gamma}u\sigma_2/2)\cosh\varphi-\delta\sigma_3]\zeta, &(2.56a)\\
\zeta_\tau &= [(\frac{3}{4}\beta p^2\delta u\sqrt{3\beta/\gamma}\cosh\varphi+\frac{p}{2}\sqrt{3\beta\gamma}\sinh\varphi)\sigma_1-\imath(\frac{3}{2}\beta\delta\sqrt{3\beta/\gamma}p^2u\sinh\varphi+ \\
&\quad p\sqrt{3\beta\gamma}\cosh\varphi)\sigma_2/2+(3\beta/2+\gamma/\delta^2)\delta\sigma_2/2]\zeta, &(2.56b)
\end{aligned}
$$

defining a $\mathrm{SU}(1,1)$-symmetry. We can extend the previous representation to higher order.

2.3.4 The orthogonal symmetry SO(3)

We seek a three-dimensional representation of the algebra. We find a solution to equation (2.29) in the form

$$X_1 = \mu_1 J_1, \tag{2.57a}$$

$$X_2 = \sqrt{3\beta/\gamma}\mu_1 J_2, \tag{2.57b}$$

$$X_3 = \sqrt{3\beta\gamma} J_3, \tag{2.57c}$$

which satisfy the commutation relations

$$[X_1, X_2] = X_3, \tag{2.58a}$$

$$[X_2, X_3] = X_1, \tag{2.58b}$$

$$[X_3, X_1] = X_2, \tag{2.58c}$$

with

$$J_1 = \begin{pmatrix} 0 & 0 & O \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.59}$$

The matrices J_i (i=1, 2, 3) are the generators of SO(3)-symmetry. Considering $\nu\mu_1 = -\gamma$, the pseudo-conservation laws to equation (2.29) can then be written as

$$\mathrm{d}\Omega = \mathrm{d}\zeta - (\mu_1 J_1 + \sqrt{3\beta/\gamma}\mu_1 J_2)\mathrm{d}\xi - (3\beta p^2 u\mu_1\sqrt{3\beta/\gamma}J_2/2 + p\sqrt{3\beta\gamma}J_3)\mathrm{d}\tau. \tag{2.60}$$

The related Lax-representation of the system (2.30) is written as

$$\zeta_\xi = \mu_1 J_1 + \sqrt{3\beta/\gamma}\mu_1 J_2, \tag{2.61a}$$

$$\zeta_\tau = 3p_2 u\mu_1\sqrt{3\beta/\gamma}J_2/2 + p\sqrt{3\beta\gamma}J_3. \tag{2.61b}$$

The analysis of system (2.30) using Wahlquist and Estrabrouk viewpoint give us several structural symmetries namely, the special linear symmetry $\mathrm{SL}(2,\mathbb{R})$, the special unitary symmetry SU(2), SU(1, 1) and special orthogonal symmetry SO(3). As well, we found the Lax-representations of our system corresponding to different structural symmetries denoting the existence of multi-valued soliton solutions to the previous system.

2.3.5 Extension to an $\mathrm{SL}(3,\mathbb{R})$ connection

Let us pay interest to the $\mathrm{SL}(2,\mathbb{R})$-symmetry derived previously. By choosing $d = -2\imath\lambda$ and $e = \lambda$, we find $3\beta = \gamma$ such that the corresponding connection Ω is given by

$$\Omega \;=\; \mathrm{d}\zeta - \imath\lambda(u\sigma_2 + \sigma_3)\mathrm{d}\xi - \imath(\gamma/2)[p\sigma_1 + \lambda p^2 u\sigma_2 + (\lambda p^2 - 1/2\lambda)\sigma_3]\mathrm{d}\tau, \quad (2.62)$$

from which the inverse scattering formulation in the form of Wadati-Konno-Ichikawa [1] can be easily found. Accordingly, F and G can be extended to the one-dimensional representation

$$F \;=\; -\lambda(v + 2\imath\zeta - u\zeta^2), \quad (2.63a)$$
$$G \;=\; -(C - 2A\zeta - B\zeta^2), \quad (2.63b)$$

such that for $v = -u$, equation (2.30) is derived with the following scattering data

$$A \;=\; \imath/4\lambda - \imath p^2/2, \quad (2.64a)$$
$$B \;=\; -\imath p/2 + \lambda p^2 u/2, \quad (2.64b)$$
$$C \;=\; -\imath p/2 - \lambda p^2 u/2, \quad (2.64c)$$

without any loss of generality, the value $\gamma = 1$ has been considered. Equation (2.63) shows that the connection on the fiber bundle E is quadratic in "ζ" with $-w_i$ (i=0, 1, 2) as the factors of ζ^i (i= 1, 2). Curvature form ϑ_0, ϑ_1 and ϑ_2 are given by [104]

$$\vartheta^1_{11} \;=\; \mathrm{d}\omega^1_{11} + \omega^2_{11} \wedge \omega^1_{12}, \quad (2.65a)$$
$$\vartheta^1_{12} \;=\; \mathrm{d}\omega^1_{12} + 2\omega^1_{12} \wedge \omega^1_{11}, \quad (2.65b)$$
$$\vartheta^2_{11} \;=\; \mathrm{d}\omega^2_{11} + 2\omega^1_{11} \wedge \omega^2_{11},, \quad (2.65c)$$

where

$$\omega^1_{11} \;=\; -\imath\lambda\mathrm{d}x + A\mathrm{d}t, \quad \omega^1_{12} = \lambda u\mathrm{d}x + B\mathrm{d}t, \quad (2.66a)$$
$$\omega^2_{11} \;=\; \lambda v\mathrm{d}x + C\mathrm{d}t, \quad \omega^2_{12} = \imath\lambda\mathrm{d}x - A\mathrm{d}t, \quad (2.66b)$$

Extending equation (2.65) to a more general system as

$$\vartheta^1_{11} = \mathrm{d}\omega^1_{11} + 2\omega^1_0 \wedge \omega^1_{211} + \omega^2_0 \wedge \omega^1_{212} + \omega^2_{11} \wedge \omega^1_{12}, \tag{2.67a}$$

$$\vartheta^1_{211} = \mathrm{d}\omega^1_{211} + \omega^1_{11} \wedge \omega^1_{211} + \omega^2_{11} \wedge \omega^1_{212}, \tag{2.67b}$$

$$\vartheta^1_{0} = \mathrm{d}\omega^1_{0} + \omega^1_{0} \wedge \omega^1_{11} + \omega^2_{0} \wedge \omega^1_{12}, \tag{2.67c}$$

$$\vartheta^1_{212} = \mathrm{d}\omega^1_{212} + \omega^1_{12} \wedge \omega^1_{211} + \omega^2_{12} \wedge \omega^1_{212}, \tag{2.67d}$$

$$\vartheta^2_{0} = \mathrm{d}\omega^2_{0} + \omega^2_{0} \wedge \omega^2_{12} + \omega^1_{0} \wedge \omega^2_{11}, \tag{2.67e}$$

$$\vartheta^1_{12} = \mathrm{d}\omega^1_{12} + \omega^1_{0} \wedge \omega^1_{212} + \omega^1_{12} \wedge \omega^1_{11} + \omega^2_{12} \wedge \omega^1_{12}, \tag{2.67f}$$

$$\vartheta^2_{11} = \mathrm{d}\omega^2_{11} + \omega^2_{0} \wedge \omega^2_{212} + \omega^2_{11} \wedge \omega^2_{12} + \omega^1_{11} \wedge \omega^2_{11}, \tag{2.67g}$$

$$\vartheta^2_{12} = \mathrm{d}\omega^2_{12} + 2\omega^2_{0} \wedge \omega^2_{222} + \omega^1_{0} \wedge \omega^2_{212} + \omega^1_{12} \wedge \omega^2_{11}, \tag{2.67h}$$

with

$$\omega^1_0 = \lambda r_\xi \mathrm{d}\xi + C\mathrm{d}\tau, \quad \omega^2_0 = \lambda s_\xi \mathrm{d}\xi + F\mathrm{d}\tau, \tag{2.68a}$$

$$\omega^1_{11} = 2\imath\lambda \mathrm{d}\xi - (A-D)\mathrm{d}\tau, \quad \omega^2_{222} = \omega^1_{212}, \tag{2.68b}$$

$$\omega^2_{12} = 2\imath\lambda \mathrm{d}\xi - (A-I)\mathrm{d}\tau, \quad \omega^1_{211} = \omega^2_{212}, \tag{2.68c}$$

$$\omega^1_{12} = \lambda u_\xi \mathrm{d}\xi + G\mathrm{d}\tau, \quad \omega^2_{11} = \lambda v_\xi \mathrm{d}\xi + H\mathrm{d}\tau, \tag{2.68d}$$

$$\omega^1_{211} = -(\lambda q_\xi \mathrm{d}\xi + B\mathrm{d}\tau), \tag{2.68e}$$

$$\omega^2_{222} = -(\lambda p_\xi \mathrm{d}\xi + E\mathrm{d}\tau), \tag{2.68f}$$

and quantities u, v, p, q and r standing for observable, while A, B, C, D, E, F, G, H and L refer to scattering data, the one-form

$$\Omega^1 = \omega^1_0 + \omega^1_{11}\zeta^1 + \omega^1_{12}\zeta^2 + \omega^1_{211}(\zeta^1)^2 + \omega^1_{212}\zeta^1\zeta^2, \tag{2.69a}$$

$$\Omega^2 = \omega^2_0 + \omega^2_{11}\zeta^1 + \omega^2_{12}\zeta^2 + \omega^2_{222}(\zeta^2)^2 + \omega^2_{212}\zeta^1\zeta^2, \tag{2.69b}$$

define a Cartan-Ehresman connection [105] with the structure group $\mathrm{SL}(3,\mathbb{R})$. Hence, by selecting

$$A = k/\lambda + \imath qr\lambda, \tag{2.70a}$$

$$B = E = -\imath q/2 - qq_\xi r\lambda, \tag{2.70b}$$

$$C = F = -\imath r/2 - qrr_\xi\lambda, \tag{2.70c}$$

$$G = H = -\imath qr\lambda/2, \tag{2.70d}$$

$$D = L = (K - \imath/2)/\lambda - \imath qr\lambda/2, \tag{2.70e}$$

it comes

$$\begin{aligned} q_{\xi\tau} &= q - (rqq_\xi)_\xi, && (2.71a)\\ r_{\xi\tau} &= r - (qrr_\xi)_\xi, && (2.71b) \end{aligned}$$

which refers to a two- component counterpart of the system (2.30). Its Lax-representation can be easily derived as [105]

$$\xi_x = \lambda \begin{pmatrix} -\imath & q_x & q_x \\ r_x & \imath & 0 \\ r_x & 0 & \imath \end{pmatrix} \xi, \quad \xi_t = \begin{pmatrix} A & B & B \\ C & D & G \\ C & G & D \end{pmatrix} \xi. \qquad (2.72)$$

The previous Lax-representation (2.72) is actually useful in constructing soliton solutions following an inverse scattering transform.

2.3.6 The general case of $\mathrm{SL}(n, \mathbb{R})$

The multi-component form of the system (2.29) can be derived by considering the following quadratic forms

$$\Omega^\alpha = \mathrm{d}\zeta^\alpha - (\omega_0^\alpha + \omega_{1\beta}^\alpha \zeta^\beta + \omega_{2\beta\gamma}^\alpha \zeta^\beta \zeta^\gamma), \qquad (2.73)$$

and quadratic connection forms

$$\begin{aligned} \omega_0^\alpha &= \lambda r_\xi^\alpha \mathrm{d}\xi + C^\alpha \mathrm{d}\tau, && (2.74a)\\ \omega_{1\beta}^\alpha &= 2\imath\lambda\delta_\beta^\alpha \mathrm{d}\xi - (A\delta_\beta^\alpha - D_\beta^\alpha)\mathrm{d}\tau, && (2.74b)\\ \omega_{2\beta\gamma}^\alpha &= -\left[\lambda(q_{\beta\xi}\delta_\gamma^\alpha + q_{\gamma x}\delta_\beta^\alpha)\mathrm{d}\xi + (B_\beta\delta_\gamma^\alpha + B_\gamma\delta_\beta^\alpha)\mathrm{d}\tau\right]/2, && (2.74c) \end{aligned}$$

which define a Cartan-Ehresman connection [104] with structure group $\mathrm{SL}(n, \mathbb{R})$. It should be noted here that the convention of Einstein is applied on repeated indices, and α, β, γ are integers less than $n \in \mathbb{N}$. From equation (2.73), we determine the following two-forms

$$\begin{aligned} \vartheta_0^\alpha &= \mathrm{d}\omega_0^\alpha + \omega_0^\beta \wedge \omega_{1\beta}^\alpha, && (2.75a)\\ \vartheta_{1\beta}^\alpha &= \mathrm{d}\omega_{1\beta}^\alpha + \omega_{1\beta}^\gamma \wedge \omega_{1\gamma}^\alpha + 2\omega_0^\alpha \wedge \omega_{2\gamma\beta}^\alpha, && (2.75b)\\ \vartheta_{2\beta\gamma}^\alpha &= \mathrm{d}\omega_{2\beta\gamma}^\alpha + 2\omega_{1\gamma}^\nu \wedge \omega_{2\beta\nu}^\alpha + \omega_{2\gamma\beta}^\nu \wedge \omega_{1\nu}^\alpha. && (2.75c) \end{aligned}$$

Expanding the scattering data in forms of equation (2.61), the following multi-component system is derived

$$q_{\alpha\tau\xi} = q_\alpha - (q_\nu r^\nu q_{\alpha\xi} + r^\beta q_{\beta\xi} q_\alpha)_\xi/4, \tag{2.76a}$$

$$r^\alpha_{\xi\tau} = r^\alpha - (q_\nu r^\nu r^\alpha_\xi + r^\alpha r^\beta_\xi q_\beta)_\xi/4. \tag{2.76b}$$

The inverse scattering problem which results from the prolongation structure (2.64) is given by

$$\zeta_x = \lambda \begin{pmatrix} -\imath & \cdot & [q_{\alpha x}] \\ \cdot & \cdot & \cdot \\ [r^\alpha_x] & \cdot & \imath I \end{pmatrix} \zeta, \tag{2.77}$$

and

$$\zeta_\tau = \begin{pmatrix} A & \cdot & [B_\alpha] \\ \cdot & \cdot & \cdot \\ [C^\alpha] & \cdot & [D^\alpha_\beta] \end{pmatrix} \zeta, \tag{2.78}$$

where symbol I refers to identify matrix from the generalized systems (2.66) derived above. We have investigated the equation derived starting with the Klein-Gordon model, from the view point of Wahlquist and Estabrook [1]. We have shown by means of this method shown that the system is integrable and admits an infinity number of conservation laws and consequently structural symmetries, namely, the special linear symmetry $SL(2, \mathbb{R})$, the special unitary symmetries $SU(1, 1)$ and $SU(2)$, and the special orthogonal symmetry $SO(3)$. For each of the previous symmetries, we have provided Lax-representation which constituted the starting point of the application of inverse scattering transform method.

2.4 Inverse scattering problem

Let us consider the equation of motion of a thin vortex filament from the following modified Wadati-Konno-Ichikawa equation:

$$i\frac{\partial^2 q}{\partial x \partial t} + sgn\left(\frac{dx}{ds}\right)\frac{\partial^2}{\partial x^2}\left(\frac{\frac{\partial q}{\partial x}}{\Phi}\right) = 0, \tag{2.79}$$

where $\Phi = \sqrt{1 + \left|\frac{\partial q}{\partial x}\right|^2}$ and

Here we introduced the sign function $sgn(\frac{dx}{ds})$ where ds is the element of the arc length along of solution curves:

$ds = \sqrt{(dx)^2 + |dq|^2}$. With the inverse scattering method, we solve equation (2.79) under boundary conditions

$$\left.\begin{array}{l} q \longrightarrow 0, \\ \frac{\partial q}{\partial x} \longrightarrow 0, \end{array}\right\} \quad as \quad x \longrightarrow \infty. \tag{2.80}$$

The eigenvalue problem is given by

$$\frac{\partial v_1}{\partial x} + i\lambda v_1 = \lambda \frac{\partial q}{\partial x} v_2, \tag{2.81a}$$

$$\frac{\partial v_2}{\partial x} - i\lambda v_2 = -\lambda \frac{\partial q^\star}{\partial x} v_1, \tag{2.81b}$$

where $\star$ denotes complex conjugate and the time dependent of eigenfunctions has the forms

$$\frac{\partial v_1}{\partial t} = Av_1 + Bv_2, \tag{2.82a}$$

$$\frac{\partial v_2}{\partial t} = Cv_1 - Av_2, \tag{2.82b}$$

in which

$$A = sgn\left(\frac{dx}{ds}\right)\left(\frac{2i}{\Phi}\lambda^2\right), \tag{2.83a}$$

$$B = sgn\left(\frac{dx}{ds}\right)\left(2\frac{\frac{\partial q}{\partial x}}{\Phi}\lambda^2 + i\frac{\partial}{\partial x}\left(\frac{\frac{\partial q}{\partial x}}{\Phi}\right)\lambda\right), \tag{2.83b}$$

$$C = sgn\left(\frac{dx}{ds}\right)\left(-2\frac{\frac{\partial q^\star}{\partial x}}{\Phi}\lambda^2 + i\frac{\partial}{\partial x}\left(\frac{\frac{\partial q^\star}{\partial x}}{\Phi}\right)\lambda\right). \tag{2.83c}$$

The Gel'fand-Levitan equation can be obtained following process, we define the Jost-functions

$$\left.\begin{array}{l} \phi \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} exp(-i\lambda x), \\ \bar{\phi} \longrightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} exp(i\lambda x), \end{array}\right\} \quad as \quad x \longrightarrow -\infty, \tag{2.84}$$

and

$$\left.\begin{array}{l} \psi \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} exp(i\lambda x), \\ \bar{\psi} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} exp(-i\lambda x), \end{array}\right\} \quad as \quad x \longrightarrow \infty. \tag{2.85}$$

The scattering coefficients are giving by :

$$\phi = a\bar{\psi} + b\psi, \tag{2.86a}$$

$$\bar{\phi} = -\bar{a}\psi + \bar{b}\bar{\psi}, \tag{2.86b}$$

where

$$a\bar{a} + b\bar{b} = 1. \tag{2.87}$$

In order to examine the analytic properties of the Jost function to large $|\lambda|$ we introduce [87]

$$\phi_1 = exp\{-i\lambda x + \int_{-\infty}^{x} \sigma(\lambda, x^{'})dx^{'}\}. \tag{2.88}$$

Substitution of equation (**??**) into equation (2.80) together with equation (2.89) then yields

$$\frac{\partial \sigma}{\partial t} = \frac{\partial}{\partial x}\left(A + \frac{\sigma}{\lambda \frac{\partial q}{\partial x}}\right). \tag{2.89}$$

If we expand σ as an inverse power of series in λ of the form

$$\sigma = \sum_{n=-1}^{\infty} \frac{\sigma_n}{(i\lambda)^n}, \tag{2.90}$$

we obtain an infinite number of conserved quantities by substituting equation (2.89) into equation (2.88). The following first to conserved quantities play a crucial role to express the asymptotic behaviour of the Jost function:

$$\sigma_{-1} = 1 - sgn\left(\frac{dx}{ds}\right)\Phi, \tag{2.91a}$$

$$\sigma_0 = \frac{\frac{\partial^2 q}{\partial^2 x}}{2\frac{\partial q}{\partial x}}\left\{sgn\left(\frac{dx}{ds}\right)\frac{1}{\Phi} - 1\right\} - \frac{1}{2}\frac{\partial}{\partial x}\log\Phi, \tag{2.91b}$$

which vanish for $|x| \longrightarrow \infty$. Then, the asymptotic form of ϕ for large $|\lambda|$ and the analytic property of scattering coefficient a are written

$$\phi = \begin{pmatrix} 1 \\ i\sigma_{-1}/\frac{\partial q}{\partial x} \end{pmatrix} exp\{-i\lambda x + i\lambda\varepsilon_- + \mu_-\} + 0\Big(\frac{1}{\lambda}\Big), \tag{2.92a}$$

$$a = exp(i\lambda\varepsilon + \mu) + 0\Big(\frac{1}{\lambda}\Big), \tag{2.92b}$$

where

$$\varepsilon = \varepsilon_+ + \varepsilon_- = \int_{-\infty}^{+\infty} \sigma_{-1}dx, \quad \varepsilon_- = \int_{-\infty}^{x} \sigma_{-1}dx, \quad \varepsilon_+ = \int_{x}^{+\infty} \sigma_{-1}dx, \tag{2.93a}$$

$$\mu = \mu_+ + \mu_- = \int_{-\infty}^{+\infty} \sigma_0 dx, \quad \mu_- = \int_{-\infty}^{x} \sigma_0 dx, \quad \mu_+ = \int_{x}^{+\infty} \sigma_0 dx. \tag{2.93b}$$

In the same way, we can obtain the asymptotic behaviour of $\bar{\phi}$, $\bar{\psi}$ and ϕ. Use the fact that $\phi exp\{i\lambda(x-\varepsilon_-)\}$, $\bar{\phi} exp\{-i\lambda(x-\varepsilon_-)\}$, $\psi exp\{-i\lambda(x+\varepsilon_+)\}$, $\bar{\psi} exp\{i\lambda(x+\varepsilon_+)\}$, and $a\ exp(-i\lambda\varepsilon)$ are entire functions of λ and introduce the kernels K_1 and K_2 :

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} exp\{i\lambda(x + \varepsilon_+(x) - \mu_+^\star(x))\} \tag{2.94}$$

$$+ \int_x^{+\infty} \begin{pmatrix} \lambda K_1(x,z) exp\{-\mu_+(x)\} \\ K_2(x,z) exp\{-\mu_+^\star(x)\} \end{pmatrix} exp\{i\lambda(z + \varepsilon_+(x))\} dz,$$

where $\star$ denotes complex conjugate, we shall assume

$$\lim_{z\to\infty} K_1(x,z) = 0, \quad \lim_{z\to\infty} K_2(x,z) = 0. \tag{2.95}$$

We have

$$K_1(x,x) = \frac{\sigma_{-1}}{\frac{\partial q^\star}{\partial x}} exp(\mu_+ + \mu_-^\star), \tag{2.96}$$

and the Gel'fand-Levitan equation for $x \geq y$:

$$K_1^\star(x,y) - F^\star(x+y) - \int_x^\infty F^\star(y+z) K_2^\star(x,z) dz = 0, \tag{2.97a}$$

$$K_2^\star(x,y) - \int_x^\infty F^{''}(y+z) K_1(x,z) dz = 0. \tag{2.97b}$$

Where

$$F(z) = \frac{1}{2\pi} \int_c \frac{b(\lambda)}{\lambda a(\lambda)} exp\{i\lambda(z + 2\varepsilon_+(x))\} d\lambda, \tag{2.98a}$$

$$F^{''}(z) = \frac{\partial^2 F}{\partial x^2} = \frac{1}{2\pi} \int_c \frac{\lambda b(\lambda)}{a(\lambda)} exp\{i\lambda(z + 2\varepsilon_+(x))\} d\lambda. \tag{2.98b}$$

Here, the contour c is defined to be the contour in the complex λ plane starting from $-\infty + i0^+$, passing over all zeros of a and ending at $+\infty + 0^+$. Time dependent of the scattering data is given by

$$\begin{aligned} a(\lambda,t) &= a(\lambda,0), && (2.99a)\\ b(\lambda,t) &= b(\lambda,0)exp(4i\lambda^2 t). && (2.99b) \end{aligned}$$

When all the zeros of $a(\lambda)$ in the upper half plane are simple, $F(z)$ can be expressed as

$$F(z) = \sum_{k=1}^{N} \frac{C_k(t)}{\lambda_k} exp\{i\lambda_k(z + 2\varepsilon_+(x))\} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\rho(\lambda,t)}{\lambda} exp\{i\lambda_k(z+2\varepsilon_+(x))\}d\lambda, \quad (2.100)$$

where

$$\begin{aligned} C_k(t) &= C_k(0)exp(4i\lambda_k^2 t), && (2.101a)\\ \rho(\lambda,t) &= \rho(\lambda,0)exp(4i\lambda_k^2 t). && (2.101b) \end{aligned}$$

Given the scattering data $\{\rho(\lambda,0), \lambda, C_k(0), \lambda_k, k = 1, 2, \ldots N\}$, we can determine $F(z)$ and solve $K_1(x,x)$ with the Gel'fand-Levitan equations. We then obtain the solution by using the relation (2.55).

2.4.1 N-soliton solution

The N soliton solution is obtained in the conditions

$$\begin{aligned} (1) \quad \rho(\lambda,t) &= 0, && (2.102a)\\ (2) \quad \lambda_k, \quad k &= 1, 2, \ldots, N, && (2.102b) \end{aligned}$$

then, $F(z,t)$ in (2.97) reduces to

$$F(z) = \sum_{k=1}^{N} \frac{C_k(t)}{\lambda_k} exp\{i\lambda_k(z+2\varepsilon_+(x))\}. \quad (2.103)$$

In order to solve the Gel'fand-Levitan equations, we introduce the representation

$$\begin{aligned} K_1 &= \sum_{k=1}^{N} A_k(x) exp\{-i\lambda_k^\star(x+y+2\varepsilon_+(x))\}, && (2.104a)\\ K_2 &= \sum_{k=1}^{N} B_k(x) exp\{-i\lambda_k^\star(x+y+2\varepsilon_+(x))\}. && (2.104b) \end{aligned}$$

where $\star$ denotes complex conjugate. Substituting equation (2.102) and equation (2.103) into equation (2.99), we obtain

$$A_k + i\frac{C_k^\star}{\lambda_k^\star}\sum_{i=1}^{N} B_i^\star \frac{exp\{2i\lambda_i(x+\varepsilon_+(x))\}}{\lambda_i - \lambda_k^\star} = \frac{C_k^\star}{\lambda_k^\star}, \tag{2.105a}$$

$$B_k^\star + iC_k\lambda_k^\star \sum_{i=1}^{N} A_i \frac{exp\{-2i\lambda_i^\star(x+\varepsilon_+(x))\}}{\lambda_i - \lambda_k^\star} = 0. \tag{2.105b}$$

2.4.2 One-soliton solution

For one-soliton solution with an eigenvalue $\lambda = \xi + i\eta$, q and ε_+ are given by

$$q = i\frac{C^\star}{\lambda^\star}\frac{exp(2i\lambda^\star s)}{1+\frac{|C_1|^2 exp\{2i(\lambda-\lambda^\star)s\}}{(\lambda-\lambda^\star)^2}}, \tag{2.106}$$

$$\varepsilon_+ = \frac{i\frac{|C|^2 exp\{2i(\lambda-\lambda^\star)s\}}{|\lambda|^2(\lambda-\lambda^\star)}}{1+\frac{|C|^2 exp\{2i(\lambda-\lambda^\star)s\}}{(\lambda-\lambda^\star)^2}}, \tag{2.107}$$

where $\star$ denotes complex conjugate with C given as follow

$$C = C(0)exp(4i\lambda^2 t). \tag{2.108}$$

The maximum amplitude of soliton is given by $\eta/(\xi^2+\eta^2) = -\xi/2$. The velocity v_s of the soliton in s coordinate is given by constant -4ξ. However the velocity in x coordinate is not constant. It take different values at different position of the soliton and tends to be equal to v_s as $|x| \to \infty$ where ε_+ become constant. The solution changes it form with the period $\tau = \pi/2(\xi^2+\eta^2)$.

2.4.3 Two-soliton solution

Two soliton solution is given by

$$q = -i\frac{S}{R}, \tag{2.109a}$$

$$\varepsilon_+ = \frac{T}{R}, \tag{2.109b}$$

where

$$R = 1 - \frac{|C_1|^2 exp\{2i(\lambda_1 - \lambda_1^\star)s\}}{(\lambda_1 - \lambda_1^\star)^2} - \frac{|C_2|^2 exp\{2i(\lambda_2 - \lambda_2^\star)s\}}{(\lambda_2 - \lambda_2^\star)^2} - \frac{C_1 C_2^\star exp\{2i(\lambda_1 - \lambda_2^\star)s\}}{(\lambda_1 - \lambda_2^\star)^2} - \frac{C_1^\star C_2 exp\{2i(\lambda_2 - \lambda_1^\star)s\}}{(\lambda_2 - \lambda_1^\star)^2} + \frac{|C_1|^2|C_2|^2|\lambda_1 - \lambda_2|^4 exp\{2i(\lambda_1 - \lambda_1^\star + \lambda_2 - \lambda_2^\star)s\}}{\{(\lambda_1 - \lambda_1^\star)(\lambda_1 - \lambda_2^\star)(\lambda_2 - \lambda_1^\star)(\lambda_2 - \lambda_2^\star)\}^2} \tag{2.110}$$

$$S = \frac{C_1^\star exp(-2i\lambda_1^\star s)}{\lambda_1^2} + \frac{C_2^\star exp(-2i\lambda_2^\star s)}{\lambda_2^2} + \frac{|C_1|^2 C_2^\star \lambda_1^2 (\lambda_1^\star - \lambda_2^\star)^2 exp\{2i(\lambda_1 + \lambda_1^\star - \lambda_2^\star)s\}}{\{\lambda_1^\star \lambda_2^\star (\lambda_1 - \lambda_1^\star)(\lambda_1 - \lambda_2^\star)\}^2} + \frac{|C_2|^2 C_1^\star \lambda_2^2 (\lambda_1^\star - \lambda_2^\star)^2 exp\{2i(\lambda_2 + \lambda_1^\star - \lambda_2^\star)s\}}{\{\lambda_1^\star \lambda_2^\star (\lambda_2 - \lambda_1^\star)(\lambda_2 - \lambda_2^\star)\}^2}, \tag{2.111}$$

and

$$T = -i\frac{|C_1|^2 exp\{2i(\lambda_1 - \lambda_1^\star)s\}}{|\lambda_1|^2(\lambda_1 - \lambda_1^\star)} - i\frac{|C_2|^2 exp\{2i(\lambda_2 - \lambda_2^\star)s\}}{|\lambda_2|^2(\lambda_2 - \lambda_2^\star)} - i\frac{C_1 C_2^\star exp\{2i(\lambda_1 - \lambda_2^\star)s\}}{\lambda_1 \lambda_2^\star (\lambda_1 - \lambda_2^\star)} - i\frac{C_2 C_1^\star exp\{2i(\lambda_2 - \lambda_1^\star)s\}}{\lambda_2 \lambda_1^\star (\lambda_2 - \lambda_1^\star)} + i\frac{|C_1|^2|C_2|^2|\lambda_1|^2(\lambda_2 - \lambda_2^\star) + |\lambda_2|^2(\lambda_1 - \lambda_1^\star)|\lambda_1 - \lambda_2|^4 exp\{2i(\lambda_1 - \lambda_1^\star + \lambda_2 - \lambda_2^\star)s\}}{|\lambda_1|^2|\lambda_2|^2(\lambda_1 - \lambda_1^\star)(\lambda_1 - \lambda_2^\star)(\lambda_2 - \lambda_1^\star)(\lambda_2 - \lambda_2^\star)^2}, \tag{2.112}$$

with

$$C_1 = C_1(0) exp(4i\lambda_1^2 t), \tag{2.113a}$$

$$C_2 = C_2(0) exp(4i\lambda_2^2 t). \tag{2.113b}$$

2.5 Jacobi elliptic functions expansion method

Jacobi elliptic functions play an important role to find the exact solutions of nonlinear wave equations in nonlinear problems. However, with this method, one can obtain solitary and periodic wave solutions of nonlinear wave equations. In this section, the Jacobi elliptic function expansion method is proposed and applied to some nonlinear wave equations.

Elliptic integrals: An integral of the form $\int R(x, y)dx$, where $R(x, y)$ is a rational function of x and y, $y^2 = P(x)$ where P is a polynomial of degree 3 or 4, is called an elliptic integral [106]. Legendre's elliptic integral of the first kind, with amplitude φ and parameter m, is defined [106] as

$$F(\varphi|m) = \int_0^{sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}. \tag{2.114}$$

The parameter m can be taken as real with $0 \leq m \leq 1$, and the complementary parameter is

$$m_1 = 1 - m \tag{2.115}$$

$F(\varphi|m)$ is often abbreviated to $F(\varphi)$, when the parameter m is to be understood. Legendre's complete elliptic integral of the First Kind [106]is

$$K(m) = F(\frac{\pi}{2}|m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} \tag{2.116}$$

As $m \to 1$, then $K(m) \to \infty$ Legendre's Complementary Complete Elliptic Integral of the First Kind is defined as

$$\acute{K}(m) = K(m_1) = K(1-m), \tag{2.117}$$

$K(m)$ and $\acute{K}(m)$ are often abbreviated to K and $\acute{K}$, when the parameter m is to be understood

In the definition (2.114) of $F(\varphi|m)$ the integrand has branch-points at $t = \pm 1$ at $t = 1/\sqrt{m}$ and at $t = -1/\sqrt{m}$. If the integral in (2.114) is taken as a Riemann integral on the real interval $(-1, 1)$, then it is single-valued; but with complex limit $\sin(\varphi)$, the Cauchy line integral has infinitely many values, depending on the number of loops made around each of the branch-points.

2.5.1 Elliptic functions

The theory of elliptic integrals, as developed by Fagnano, Euler and Legendre, was exceedingly complicated, involving infinitely many values for each elliptic integral. Thereafter, Abel simplified the subject immensely by inverting elliptic integrals to get elliptic functions, and showed that elliptic functions are doubly-periodic single-valued functions [107]. If g is a doubly-periodic function with ξ a period of least modulus, and with χ a period of least modulus which is not an integral multiple of χ, then the pair of periods (ξ, χ) are called fundamental periods of g [108].

The doubly periodic function $g(u)$ over the primitive period parallelogram which is generated by the vectors ξ and χ from 0 in the complex u-plane, gives a full representation of $g(u)$; for the entire complex plane could be tiled with copies of that parallelogram and the values of $g(u)$ over it. Indeed, any parallelogram with sides equal and parallel to those vectors, centred anywhere in the complex plane, could be taken as a basic period parallelogram for g, which could be copied to tile the entire complex plane with g. As functions of the complex variable u, the Jacobi elliptic functions $sn(u)$, $cn(u)$ and $dn(u)$ are doubly-periodic single-valued functions of u.

Jacobi elliptic function $sn(u)$

The inverse function of the Legendre elliptic function F is $\varphi = F^{-1}(u)$, and the Jacobi elliptic function $sn(u) = \sin(\varphi)$ (or $sn(u|m)$) is single-valued for all complex parameters [106], with

$$u = \int_0^{\sin u} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}, \tag{2.118}$$

and $sn(u)$ is an odd single-valued function of u. For real u, the function sn has real period $4K(m)$ and range [-1, 1], with $sn(0) = 0$, $sn(K) = 1$, $sn(2K) = 0$, $sn(3K) = -1$ and $sn(4K) = 0$. Let $t = sn(u)$, so that

$$sn^{-1}\tau = \int_0^{\tau} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = F(sn^{-1}\tau|m). \tag{2.119}$$

Using the principal branch of the function $sin^{-1}\tau$, through the origin. On the real interval $-K \leq u \leq K$ the function sn increases monotonically from -1 to 1, and so for real $\tau \in [-1, 1]$ the function $sn^{-1}\tau$ has a single value in the real interval $[-K, K]$.

Jacobi elliptic function $cn(u)$

The Jacobi elliptic function cn is defined by

$$cn(u) = cn(u|m) = \cos(\phi) \tag{2.120}$$

so that

$$cn(u) = \sqrt{1 - sn^2(u)} \tag{2.121}$$

and $cn(u)$ is an even single-valued function of u. The branch of the square root function in (2.121) is determined by (2.120). For real u, the function cn has real period $4K(m)$ and range $[-1,1]$, with $cn(0)=1, cn(K)=0, cn(2K)=-1, cn(3K)=0$ and $cn(4K)=1$ [106]. On the real interval $0 \leq u \leq 2K$ the function cn decreases monotonically from 1 to -1, and so for real $r \in [-1,1]$ the function $cn^{-1}(r)$ has a single value in the real interval $[0, 2K]$.

Jacobi elliptic function $dn(u)$

And so $dn(u)$ is an even single-valued function of u. For real u, the function dn has real period $2K(m)$ and range $[\sqrt{m_1}, 1]$, with $dn(0)=1$, $dn(K)=\sqrt{m_1}$ and $dn(2K)=1$ [106]. On the real interval $0 \leq u \leq K$ the function dn decreases monotonically from 1 to $\sqrt{m_1}$, and so for real $r \in [\sqrt{m_1}, 1]$ the function $dn^{-1}r$ has a single value in the real interval $[0, K]$.

Additionnal formula for elliptic functions

For the extreme values $m=0$ and $m=1$, those elliptic functions reduce respectively [106] to trigonometric and hyperbolic functions:

$$sn(u|0) = sin(u), \quad cn(u|0) = cos(u), \quad dn(u|0) = 1, \tag{2.122a}$$

$$sn(u|1) = tanh(u), \quad cn(u|1) = dn(u|1) = sech(u). \tag{2.122b}$$

2.5.2 Application of Jacobi elliptic function expansion method

We consider the Kraenkel-Mana-Merle system describing the propagation of short-wave in nanoscale saturated ferromagnetic materials which derived from Maxwell equations and Landau-Lifshitz-Gilbert equations given as follows

$$u_{xt} - uv_x = 0, \tag{2.123a}$$

$$v_{xt} + uu_x = 0, \tag{2.123b}$$

where $u=u(x,t)$ and $v=v(x,t)$ stand for the magnetization and the external magnetic fields related to the ferrite, respectively. x and t represent the space-like and time-like variable, respectively. In order to investigate the solutions for the KMM system (2.94), we introduce the following traveling wave transformations

$$\xi = kx + \omega t + \xi_0, \tag{2.124a}$$

$$u(x,t) = u(\xi), \tag{2.124b}$$

$$v(x,t) = v(\xi), \tag{2.124c}$$

where k and w are non-zero constants, ξ_0 is an arbitrary constant. Then from (2.95), we have

$$u_x = ku^{'}(\xi), \quad u_{xt} = kwu^{''}(\xi), \quad v_x = kv^{'}(\xi), \quad v_{xt} = kwv^{'}(\xi). \tag{2.125}$$

Substituting (2.95) and (2.96) into Equation (2.94), we have

$$wu^{''} - uv^{'} = 0, \tag{2.126a}$$

$$wv^{''} = -uu^{'}, \tag{2.126b}$$

noting equation (2.126)b, and integrating once with respect to ξ, it becomes

$$wv^{'}(\xi) = -\frac{1}{2}u^2(\xi) + c_0, \tag{2.127}$$

where c_0 is an integration constant. Substituting (2.127) into (2.126), it yields

$$u^{''}(\xi) - \frac{c_0}{w^2}u(\xi) + \frac{1}{2w^2}u^3(\xi) = 0. \tag{2.128}$$

Now, we introduce a transformation as follows

$$\frac{du}{d\xi} = y, \tag{2.129a}$$

$$u^{''}(\xi) = y\frac{du}{d\xi}. \tag{2.129b}$$

Substituting equation (2.129) into equation (2.128), we get

$$y\frac{dy}{du} = \frac{1}{w^2}(c_0 u - \frac{1}{2}u^3). \tag{2.130}$$

Then, it follows

$$y^2 = \frac{1}{w^2}(c_0 u^2 - \frac{1}{4}u^4) + 2h, \tag{2.131}$$

where h is an integration constant. Note that equations (2.129) and (2.131) can be rewritten as

$$[u^{'}(\xi)]^2 = 2h + \frac{c_0}{w^2}u^2 - \frac{1}{4w^2}u^4. \tag{2.132}$$

Referring to the auxiliary equation method [109–115], solving equation (2.132) leads that some solutions $u(\xi)$ are obtained under different conditions as follows

$$\textbf{Case} \quad \textbf{a}: \quad u(\xi) = cn(\xi), if \quad w = \pm\frac{1}{1m}, c_0 = \frac{1m^2 - 1}{4m^2}, h = \frac{1 - m^2}{2}, \tag{2.133a}$$

$$\textbf{Case} \quad \textbf{b}: \quad u(\xi) = dn(\xi), if \quad w = \pm\frac{1}{2}, c_0 = \frac{2 - m^2}{4}, h = \frac{m^2 - 1}{2}, \tag{2.133b}$$

$$\textbf{Case} \quad \textbf{c}: \quad u(\xi) = mcn(\xi) + dn(\xi), if \quad w = \pm 1, c_0 = \frac{1 + m^2}{2}, h = -\frac{1}{8}(1 - m^2)^2, \tag{2.133c}$$

$$\textbf{Case} \quad \textbf{d}: \quad u(\xi) = \frac{1}{dn(\xi)}, if w = \pm\frac{1}{2\sqrt{1 - m^2}}, c_0 = \frac{2 - m^2}{4(1 - m^2)}, h = -\frac{1}{2}, \tag{2.133d}$$

$$\textbf{Case} \quad \textbf{e}: \quad u(\xi) = \frac{sn(\xi)}{dn(\xi)}, if w = \pm\frac{1}{2m\sqrt{1 - m^2}}, c_0 = \frac{2m^2 - 1}{4m(1 - m^2)}, h = \frac{1}{2}, \tag{2.133e}$$

m in equation (2.133) represents the Jacobi elliptic function modulus, and $0 < m < 1$. According to the above cases, a series of Jacobi elliptic function solutions of the system (2.123) are obtained as follows

Case 1

$$u_{1a}(x,t) = cn(\xi), \quad v_{1a}(x,t) = \frac{1}{2m}\xi - \frac{1}{m}E(\xi,m), \quad \xi = kx + \frac{1}{m}t + \xi_0, \tag{2.134a}$$

$$u_{1b}(x,t) = cn(\xi), \quad v_{1b}(x,t) = -\frac{1}{2m}\xi + \frac{1}{m}E(\xi,m), \quad \xi = kx - \frac{1}{m}t + \xi_0, \tag{2.134b}$$

where $E(\xi,m) = \int dn^2(\xi,m)d d\xi$. Especially, when $m \to 1$, the solutions (2.134) generate the soliton solutions given as follows

$$u_{1a}(x,t) = sech(kx + \frac{1}{2}t + \xi_0), \quad v_{1a}(x,t) = \frac{1}{2}(kx + \frac{1}{2}t + \xi_0) - tanh(kx + \frac{1}{2}t + \xi_0), \tag{2.135a}$$

$$u_{1b}(x,t) = sech(kx - \frac{1}{2}t + \xi_0), \quad v_{1b}(x,t) = -\frac{1}{2}(kx - \frac{1}{2}t + \xi_0) - tanh(kx + \frac{1}{2}t + \xi_0). \tag{2.135b}$$

Case 2

$$u_{2a}(x,t) = dn(\xi), \quad v_{2a}(x,t) = \frac{2 - m^2}{2}\xi - E(\xi,m), \quad \xi = kx + \frac{1}{2}t + \xi_0, \tag{2.136a}$$

$$u_{2b}(x,t) = dn(\xi), \quad v_{2b}(x,t) = -\frac{m^2 - 1}{2}\xi + E(\xi,m), \quad \xi = kx - \frac{1}{2}t + \xi_0. \tag{2.136b}$$

Especially, when $m \to 1$, equation (2.136) also generate the same soliton solutions of equation (2.135), respectively.

Case 3

$$u_{3a}(x,t) = mcn(\xi) + dn(\xi), \quad v_{3a}(x,t) = \xi - msn(\xi) - E(\xi,m), \quad \xi = kx + t + \xi_0, \tag{2.137a}$$

$$u_{3b}(x,t) = mcn(\xi) + dn(\xi), \quad v_{3b}(x,t) = -\xi + msn(\xi) + E(\xi,m), \quad \xi = kx - t + \xi_0. \tag{2.137b}$$

When $m \to 1$, Eq.(2.136) generate the soliton solutions,

$$u_{1a}(x,t) = 2sech(kx + t + \xi_0), \quad u_{1a}(x,t) = (kx + t + \xi_0) - 2tanh(kx + t + \xi_0), \tag{2.138a}$$

$$u_{1b}(x,t) = sech(kx - t + \xi_0), \quad u_{1b}(x,t) = -(kx - t + \xi_0) + 2tanh(kx - t + \xi_0). \tag{2.138b}$$

Soliton and its dynamics play an essential role in nonlinear evolution system. So, it is necessary to further discuss the soliton solutions obtained here. In Refs. [29, 30], the authors obtained the one-soliton solution as

$$u = 2\omega sech(kx + \omega t + \eta_0), \tag{2.139a}$$

$$v = x - 2\omega(1 + tanh(kx + \omega t + \eta_0) + x_0. \tag{2.139b}$$

where k, ω, x_0 and η_0 are constants. There exists only the loop-like soliton between the dependent variable u and v versus the independent variables x and t. To form mathematically multiple-valued relation between u and v to x and t, it is critical whether there exist a linear function about x and t in v. Fortunately, the linear functions of x and t are involved in the soliton solutions in equation (2.135) and equation (2.139). For example, $\frac{1}{2}(kx+\frac{1}{2}t+\xi_0)$ is comprised in v in equation (2.135). It means that all the soliton solutions here are loop-like for u and v versus the both x and t. So the soliton solutions are different form one reported in Refs. [29, 30].

The KMM system is a model describing the propagations of the ferromagnetic particles in nano-scale ferrite materials. A series of the solutions are obtained. These solutions include periodic traveling wave solutions in Jacobi elliptic functions and soliton solutions. There exist loop-like periodic waves and loop-like solitons between the independent variables $u(x,t)$ and $v(x,t)$ versus the time and space variables. The modulus m in Jacobi elliptic functions can control the wave density for the periodic wave solutions. The Jacobi elliptic functions sn and cn are analogous to the trigonometric functions sine and cosine. They come up in applications such as nonlinear oscillations and conformal mapping. This introduction to the Jacobi elliptic, sn, cn, dn and related functions is parallel to the usual development of trigonometric functions. These functions satisfy nonlinear differential equations that appear often in physical applications, for instance in particle mechanics.

2.6 Finite difference method

One can also integrate using a finite difference approximation, in which derivatives are replaced with differences via Taylor expansion. This type of approach is well-suited to linear and nonlinear, boundary-value problems in mechanics [116]. We expect that as a mesh is made finer, the solution that is represented on it becomes better or more accurate. But we would like to quantify this notion of accuracy. For a problem having a fixed size, we expect the accuracy to depend on the size Δx of the zones that make up the mesh. The Taylor series expansion of a smooth function gives us a way to make this quantification more accurate. Thus say that we have a sufficiently differentiable function $u(x)$ in one variable x for which we know the derivatives $u_x(x_i), u_{xx}(x_i), u_{xxx}(x_i), u_{xxxx}(x_i) \cdots$ at the origin $x = x_0$. As we increase the number of derivatives, we can increase the accuracy with which we can predict $u(x_i + \Delta x)$ a small distance Δx away from the origin. We thus have

$$u(x_i + \Delta x) = u(x_i) + u_x(x_i)\Delta x + \frac{1}{2}u_{xx}(x_i)\Delta x^2 + \frac{1}{6}u_{xxx}(x_i)\Delta x^3 + \frac{1}{24}u_{xxxx}(x_i)\Delta x^4 + \cdots \tag{2.140}$$

We know from calculus that as the terms of the Taylor series are extended, our predicted solution also becomes more accurate [116]. We want to carry that concept of accuracy over to our discrete numerical representation. Let us, therefore, take the origin at the i^{th} mesh point of a uniform one-dimensional mesh, shows the continuous curve that we wish to specify at a set of mesh points $\cdots, x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots$. We do that by specifying the mesh function $\cdots, u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}, \cdots$ which, for the simple finite difference approximations that we are exploring here, is just the set of values taken by the function at the specified mesh points. The $(i+1)^{th}$ mesh point is located at $x_i + \Delta x$ and the $(i-1)^{th}$ mesh point is located at $x_i - \Delta x$. Using our formula for the Taylor series we get

$$u_{i+1} = u(x_i + \Delta x) = u(x_i) + u_x(x_i)\Delta x + \frac{1}{2}u_{xx}(x_i)\Delta x^2 + \frac{1}{6}u_{xxx}(x_i)\Delta x^3 + \frac{1}{24}u_{xxxx}(x_i)\Delta x^4 + \cdots \quad (2.141a)$$

$$u_i = u(x_i) \quad (2.141b)$$

$$u_{i-1} = u(x_i - \Delta x) = u(x_i) - u_x(x_i)\Delta x + \frac{1}{2}u_{xx}(x_i)\Delta x^2 - \frac{1}{6}u_{xxx}(x_i)\Delta x^3 + \frac{1}{24}u_{xxxx}(x_i)\Delta x^4 + \cdots \quad (2.141c)$$

Note that equation (2.141) implicitly assumes that the data is specified on a uniform mesh with a distance Δx between mesh points. Subtracting the third equation above from the first and dividing by $2\Delta x$ gives

$$u_x(x_i) = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{1}{6}u_{xxx}(x_i)\Delta x^2 + \cdots . \quad (2.142)$$

Notice from equation (2.142) that $u_x(x_i + \Delta x)$ is the actual first derivative that we seek. The term $\frac{u_{i+1}-u_{i-1}}{2\Delta x}$ in equation (2.142) is referred to as the finite difference approximation of the first derivative. It does not furnish an exact representation of $u_x(0)$ as shown by the higher order terms in equation (2.142). The second term on the right hand side of equation (2.142) is given by $u_{xxx}(x_i)\Delta x^2/6$ and shows the truncation error in our finite difference approximation. It is the term that dominates the error in the first derivative as $\Delta x \to 0$. Notice from equation (2.142) that our finite difference approximation is second order accurate owing to the Δx^2 dependence in the truncation error. Realize too that the mesh function is only capable of giving us a finite difference approximation. The finite difference approximation will necessarily have an associated truncation error whose magnitude we can estimate with the use of calculus. We can make a further illustration for the second derivative by using the three equations in equation (2.141) to get

$$u_{xx}(x_i) = \frac{u_{i+1} - u_i + u_{i-1}}{\Delta x^2} - \frac{1}{12}u_{xxxx}(x_i)\Delta x^2 + \cdots . \quad (2.143)$$

We see equation (2.143) that $u_{xx}(x_i)$ has been approximated to second order of accuracy

The diffusive problem

We consider the following diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(D\frac{\partial u}{\partial x}), \quad (2.144)$$

where u represents the temperature through the wire and D the conductivity. The conservation of heat-energy inside a volume is applied, the change in the energy (so that in the temperature) inside a volume equals the flux of heat, if D is constant we have

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{2.145}$$

In order to solve numerically this diffusion equation, we need the boundary and initial conditions which are given as follows. Boundary conditions $u(0,t) = 1$ and $u(L,t) = 0$ Initial condition $u(x,0) = sin(\pi x)$. Forward Euler, and central difference representation are used but still using only quantities known at time step j scheme and we obtain the following relation

$$\frac{\partial u}{\partial t} = \frac{u_{n,j+1} - u_{n,j}}{\Delta t}, \tag{2.146a}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{(\Delta x)^2}, \tag{2.146b}$$

which give the following relation

$$\frac{u_{n,j+1} - u_{n,j}}{\Delta t} = D\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{(\Delta x)^2}. \tag{2.147}$$

We depict in Figure (5) the temperature through the wire at times $t = 0$, $t = 0.0396$, $t = 0.1196$ and $t = 0.2396$ given by equation (2.147).

Conclusion

In this chapter where the question was to present some methods being able to show the effect of inhomogeneity and dissipation on the waves propagating in ferrites, the Wahlquist and Estabrook formalism while investigating the prolongation structure of $(1 + 1)$-dimensional evolution equation which must lead to the Lax Pair has been presented. We illustrate this method by applying it to the Schäfer-Wayne equation. We pursue in this chapter by exposing the inverse scattering transform method and also Jacobi elliptic function expansion method. The numerical shutter is not remainder in this section with the presentation of finite difference method which has been illustrated on the diffusion equation. Various methods thus presented are direct technics of finding exact solutions to a large number of nonlinear evolution systems among which the KMM-system

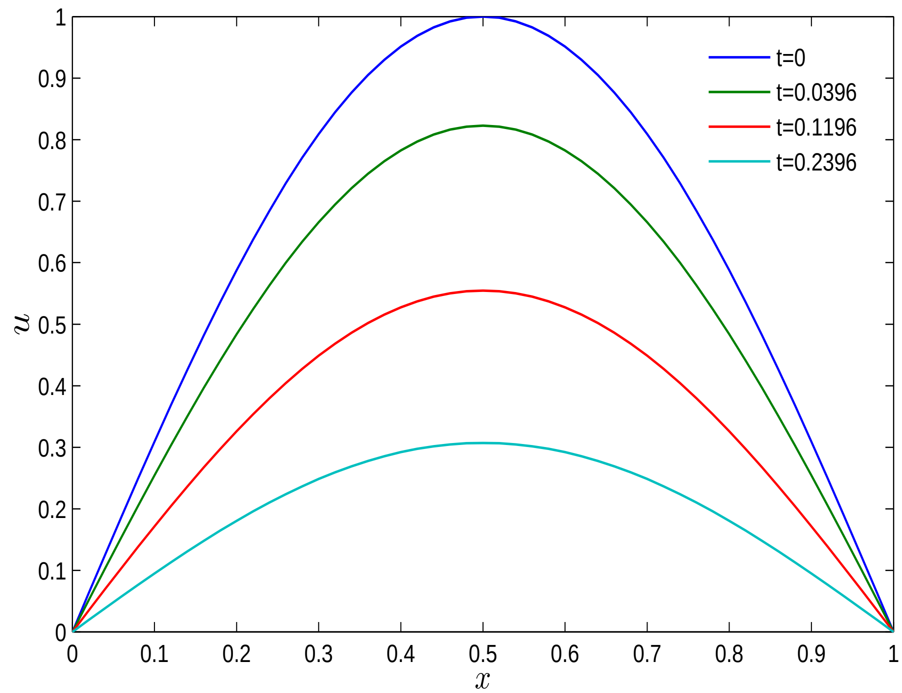

Figure 5: Numerical solution u of the diffusion Eq.(2.147) for $L = 1$ the length of the wire, at times $t = 0$, $t = 0.0396$, $t = 0.1196$ and $t = 0.2396$.

describing the waves propagating in ferrites. In this look, we will use these methods to generate soliton solutions to the KMM-system while investigating impact of inhomogeneous exchange effects and damping effects on the characteristics of the wave moving in the ferrite in the forthcoming chapter.

Chapter 3

Results and Discussions

Introduction

Nonlinear media have attracted over years so many attention and still continuing to do so. The reason is that, these media often present some particularities that are of technological importance. Studying these media, there are nonlinear model equations that come out describing their dynamic under some perturbation. Thus, there are number of evolution equations in the literature that describe wide range of physical phenomena such as, the Korteweg-de Vries equation [1] describing the propagation of water waves, the nonlinear Schrödinger equation [14] describing the propagation of waves in optical fibers under the slowly varying envelope approximation, the stretched rope equation [2, 99] describing the propagation of transverse oscillation of elastic beam under tension, just to name a few. Constructing such nonlinear evolution equation, a challenge often arises, the one of investigating the integrability of these systems, useful for deep comprehension of the dynamic of the media they model. To respond to this challenge, number of approaches have been proposed including the Painlevé analysis [24], Hirota's bilinearization [22], collective coordinate theory [100], inverse scattering transform (IST) [15]. As far as we are concerned with wave possessing vanishing tails, the IST method appears to be one of the most important discovery in soliton theory. There are number of nonlinear evolution equations that are integrable by means of this technique, most of them being gathered in two principal classes: the Ablowitz Kaup Newell and Segur (AKNS) systems [17] and the

Wadati Konno and Ichikawa (WKI) system [19] to which belongs the Kraenkel-Manna-Merle (KMM) system, describing the propagation of magnetic field in ferromagnet with zero conductivity. The KMM-system has been investigated under some tractable methods including the Hirota's billinearization [30], inverse scattering transform [29] and most recently, the generalized $G'/G-espansion$ method [27] and auxiliary equation method [28]. In each of the cases, soliton solutions with loop-profile and singularity have been constructed and studied in details. In Ref. [30] the KMM-system has been generated while taking into account the Landau Lifshitz Gilbert damping. To our Knowledge, the contribution of this damping effect, characterized by a parameter s on the dynamic of magnetic waves in ferrites has only been investigated via the phase portraits analysis. The object of the present Chapter is to investigate analytically the influence of the damping and inhomogeneous effects on the dynamic of short waves in ferrites and to confirm analytical results while proceeding to some numerical simulations. Indeed, to carry out our gold, the mathematical toll described by Konno [101] is used to deal with the damping system. Since to our knowledge the Lax-pairs of this system is unknown till now, we provide a system that is Lax integrable and that approximate in some conditions which will be specify, the damped-KMM system. Under this conditions, we provide analytical expression of the solutions to the damped-KMM system. We investigate, numerical simulations to confirm the results obtained analytically and the predictions that have come out at the end of phase portraits analysis. Therefore, in order to take into account inhomogeneous effects on the dynamic of short waves in ferrites, we construct new traveling wave solution to the inhomogeneous system of our interest using Jacobi elliptic expansion method. We discuss the influence of the inhomogeneous exchange effects on the traveling waves and we address some physical implications. Finally, we then, deduce and approximated analytical solution of the system taking into account the combined effects of damping and inhomogeneous exchange. we proceed to some numerical solution to complete the analytical analysis of the system taking into account the combined effects of damping and inhomogeneous exchange.

3.1 Influence of damping effects

3.1.1 Construction of approximate system

While investigating the propagation of short-waves in saturated ferromagnetic materials with zero-conductivity in the presence of an external field, we have constructed in Chapter one the following nonlinear evolution system

$$\begin{cases} B_{xt} = BC_x - sB_x, \\ C_{xt} = -BB_x, \end{cases} \tag{3.1}$$

from the Maxwell's equations supplemented by the Landau-Lifshitz-Gilbert equation [25, 26, 30, 31], where the quantities B and C represent two physical observables standing for the magnetization density and the magnetic induction related to the ferrite respectively. The parameter s stands for the damping effects, while the subscripts t and x are denoting partial derivatives according to the time-like and space-like variables respectively. Considering the zero damping effects ($s = 0$), Kraenkel et al. [25] have provided via some transformations, one-soliton solution to the above system. Going forward in Ref. [30] and separately in [26, 33], after having investigated the phase portrait analysis of the system (3.1), these authors have investigated its integrability properties under the framework of prolongation structure and have provided associated Lax-pairs, opening the way in looking for soliton solutions via the inverse scattering transform method. But, instead of using the previously mentioned method, they derived related Hirota's bilinearization to system (3.1), they constructed its one- and two-soliton solutions, and studied in details their scattering properties. Therefore, the study of the Lax-pairs derived in Ref. [26,33] is worth underlying. Investigating the prolongation structure of the system (3.1), Kuetche et al. [26] have shown that the system is integrable under the condition ($s = 0$) while separately in Ref. [27] rich soliton solutions have also been provided under the same condition. The associated Lax-pairs being given as

$$y_x = \imath\lambda \begin{pmatrix} C_x & B_x \\ B_x & -C_x \end{pmatrix} y, \tag{3.2a}$$

$$y_t = \begin{pmatrix} 1/4\imath\lambda & -B/2 \\ B/2 & -1/4\imath\lambda \end{pmatrix} y, \tag{3.2b}$$

which give rise to Eq.(3.1) under the zero-curvature formulation, implying that the spectral parameter λ is a constant and $s = 0$. In order to take into account the damping effects, we then propose phenomenologically a new system, that is close to Eq.(3.1) as follows

$$B_{xt} = BC_x - sB_x, \tag{3.3a}$$

$$C_{xt} = -\alpha^2 BB_x, \qquad \alpha = \exp(st), \tag{3.3b}$$

with $s \neq 0$. α can be developed in Taylor series near the initial point as follows:

$$\exp(st) = 1 + st + \frac{(st)^2}{2} + \cdots\cdots \tag{3.4}$$

While considering small values of st that correspond to ultra-fast processes assumption, α may be restricted to the first order as, $\alpha \simeq 1$. At this condition, the system Eq.(3.3) appears to be a good approximation to the system Eq.(3.1). To investigate analytically the influence of such a parameter on the dynamic of waves in Ferrites, this equation will constitute our starting point along with the approximation condition on α.

It is easy to show that the above system Eq.(3.3) is completely integrable via some methods including prolongation structures analysis [102]. The proof of this integrability lies in its associated Lax-pairs which are expressed as

$$y_x = \imath\lambda \begin{pmatrix} C_x & \alpha B_x \\ \alpha B_x & -C_x \end{pmatrix} y, \tag{3.5a}$$

$$y_t = \begin{pmatrix} 1/4\imath\lambda & -\alpha B/2 \\ \alpha B/2 & -1/4\imath\lambda \end{pmatrix} y. \tag{3.5b}$$

These Lax-pairs will constitute the starting point to the investigation of the inverse scattering transform method.

If we take a varying magnetic field in a conductor, then there appears a rotating electric field around the magnet. In the Maxwell-Faraday equation, the rotational of electric field is proportional to the variation of magnetic field B. Indeed, it is the variation of the magnetic field that generates an electric field and not the magnetic field alone. If you place a magnet in a coil, nothing happens. On the other hand, if you move the magnet an electric field is created around it which itself will generate an electric current in the wire. That is why our bike's dynamo only powers the lights when you are driving and not at a standstill.

In this chapter, we derive two new equations starting with the nonlinear Klein-Gordon model describing the waves propagating in the positive and negative directions, respectively. We analyse the equation describing the waves propagating in the negative direction using the prolongation structure theory proposed by Wahlquist and Estabrook [1], which is a very useful and effective tool in the analysis of $(1+1)$-dimensional system.

3.1.2 Inverse scattering transform and soliton solution of damping KMM-system

According to the expression of the above Lax-pairs, we suitably follow the scheme described by Konno [101].

Indeed, we define the associated Jost functions

$$\left.\begin{array}{l} \phi \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-\imath\lambda x), \\ \bar{\phi} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(\imath\lambda x), \end{array}\right\} \texttt{as} \quad x \rightarrow -\infty, \tag{3.6}$$

and

$$\left.\begin{array}{l} \psi \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(\imath\lambda x), \\ \bar{\psi} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-\imath\lambda x) \end{array}\right\} \texttt{as} \quad x \rightarrow +\infty. \tag{3.7}$$

The functions $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$ being related by scattering coefficients as follows

$$\phi \;=\; a\bar{\psi} + b\psi, \qquad \bar{\phi} = -\bar{a}\psi + \bar{b}\bar{\psi}, \tag{3.8}$$

where $a, \bar{a}, b, \bar{b}$ verify

$$a\bar{a} + b\bar{b} = 1. \tag{3.9}$$

It is important to mention that for complex λ we have

$$\begin{pmatrix} \bar{\phi}_1(\lambda) \\ \bar{\phi}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \phi_2^\star(\lambda^\star) \\ -\phi_1^\star(\lambda^\star) \end{pmatrix}, \quad \begin{pmatrix} \bar{\psi}_1(\lambda) \\ \bar{\psi}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \psi_2^\star(\lambda^\star) \\ -\psi_1^\star(\lambda^\star) \end{pmatrix}, \tag{3.10}$$

from which one deduces naturally the relations

$$\bar{a}(\lambda) = a^\star(\lambda^\star), \qquad \bar{b}(\lambda) = b^\star(\lambda^\star). \tag{3.11}$$

We now investigate the asymptotic behavior of the Jost functions for large λ under the following boundary conditions

$$\left.\begin{array}{l} C_x \to 1 \\ B_x \to 0 \end{array}\right\} \quad as \quad |x| \to \infty. \tag{3.12}$$

Defining the quantity

$$\phi_1 = \exp\left(-\imath\lambda x + \int_{-\infty}^{x} \sigma(\lambda, l)\mathrm{d}l\right), \tag{3.13}$$

and substituting Eq.(3.13) into Eq.(3.5), we obtain

$$\sigma_t = \frac{1}{2\alpha}\left[\left(C_x + 1 - \frac{\sigma}{\imath\lambda}\right)\frac{B}{B_x}\right]_x, \tag{3.14a}$$

$$\sigma_x + \sigma^2 - 2\imath\lambda\sigma - \lambda^2 = \frac{B_{xx}}{B_x}\sigma - \imath\lambda B_x\left(\frac{C_x+1}{B_x}\right)_x - \lambda^2(C_x^2 + \alpha^2 B_x^2), \tag{3.14b}$$

where it appears clearly that σ is a conserved quantity. Let us now expand σ in power series of $\imath\lambda$ as follows

$$\sigma = \sum_{j=-1}^{\infty} \sigma_j (\imath\lambda)^{-j}, \tag{3.15}$$

which, substituted in Eq.(3.14) yields

$$\begin{aligned} \sigma_{n,x} + \sum_{j=-1}^{n+1} \sigma_j\sigma_{n-j} - 2\sigma_{n+1} - \frac{B_{xx}}{B_x}\sigma_n + B_x\left(\frac{C_x+1}{B_x}\right)_x \delta_{n,-1} \\ = (C_x^2 + \alpha^2 B_x \alpha^2 - 1)\delta_{n,-2}. \end{aligned} \tag{3.16}$$

Infinite number of conserved quantities can hence be generated, the first two of them being given as follows

$$\sigma_{-1} \;=\; 1 - \epsilon\sqrt{C_x^2 + \alpha^2 B_x^2}, \tag{3.17a}$$

$$\sigma_0 \;=\; \frac{B_x}{2(1-\sigma_{-1})}\left(\frac{\sigma_{-1} - C_x - 1}{B_x}\right)_x, \tag{3.17b}$$

with $\epsilon = \pm 1$. We then provide the asymptotic behavior of the function ϕ as to be

$$\phi = \begin{pmatrix} 1 \\ -\frac{C_x + \sqrt{C_x^2 + \alpha^2 B_x^2}}{\alpha B_x} \end{pmatrix} \exp\left[-\imath\lambda x + \imath\lambda \int_{-\infty}^{x} \sigma_{-1}\mathrm{d}l + \int_{-\infty}^{x} \sigma_0 \mathrm{d}l\right] + o(1/\lambda), \tag{3.18}$$

while the analytical property of the scattering coefficient a is written as

$$a = \lim_{x\to\infty} \phi_1 \exp(\imath\lambda x) = \exp\left[\imath\lambda \int_{-\infty}^{+\infty} \sigma_{-1}\mathrm{d}l + \int_{-\infty}^{+\infty} \sigma_0 \mathrm{d}l\right] + o(1/\lambda). \tag{3.19}$$

In the same way, we provide for $\bar{\phi}$, $\bar{\psi}$ and ψ the following asymptotic behavior, summarized as follows

$$\begin{aligned}
\phi &= \begin{pmatrix} 1 \\ -\frac{C_x+\sqrt{C_x^2+\alpha^2 B_x^2}}{\alpha B_x} \end{pmatrix} \exp(-\imath\lambda x + \imath\lambda\varepsilon_- + \mu_-) + o(1/\lambda), && (3.20a)\\
\bar{\phi} &= \begin{pmatrix} -\frac{C_x+\sqrt{C_x^2+\alpha^2 B_x^2}}{\alpha B_x} \\ -1 \end{pmatrix} \exp(\imath\lambda x - \imath\lambda\varepsilon_- + \mu_-) + o(1/\lambda), && (3.20b)\\
\psi &= \begin{pmatrix} \frac{C_x+\sqrt{C_x^2+\alpha^2 B_x^2}}{\alpha B_x} \\ 1 \end{pmatrix} \exp(\imath\lambda x + \imath\lambda\varepsilon_+ - \mu_+) + o(1/\lambda), && (3.20c)\\
\bar{\psi} &= \begin{pmatrix} 1 \\ -\frac{C_x+\sqrt{C_x^2+\alpha^2 B_x^2}}{\alpha B_x} \end{pmatrix} \exp(-\imath\lambda x - \imath\lambda\varepsilon_+ - \mu_+) + o(1/\lambda), && (3.20d)\\
a &= \exp(\imath\lambda\varepsilon + \mu) + o(1/\lambda), && (3.20e)
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_- &= \int_{-\infty}^{x} \sigma_{-1}\mathrm{d}l, \qquad \varepsilon_+ = \int_{x}^{\infty} \sigma_{-1}\mathrm{d}l, && (3.21a)\\
\varepsilon &= \varepsilon_- + \varepsilon_+ = \int_{-\infty}^{\infty} \sigma_{-1}\mathrm{d}l, && (3.21b)\\
\mu_- &= \int_{-\infty}^{x} \sigma_0 \mathrm{d}l, \qquad \mu_+ = \int_{x}^{\infty} \sigma_0 \mathrm{d}l, && (3.21c)\\
\mu &= \mu_- + \mu_+ = \int_{-\infty}^{\infty} \sigma_0 \mathrm{d}l. && (3.21d)
\end{aligned}$$

Considering that B_x and C_x have compact support, implying $\phi\exp[\imath\lambda(x-\varepsilon_-)]$, $\bar{\phi}\exp[-\imath\lambda(x-\varepsilon_-)]$, $\psi\exp[-\imath\lambda(x+\varepsilon_+)]$, $\bar{\psi}\exp[\imath\lambda(x+\varepsilon_+)]$ and $a\exp(-\imath\lambda\varepsilon)$ are entire functions of λ, We introduce the integral

$$I = \int_\Gamma \frac{\mathrm{d}\lambda'}{\lambda' - \lambda} \frac{1}{a(\lambda')\exp(-\imath\lambda'\varepsilon)} \begin{pmatrix} \phi_1(\lambda') \\ \phi_2(\lambda') \end{pmatrix} \exp[\imath\lambda'(x-\varepsilon_-)], \tag{3.22}$$

where (Γ) is standing for the contour in the complex λ' plane, starting from $\lambda' = -\infty + 0^+$, passing over all the zeros of $\mathsf{a}(\lambda')$ and ending at $\lambda' = +\infty + 0^+$ for λ' below (Γ). We now aim at deriving the Gel'fand-Levitan equations linked to our problem. From equation (3.7) combined with the last equation, yields

$$\begin{aligned} I &= \int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\varepsilon_+)] + \\ & \int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\varepsilon_+)]. \end{aligned} \tag{3.23}$$

The left-hand side of Eq.(3.23) reduces to

$$LHS = -\imath\pi\begin{pmatrix}1\\ -\frac{C_x+\sqrt{C_x^2+\alpha^2B_x^2}}{\alpha B_x}\end{pmatrix}\exp(-\mu_+), \tag{3.24}$$

and the right-hand side to

$$\begin{aligned} RHS &= -2\imath\pi\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\varepsilon_+)] \\ & +\imath\pi\begin{pmatrix}1\\ -\frac{C_x+\sqrt{C_x^2+\alpha^2B_x^2}}{\alpha B_x}\end{pmatrix}\exp(-\mu_+) \\ & +\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\varepsilon_+)]. \end{aligned} \tag{3.25}$$

Therefore,

$$\begin{aligned} \begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')\end{pmatrix}\exp[\imath\lambda(x+\varepsilon_+)] &= \begin{pmatrix}1\\ -\frac{C_x+\sqrt{C_x^2+\alpha^2B_x^2}}{\alpha B_x}\end{pmatrix}\exp(-\mu_+) \\ & +\frac{1}{2\imath\pi}\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\varepsilon_+)]. \end{aligned} \tag{3.26}$$

Because of their usefulness, the kernels K_1 and K_2 are introduced as follows

$$\begin{aligned} \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix}0\\ 1\end{pmatrix}\exp[\imath\lambda(x+\varepsilon_+(x))-\mu_+(x)] \\ & +\int_x^{\infty}\begin{pmatrix}\lambda K_1(x,z)\\ K_2(x,z)\end{pmatrix}\exp[\imath\lambda(z+\varepsilon_+(x))-\mu_+(x)]\mathrm{d}z, \end{aligned} \tag{3.27}$$

at the condition

$$\lim_{z\to\infty} K_1(x,z)=0, \qquad \lim_{z\to\infty} K_2(x,z)=0. \tag{3.28}$$

By a comparison of the equations (3.27) and (3.20), we obtain

$$K_1(x,x) = -\frac{C_x + \sqrt{C_x^2 + \alpha^2 B_x^2}}{\alpha B_x}. \tag{3.29}$$

Combining Eqs. (3.26) and (3.27), we obtain the Gel'Fand-Levitan equations as follows

$$K_1^\star(x,w) - F(x+w) - \int_x^\infty K_2(x,z)F(z+w)\mathrm{d}z = 0, \tag{3.30a}$$

$$K_2^\star(x,w) - \int_x^\infty K_1(x,z)F''(z+w)\mathrm{d}z = 0, \tag{3.30b}$$

at the condition $x \geq w$, where

$$F = \frac{1}{2\imath\pi}\int_C \frac{\mathrm{d}\lambda}{\lambda}\frac{b(\lambda)}{a(\lambda)} \exp[\imath\lambda(z + 2\varepsilon_+(x))], \tag{3.31a}$$

$$F'' = \frac{\partial^2 F}{\partial z^2} = \frac{1}{2\imath\pi}\int_C \lambda\mathrm{d}\lambda\frac{b(\lambda)}{a(\lambda)} \exp[\imath\lambda(z + 2\varepsilon_+(x))]. \tag{3.31b}$$

The time dependence of the scattering data is provided by Eq.(3.5) to be

$$a(\lambda,t) = a(\lambda,0), \tag{3.32a}$$

$$b(\lambda,t) = b(\lambda,0)\exp(\imath t/(2\lambda)). \tag{3.32b}$$

The bound states are given by the zeros of $a(\lambda)$ in the upper-half plane. When all the zeros of $a(\lambda)$ in the upper-half plane are simple, $F(z)$ can be expressed as

$$\begin{aligned} F(z) &= \sum_j^n \frac{c_j(t)}{\lambda_j}\exp[\imath\lambda_j(z + 2\varepsilon_+(x))] \\ &\quad + \frac{1}{2\imath\pi}\int_{-\infty}^{+\infty}\frac{\rho(\lambda,t)}{\lambda}\exp[\imath\lambda(z + 2\varepsilon_+(x))], \end{aligned} \tag{3.33}$$

where

$$c_j(t) = c_j(0)\exp(\imath t/2\lambda_j), \tag{3.34a}$$

$$\rho(\lambda,t) = \rho(\lambda,0)\exp(\imath t/2\lambda). \tag{3.34b}$$

It is then possible to solve the Gel'fand-Levitan equations after having determined $F(z)$ to obtain $K_1(x,x)$, when the scattering data $\{\rho(\lambda,0), \lambda, c_j(0), \lambda_j, j = 1, \cdots, N\}$ are given and, consequently, the solution is obtained using the Eq.(3.29).

The soliton solution to system (3.3) is obtained under the conditions $\rho(\lambda,t) = 0$. Therefore, we find for F the following expression

$$F(z,t) = \sum_{j=1}^N \frac{c_j(t)}{\imath\eta_j}\exp[-\eta_j(z + 2\varepsilon_+)], \tag{3.35}$$

where we have set $\lambda_j = \imath\eta_j$. Let us now introduce the representation of the kernels in an identical form of F in order to facilitate the resolution of the Gel'Fand-Levitan equations, as follows

$$K_1(x,z) = \sum_{j=1}^{N} A_j(x)\exp[-\eta_j(x+z+2\varepsilon_+(x))], \tag{3.36a}$$

$$K_2(x,z) = \sum_{j=1}^{N} B_j(x)\exp[-\eta_j(x+z+2\varepsilon_+(x))], \tag{3.36b}$$

where A_j and B_j are complex functions. Substituting Eqs.(3.35)-(3.36) into Eq.(3.30), yields

$$A_k - \frac{c_k}{\imath\eta_k}\sum_{j=1}^{N}\frac{B_j}{\eta_j+\eta_k}\exp[-2\eta_j(x+\varepsilon_+)] = \frac{c_k}{\imath\eta_k}, \tag{3.37a}$$

$$B_j + \imath c_j\eta_j\sum_{m=1}^{N}\frac{A_m}{\eta_j+\eta_m}\exp[-2\eta_m(x+\varepsilon_+)] = 0, \tag{3.37b}$$

which combined, gives

$$A_k + \frac{c_k}{\eta_k}\sum_{j=1}^{N}\sum_{m=1}^{N}\frac{c_j\eta_j A_m}{(\eta_j+\eta_k)(\eta_j+\eta_m)}e^{-2(\eta_j+\eta_m)(x+\varepsilon_+)} = \frac{c_k}{\imath\eta_k}. \tag{3.38}$$

Thus,

$$K_1(x,x) = \sum_{j=1}^{N} A_j\exp[-2\eta_j(x+\varepsilon_+)]. \tag{3.39}$$

From equation (3.27), we find that

$$\frac{\alpha B_x}{C_x} = \frac{2K_1(x,x)}{K_1^2(x,x)-1}. \tag{3.40}$$

Using Eqs. (3.21) and (3.17), we find the relation

$$\frac{\partial h}{\partial x} = \sqrt{C_x^2+\alpha^2 B_x^2}, \tag{3.41}$$

where we have set $h = x + \varepsilon_+(x,x)$. For K_1, B_x and C_x are all of them dependent of $x+\varepsilon_+(x,x)$, it is reasonable to transform the independent variable x to the new variable h while looking for the soliton solutions to Eq.(3.3). Making use of the new independent variable, we obtain

$$\frac{\partial(\alpha B)}{\partial h} = \frac{2K_1(h)}{1+K_1^2(h)}, \tag{3.42a}$$

$$\frac{\partial C}{\partial h} = 1 - \frac{2K_1^2(h)}{1+K_1^2(h)}, \tag{3.42b}$$

$$\frac{\partial \varepsilon_+}{\partial h} = \frac{K_1(h)^2}{1+K_1^2(h)}, \tag{3.42c}$$

where $K_1(h) = K_1(x, x)$. By integration of the previous two equations with respect to h, one obtains at the required order N, the required number of soliton solutions to the coupled system (3.3).

For the case $N = 1$ corresponding to the one-soliton solution, it is found that K_1 has the expression

$$K_1(h) = \frac{c}{\imath\eta} \frac{\exp(-2\eta h)}{1 - \frac{c^2}{4\eta^2}\exp(-4\eta h)}. \tag{3.43}$$

It comes by simple integration of (3.42), that

$$\begin{aligned} \alpha B &= -\frac{c}{\imath\eta^2}\frac{\exp(-2\eta h)}{T}, && (3.44a)\\ C &= h - \frac{2}{\eta T}, && (3.44b)\\ \varepsilon_+ &= \frac{1}{\eta T}, && (3.44c) \end{aligned}$$

with

$$\begin{aligned} c &= c_0 \exp(-t/2\eta), && (3.45a)\\ T &= 1 - \frac{c^2}{4\eta^2}\exp(-4\eta h). && (3.45b) \end{aligned}$$

Eq.(3.44) is rewritten as

$$\begin{aligned} B &= \frac{e^{-st}}{\eta} sech(\varphi + \varphi_0), && (3.46a)\\ C &= h - \frac{1}{\eta}(tanh(\varphi + \varphi_0) + 1), && (3.46b)\\ \varepsilon_+ &= \frac{1}{2\eta}(tanh(\varphi + \varphi_0) + 1), && (3.46c)\\ \varphi &= 2\eta\left(h + \frac{t}{4\eta^2}\right), && (3.46d)\\ \varphi_0 &= -\frac{1}{2\eta}\ln\left(\frac{\imath c_0}{2\eta}\right) && (3.46e) \end{aligned}$$

which stands for the one soliton solution to the system Eq.(3.3), the envelop moving at the velocity $V = 1/(4\eta^2)$. We recall that solution to Eq.(3.1) is obtained when considering the approximation $\alpha^2 \simeq 1$. The amplitude of B is an exponential function of st which guarantee the increasing or the decreasing amplitude of the wave with time, depending on the values of s and t simultaneously. We point out that when we consider $s = 0$,

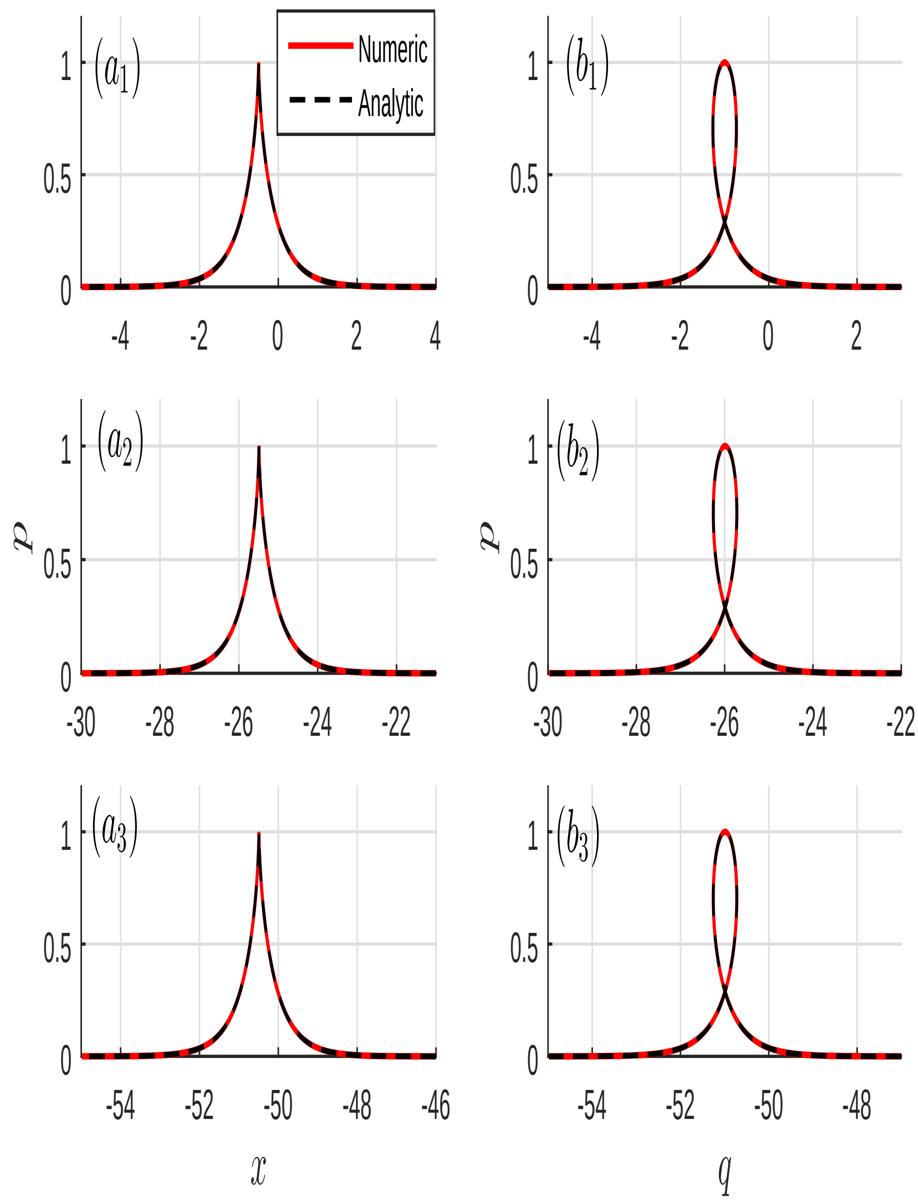

Figure 6: Propagation of one-soliton solution B to the KMM-system Eq.(3.1) for $s = 0$, with the eigenvalue $\eta = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels $(a1)$, $(a2)$ and $(a3)$) and versus the Magnetization induction C (panels $(b1)$ $(b2)$ and $(b3)$). The analytical and numerical solutions that show a left moving wave, that conserves its profile as predicted in Refs. [18, 30]

we recover the solution of the original KMM-system, free of damping effect. It appears that for $s > 0$ and while considering the ultra-fast process characterized by Eq.(3.4), the wave propagate conserving profile but not amplitude. Such a result has been predicted in Ref. [33] under phase portraits analysis.

3.1.3 Numerical simulations of KMM-system with and without damping

In order to complete and confirm the results obtained analytically, we proceed to some numerical simulation. We use as base equation the dissipative KMM-system Eq.(3.1). To achieve this goal, we use the finite difference method with initial conditions given as:

$$B(x,0) = \frac{1}{\eta}\mathtt{sech}\left(2\eta h + \varphi_0\right), \tag{3.47a}$$

$$C(x,0) = h - \frac{1}{\eta}\mathtt{tanh}\left(2\eta h + \varphi_0\right) - \frac{1}{\eta}, \tag{3.47b}$$

$$x(h,0) = h - \frac{1}{2\eta}\mathtt{tanh}\left(2\eta h + \varphi_0\right) - \frac{1}{2\eta}. \tag{3.47c}$$

The discretization of KMM-system free of damping effects is given as follows

$$B_{i+1,j+1} = B(i,j+1) + B(i+1,j) - B(i,j) + \Delta t B(i,j)(C(i+1,j) - C(i,j)) \tag{3.48a}$$

$$C_{i+1,j+1} = C(i,j+1) + C(i+1,j) - C(i,j) - \Delta t C(i,j)(B(i+1,j) - B(i,j)) \tag{3.48b}$$

where $x = i\Delta x$ and $t = j\Delta t$.

For the case of the original KMM-system free of damping effects (characterized by $s = 0$), we present in Figure (6) a result that is in accordance with the results obtained analytically in Refs. [18, 26]. It is observed clearly on this figure that numerical and analytical results match perfectly, which proves that the above numerical simulation carried out is good. We can then pursue with similar analysis while considering damping effects.

We present in Figure (7) the propagation of waves in magnetic slab subjected to damping effects. For this aim, we present analytical results and numerical simulations for the parameter $s = 20$ (which we have chosen arbitrarily without loss of generality), of the Magnetic field B versus x and the magnetic field B versus the magnetization C. We observe also a good coincidence as for the case of the system free of damping effects. The waves evolve with amplitude that decrease with time due to the damping parameter s.

It appears clearly that the numerical analysis confirms the analytical results discussed in the previous section.

On the regards of the above results, the parameter s, originated from the Landau-Lifshitz-Gilbert damping, plays a role on the dynamic of magnetic waves in ferrites. This role is characterized by the dissipation of energy during the motion of wave in ferrite observed here on the amplitude of the wave that reduces as it moves. Since the dissipative KMM system Eq.(3.1) has been derived under the ultra fast and near adiabatic process assumption, the value of the the damping parameter s guarantee whether or not wave survive in the ferrite. As observed analytically, for $s \geq 0$ if:

(i) $s = 0$, we are in an ideal case where magnetic wave moves in ferrite conserving it entire properties, profile and amplitude.

(ii) s is not great enough so that the amplitude of the wave diminish with a rate $B_{i+1}/B_i > 0.5$, the wave propagate in the material and may disappear if the process is not fast enough

(iii) $s \to \infty$ the amplitude of the wave vanishes rapidly and the wave can not propagate in such a media.

3.2 Impact of inhomogeneous exchange effects

3.2.1 Jacobi elliptic sine function expansion

To investigate solutions to nonlinear evolution equation, one often has recourse to expansion of known solution. It is in this idea that for example in Ref. [26], G'/G-expansion method has been used to construct solution to nonlinear evolution equation. We follow a similar procedure while expanding elliptic functions to derive solutions of the following system

$$B_{xt} - BB_x = \rho B_{xx}, \tag{3.49a}$$

$$C_{xt} + BB_x = 0, \tag{3.49b}$$

which describes the propagation of magnetic waves in ferrites, the parameter ρ representing inhomogeneous parameter, u represents the magnetic field and v represents magneti-

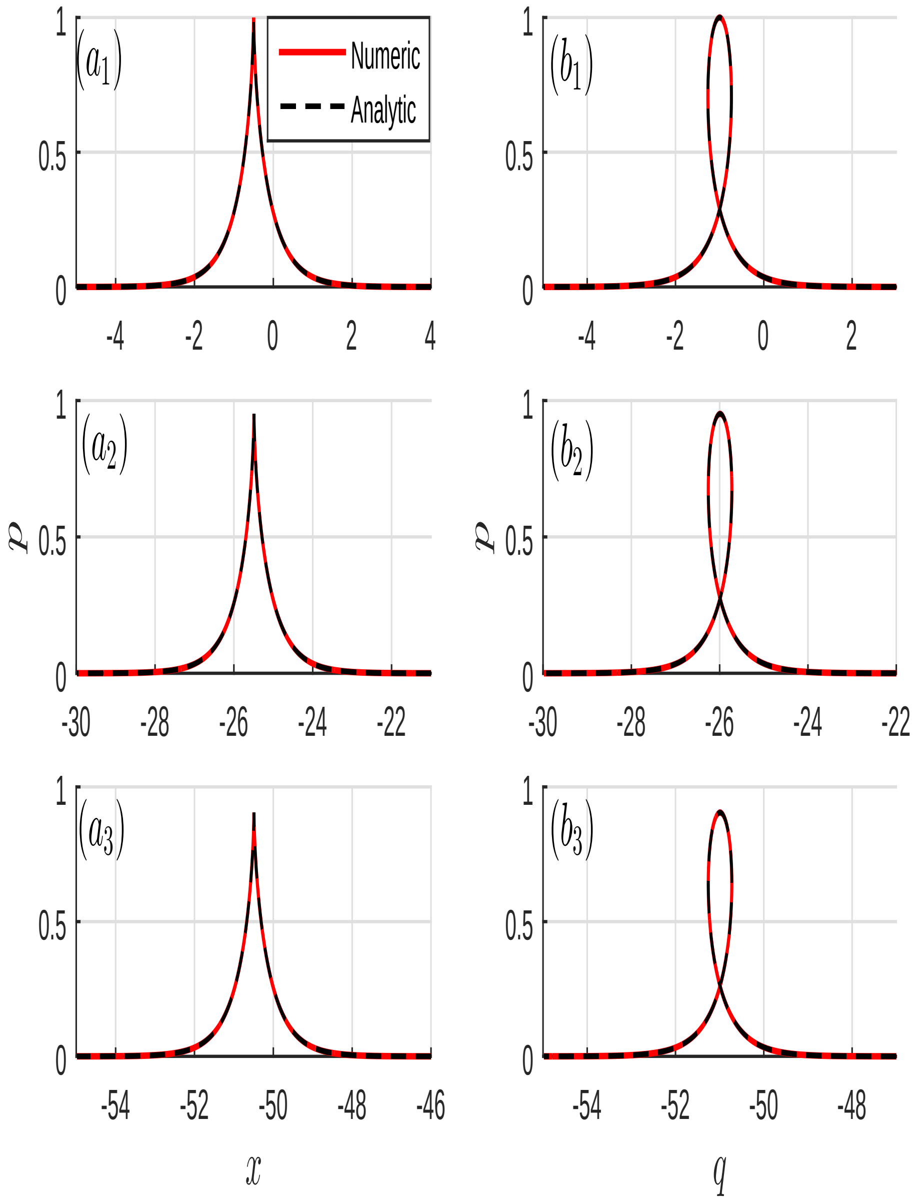

Figure 7: Propagation of one-soliton solution B to the KMM-system Eq.(3.1) for $s = 20$, with the eigenvalue $\eta = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels ($a1$), ($a2$) and ($a3$)) and versus the Magnetization induction C (panels ($b1$) ($b2$) and ($b3$)). The analytical and numerical solutions showing a left moving wave, that conserves its profile but not its amplitude as predicted in Ref. [30]

zation. This system has been derived from the Maxwell's equations supplemented by the Landau-Lifshitz-Gilbert equation [25] in which the damping effects are neglected.

Jacobi elliptic functions, are periodic functions that properties are in someway similar to the well known periodic functions. As illustration, let us consider the following integral

$$R(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-m^2t^2}}. \tag{3.50}$$

The sine elliptic function *sn* is defined as follows

$$\mathrm{sn}(x,m) = R^{-1}(x), \quad 0 \leq m \leq 1. \tag{3.51}$$

Beside the sine elliptic function is defined, the other elliptic functions satisfy the relations

$$\mathrm{cn}^2(x,m) = 1 - \mathrm{sn}^2(x,m), \tag{3.52a}$$
$$\mathrm{dn}^2(x,m) = 1 - m^2\mathrm{sn}^2(x,m), \tag{3.52b}$$

where *cn* is the cosine elliptic function while *dn* is the elliptic function of the third kind. The derivative of the above elliptic functions can be expressed as follows Refs. [73–75]:

$$\frac{d}{dx}\mathrm{sn}(x) = \mathrm{cn}(x)\mathrm{dn}(x), \tag{3.53a}$$
$$\frac{d}{dx}\mathrm{cn}(x) = -\mathrm{sn}(x)\mathrm{dn}(x), \tag{3.53b}$$
$$\frac{d}{dx}\mathrm{dn}(x) = -m^2\mathrm{sn}(x)\mathrm{cn}(x). \tag{3.53c}$$

Before proceeding to the determination of solution of the system (3.49), it is necessary to introduce the variable transformation

$$\eta = kx + \omega t + \eta_0. \tag{3.54}$$

Under this transformation, one replaces $B(x,t)$ by $B(\eta)$, $C(x,t)$ by $C(\eta)$ and the system (3.49) is rewritten as follows

$$(\omega - k\rho)B_{\eta\eta} = a_0 B - \frac{1}{2\omega}B^3, \tag{3.55a}$$
$$\omega C_{\eta\eta} = -BB_\eta, \tag{3.55b}$$

where a_0 is an integration constant. Under the new variable transformation (3.54), we now look for solutions of the system (3.55) while proceeding to elliptic function expansions.

3.2.2 Jacobi elliptic sine function

Using the Jacobi elliptic function method, we set the ansatz as a finite series of elliptic sine function as follows

$$B(\eta) = \sum_{j=0}^{N} p_j \mathrm{sn}^j(\eta). \tag{3.56}$$

According to the homogeneous balance principle, ne know $N = 1$. So we have

$$B(\eta) = p_0 + p_1 \mathrm{sn}(\eta). \tag{3.57}$$

replacing Eq.(3.57) into Eq.(3.55) we obtain as a result

$$p_0 = 0, \tag{3.58a}$$

$$p_1 = \pm 2m\sqrt{\omega(k\rho - \omega)}, \tag{3.58b}$$

$$a_0 = (1 + m^2)(k\rho - \omega), \tag{3.58c}$$

and the explicit form of the solution are given as follows

$$B(\eta) = \pm 2m\sqrt{\omega(k\rho - \omega)}\mathrm{sn}(\eta), \tag{3.59a}$$

$$C(\eta) = (m^2 - 1)(k\rho - \omega)\eta + 2(k\rho - \omega)E(\eta, m), \tag{3.59b}$$

$$E(\eta, m) = \int_0^{\eta} \mathrm{dn}^2(s)ds. \tag{3.59c}$$

where ω, k and ρ are chosen in such a way that $\omega(k\rho - \omega) \geq 0$. When $m \longrightarrow 1$, Eq.(3.59) reduces to

$$B(x, t) = \pm 2\sqrt{\omega(k\rho - \omega)}\tanh(kx + \omega t + \eta_0), \tag{3.60a}$$

$$C(x, t) = 2(k\rho - \omega)\tanh(kx + \omega t + \eta_0). \tag{3.60b}$$

From this Eq.(3.60), it is clear that while drawing B versus C, one will obtain a straight line.

3.2.3 Jacobi elliptic cosine function expansion

After the sine elliptic function expansion, we can also set an ansatz constituted of expansion of elliptic cosine function as follow

$$B(\eta) = \sum_{j=0}^{N} q_j \mathrm{cn}^j(\eta). \tag{3.61}$$

As for the previous analysis, we restrict our attention to the first order, that is

$$B(\eta) = q_0 + q_1 \mathrm{cn}(\eta). \tag{3.62}$$

Replacing Eq.(3.62) into Eq.(3.55), one obtains

$$q_0 = 0, \tag{3.63a}$$
$$q_1 = \pm 2m\sqrt{\omega(\omega - k\rho)}, \tag{3.63b}$$
$$a_0 = (2m^2 - 1)(\omega - k\rho), \tag{3.63c}$$

solution being explicitly expressed as

$$B(\eta) = \pm 2m\sqrt{\omega(\omega - k\rho)}\mathrm{cn}(\eta), \tag{3.64a}$$
$$C(\eta) = (\omega - k\rho)\eta - 2(\omega - k\rho)E(\eta, m). \tag{3.64b}$$

For $m \longrightarrow 1$, Eq.(3.64) reduces to

$$B(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{sech}(kx + \omega t + \eta_0), \tag{3.65a}$$
$$C(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). \tag{3.65b}$$

3.2.4 Jacobi elliptic function of the third kind method

Similarly to the previous analysis, we set the following ansatz that is an expansion of the elliptic function of the third kind as follows

$$B(\eta) = \sum_{j=0}^{N} r_j \mathrm{dn}^j(\eta). \tag{3.66}$$

Restriction made to the first order, u gives

$$B(\eta) = r_0 + r_1 \mathrm{dn}(\eta). \tag{3.67}$$

By replacing restriction into Eq.(3.55), we obtain as a result

$$r_0 = 0, \tag{3.68a}$$
$$r_1 = \pm 2\sqrt{\omega(\omega - k\rho)}, \tag{3.68b}$$
$$a_0 = (2 - m^2)(\omega - k\rho), \tag{3.68c}$$

which help writing u and v in their explicit form

$$B(\eta) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{dn}(\eta), \tag{3.69a}$$

$$C(\eta) = (2 - m^2)(\omega - k\rho)\eta - 2(\omega - k\rho)E(\eta, m). \tag{3.69b}$$

For $m \longrightarrow 1$, Eq.(3.69) reduces to

$$B(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{sech}(kx + \omega t + \eta_0), \tag{3.70a}$$

$$C(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). \tag{3.70b}$$

3.2.5 Jacobi elliptic function $cs(\eta)$ method

We now proceed to the expansion of the $cs(\eta)$ elliptic function as follows

$$B(\eta) = \sum_{j=0}^{N} \theta_j \mathrm{cs}^j(\eta), \tag{3.71}$$

where $cs(\eta) = cn(\eta)/sn(\eta)$. Restriction to the first order allow us to write

$$B(\eta) = \theta_0 + \theta_1 \mathrm{cs}(\eta). \tag{3.72}$$

As for the previous cases, while inserting the previous Eq.(3.72) into Eq.(3.55) one obtains

$$\theta_0 = 0, \tag{3.73a}$$

$$\theta_1 = \pm 2\sqrt{\omega(\omega - k\rho)}, \tag{3.73b}$$

$$a_0 = (2 - m^2)(\omega - k\rho), \tag{3.73c}$$

and with the help of this parameters, the solutions are expressed as follows

$$B(\eta) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{cs}(\eta), \tag{3.74a}$$

$$C(\eta) = (2 - m^2)(\omega - k\rho)\eta - 2(\omega - k\rho)\int \mathrm{cs}^2(\eta) d\eta. \tag{3.74b}$$

When $m \longrightarrow 1$, Eq.(3.74) reduces to

$$B(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{csch}(kx + \omega t + \eta_0), \tag{3.75a}$$

$$C(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) + 2\frac{\omega - k\rho}{\tanh(kx + \omega t + \eta_0)}. \tag{3.75b}$$

3.2.6 Rational function solution

To determine algebraic solution to the system of our interest, we set $a_0 = 0$. Multiplying Eq.(3.55) with u_η one obtains

$$
\begin{aligned}
(\omega - k\rho)B_\eta B_{\eta\eta} &= -\frac{1}{2\omega}B_\eta B^3, && (3.76a)\\
\omega C_{\eta\eta} &= -BB_\eta, && (3.76b)
\end{aligned}
$$

and by simple integration we obtain

$$
\begin{aligned}
B(x,t) &= \pm 2\frac{\sqrt{\omega(k\rho - \omega)}}{kx + \omega t + \eta_0}, && (3.77a)\\
C(x,t) &= 2\frac{k\rho - \omega}{kx + \omega t + \eta_0} + C_0, && (3.77b)
\end{aligned}
$$

which, when depicting u versus v with $C_0 = 0$, one will obtain a straight line as for limiting case obtained in Eq.(3.60). All the solutions obtained above are new and deserve many attentions in investigating the influence of inhomogeneous exchange effect on the dynamics of magnetic wave in ferrites. It is also important to point out the fact that solutions given by Eqs.(3.60) and (3.77) are particular because they are due to the presence of inhomogeneities, while the other solutions can exist even in the absence of inhomogeneities.

3.2.7 Influence of the inhomogeneous exchange effects on the dynamics of magnetic waves in ferrites

As pointed out before, the expansion of Jacobi elliptic functions has allow to construct a series of nonlinear wave solution and periodic solution to the system Eq.(3.49). We now aim at using some of these solutions to describe the propagation of magnetic waves in inhomogeneous ferrites. The solutions that particularly retain our attention are

$$
\begin{aligned}
B(x,t) &= \pm 2\sqrt{\omega(\omega - k\rho)}\operatorname{sech}(kx + \omega t + \eta_0), && (3.78a)\\
C(x,t) &= (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). && (3.78b)
\end{aligned}
$$

These solutions are different from the one obtained in Ref. [30], where solutions are given under the form

$$
\begin{aligned}
B(x,t) &= \pm 2\sqrt{\omega/k}\operatorname{sech}(kx + \omega t + \eta_0), && (3.79a)\\
C(x,t) &= x - (2/k)\tanh(kx + \omega t + \eta_0). && (3.79b)
\end{aligned}
$$

Its comes clearly that to the expression given in Eq.(3.79) one must add to x a time-like parameter. There is an information that does not appears directly through this method, that is the dispersion relation. Such a relation is important to choose appropriately the parameters k standing for wave number, and the parameter ω standing for frequency. But in Ref. [30], such a relation has been given that from the Hirota's bilinear scheme. From the results obtained above, it is possible to derive this dispersion relation. Indeed, let us recall the system of our interest:

$$\begin{aligned}(\omega - k\rho)u_{\eta\eta} &= a_0 B - \frac{1}{2\omega}B^3,\\ \omega C_{\eta\eta} + BB_\eta &= 0.\end{aligned}$$

By simple integration, the second of the above system can be rewritten as follows

$$C_\eta = -\frac{1}{2\omega}B^2 + a_0, \tag{3.80}$$

where a_0 is the constant of integration. Using the solutions given in Eq.(3.70), we easily obtain the relation

$$\omega - k\rho = a_0. \tag{3.81}$$

Considering in particular that $a_0 = 1/k$, Eq.(3.81) becomes

$$k\omega - k^2\rho = 1, \tag{3.82}$$

which is the dispersion relation that has been obtained in Ref. [30]. We can choose arbitrarily without loss of generality $\omega = 1$, to go further with our analysis. Therefore, using this value of ω, solution provided in Eq.(3.70) can be rewritten as follows

$$B(x,t) = \pm 2\sqrt{1-k\rho}\,\mathrm{sech}(kx + t + \eta_0), \tag{3.83a}$$

$$C(x,t) = (1-k\rho)(kx+t+\eta_0) - 2(1-k\rho)\tanh(kx+t+\eta_0). \tag{3.83b}$$

Deriving this solution with respect to η, one obtains

$$B_\eta = \pm\sqrt{1-k\rho}\,\sinh(\eta)\,\mathrm{sech}^2(\eta), \tag{3.84a}$$

$$C_\eta = (1-k\rho)(1-2\,\mathrm{sech}^2(\eta)). \tag{3.84b}$$

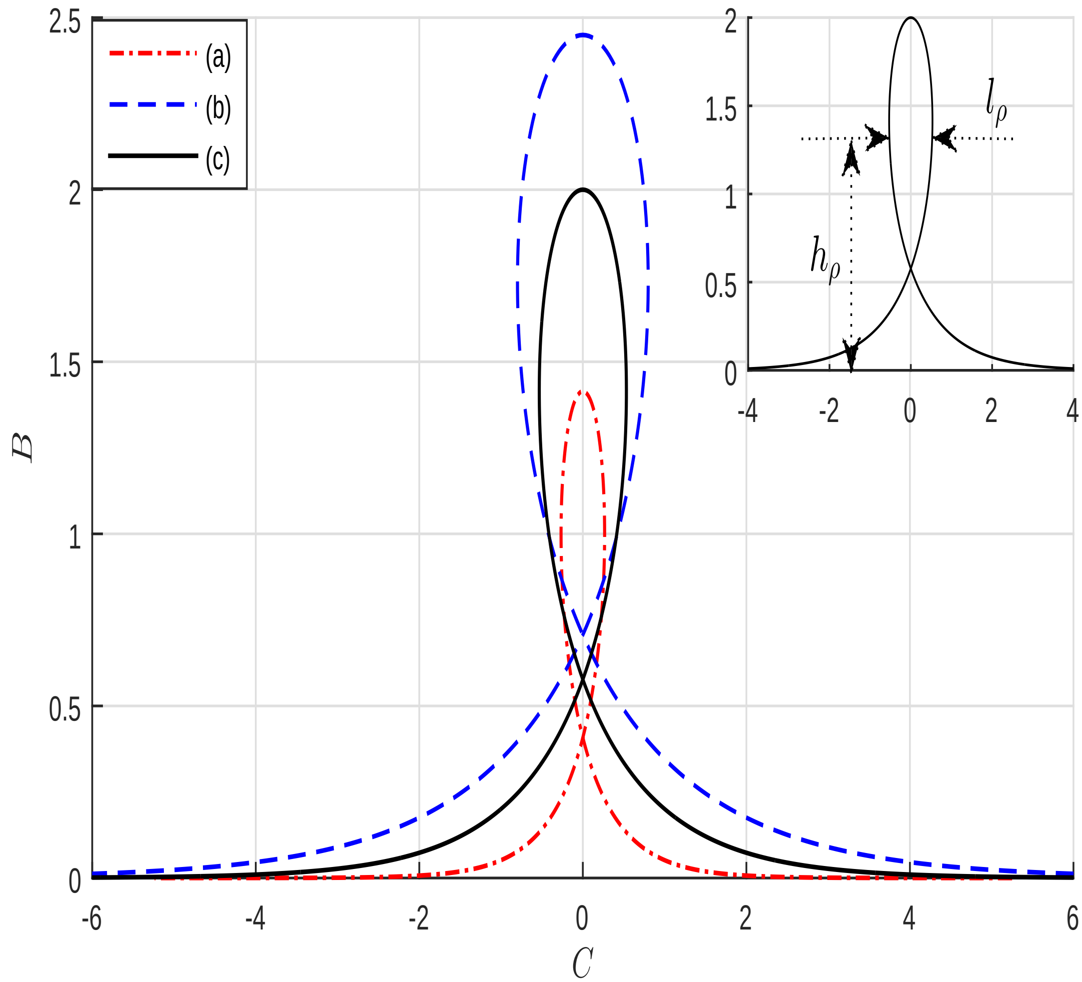

Figure 8: One-soliton solution B to the system Eq.(3.49), with the wave number $k = 0.5$, (a) $\rho = 1$ (dashed-dotted line), (b) $\rho = -1$ (dashed line) and (c) $\rho = 0$ (solid line) at times $t = 0$, depicted versus the magnetization v. One observes that, inhomogeneous term contribute by increasing the amplitude of the wave when ρ takes negative values, or contribute by slowing down this amplitude when ρ is taken positive. One also observe that the parameter ρ also has a contribution on the maximum width of the loop soliton l_ρ, that increases or decreases as the amplitude of the wave increases or decreases. The hight h_ρ at which this maximum width of the wave occurs is also subjected to changes in the same way as maximum width.

From this system Eq.(3.84), it comes that B_η is positive definite for $\eta \in]-\infty, 0]$ and negative definite for $\eta \in [0, +\infty[$. It also comes that $C_\eta = 0$ admits two solutions η_1 and η_2. Then, v_η is positive definite for $\eta \in]-\infty, \eta_1] \cup [\eta_2, +\infty[$ and negative definite for $\eta \in]\eta_1, \eta_2[$. It is then easy to conclude from the above analysis that v admits two extrema and changes direction twice, while in the same time u admits one extremum and changes direction ones. Therefore, by depicting u versus v, one obtains a loop soliton as presented in Fig.(**??**).

In Eq.(3.84), one observes clearly that the wave amplitude depends on the inhomogeneous parameter ρ. To complete our analysis on the impact of the inhomogeneous term on the dynamics of the wave, we define the parameter l_ρ as the maximum width of the wave and h_ρ the hight at which it occurs (see Figure (8)). Then by direct calculus, these parameters can be expressed as follows

$$l_\varrho = 2(1-k\rho)\left(\sqrt{2} - \text{arctanh}\left(\frac{\sqrt{2}}{2}\right)\right), \tag{3.85a}$$

$$h_\varrho = \sqrt{2(1-k\rho)}, \tag{3.85b}$$

which allow the following discussion:

(a) By setting $\rho = 0$ one obtains the solution of the system free of inhomogeneous term. Such a solution has been provided in Refs. [14, 16, 18, 29, 71] under some tractable methods including the Hirota's billinearization method, inverse scattering transform, expansion functions, just to name a few. This case then constitutes the reference case that serves to evaluate the impact of the inhomogeneous parameter on the dynamics of the wave.

(b) Choosing ρ to be negative ($\rho < 0$), the amplitude of the wave increases. The maximum width l_ρ of the wave also increases along with the height h_ρ at which this maximum width occurs. The inhomogeneous exchange effects act here by amplifying the pulse in ferrites.

(c) While selecting ρ to be positive valued ($\rho > 0$), the amplitude of the wave decreases along with l_ρ and h_ρ. This means that, inhomogeneous exchange effects contribute by reducing the pulse profile in ferrite.

As illustration, we depict in Figure (8) the loop-like soliton solution of the system (3.49) for different values of the parameter ρ where all the conclusions that have emerged in the above items, are observed.

In fact, inhomogeneities known as deformations in a system, can be due to external fields or to the presence of defects, voids and gaps in the materials, and has a significant effect on the magnetization dynamics of the ferromagnet. The soliton responsible for the localization is deformed by the presence of inhomogeneities and in particular its structure. The presence of inhomogeneity delivers dissipation into the system which supports soliton excitation, but can also inject energy into it.

3.3 Analytical investigation of the solutions to KMM-system taking into account inhomogeneous exchange and damping effects

3.3.1 The KMM-system

While investigating the propagation of short-waves in saturated ferromagnetic materials with zero-conductivity in the presence of an external field, Kraenkel and co-authors [25] have constructed the following nonlinear evolution system

$$B_{xt} = BC_x, \tag{3.86a}$$

$$C_{xt} = -BB_x, \tag{3.86b}$$

from the Maxwell's equations, where B is standing for the magnetic field, when C is standing for the external magnetization. The subscripts x and t denoting partial derivatives according to space-like and time-like variables respectively. Integrability properties of such a system has been investigated intensively under some framework including prolongation structure analysis, Hirota's bilinearization [26], and inverse scattering transform [27]. As a result, multi-soliton solutions have been predicted and constructed, the expression of the one soliton solution being given as:

$$\begin{aligned} B &= -\frac{c}{\imath\eta^2}\frac{\exp(-2\eta h)}{T}, && (3.87a)\\ C &= h-\frac{2}{\eta T}, && (3.87b)\\ \varepsilon_+ &= \frac{1}{\eta T}, && (3.87c) \end{aligned}$$

where ε_+ is an explicit function coming from a variable change connected to x as: $x = h-\varepsilon_+$, details of such a result being provided in Ref. [29] where the full inverse scattering method has been helpful in constructing such a result. The functions c and T are given as follows

$$\begin{aligned} c &= c_0\exp(-t/2\eta), && (3.88a)\\ T &= 1-\frac{c^2}{4\eta^2}\exp(-4\eta h). && (3.88b) \end{aligned}$$

The envelop moving left at the constant velocity $\mathfrak{v} = 1/(4\eta^2)$. This solution has been shown to be of loop-shape when depicting B versus C [26, 29] and of spike shape when depicting B versus x [29]. The solution Eq.(3.87) will serve to discuss the contribution of additional terms that occur in the system taking into account inhomogeneous exchange and damping effects.

3.3.2 The KMM-system with damping effects

In the windings of Kraenkel, Manna and Merle ideas, Nguepjouo and coworkers [30] have derived, while considering the Landau-Lifschitz-Gilbert damping, the system

$$\begin{aligned} B_{xt} &= BC_x - sB_x, && (3.89a)\\ C_{xt} &= -BB_x, && (3.89b) \end{aligned}$$

where s is a constant parameter that translates the damping effects on the dynamic of waves in ferrites. The initial value problem of this equation has also been investigated in details by the same authors and, as conclusion, it has appeared that the energy of the system decreases as time evolves. But till now, analytical expression of the associated solution has not been provided. It has not even been proven yet that this equation is integrable or not. Paying attention of such a question of integrability, we propose

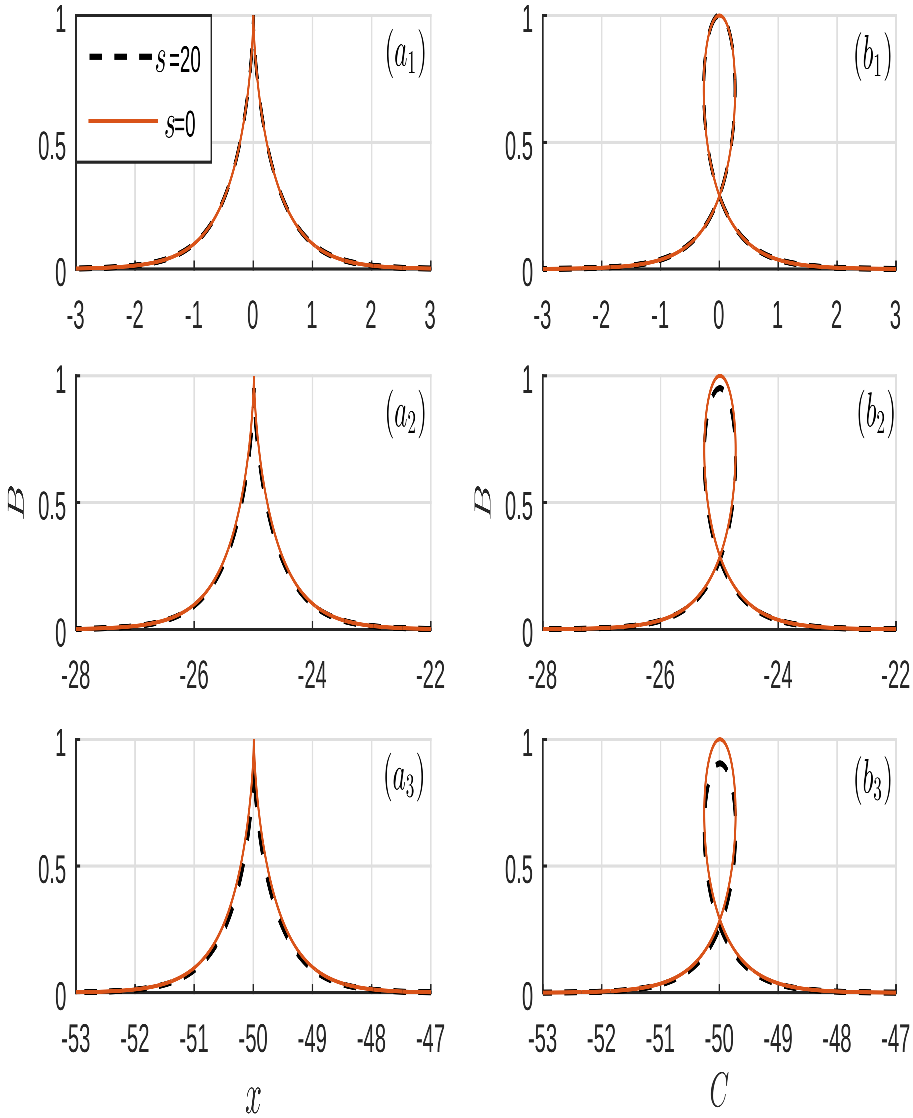

Figure 9: Propagation of one-soliton solution B to the system Eq.(3.89) for $s = 0$ (solid line) and $s = 20$ (dashed line), with the eigenvalue $\eta = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). The solid line ($s = 0$) is the wave solution of the KMM-system Eq.(3.86) while the dashed line (s=20) is standing for the approximated solution of Eq.(3.89). It is observed on this figure that damping acts principally on the amplitude of the wave while its influence on the width is very low.

to investigate solution to a nonlinear system from which, through and approximation one obtains the solutions to the system Eq.(3.88). The system we are talking about is expressed as follows:

$$B_{xt} = BC_x - sB_x, \tag{3.90a}$$

$$C_{xt} = -\alpha^2 BB_x, \qquad \alpha = \exp(st), \tag{3.90b}$$

with $s \neq 0$. α can be developed in Taylor series near the initial point as follows:

$$\exp(st) = 1 + st + \frac{(st)^2}{2} + \cdots\cdots \tag{3.91}$$

This equation Eq.(3.90) has the advantage to possess associated Lax-pairs:

$$y_x = \imath\lambda \begin{pmatrix} C_x & \alpha B_x \\ \alpha B_x & -C_x \end{pmatrix} y, \tag{3.92a}$$

$$y_t = \begin{pmatrix} 1/4\imath\lambda & -\alpha B/2 \\ \alpha B/2 & -1/4\imath\lambda \end{pmatrix} y. \tag{3.92b}$$

Following the inverse scattering transform method, while considering $\alpha^2 \simeq 1$, one easily obtains a solution to the system Eq.(3.89). This approximated solution is given as [65]

$$B = \frac{e^{-st}}{\eta}\mathrm{sech}(\varphi + \varphi_0), \tag{3.93a}$$

$$C = h - \frac{1}{\eta}(\tanh(\varphi + \varphi_0) + 1), \tag{3.93b}$$

$$\varepsilon_+ = \frac{1}{2\eta}(\tanh(\varphi + \varphi_0) + 1), \tag{3.93c}$$

$$\varphi = 2\eta\left(h + \frac{t}{4\eta^2}\right), \tag{3.93d}$$

$$\varphi_0 = -\frac{1}{2\eta}\ln\left(\frac{\imath c_0}{2\eta}\right), \tag{3.93e}$$

which moves at the velocity $\mathfrak{v} = 1/(4\eta^2)$. Observing solutions Eq.(3.93) and the one given in Eq.(3.87), it appears that the damping effects contributes on the dynamic of wave by decreasing its amplitude as wave evolve. The shapes of the waves remaining the same as for the case of wave solution to the KMM-system. Influence of such a parameter is depicted in Figure (9) where it is well observe that wave decreases as time evolve. The impact of damping effects stands to act principally on the amplitude of the wave. It also has an impact on the width of the wave but, this impact on the width is too weak that it can be neglected, as shown in Figure (9), due to ultra fast process assumption occurring in ferrite during re-magnetization. Since influence of damping effects has been investigated, we then further discuss the influence of the inhomogeneous exchange effects.

3.3.3 Inhomogeneous exchange influence on the propagation of waves in ferrites

Known as deformations, inhomogeneity in general represents variations of the physical interactions due to spatial distortion of the crystal lattice [55, 117]. It also represents the symmetry breaking or disorder in crystal systems [60]. Inhomogeneities can occur in materials due to external fields, presence of defects, vacuum or gaps. On the other hand, inhomogeneity can also be the result of spin interactions. Indeed, knowing that the magnetization is treated in terms of spin waves and the Heisenberg model allows to directly treat a set of spins relative interaction depends on the distance between the nearest neighbors [118]. The usual picture of the two-magnon model is that inhomogeneities introduce weak interactions between the spin-wave modes that allow the energy of the uniform precession to leak into a number of other modes, providing an effective damping of the uniform mode. Alternatively, the effect of the inhomogeneity may be regarded as a mixing of the eigenmodes of the uniform film in a way that distributes the ferromagnetic resonance intensity over a number of eigenmodes, resulting in an ferromagnetic resonance peak composed of a number of overlapping resonances [119, 120]. In certain cases, due to presence of imperfection, intra-sublattice interaction becomes comparable with inter-sublattice interaction. In such a situation disorder and frustration takes place in the spin subsystem. Similarly, inhomogeneity can also be simulated by the deliberate introduction of imperfections (impurities or organic complexes) in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion [117, 120]. These defects cause distortions in atomic shells and induce deformation of the materials. Inhomogeneities affect significantly the dynamics of magnetization of ferrites [55]. In order to investigate the inhomogeneous exchange effects on the dynamic of magnetic waves in ferrites, Kuetche and coworkers [33] have consider the Landau-Lifschitz-Gilbert damping

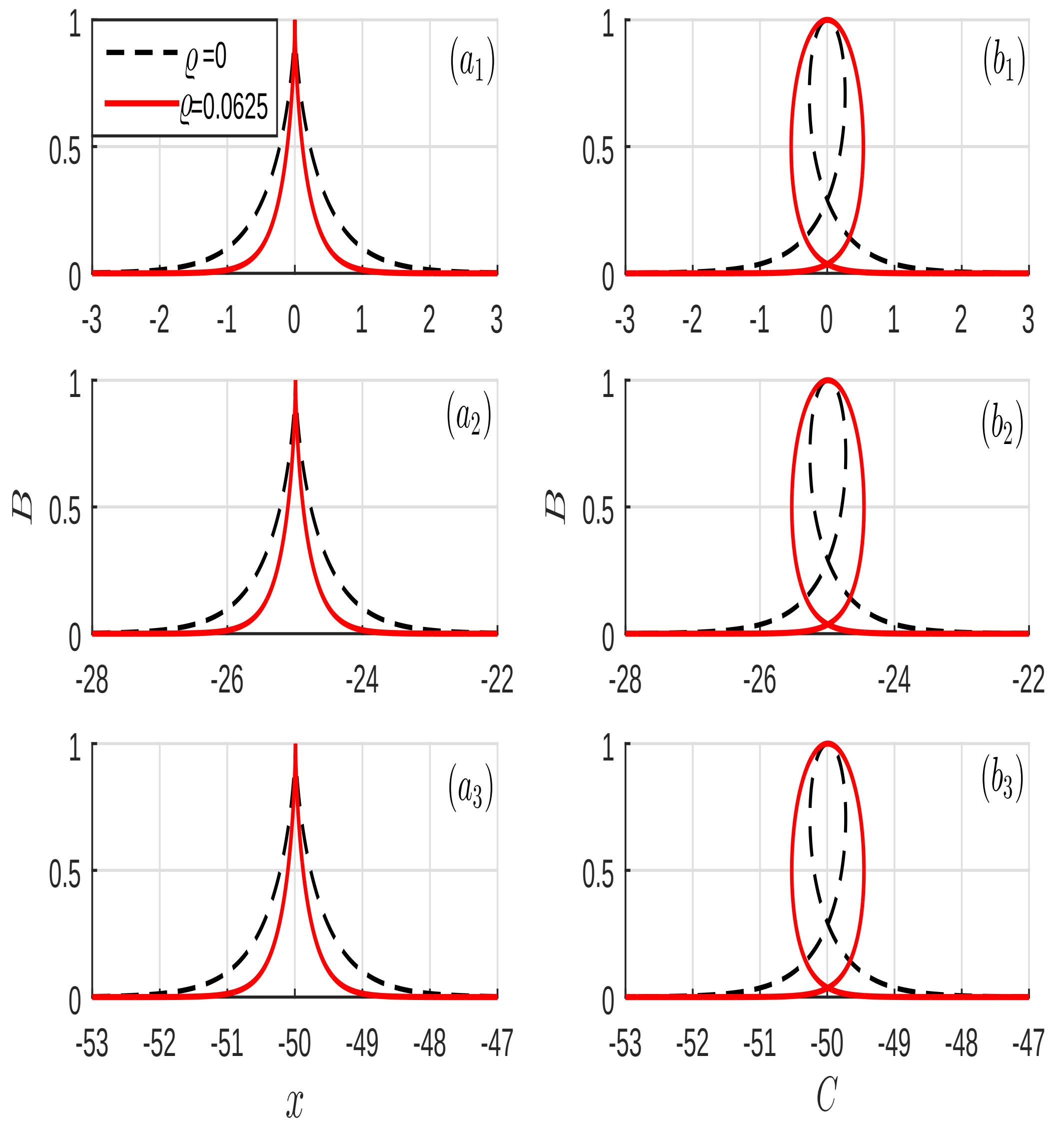

Figure 10: Propagation of one-soliton solution B to the system Eq.(3.94) for $\varrho = 0$ (dashed line) and $\varrho = 0.0625$ solid line, with the eigenvalue $\eta = 1$, $s = 0$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). The dashed line ($\varrho = 0$) is the wave solution of the KMM-system Eq.(3.86) while the continuous line ($\varrho = 0.0625$) is standing for exact one soliton solution of Eq.(3.94) for ($s = 0$). It is observed on this figure that inhomogeneous effects act not on the amplitude, but on the width of the waves. For spike soliton the width of the wave decreases as observed in panels (a_1), (a_2) and (a_3) while the width of the loop soliton increases as observed on panels (b_1) (b_2) and (b_3).

at higher order and have derived the following system

$$B_{xt} = BC_x - sB_x + \varrho B_{xx}, \tag{3.94a}$$

$$C_{xt} = -BB_x, \tag{3.94b}$$

where B_{xx} stands to translate the inhomogeneous exchange within ferrites. We first consider the case where the damping effects are neglected (s=0) to investigate the contribution of this inhomogeneous exchange on the propagating wave only. Looking for solutions to such a nonlinear system, Kuetche [34] has derived via Hirota's bilinear method soliton solutions and has depicted B versus C. We aim in this work at constructing in addition to B versus C, B versus x. Indeed, we consider that solution to Eq.(3.94) can be written as follows

$$B = k\mathrm{sech}\gamma, \tag{3.95a}$$

$$C = h - p\tanh\gamma, \tag{3.95b}$$

$$x = h - r\tanh\gamma, \tag{3.95c}$$

$$\gamma = nh + mt, \tag{3.95d}$$

the constant parameters k, n, m, p and r are to be determine. Introducing Eq.(3.95) into Eq.(3.94) we obtain the following system

$$n^2\varrho - mn + 1 = 0, \tag{3.96a}$$

$$np - 2nm = 0, \tag{3.96b}$$

$$2mp - k^2 = 0, \tag{3.96c}$$

The first of the above equation Eq.(3.96a) corresponds directly to the dispersion relation provided in Ref. [34]. Solving this system, we obtain

$$k = \pm 2m, \tag{3.97a}$$

$$p = 2m, \tag{3.97b}$$

$$n = \frac{m \pm \sqrt{m^2 - 4\varrho}}{2\varrho}, \tag{3.97c}$$

$$r = \frac{2\varrho}{m \pm \sqrt{m^2 - 4\varrho}}, \tag{3.97d}$$

with $m^2-4\varrho \geq 0$. Considering propagation of waves in ferrites without any inhomogeneous exchange processes, one has to set $\varrho = 0$ before solving Eq.(3.94). As a result, one recovers the solutions obtained for the case of KMM-system.

Case of $m^2 - 4\varrho = 0$

Explicit expression of the analytical solution of the system Eq.(3.94) is written as

$$B = \frac{1}{\eta}\mathrm{sech}\gamma, \tag{3.98a}$$

$$C = h - \frac{1}{\eta}\tanh\gamma, \tag{3.98b}$$

$$x = h - \frac{1}{4\eta}\tanh\gamma, \tag{3.98c}$$

$$\gamma = 4\eta h + \frac{t}{2\eta}, \tag{3.98d}$$

$$\varrho = \left(\frac{1}{4\eta}\right)^2, \tag{3.98e}$$

where we have replaced m by $1/2\eta$ for convenience and have chosen $\varrho = \left(\frac{1}{4\eta}\right)^2$ for simplification in the analysis. Eq.(3.98) stands for exact one-soliton solution to the system Eq.(3.94) that allows us to depict B vs C as in Ref. [34] and B versus x. To discuss the contribution of the parameter ϱ on the dynamic of the waves, we evaluate bandwidth at half height of the wave, the maximum width of the loop wave and the height at which this maximum width occurs. Indeed we define the quantities L_0 and L_ϱ as the maximum width of loop solitons, H_0 and H_ϱ the height at which it occurs and the quantities l_0 and l_ϱ the width at half maximum height of the wave when $\varrho = 0$ and when $\varrho \neq 0$. It appears

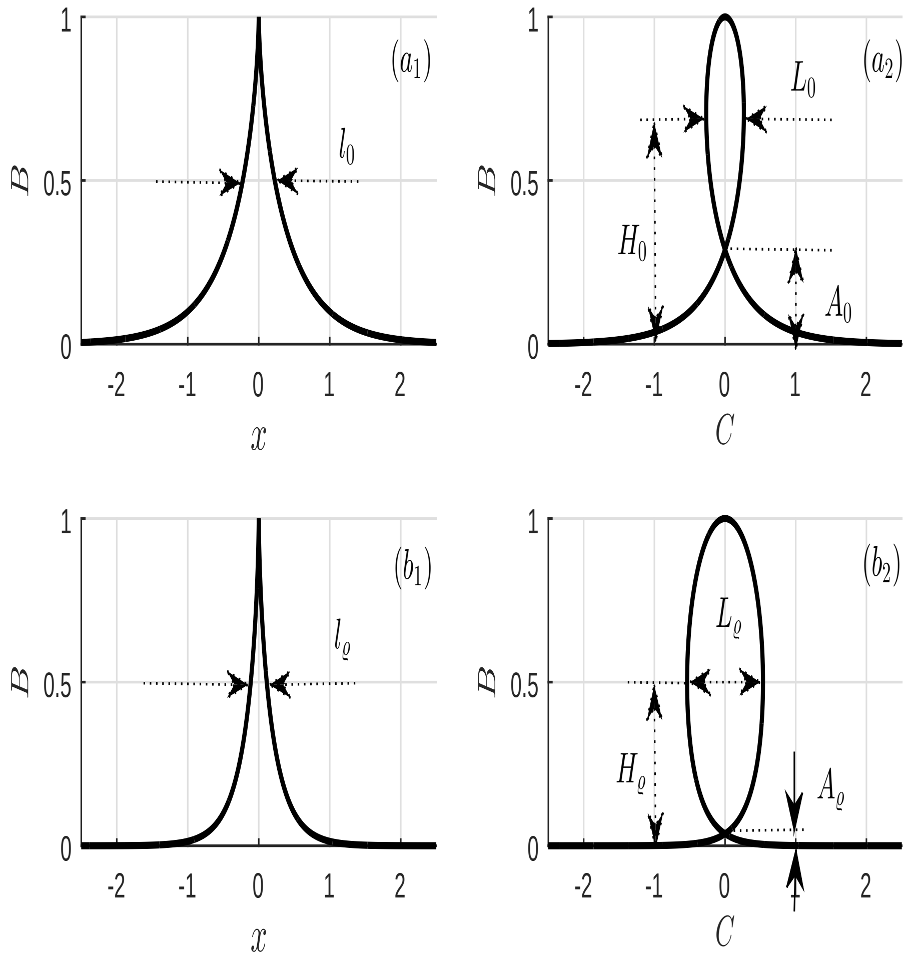

Figure 11: Propagation of one-soliton solution B to the system Eq.(3.94) for $s = 0$, with the eigenvalue $\eta = 1$, $\varrho = 0.0625$ at times $t = 0$, depicted versus x and versus C (panels (a_1), (a_2)) and the one-soliton solution B to the KMM-system Eq.(3.86) versus versus x and versus C (panels (b_1), (b_2)). The parameters l_0 and l_ϱ stand for the width at half maximum height, L_0 and L_ϱ stand for the maximum width of the loop solitons, H_0 and H_ϱ represent the magnitude at which the width of the loops solitons are maximal and A_0 and A_ϱ represent the magnitude at which the cross over occurs.

that

$$
\begin{aligned}
L_\varrho &= \frac{1}{\eta}\left(\sqrt{3}-\frac{1}{2}\text{arctanh}\left(\frac{\sqrt{3}}{2}\right)\right), && (3.99a)\\
L_0 &= \frac{1}{\eta}\left(\sqrt{2}-\text{arctanh}\left(\frac{\sqrt{2}}{2}\right)\right), && (3.99b)\\
l_\varrho &= \frac{1}{4\eta}\left(2\ln(2+\sqrt{3})-\sqrt{3}\right), && (3.99c)\\
l_0 &= \frac{1}{2\eta}\left(2\ln(2+\sqrt{3})-\sqrt{3}\right), && (3.99d)\\
H_\varrho &= \frac{1}{2\eta}, && (3.99e)\\
H_0 &= \frac{\sqrt{2}}{2\eta}, && (3.99f)
\end{aligned}
$$

From these expressions it is shown clearly that $\Delta = L_\varrho - L_0 > 0$ and $\delta = l_\varrho - l_0 < 0$. The influence of inhomogeneous exchange within ferrites begin to give some ideas since for the same amplitude, the additional dispersive term increase the maximum width of the loop soliton while the bandwidth at half maximum height of the spike wave decreases as presented in Figures.(10) and (11), comforted with expressions given in Eq.(3.99). To investigate more deeply the influence of inhomogeneous exchange effects on the dynamics of magnetic waves in ferrites, further investigations are needed while considering the case where $m^2 - 4\varrho > 0$.

Investigating solutions to the system Eq.(3.94), since it is not proven integrable, it is possible to find a solution that, under some approximation, verify Eq.(3.94). On the regards of the above solutions provided for $s = 0$, $\varrho \neq 0$ given in Eq.(3.98) and the solution obtained for $s \neq 0$, $\varrho = 0$, we deduce an approximated solution to Eq.(3.94), written as follow

$$
\begin{aligned}
B &= \frac{e^{-st}}{\eta}\text{sech}\gamma, && (3.100a)\\
C &= h - \frac{1}{\eta}\tanh\gamma, && (3.100b)\\
x &= h - \frac{1}{4\eta}\tanh\gamma, && (3.100c)\\
\gamma &= 4\eta h + \frac{t}{2\eta}, && (3.100d)\\
\varrho &= \left(\frac{1}{4\eta}\right)^2, && (3.100e)
\end{aligned}
$$

provided st is very small that $e^{-2st} \simeq 1$. This consideration is in accordance with the ultra fast processes in ferrites which practically can be refer to data inputs which undergo some fast re-magnetization process within magnetic memory devices [34].

Case of $m^2 - 4\varrho > 0$

In this case, while considering the system free of damping effects, Eq.(3.98) takes the following expression

$$
\begin{aligned}
B &= \frac{1}{\eta}\operatorname{sech}\gamma, && (3.101a)\\
C &= h - \frac{1}{\eta}\tanh\gamma, && (3.101b)\\
x &= h - \frac{4\eta\varrho}{1+\epsilon\sqrt{1-16\eta^2\varrho}}\tanh\gamma, && (3.101c)\\
\gamma &= \frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}h + \frac{t}{2\eta}, && (3.101d)
\end{aligned}
$$

which alow construction of magnetic waves with arbitrarily chosen non zero ϱ. Similarly to the previous analysis, at $t=0$ we evaluate H_ϱ, l_ϱ and L_ϱ to be expressed as follows

$$
\begin{aligned}
H_\varrho &= \sqrt{\frac{4\varrho}{1+\epsilon\sqrt{1-16\eta^2\varrho}}}, && (3.102a)\\
l_\varrho &= \frac{4\eta\varrho}{1+\epsilon\sqrt{1-16\eta^2\varrho}}\left(2\ln(2+\sqrt{3})-\sqrt{3}\right), && (3.102b)\\
L_\varrho &= \frac{2}{\eta}\sqrt{\frac{1-4\eta^2\varrho+\epsilon\sqrt{1-16\eta^2\varrho}}{1+\epsilon\sqrt{1-16\eta^2\varrho}}}\\
&\quad -\frac{8\eta\varrho}{1+\epsilon\sqrt{1-16\eta^2\varrho}}\operatorname{arctanh}\left(\sqrt{\frac{1-4\eta^2\varrho+\epsilon\sqrt{1-16\eta^2\varrho}}{1+\epsilon\sqrt{1-16\eta^2\varrho}}}\right), && (3.102c)
\end{aligned}
$$

where $\epsilon = \pm 1$. According to the parameter ϵ, we observe different behavior for the same value of ϱ. Then, the parameter ϵ characterizes two types of inhomogeneities that can occur in ferrite having different effects on the parameter of the wave. Such information is illustrated in Fig.(12) and confirmed in Fig.(13). When considering the case with $s \geq 0$,

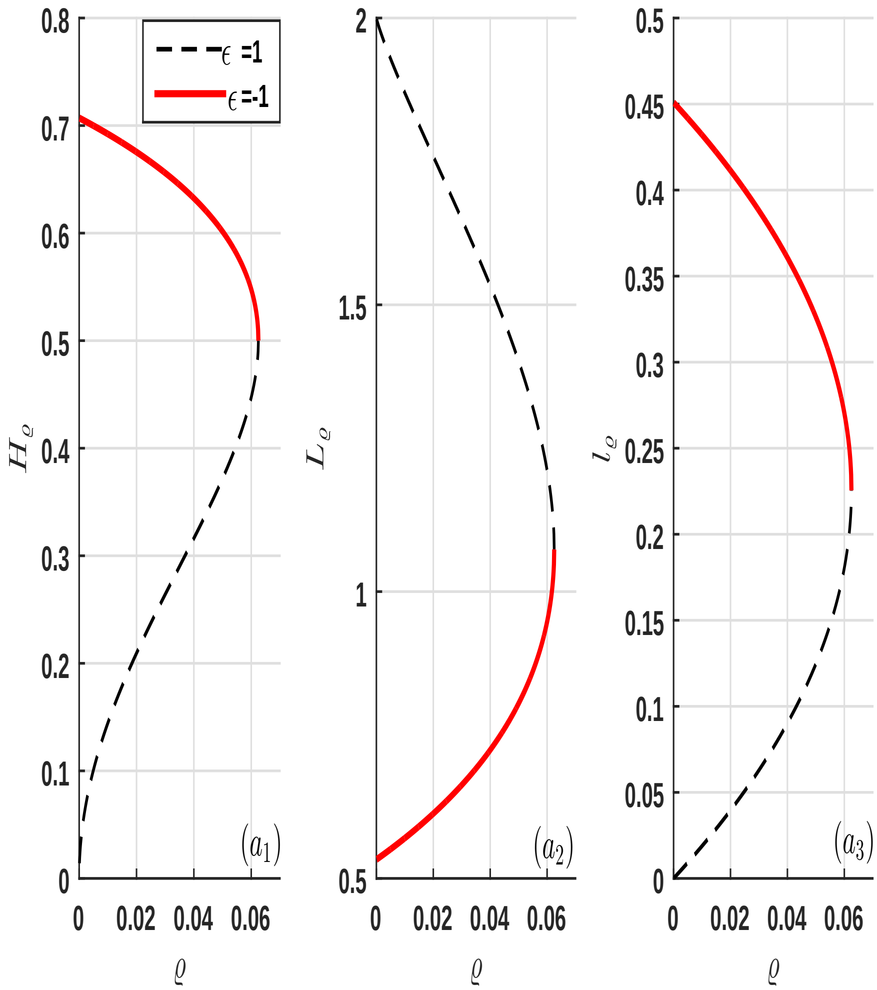

Figure 12: Evolution of height H_ϱ at which maximum width of loop occurs versus ϱ (panel a_1), evolution of the width L_ϱ versus ϱ (panel a_2) and variation of width at half maximum l_ϱ versus ϱ (panel a_3) For different values of ϵ.

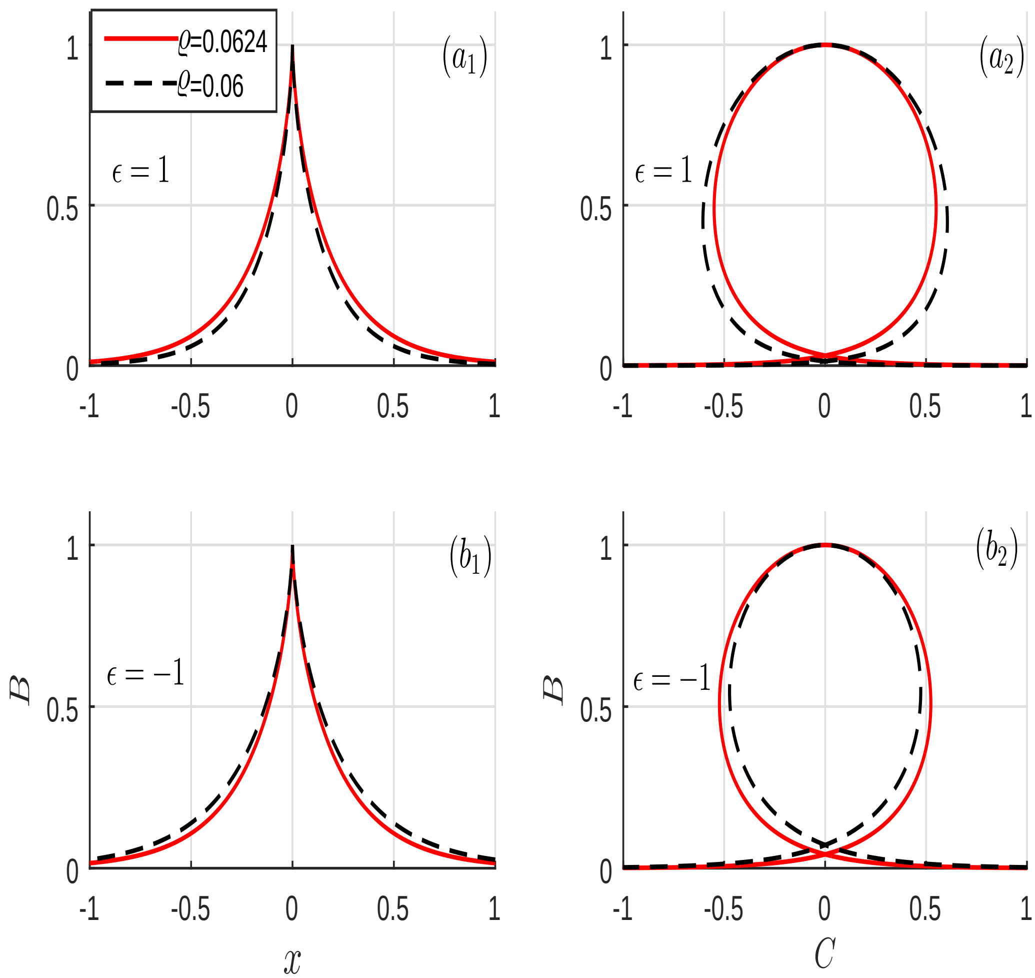

Figure 13: One-soliton solution of the system Eq.(3.94). For the case $\epsilon = +1$ (panels a_1 and a_2) one observe that H_ϱ and L_ϱ decrease as ϱ increase for the loop (panel a_2) while the bandwidth increase as ϱ increase for the spike (panels a_1). When $\epsilon = -1$ the previous process reverse (panels b_1 and b_2).

one obtains

$$
\begin{aligned}
B &= \frac{e^{-st}}{\eta}\mathrm{sech}\gamma, &\text{(3.103a)}\\
C &= h - \frac{1}{\eta}\tanh\gamma, &\text{(3.103b)}\\
x &= h - \frac{4\eta\varrho}{1+\epsilon\sqrt{1-16\eta^2\varrho}}\tanh\gamma, &\text{(3.103c)}\\
\gamma &= \frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}h + \frac{t}{2\eta}, &\text{(3.103d)}
\end{aligned}
$$

which approaches the solution of the system Eq.(3.94) under the ultra fast process assumption. Under this approximation, the values of L_ϱ, l_ϱ and H_ϱ do not vary significantly. Such a result is in accordance with the one obtained in Ref. [121] for YIG film. It appears that the damping effects are not only characteristic of the properties of the material, but must have something to do also with some parameters of the wave itself Ref. [55]. To bring further information about the contribution of damping effects and inhomogeneous exchange processes on the dynamics of waves in ferrites we provide the following equation

$$
\begin{aligned}
E &= \frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{16\eta^4\varrho}\frac{\sqrt{1-\mathrm{sech}^2\gamma}}{1-\mathrm{sech}^2\gamma}\left(2s\eta+\sqrt{1-\mathrm{sech}^2\gamma}\right)\mathrm{sech}^2\gamma\\
&\quad -\frac{1}{4\eta^3}\frac{\mathrm{sech}^2\gamma}{1-\mathrm{sech}^2\gamma}\left(\eta-\frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}\mathrm{sech}^2\gamma\right)\\
&\quad +\frac{1}{2\eta^2}\left(\frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}\right)^2(1-\mathrm{sech}^2\gamma)\mathrm{sech}^2\gamma, \qquad \text{(3.104)}
\end{aligned}
$$

which stands for the energy density of the moving wave given in Eq.(3.103). In all of the cases $\epsilon = 1$ or $\epsilon = -1$ this energy density E decreases as x increases and, when ϱ increases, energy density decreases for $\epsilon = 1$ and increases for $\epsilon = -1$. Presence of inhomogeneities cause dissipation in the system exciting the soliton or can also bring in energy. We depict in Fig.(12) the evolution of height H_ϱ versus ϱ alongside with the evolution of the width of the spike l_ϱ versus ϱ. This figures show that according to the parameter ϵ these quantities behave in the same manner, increasing when $\epsilon = +1$ and decreasing when $\epsilon = -1$. This result is coherent with the result obtained in Eq.(3.101) and depicted in Fig.(12) for different values of ϱ. For $\epsilon = +1$ it appears clearly that L_ϱ decreases as ϱ increases and for $\epsilon = -1$ L_ϱ increases as ϱ increases. It was not evident to provide analytical expression of A_ϱ. Meanwhile, it appears in Fig.(13) that A_ϱ increases as ϱ increases for $\epsilon = 1$ and A_ϱ decreases as ϱ increases for $\epsilon = -1$.

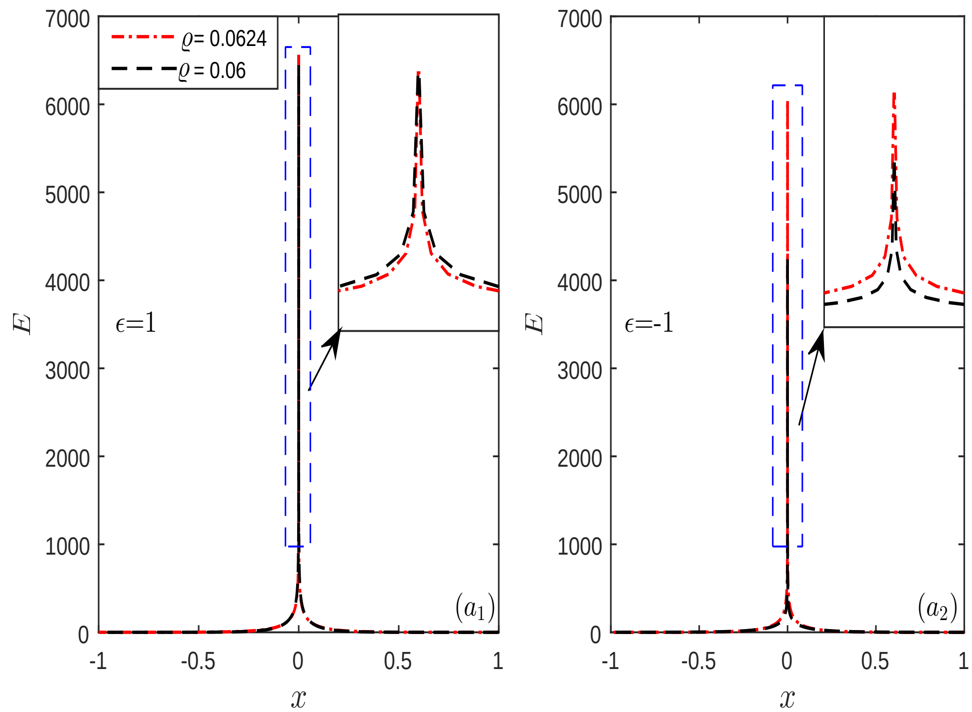

Figure 14: Energy densities Eq.(3.104) depicted for the case $\epsilon = +1$ (panels a_1) where one observes that energy density increases when ϱ increases. For the case $\epsilon = -1$ (panels a_2) a reverse process is observed where energy density decrease when ϱ increases. Such a result complete and confirms the one presented in Fig.(12)

Now that we have investigated analytically the solution of the system Eq.(3.94), and have constructed its solution without damping effects and, have also constructed approximated solution while considering zero inhomogeneous exchange effects under the ultra-fast process assumption, it is thus important to verify if this solution can be confirmed numerically.

3.3.4 Numerical simulations

Investigating solutions to nonlinear evolution equations, one often derive when possible exact analytical solutions when equation is proven integrable. When on the contrary it appears difficult to solve exactly the nonlinear system under interest, it sometimes appears that approximated solutions can be derived and, in this case, further investigations are necessary. As far as we are concerned with inhomogeneous exchange within ferrite, we found it difficult to solve exactly the nonlinear system Eq.(3.94), but we have been able to propose a solution that approximates the solutions to the aforementioned system under a condition that translate ultra fast processes in ferrites. We now pursue with numerical investigations to confirm the results that have come out from analytical investigations. Indeed, we use the finite difference method with initial conditions given as:

$$\begin{aligned} B(x,0) &= \frac{1}{\eta}\mathrm{sech}\left(2\eta h+\varphi_0\right), && (3.105a)\\ C(x,0) &= h-\frac{1}{\eta}\tanh\left(2\eta h+\varphi_0\right), && (3.105b)\\ x(h,0) &= h-\frac{1}{2\eta}\tanh\left(2\eta h+\varphi_0\right). && (3.105c)\end{aligned}$$

We present in Fig.(15) the evolution of loop and spike soliton, solutions of Eq.(3.94). On this figure, it appears that numerical simulation and analytical results are matching very well. Such a result confirms the investigations carried out in the previous section. Then, going forward in the windings of the previous analysis, we discuss numerically the case where the damping effect is taken into account ($s \neq 0$). We depict in Fig.(15) the solution of the system Eq.(3.94) in which one observe that once more, analytical and numerical simulations are matching. It appears that the combined effects of damping an inhomogeneous exchange contributions on the dynamics of the wave are: decreasing the amplitude of the wave, increase the width of the loop wave and decrease the bandwidth

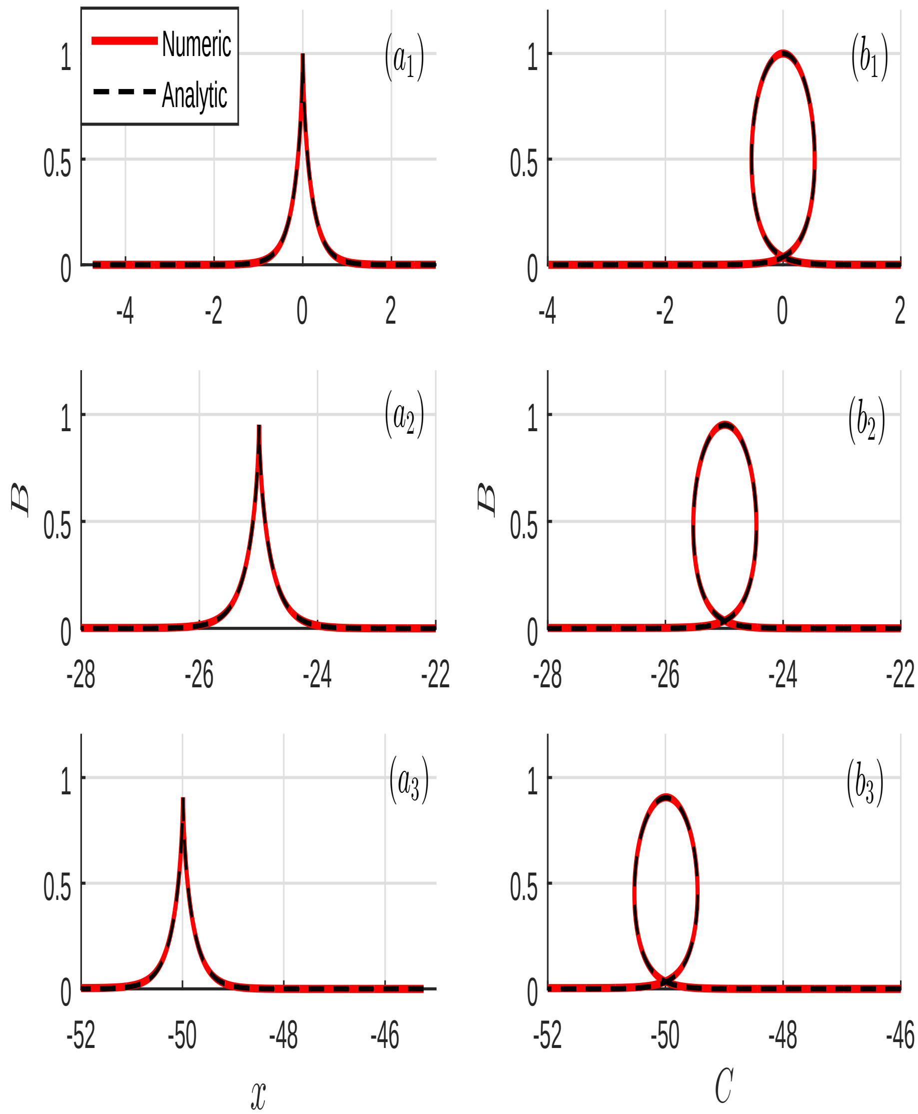

Figure 15: Propagation of one-soliton solution B to the system Eq.(3.94) for $s = 20$, with the eigenvalue $\eta = 1$, $\varrho = 0.0625$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). One observes that, for the combine effects of damping and inhomogeneous exchange, analytical and numerical solutions match.

at half maximum amplitude of the wave.

The numerical simulations then confirm the analytical result.

3.3.5 Discussion of the results

We have investigated the influence of inhomogeneous exchange and damping effects on the dynamics of wave in ferrites. Indeed, we have constructed analytical solution of the system Eq.(3.94) that model the dynamics of such waves in which involve parameters s and ϱ that model damping and inhomogeneous exchange respectively. As a result, the following conclusions have risen out

(a) For the case where $\varrho = 0$, we observe that for $s > 0$, the damping effects act on the wave which decrease as time evolve. The bandwidth not changing significantly in the case of ultra fast process assumption as observed in Fig.(**??**). Such a result has been predicted and obtained in Ref. [121] and references therein.

(b) When considering the inhomogeneous exchange only ($\varrho \neq 0$), it appears that two cases came out according to a parameter ϵ that takes two different values which are $\epsilon = +1$ and $\epsilon = -1$. The parameters that help characterizing these inhomogeneous exchange effects being the maximum width of the loop wave L_ϱ, the height H_ϱ at which this maximum width occurs and the width l_ϱ at half maximum as provided in Fig.(**??**).

* When $\epsilon = +1$ it has appeared that the maximum width L_ϱ of the loop soliton decreases as ϱ increases and the height H_ϱ at which its occurs increases as ϱ increases . As illustration we have depicted in Fig.(12) (dashed line) and in Fig.(13) (panels (a_1) and (a_2)) the evolution of H_ϱ and L_ϱ versus ϱ. For the spike soliton solution that are originated from the depiction of B versus x, one has observed reverse processes where the width at half maximum l_ϱ increase as ϱ increases as depicted Fig.(12) (panels (a_3)).

* When $\epsilon = -1$ One observe a reverse process that is, L_ϱ increases while H_ϱ decreases as ϱ increases, while l_ϱ decreases as ϱ increases as observed in Figs.(12) and (13). In both cases ($\epsilon = +1$ and $\epsilon = -1$), energy densities have been

provided in Eq.(3.104) and depicted in Fig.(14) for two different values of ϱ, which complete and confirm the previous results. All the behaviors described here are simply due to the type of inhomogeneity present in the ferrite.

(c) When considering the combined effects of damping an inhomogeneous exchange process within ferrites, the evolution of waves are subjected to loss of energy with time, characterized by the decreasing of the wave amplitude and, according to the value taken by ϵ the bandwidth of the wave increase or decrease with ϱ.

(d) Further investigations have been carried out to confirm the above analytical results. Indeed we have proceeded to some numerical simulations. We have followed the finite difference method and, as a result, it has come out that the analytical and numerical simulations are matching very well. We have depicted in Fig.(15) the analytical and numerical results where the combined effects of damping and inhomogeneous exchange have been taken into account.

In ferrites made of polycrystal, inhomogeneous exchange process and damping effects must be taken into account when investigating the propagation of wave in such media. The system Eq.(3.94) thus respond to this demand. Investigating the solution of such equation, it appears that two different types of inhomogeneities occur that increase or decrease the width of the wave according to the parameter that characterizes the inhomogeneity. One can also think about ferrites that combine the two different types of inhomogeneities that come periodically. Therefore combined effects will occur that is, width of the wave will increase and decrease alternatively and one will observe a sort of wave that breath as time evolves. Another scenario can occur that is, the combined types of inhomogeneities in the same material may help in reducing the effects of inhomogeneous exchange within ferrite which will result to a stable profile that is near to the result of KMM-system, that does not takes into consideration these combined effects.

Conclusion

In this chapter, we have firstly studied analytically the influence of the damping effects on the dynamic of short waves in ferrites and confirmed analytical results while proceeding to some numerical simulations throughout finite difference method. Indeed, to carry out our gold, the mathematical toll described by Konno [101] has been used to deal with the system under current interest. We have constructed a system that is Lax integrable and that approximates in some conditions that have been specify, the damped-KMM system. Under this conditions, we have generated analytical expressions of the solutions to the damped-KMM system. We have investigated, numerical simulations to confirm the results obtained analytically.

Secondly, we have constructed new traveling wave solutions to the inhomogeneous system of our interest using Jacobi elliptic expansion method. As a result, it appeared that inhomogeneity increases or decreases the amplitude of travelling wave due to the value taken by the inhomogeneous parameter.

Finally, we have presented the analytical solution of the KMM-system taking into account the both effects of damping and inhomogeneous exchange. Pursuing our analysis, we have discussed the case of inhomogeneous exchange effects and have proposed analytical one soliton solution to the system Eq.(3.94) where damping effects are neglected ($s = 0$). It appeared as a result that, for a leading case verified by ϱ, inhomogeneous exchange parameter acts on the wave by reducing it bandwidth for the case of B versus x, or increasing the maximum width of the loop soliton for B depicted versus C. The amplitude of the wave and its shape are conserved in this case. In regards to the previous solutions, we have proposed an approximated solution to the system Eq.(3.94) that takes into account the combined effects of damping and inhomogeneous exchange. Then, through this solution, we have characterized the wave moving in such medium while evaluating the maximum width of the loop soliton L_ϱ, the height H_ϱ at which it occurs and, for the spike soliton we have evaluated the width at half maximum of the wave l_ϱ. It has appeared as a result that, according to a parameter ϵ, l_ϱ, the height H_ϱ decrease as ϱ increases when in the same time L_ϱ increases ($\epsilon = -1$). A reverse process has been obtained for the case ($\epsilon = +1$). We have illustrated the different situations in Fig.(12) and Fig.(13). To complete these analysis, we have provided the energy densities for both cases of ϵ in Eq.(3.104) and depicted these energies in Fig.(14) where for two different values of ϱ one observe in each case how energy is influenced by inhomogeneity. To further investigate the analytical results obtained, we have proceeded to some numerical simulations. We have followed indeed the finite difference method. Conclusion that appeared at the end of such analysis is that: numerical and analytical results are matching very well. Illustration is given in Fig.(15), where we have depicted analytical solution Eq.(3.103) and numerical solution obtained under the initial conditions provided in Eq.(3.105).

General Conclusion

In this work, our main goal was to investigate the effects of damping effects and inhomogeneous exchange on the propagation of magnetic waves in ferrites. In order to answer this preoccupation, our approach included three important steps. We have investigated at first the damping effect, later on the effect of inhomogeneous exchange and finally the combined effects of damping and inhomogeneous exchange on the propagation of waves in ferrites.

In regards to damping effect on the wave propagation in ferrites, we paid attention on the dissipative KMM-system that describes the propagation of short-waves in magnetic insulators. Since this equation has not been proven completely integrable and solution has not been provided, we have proposed a nonlinear integrable system with associated Lax-pairs which, through an approximation that translate ultra-fast process, one easily recover the dissipative KMM-system of our interest. Investigating via the inverse scattering transform method the integrable system following particularly the WKI-scheme, we have constructed, while considering an ultra-fast process approximation, soliton solutions to the dissipative KMM-system and have presented analytical expressions in Eq.(3.46). This analytical expressions show that amplitude of the wave is proportional to an exponential function that decreases as time evolve. It then appeared that the wave survive only if we are on an ultra-fast process. In order to confirm predictions of analytical results obtained, we have proceeded to some numerical simulations. We have followed indeed the finite difference method and, as a result, we have depicted in Figure 6 numerical and analytical results showing the dynamics of magnetic wave in ideal ferrite (free of damping effects). It appeared that analytical and numerical results match perfectly, showing that numerical analysis has been well carried out. The results presented for the non-dissipative KMM-system is in accordance with results provided by Kuetche et al. and Tchokouansi et al. [26, 29]. While considering $s \neq 0$ and $s > 0$, the damping effect on the dynamic of magnetic wave appeared to behave very similarly with the results obtained analytically. We have then depicted in Figure 7 numerical solutions superposed on analytical solution and have observed that the solutions also coincide here. This Figure 7 shows that analytical results carried out previously is in accordance with predictions. The damping parameter on the propagation of waves in ferrite acts just on the amplitude of the waves during their motion and not on the profile. A question then arises to know if there are parameters that may act simultaneously on the profile and amplitude of the wave during it motion in ferrites.

In regards to inhomogeneous exchange effect, we have paid attention to a nonlinear system describing the propagation of short-waves in ferrite, in which inhomogeneous exchange effects have been taken into account. This system, namely Eq.(3.49), has been investigated under the Hirota's bilinear method and multi-soliton solutions have been provided, which are of loop types. The present work was devoted first of all in constructing new solutions to the same system (3.49). Indeed, the Jacobi elliptic functions expansion method has been used and, as a result, number of new exact solutions have been derived which are periodic functions. Their limiting cases have been provided where the modulus of the Jacobi's elliptic functions tend to one ($m \longrightarrow 1$). These limiting case solutions appeared to be localized solutions. One solution has particularly retain our attention, which is the one given in Eqs.(3.65) and (3.69) which, compare to the result obtained by Kuetche et al. [26] possess an additional term of time in the v solution. The second step was to investigate the influence of inhomogeneous terms in ferrites. To achieve such a task, we have used the solution given in Eqs.(3.65) and (3.69). From this equation, we have evaluated the maximum width of the loop wave and the height at which it occurs. With their explicit expressions at hand, it was possible to conclude on the influence of inhomogeneous exchange on the dynamics of the wave in ferrite. It appeared that for ρ negative, the amplitude of the wave increases along with the maximum width l_ρ and the height h_ρ at which it occurs. When ρ is taken positive, the opposite phenomenon is observed. It is then clear that, under the light cone of the analysis made here and the solutions obtained, there are two types of inhomogeneities in ferrites that are represented with the values taken by ρ. One cannot exclude the fact that in ferrite, the two types of inhomogeneities can appear periodically or can be randomly distributed. In the periodical appearance of inhomogeneities, one may observe a wave that moves while breathing due to the compensation between inhomogeneities such that the mean wave can move as if there were no inhomogeneous exchange effects in the considered material. For the case were the inhomogeneities appear randomly, the wave moves while breathing irregularly in function of the type of inhomogeneities encounter.

In regards to combined effect of damping and inhomogeneous exchange, we have considered the KMM-system that appears to describe propagation of wave in ferrites free of damping and inhomogeneous exchanges. We have recalled the analytical solutions obtained for the previous equation and have used the associated one-soliton for comparison purpose. Aiming at investigating the influence of damping effects in the dynamic of waves in such a medium we have derived a solution that, under some approximation (approximation that approves the ultra-fast process assumption) verify the system Eq.(3.89). As a result, it appears that damping has as effect to decrease the amplitude of the wave as it evolves. Pursuing our analysis, we have discussed the case of inhomogeneous exchange effects and have proposed analytical one-soliton solution to the system Eq.(3.94) where damping effects are neglected ($s = 0$). It appeared as a result that, for a leading case verified by ϱ, inhomogeneous exchange parameter acts on the wave by reducing it bandwidth for the case of B versus x, or increasing the maximum width of the loop soliton for B depicted versus C. The amplitude of the wave and its shape are conserved in this case. In regards to the previous solutions, we have proposed an approximated solution to the system Eq.(3.94) that takes into account the combined effects of damping and inhomogeneous exchange. Going over the leading case verified by ϱ, we have investigated the influence of the inhomogeneous exchange process in ferrites of different homogeneities. Indeed, we constructed solution of Eq.(3.94) without damping, that is expressed explicitly as function of ϱ. Then, through this solution, we have characterized the wave moving in such medium while evaluating the maximum width of the loop soliton L_ϱ, the height H_ϱ at which it occurs and, for the spike soliton we have evaluated the width at half maximum of the wave l_ϱ. It has appeared as a result that, according to a parameter ϵ, l_ϱ, the height H_ϱ decrease as ϱ increases when in the same time L_ϱ increases ($\epsilon = -1$). A reverse process has been obtained for the case ($\epsilon = +1$). We have illustrated the different situations in

Figure (12) and (13). To complete these analysis, we have provided the energy densities for both cases of ϵ in Eq.(3.104) and depicted these energies in Figure (14) where for two different values of ϱ one observe in each case how energy is influenced by inhomogeneity.

From the results obtained throughout as inhomogeneous effects, it has appeared two types of inhomogeneities that can be recognized by their effects on the waves. These effects consisting of increasing or decreasing the width of the waves as it moves. Each effect will then be observed in ferrites made of one crystal with particular inhomogeneity. But one may think about poly crystals in which the two different types of inhomogeneities are structured periodically, then one will observe increasing and decreasing of width of the wave leading to a sort of wave that breath as it evolve. One may also think about reducing the effects of inhomogeneous exchange within ferrite by introducing the two types of inhomogeneities so that the effect of these two types of inhomogeneities will compensate each order. In this case, the mean wave will evolve as if there is no inhomogeneity in the medium as described by the KMM-system. To further investigate the analytical results obtained, we have proceeded to some numerical simulations. We have followed indeed the finite difference method. Conclusion that appeared at the end of such analysis is that: numerical and analytical results are matching very well. Illustration is given in Fig.(3.92), where we have depicted analytical solution Eq.(3.103) and numerical solution obtained under the initial conditions provided in Eq.(3.102). This result sustains the analytical results obtained in section 3.3. Now that numerical simulations have been carried out along with analytical results that are not in contradiction, the further investigation should consist of experimental investigations.

We have obtained in this work solutions that are loop-shaped for the case of magnetic field depicted against the magnetization. We have not heard as far as we are concerned with such a loop-soliton that it has been obtained experimentally. It seems then judicious to look for alternative model equations which solution may be single-valued. Such systems may help understanding very deeply the dynamics of waves in magnetic materials, and one will have as advantage that single valued solutions have been obtained experimentally. Such an approach has recently been carried out by Saravanan et al. [56] where a modified KdV equation has been shown to describe propagation of short-waves in ferrites. Breathers as solutions have been constructed. Similar investigations can be made under some different conditions and one may obtain novel equation that, we are sure, will be helpful in describing the propagation of single-valued short waves in inhomogeneous ferrites. There are in the literature so many equations of physical implication that have not been proven integrable and there are poor knowledge on the analytical description of the dynamics of their wave solutions. One of such example is the vector short pulse equation describing the propagation of ultra-short waves in optical fibers. Such systems deserve deep investigation both analytically and numerically. These problems will constitute the matter of future investigations.

List of Publications

1. **"Influence of damping effects on the propagation of magnetic waves in ferrites": Robert Tamwo Tchidjo**, Hermann T. Tchokouansi, E. Tchomgo Felenou, Victor K. Kuetche and Thomas B. Bouetou , Chaos Solitons and Fractals **119** (2019) 203.

2. **"Traveling magnetic wave motion in ferrites: Impact of inhomogeneous exchange effects":** Hermann T. Tchokouansi, E. Tchomgo Felenou, **Robert Tamwo Tchidjo** Victor K. Kuetche and Thomas B. Bouetou, Chaos Solitons and Fractals **121** (2019) 1.

3. **"On the dynamics of magnetic wave in ferrites: Influence of damping and inhomogeneous exchange effects": Robert Tamwo Tchidjo**, Hermann T. Tchokouansi, E. Tchomgo Felenou, Victor K. Kuetche and Thomas B. Bouetou , Journal of Magnetism and Magnetic Materials **484** (2019) 382.

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Influence of damping effects on the propagation of magnetic waves in ferrites

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ABSTRACT

In this paper, we study in details the propagation of ultra short waves in magnetic insulators. We discuss the Kraenkel–Manna–Merle (KMM) system, derived from the Maxwell's equations in which damping effects and nonlinearity set in from the first order Landau–Lifshitz–Gilbert equation. We investigate this system analytically, using the well developed inverse scattering transformed method and, as a result the impact of damping effects on the propagation of magnetic waves in ferrites is revealed. We launch our investigations with an integrable system that, under the ultra fast process assumption, approximates the one of our interest which shows that damping acts mostly on the energy of the wave that decrease as time evolves. Such a result is confirmed through some numerical simulations in which analytical and numerical results are matching very goodly. We then conclude on the influence of damping effects on the propagation of waves in magnetic insulators.

1. Introduction

Nonlinear media have attracted over years so many attention and this situation is still going on. The reason is that, these media often present some particularities that are of technological importance. Studying these media, there are nonlinear model equations that come out describing their dynamic under some perturbation. Thus, there are number of evolution equations in the literature that describe wide range of physical phenomena such as, the Korteweg-de Vries equation [1] describing the propagation of water waves, the nonlinear Schrödinger equation [2–4] describing the propagation of waves in optical fibers under the slowly varying envelope approximation, the stretched rope equation [5,6] describing the propagation of transverse oscillation of elastic beam under tension, just to name a few. Constructing such nonlinear evolution equation, a challenge often arises, the one of investigating the integrability of these systems, useful for deep comprehension of the dynamic of the media they model. To respond to this challenge, number of approaches have been proposed including the Painlevé analysis [7], Hirota's bilinearization [8–15], collective coordinate theory [16], Riccati mapping method [17], Darboux transformation [18,19], Bäcklund transformation [20] and inverse scattering transform (IST) [21–23]. As far as we are concerned with wave possessing vanishing tails, the IST method appears to be one of the most important discovery in soliton theory. There are number of nonlinear evolution equations that are integrable by means of this technique, most of them being gathered in two principal classes: the Ablowitz–Kaup–Newell–Segur (AKNS) systems [24] and the Wadati Konno and Ichikawa (WKI) system [25] to which belongs the Kraenkel–Manna–Merle (KMM) system, describing the propagation of magnetic field in ferromagnet with zero conductivity.

The KMM-system has been investigated under some tractable methods including the Hirota's billinearization [26–30], inverse scattering transform [31] and most recently, the generalized $G'/G-$ *expansion* method [32] and auxiliary equation method [33]. In each of the cases, soliton solutions with loop-profile and singularity have been constructed and studied in details.

In Ref. [26] the KMM-system has been generated while taking into account the Landau–Lifshitz–Gilbert damping at first order. To our Knowledge, the contribution of this damping effect,

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characterized by a parameter s on the dynamic of magnetic waves in ferrites has only been investigated via the phase portraits analysis.

The object of the present work is to investigate analytically the influence of the damping effects on the dynamic of short waves in Ferrites and to confirm analytical results while proceeding to some numerical simulations. Indeed, to carry out our gold, the mathematical tool described by Konno [34] is used in Section 2 to deal with the system under current interest. Since to our knowledge the Lax-pairs of this system is unknown till now, we provide a system of equations that is Lax integrable and that approximate in some conditions which will be specify, the damped-KMM system. Under these conditions, we provide analytical expression of the solutions to the damped-KMM system which, to our knowledge, has not yet been constructed in previous work. In Section 3, the numerical solutions are used to confirm the consistency of the analytical solutions. We end this work with a brief summary and perspective.

2. Soliton solution of the damped-KMM system

While investigating the propagation of short-waves in saturated non-conductive ferromagnetic materials immersed in an external field, Nguepjouo and coworkers [26,35] have constructed from the Maxwell's system, the following nonlinear system

$$p_{xt} = pq_x - sp_x, \tag{1a}$$

$$q_{xt} = -pp_x, \tag{1b}$$

where the Landau–Lifshitz–Gilbert equation [26,35–37] has been taken into account and where the quantities p and q represent the magnetization density and the magnetic induction respectively. The parameter s stands for the contribution of damping effects, when subscripts t and x are partial derivatives which stand for the time-like and space-like variables respectively. Kraenkel and coauthors [36], while neglecting damping effects ($s = 0$) have found a link of the above system with the famous sine-Gordon equation which multi-solitons are well known. From this solutions, the analytical expression of the KMM system has been provided. The system (1a) has been investigated by many methods. In Refs. [26,32,33,35,38] this has been investigated under many methods and, apart from the results provided in Ref. [26] where the phase portraits provide some information on the dynamics of magnetic waves in ferrites, in all the other paper it has been difficult to give explicit solution of the system (1a) in which involve the parameter s different from zero, that will set it clear the influence of damping effects on moving magnetic waves in ferrites.

Evaluating a zero-curvature formulation while restricting attention to the $SL(2, R)$ symmetry [35], it has been shown that the system (1a) is integrable under the condition ($s = 0$). This result has been confirmed in Ref. [32] where rich soliton solutions have also been provided under the same condition. Lax-pairs of the system (1a) free of damping effects ($s = 0$) is given as

$$\zeta_x = \imath\lambda \begin{pmatrix} q_x & p_x \\ p_x & -q_x \end{pmatrix} \zeta, \tag{2a}$$

$$\zeta_t = \begin{pmatrix} 1/4\imath\lambda & -p/2 \\ p/2 & -1/4\imath\lambda \end{pmatrix} \zeta. \tag{2b}$$

In order to take into account the damping effects, we then propose phenomenologically a new system, that is close to Eq. (1a) as follow

$$p_{xt} = pq_x - sp_x, \tag{3a}$$

$$q_{xt} = -\alpha^2 pp_x, \qquad \alpha = \exp(st), \tag{3b}$$

with $s \neq 0$. α can be developed in Taylor series near the initial time as follow:

$$\exp(st) = 1 + st + \frac{(st)^2}{2} + \cdots\cdots \tag{4}$$

While considering very small values of st that correspond to ultra fast processes assumption, α^2 may be restricted to the first order as, $\alpha^2 \simeq 1$. At this condition, the system Eq. (3) appears to be a good approximation to the system Eq. (1a). To investigate analytically the influence of such a parameter on the dynamic of waves in Ferrites, this equation will constitute our starting point along with the approximation condition on α.

It is easy to show that the system Eq. (3) is completely integrable via some tractable methods including prolongation structures analysis [39]. The proof of this integrability lies in its associated Lax-pairs expression:

$$\varsigma_x = \imath\lambda \begin{pmatrix} q_x & \alpha p_x \\ \alpha p_x & -q_x \end{pmatrix} \varsigma, \tag{5a}$$

$$\varsigma_t = \begin{pmatrix} 1/4\imath\lambda & -\alpha p/2 \\ \alpha p/2 & -1/4\imath\lambda \end{pmatrix} \varsigma. \tag{5b}$$

These Lax-pairs will constitute the starting point to the investigation of the inverse scattering transform method. According to the expressions of the above Lax-pairs, we suitably follow the scheme described in Refs. [31,34].

It seems important talking about the scheme described by Konno [34] that, it has been developed for the case where the Lax-pairs are not explicit functions of independent variables. To solve exactly the above system Eq. (3), it will then be important to develop the Konno's [34] scheme in order to derive the solutions of nonlinear evolution equations for systems, which associated Lax-pairs are explicit functions of independent variable. Since Eq. (3) is not the one we are focussing our attention on, throughout this paper, we will follow the Konno's [34] scheme associated with the approximation given in Eq. (4), restriction being made at the first order, to derive solution of the system Eq. (1a) of our interest.

We first define associated Jost functions given as

$$(\phi, \bar{\phi}) \to \begin{pmatrix} \exp(-\imath\lambda x) & 0 \\ 0 & -\exp(\imath\lambda x) \end{pmatrix} \quad as'' \quad x \to -\infty, \tag{6}$$

and

$$(\bar{\psi}, \psi) \to \begin{pmatrix} \exp(-\imath\lambda x) & 0 \\ 0 & \exp(\imath\lambda x) \end{pmatrix} \quad as'' \quad x \to +\infty. \tag{7}$$

The functions $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$ being related by scattering coefficients as follows

$$(\phi, \bar{\phi}) = (\bar{\psi}, \psi) \begin{pmatrix} a & \bar{b} \\ b & -\bar{a} \end{pmatrix}, \tag{8}$$

where the data $a, \bar{a}, b, \bar{b}$ verify

$$\begin{vmatrix} a & \bar{b} \\ b & -\bar{a} \end{vmatrix} = -1. \tag{9}$$

Considering λ to be complex-valued we have

$$\begin{pmatrix} \bar{\phi}_1(\lambda) \\ \bar{\phi}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \phi_2^\star(\lambda^\star) \\ -\phi_1^\star(\lambda^\star) \end{pmatrix}, \qquad \begin{pmatrix} \bar{\psi}_1(\lambda) \\ \bar{\psi}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \psi_2^\star(\lambda^\star) \\ -\psi_1^\star(\lambda^\star) \end{pmatrix}, \tag{10}$$

from which we one obtains

$$\bar{a}(\lambda) = a^\star(\lambda^\star), \qquad \bar{b}(\lambda) = b^\star(\lambda^\star). \tag{11}$$

For large λ, asymptotic behavior of the Jost functions can be investigated under the boundary conditions

$$\left.\begin{matrix} q_x \to 1 \\ p_x \to 0 \end{matrix}\right\} \quad as \quad |x| \to \infty. \tag{12}$$

We define the quantity

$$\phi_1 = \exp\left(-\imath\lambda x + \int_{-\infty}^{x} \theta(\lambda, l)\mathrm{d}l\right), \tag{13}$$

and substituting Eq. (13) into Eq. (5), we obtain

$$\theta_t = \frac{1}{2}\left[\left(q_x + 1 - \frac{\theta}{\imath\lambda}\right)\frac{p}{p_x}\right]_x, \tag{14a}$$

$$\theta_x + \theta^2 - 2\imath\lambda\theta - \lambda^2 = \frac{p_{xx}}{p_x}\theta - \imath\lambda p_x\left(\frac{q_x+1}{p_x}\right)_x - \lambda^2(q_x^2 + \alpha^2 p_x^2), \tag{14b}$$

where it appears clearly that θ is a conserved quantity. Let us now expand θ in power series of $\imath\lambda$ as follows

$$\theta = \sum_{j=-1}^{\infty} \frac{\theta_j}{(\imath\lambda)^j}, \tag{15}$$

which, substituted in Eq. (14) yields

$$\theta_{n,x} + \sum_{j=-1}^{n+1}\theta_j\theta_{n-j} - 2\theta_{n+1} - \frac{p_{xx}}{p_x}\theta_n + p_x\left(\frac{q_x+1}{p_x}\right)_x\delta_{n,-1}$$
$$= (q_x^2 + \alpha^2 p_x - 1)\delta_{n,-2}, \tag{16}$$

where $\delta_{u,v} = 1$ for $u = v$ and $\delta_{u,v} = 0$ for $u \neq v$. Infinite number of conserved quantities can hence be generated, the first two of them being given as follows

$$\theta_{-1} = 1 \pm \sqrt{q_x^2 + \alpha^2 p_x^2}, \tag{17a}$$

$$\theta_0 = \frac{p_x}{2(1-\theta_{-1})}\left(\frac{\theta_{-1} - q_x - 1}{p_x}\right)_x. \tag{17b}$$

The asymptotic behaviors of the functions ϕ, $\bar{\phi}$, $\bar{\psi}$, ψ and a are given as

$$\phi = \begin{pmatrix} 1 \\ -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}\end{pmatrix}\exp(-\imath\lambda x + \imath\lambda\epsilon_- + \nu_-) + o(1/\lambda), \tag{18a}$$

$$\bar{\phi} = \begin{pmatrix} -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x} \\ -1\end{pmatrix}\exp(\imath\lambda x - \imath\lambda\epsilon_- + \nu_-) + o(1/\lambda), \tag{18b}$$

$$\psi = \begin{pmatrix} \frac{q_x+\sqrt{p_x^2+\alpha^2p_x^2}}{\alpha p_x} \\ 1\end{pmatrix}\exp(\imath\lambda x + \imath\lambda\epsilon_+ - \nu_+) + o(1/\lambda), \tag{18c}$$

$$\bar{\psi} = \begin{pmatrix} 1 \\ -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}\end{pmatrix}\exp(-\imath\lambda x - \imath\lambda\epsilon_+ - \nu_+) + o(1/\lambda), \tag{18d}$$

$$a = \exp(\imath\lambda\epsilon + \nu) + o(1/\lambda), \tag{18e}$$

where

$$\epsilon_- = \int_{-\infty}^{x}\theta_{-1}\mathrm{d}l, \qquad \epsilon_+ = \int_{x}^{\infty}\theta_{-1}\mathrm{d}l, \tag{19a}$$

$$\epsilon = \epsilon_- + \epsilon_+ = \int_{-\infty}^{\infty}\theta_{-1}\mathrm{d}l, \tag{19b}$$

$$\nu_- = \int_{-\infty}^{x}\theta_0\mathrm{d}l, \qquad \nu_+ = \int_{x}^{\infty}\theta_0\mathrm{d}l, \tag{19c}$$

$$\nu = \nu_- + \nu_+ = \int_{-\infty}^{\infty}\theta_0\mathrm{d}l. \tag{19d}$$

Following the procedure, we consider that p_x and q_x have compact support, that is, $\phi\exp[\imath\lambda(x-\epsilon_-)]$, $\bar{\phi}\exp[-\imath\lambda(x-\epsilon_-)]$, $\psi\exp[-\imath\lambda(x+\epsilon_+)]$, $\bar{\psi}\exp[\imath\lambda(x+\epsilon_+)]$ and $a\exp(-\imath\lambda\epsilon)$ can be expressed as functions of the eigenvalue λ. Introducing the following equations

$$I = \int_C \frac{\mathrm{d}\lambda'}{\lambda' - \lambda}\frac{1}{a(\lambda')\exp(-\imath\lambda'\epsilon)}\begin{pmatrix}\phi_1(\lambda')\\ \phi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x-\epsilon_-)], \tag{20}$$

C being the contour in the complex λ' plane, that goes from $\lambda' = -\infty + 0^+$, englobing all the zeros of $a(\lambda')$ and ending at $\lambda' = \infty + 0^+$ for λ' below C.

After all of this is settled, we pursue by deriving a system that constitutes the heart of inverse scattering transform method. Indeed, from Eqs. (8) and (20) we derive:

$$I = \int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\epsilon_+)] +$$
$$\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\epsilon_+)]. \tag{21}$$

which can be rewritten as follows

$$I_{left} = -\imath\pi\begin{pmatrix}1\\ -\frac{C_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}\end{pmatrix}\exp(-\nu_+), \tag{22a}$$

$$I_{right} = -2\imath\pi\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\epsilon_+)]$$
$$+\imath\pi\begin{pmatrix}1\\ -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}\end{pmatrix}\exp(-\nu_+)$$
$$+\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')\end{pmatrix}\exp[\imath\lambda'(x+\epsilon_+)]. \tag{22b}$$

We then deduce,

$$\bar{\psi}_1(\lambda')\exp[\imath\lambda(x+\epsilon_+)] = \exp(-\nu_+)$$
$$+\frac{1}{2\imath\pi}\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi_1(\lambda')\exp[\imath\lambda'(x+\epsilon_+)] \tag{23a}$$

$$\bar{\psi}_2(\lambda')\exp[\imath\lambda(x+\epsilon_+)] = -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}\exp(-\nu_+)$$
$$+\frac{1}{2\imath\pi}\int_C \frac{\mathrm{d}\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi_2(\lambda')\exp[\imath\lambda'(x+\epsilon_+)]. \tag{23b}$$

To derive solutions to the system of our interest, we need at this point to introduce two functions g_1 and g_2 as follows

$$\psi_1 = \int_x^{\infty}\lambda g_1(x,z)\exp[\imath\lambda(z+\epsilon_+(x)) - \nu_+(x)]\mathrm{d}z, \tag{24a}$$

$$\psi_2 = \exp[\imath\lambda(x+\epsilon_+(x)) - \nu_+(x)]$$
$$+\int_x^{\infty} g_2(x,z)\exp[\imath\lambda(z+\epsilon_+(x)) - \nu_+(x)]\mathrm{d}z, \tag{24b}$$

g_1 and g_2 verifying the condition

$$\lim_{z\to\infty}\begin{pmatrix}g_1(x,z)\\ g_2(x,z)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{25}$$

Proceeding to some comparison between (24a) and (18) it comes out that

$$g_1(x,x) = -\frac{q_x+\sqrt{q_x^2+\alpha^2p_x^2}}{\alpha p_x}. \tag{26}$$

It also comes straightforward, while combining Eqs. (23) and (24a), the following system

$$g_1^{\star}(x,v) - y(x+v) - \int_x^{\infty} g_2(x,z)y(z+v)\mathrm{d}z = 0, \tag{27a}$$

$$g_2^{\star}(x,v) - \int_x^{\infty} g_1(x,z)y''(z+v)\mathrm{d}z = 0, \tag{27b}$$

provided $x \geq v$ and where y and its derivative are expressed as follows

$$y = \frac{1}{2\imath\pi}\int_C \frac{d\lambda}{\lambda}\frac{b(\lambda)}{a(\lambda)}\exp[\imath\lambda(z+2\epsilon_+(x))], \quad (28a)$$

$$y'' = \frac{\partial^2 y}{\partial z^2} = \frac{1}{2\imath\pi}\int_C \lambda d\lambda \frac{b(\lambda)}{a(\lambda)}\exp[\imath\lambda(z+2\epsilon_+(x))]. \quad (28b)$$

Eq. (27) is often referred to as the Gel'fand–Levitan equation, which is the central equation that emerge from the direct scattering and from which starts the inverse scattering procedure to derive solutions of nonlinear evolution equation. The time dependence of the scattering data is provided by Eq. (5) to be

$$a(\lambda, t) = a(\lambda, 0), \quad (29a)$$

$$b(\lambda, t) = b(\lambda, 0)\exp(\imath t/(2\lambda)). \quad (29b)$$

A more simple expression of $y(z)$ can be given when considering that the bound state of $a(\lambda)$ is restricted in the upper-half plane and its zeros in this plane are simple. In this case, $F(z)$ can be expressed as

$$y(z) = \sum_j^n \frac{c_j(t)}{\lambda_j}\exp[\imath\lambda_j(z+2\epsilon_+(x))] + \frac{1}{2\imath\pi}\int_{-\infty}^{+\infty}\frac{\varrho(\lambda,t)}{\lambda}\exp[\imath\lambda(z+2\epsilon_+(x))], \quad (30)$$

with

$$c_j(t) = c_{0j}\exp(\imath t/2\lambda_j), \quad (31a)$$

$$\varrho(\lambda, t) = \varrho(\lambda, 0)\exp(\imath t/2\lambda). \quad (31b)$$

To solve the Gel'fand–Levitan Eq. (27), one foremost has to determine $y(z)$ to obtain $g_1(x, x)$, when the time dependent data $\{\rho(\lambda, 0), \lambda, c_j(0), \lambda_j, j = 1, \cdots, N\}$ are given and, as a result, multi soliton solutions are derived from Eq. (26).

Looking for soliton solution to the above system (3), we restrict our attention on discrete spectrum that is, we consider $\rho(\lambda, t) = 0$. Therefore, the expression of y becomes

$$y(z,t) = \sum_{j=1}^N \frac{c_j(t)}{\imath\omega_j}\exp[-\omega_j(z+2\epsilon_+)], \quad (32)$$

where we have set $\lambda_j = \imath\omega_j$. To the resolution of the Gel'Fand–Levitan equations being easier, we chose to express the Kernels $g_1(x, x)$ and $g_2(x, x)$ as follows

$$g_1(x,z) = \sum_{j=1}^N R_j(x)\exp[-\omega_j(x+z+2\epsilon_+(x))], \quad (33a)$$

$$g_2(x,z) = \sum_{j=1}^N S_j(x)\exp[-\omega_j(x+z+2\epsilon_+(x))], \quad (33b)$$

with R_j and S_j being complex functions. The substitution of Eqs. (32) and (33) into Eq. (27), yields

$$R_k - \frac{c_k}{\imath\omega_k}\sum_{j=1}^N \frac{S_j}{\omega_j+\omega_k}\exp[-2\omega_j(x+\epsilon_+)] = \frac{c_k}{\imath\omega_k}, \quad (34a)$$

$$S_j + \imath c_j\omega_j \sum_{m=1}^N \frac{R_m}{\omega_j+\omega_m}\exp[-2\omega_m(x+\epsilon_+)] = 0. \quad (34b)$$

Thus,

$$g_1(x,x) = \sum_{j=1}^N R_j\exp[-2\omega_j(x+\epsilon_+)]. \quad (35)$$

From Eq. (26), we find that

$$\alpha p_x = 2g_1(x,x), \quad (36a)$$

$$q_x = g_1^2(x,x) - 1. \quad (36b)$$

From Eqs. (21) and (19), we find the relation

$$\frac{\partial u}{\partial x} = \sqrt{q_x^2+\alpha^2 p_x^2}, \quad (37)$$

where we have chosen $u = x + \epsilon_+(x, x)$. For g_1, p_x and q_x being functions that depend on $x + \epsilon_+(x, x)$, this justifies the idea of using the variable u in replacement of x while investigating analytical solutions to the system (3). Under this new variable u, p and q can be rewritten as follows

$$\frac{\partial(\alpha p)}{\partial u} = \frac{2g_1(u)}{1+g_1^2(u)}, \quad (38a)$$

$$\frac{\partial q}{\partial u} = 1 - \frac{2g_1^2(u)}{1+g_1^2(u)}, \quad (38b)$$

$$\frac{\partial \epsilon_+}{\partial u} = \frac{g_1(u)^2}{1+g_1^2(u)}, \quad (38c)$$

where $g_1(u) = g_1(x, x)$. By integration of the previous two equations with respect to u, one obtains at order N, multi-soliton solutions to the system (3).

For the case $N = 1$ corresponding to the one-soliton solution, it is found that g_1 has the expression

$$g_1(u) = \frac{c}{\imath\omega}\frac{\exp(-2\omega u)}{1-\frac{c^2}{4\omega^2}\exp(-4\omega u)}. \quad (39)$$

By simply evaluating the integral of Eq. (38), it easily comes

$$\alpha p = -\frac{c}{\imath\omega^2}\frac{\exp(-2\omega u)}{K}, \quad (40a)$$

$$q = u - \frac{2}{\omega K}, \quad (40b)$$

$$\epsilon_+ = \frac{1}{\omega K}, \quad (40c)$$

with

$$c = c_0\exp(-t/2\omega), \quad (41a)$$

$$K = 1 - \frac{c^2}{4\omega^2}\exp(-4\omega u). \quad (41b)$$

Eq. (40) can be rewritten as

$$p = \frac{\exp(-st)}{\omega}sech''(\varphi+\varphi_0), \quad (42a)$$

$$q = u - \frac{1}{\omega}(tanh''(\varphi+\varphi_0)+1), \quad (42b)$$

$$\varepsilon_+ = \frac{1}{2\omega}(tanh''(\varphi+\varphi_0)+1), \quad (42c)$$

$$\varphi = 2\omega\left(u+\frac{t}{4\omega^2}\right), \quad (42d)$$

$$\varphi_0 = -\frac{1}{2\omega}ln''\left(\frac{\imath c_0}{2\omega}\right), \quad (42e)$$

which is the one soliton solution to the system Eq. (3), the envelop moving at the velocity $V = 1/(4\omega^2)$. We recall that solution to Eq. (1a) is obtained when considering the approximation $\alpha^2 \simeq 1$. The amplitude of p is an exponential function of st which guarantee the increasing or the decreasing amplitude of the wave with time, depending on the values of s and t simultaneously. We point out that when we consider $s = 0$, we recover the solution of the original KMM-system, free of damping effect as given in Refs.

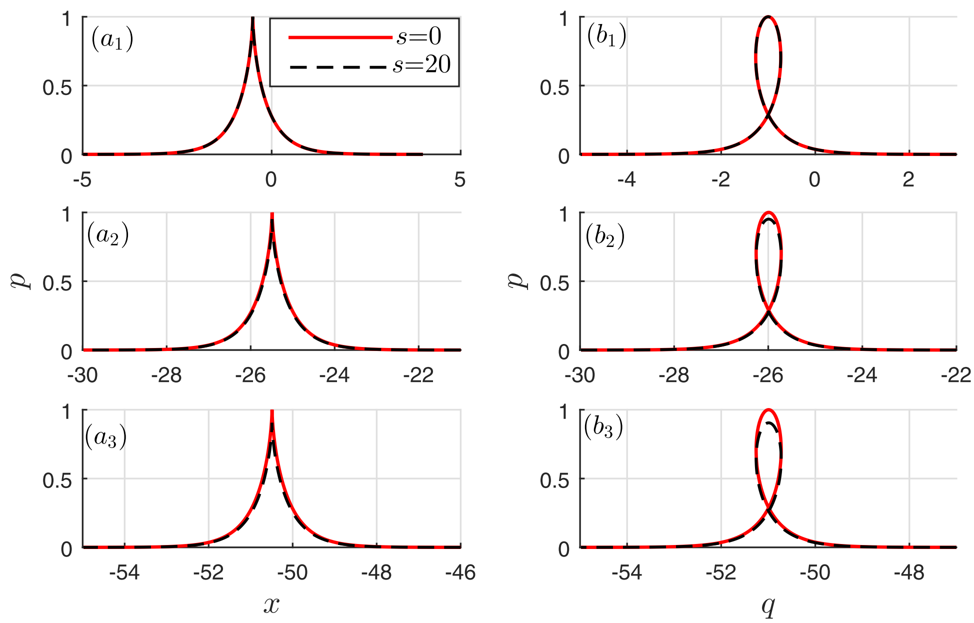

Fig. 1. Propagation of one-soliton solution p to the KMM-system Eq. (1a) for $s = 0$ (solid line) and $s = 20$ (dashed line), with the eigenvalue $\omega = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction q (panels (b_1) (b_2) and (b_3)). One observe the influence of damping on the amplitude of the left moving wave as predicted in Refs. [26].

[31,35]. It appears that for $s > 0$ and while considering the ultra fast process assumption characterized by Eq. (4), the wave propagate conserving profile but not amplitude. Such a result has been predicted in Ref. [38] under phase portraits analysis and obtained experimentally in Ref. [40] and references therein.

3. Numerical simulations

In order to complete and confirm the results obtained analytically, we proceed to some numerical simulation. We use as base equation the dissipative KMM-system Eq. (1a). To achieve this goal, we use the finite difference method with initial conditions given as:

$$B(x,0) = \frac{1}{\omega} sech''(2\omega u + \varphi_0), \tag{43a}$$

$$C(x,0) = u - \frac{1}{\omega} tanh''(2\omega u + \varphi_0) - \frac{1}{\omega}, \tag{43b}$$

$$x(u,0) = u - \frac{1}{2\omega} tanh''(2\omega u + \varphi_0) - \frac{1}{2\omega}. \tag{43c}$$

For the case of the original KMM-system free of damping effects (characterized by $s = 0$), we present in Fig. 1 a result that is in accordance with the results obtained analytically in Refs. [31,35]. It is observed clearly on this figure that numerical and analytical results match perfectly, which proves that the above numerical simulation carried out is good. We can then pursue with similar analysis while considering damping effects.

We present in Fig. 2 the propagation of waves in magnetic slab subjected to damping effects. For this aim, we present analytical results and numerical simulations for the parameter $s = 20$ (which we have chosen arbitrarily without lost of generality), of the Magnetic field p versus x and the magnetic field p versus the magnetization q. We observe also a good coincidence as for the case of the system free of damping effects. The wave evolves with amplitude that decreases with time due to the damping parameter s. It appears clearly that the numerical analysis confirms the analytical results discussed in the previous section.

On the regards of the above results, the parameter s, originated from the Landau–Lifshitz–Gilbert damping, plays a role on the dynamic of magnetic waves in ferrites. This role is characterized by the dissipation of energy during the motion of wave in ferrite observed here on the amplitude of the wave that reduces as it moves. Since the dissipative KMM-system Eq. (1a) has been derived under the ultra fast and near adiabatic process assumption, the value of the damping parameter s guarantee whether or not wave survive in the ferrite. As observed analytically, for $s \geq 0$ if:

(i) $s = 0$, we are in an ideal case where magnetic wave moves in ferrite conserving it entire properties, profile and amplitude.
(ii) s is not great enough so that the amplitude of the wave diminishes with a rate $p_{i+1}/p_i > 0.5$, the wave propagates in the material and may disappear if the process is not fast enough. As illustration, we have chosen $s = 20$ to make it clear the influence of damping effects. But, experimental values of s is smaller than 20.
(iii) $s \to \infty$, the amplitude of the wave vanishes rapidly and the wave can not propagate in such a medium.

As the damping effects often exist in real circumstances, we investigate the effect of the damping on the wave propagation in ferrites. In fact, in ferrite which is the case of our study, the damping involves loss of energy from the macroscopic motion of the local magnetization field by transfer of kinetic and potential energies to microscopic thermal motion in the form of spin waves, lattice vibration, and thermal excitations, among others [26]. On the other hand, when the external magnetic field is not strong enough to eliminate all domain walls, the domain structure plays a dominant role in the damping and the local rate of energy loss vary by large amount within a ferromagnet [37]. From where, controlling the state of ferromagnet crystal describe by the magnetization vector stand to be fundamental in the understanding of the magnetic storage process of the data elements. It's clear that, magnetization dissipation expressed in term of the Gilbert-damping parameter s, is a key factor determining the performance of magnetic material in a host of application.

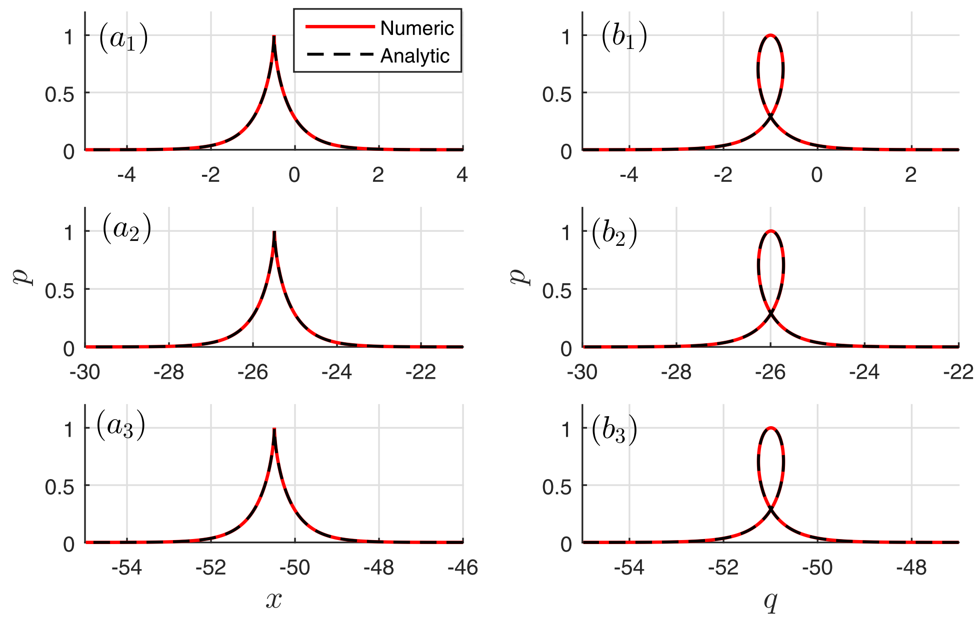

Fig. 2. Propagation of one-soliton solution p to the KMM-system Eq. (1a) for $s = 0$, with the eigenvalue $\omega = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction q (panels (b_1) (b_2) and (b_3)). The analytical and numerical solutions that show a left moving wave, that conserves its profile as predicted in Refs. [26,31].

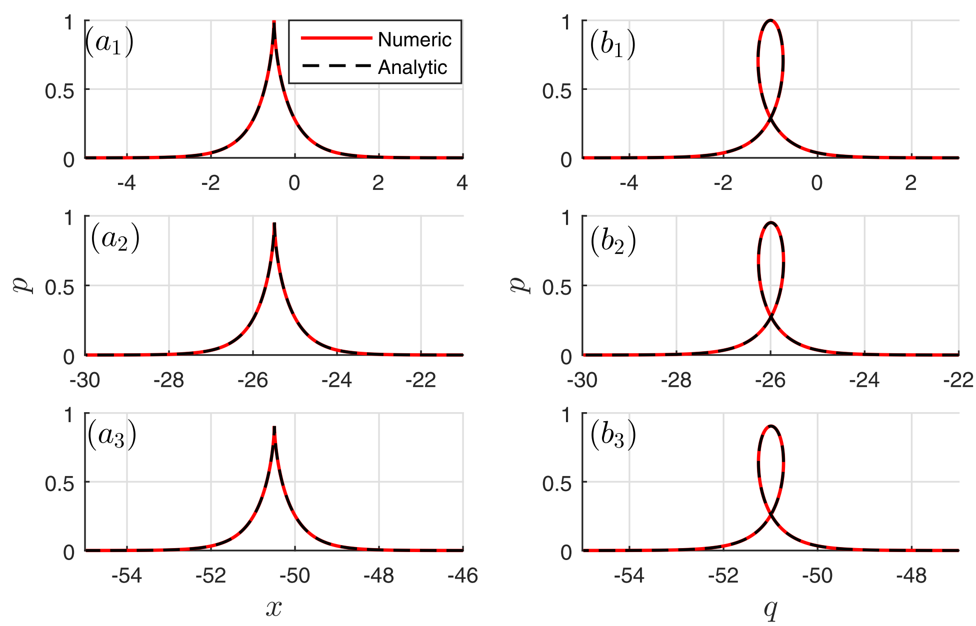

Fig. 3. Propagation of one-soliton solution p to the KMM-system Eq. (1a) for $s = 20$, with the eigenvalue $\eta = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction q (panels (b_1) (b_2) and (b_3)). The analytical and numerical solutions showing a left moving wave, that conserves its profile but not its amplitude as predicted in Ref. [26].

4. Conclusion

We paid attention on the dissipative KMM-system that describes the propagation of short-waves in magnetic insulators. Since this equation has not been proven completely integrable and solution has not been provided, we have proposed a nonlinear integrable system with associated Lax-pairs which, through an approximation that translate ultra fast process, one easily recover the dissipative KMM-system of our interest. Investigating via the inverse scattering transform method the integrable system following particularly the WKI-scheme, we have constructed, while considering an ultra fast process approximation, soliton solutions to the dissipative KMM-system and have presented analytical expressions in Eq. (42). This analytical expressions show that amplitude of the wave is proportional to an exponential function that decreases as time evolves. It then appears that the wave survive only if we are on an ultra fast process. As illustration, we depict in Fig. 1 the evolution of magnetic wave subjected to damping effects where one observe that the amplitude of the wave decreases as time evolves.

In order to confirm predictions of analytical results obtained throughout this work, we have proceeded to some numerical simulations. We have followed indeed the finite difference method and, as a result, we have depicted in Fig. 2 numerical and analytical results showing the dynamic of magnetic wave in ideal ferrite(free of damping effects). It appeared that analytical and numerical results match perfectly, showing that numerical analysis has been well carried out. The results presented for the non dissipative KMM-system is in accordance with results provided in Refs. [31,35]. While considering $s \neq 0$ and $s > 0$ the damping effect on the dynamic of magnetic wave appeared to behave very similarly with the results obtained analytically. We have then depicted in Fig. 3 numerical solutions superposed on analytical solution and have observed that the solutions also coincide here. This Fig. 3 shows that analytical results carried out previously in this work is in accordance with predictions in Refs. [26,38,41,42]. The damping parameter on the propagation of waves in ferrite acts just on the amplitude of the waves during their motion and not on the profile. such a conclusion is in accordance with experiments as pointed out in Ref. [40]. A question then arises to know if there are parameters that may act simultaneously on the profile and amplitude of the wave during it motion in ferrites. Discussing higher-order terms in Landau-Lifshitz-Gilbert damping may help answering to such a question.

There are many nonlinear systems of physical implications that deserve similar analysis both analytically and numerically. The case of a more general KMM-system derived in Ref. [38], that include damping effects and the inhomogeneous exchange process will be of particular interest in the winding of this work. Going forward while considering higher-dimensional systems, the case of Leblond and Manna system [43,44] deserves many attention. These investigation will constitute the matter of forthcoming papers.

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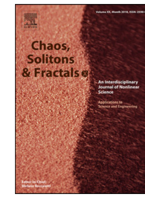

Traveling magnetic wave motion in ferrites: Impact of inhomogeneous exchange effects

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ABSTRACT

In this article, we derive new traveling wave solution to a nonlinear evolution equation, describing propagation of short wave in inhomogeneous ferrites. Applying Jacobi elliptic function, we derive as series of new exact solutions to the system of our interest which are either periodic or localized solutions. We point out the influence of the inhomogeneous exchange effects on the dynamics of traveling waves obtained. It appears that the soliton responsible of localization is deformed by the presence of inhomogeneities in particular its structure. We discuss some physical implications of these results.

1. Introduction

Propagation of wave in nonlinear media appears to be one of the most famous topic that has attracted attention of number of researchers [1–5]. Such a topic has been applied in a wide variety of media including water [6–9], optical fibers [10–14], magnetic insulators [15–18], just to name these ones. Investigations of propagation of wave in the previously mentioned media often concern experimental observations and theoretical descriptions. Concerning theoretical descriptions, it appears as an important part investigations that often come after an experimental observation and, when theoretical explanation is well carried out to generate nonlinear evolution system to model the phenomenon observed, it is possible to predict theoretically the behavior of the wave in the medium of interest under some new hypothesis before returning in laboratories to verify if the theoretical hypothesis are confirmed experimentally. Such an inter-dependance between experiment and theoretical investigation make it clear the importance of both in looking for more innovative technological devices. As far as we are concerned throughout this paper with the propagation of short wave in ferrites, since the nonlinear system, derived in Ref. [16] has been investigated intensively under some tractable methods including Hirota's bilinearization [17,19], inverse scattering transform method [5,14,15] and some expansion functions [20–23]. Beside this system, known in the literature as the KMM system [24], a new nonlinear system has been derived in Ref. [17] that takes into account the damping effects. Influence of this damping effect has been investigated under the initial value problem and, as a result, it appeared that for non zero damping, energy of the wave decreases. Going further in investigating propagation of short waves in real ferrites, the previously mentioned system that takes into account damping, has been subjected to amelioration while taking into account not only the damping effects, but also the inhomogeneous exchange effects in Ref. [25]. At fixed value of the inhomogeneous term, initial value problem of the new system has been investigated and, as a result, it appeared that energy decreases as wave evolves. In Ref. [26] soliton solutions of the inhomogeneous system have been constructed along with associated energy densities. By fixing the value of the parameter characterizing inhomogeneous exchange effects, it influence on the parameters of the propagating wave in ferrites subjected to this phenomenon is not clear. Such a question then constitutes the main aim of our investigations.

Therefore, the present paper is organized as follows: in Section 2, we construct new traveling wave solution to the inhomogeneous system of our interest using expansion of Jacobi elliptic

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functions. We discuss in Section 3 the influence of the inhomogeneous exchange effects on the traveling waves, we address some physical implications and, we end this work with a brief conclusion and perspectives.

2. Jacobi elliptic function method

To investigate solutions to nonlinear evolution equation, one often has recourse to expansion of known solution. It is in this idea that for example in Ref. [27], G'/G-expansion method has been used to construct solution to nonlinear evolution equation. We follow a similar procedure while expanding elliptic functions to derive solutions of the following system

$$u_{xt} - uv_x = \rho u_{xx}, \tag{1a}$$

$$v_{xt} + uu_x = 0, \tag{1b}$$

which describes the propagation of magnetic waves in ferrites, the parameter ρ representing inhomogeneous parameter, u represents the magnetic field and v represents magnetization. This system has been derived from the Maxwell's equations supplemented by the Landau-Lifshitz-Gilbert equation [25] in which the damping effects are neglected.

Jacobi elliptic functions, are periodic functions that properties are in someway similar to the well known periodic functions. As illustration, let us consider the following integral

$$R(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-m^2t^2}}. \tag{2}$$

The sine elliptic function *sn* is defined as follows

$$\mathrm{sn}(x, m) = R^{-1}(x), \quad 0 \le m \le 1. \tag{3}$$

Beside the sine elliptic function is defined, the other elliptic functions satisfy the relations

$$\mathrm{cn}^2(x, m) = 1 - \mathrm{sn}^2(x, m), \tag{4a}$$

$$\mathrm{dn}^2(x, m) = 1 - m^2\mathrm{sn}^2(x, m), \tag{4b}$$

where *cn* is the cosine elliptic function while *dn* is the elliptic function of the third kind. The derivative of the above elliptic functions can be expressed as follows Refs. [28–30]:

$$\frac{d}{dx}\mathrm{sn}(x) = \mathrm{cn}(x)\mathrm{dn}(x), \tag{5a}$$

$$\frac{d}{dx}\mathrm{cn}(x) = -\mathrm{sn}(x)\mathrm{dn}(x), \tag{5b}$$

$$\frac{d}{dx}\mathrm{dn}(x) = -m^2\mathrm{sn}(x)\mathrm{cn}(x). \tag{5c}$$

Before proceeding to the determination of solution of the system (1), it is necessary to introduce the variable transformation

$$\eta = kx + \omega t + \eta_0. \tag{6}$$

Under this transformation, one replaces $u(x,\ t)$ by $u(\eta)$, $v(x,\ t)$ by $v(\eta)$ and the system (1) is rewritten as follows

$$(\omega - k\rho)u_{\eta\eta} = a_0 u - \frac{1}{2\omega}u^3, \tag{7a}$$

$$\omega v_{\eta\eta} = -uu_\eta, \tag{7b}$$

where a_0 is an integration constant. Under the new variable transformation (6), we now look for solutions of the system (7) while proceeding to elliptic function expansions.

2.1. Jacobi elliptic sine function method

Using the Jacobi elliptic function method, we set the ansatz as a finite series of elliptic sine function as follows

$$u(\eta) = \sum_{j=0}^{N} p_j \mathrm{sn}^j(\eta). \tag{8}$$

According to the homogeneous balance principle, ne know $N = 1$. So we have

$$u(\eta) = p_0 + p_1\mathrm{sn}(\eta). \tag{9}$$

replacing Eq. (9) into Eq. (7) we obtain as a result

$$p_0 = 0, \tag{10a}$$

$$p_1 = \pm 2m\sqrt{\omega(k\rho - \omega)}, \tag{10b}$$

$$a_0 = (1 + m^2)(k\rho - \omega), \tag{10c}$$

and the explicit form of the solution are given as follows

$$u(\eta) = \pm 2m\sqrt{\omega(k\rho - \omega)}\mathrm{sn}(\eta), \tag{11a}$$

$$v(\eta) = (m^2 - 1)(k\rho - \omega)\eta + 2(k\rho - \omega)E(\eta, m), \tag{11b}$$

$$E(\eta, m) = \int_0^\eta \mathrm{dn}^2(s)ds. \tag{11c}$$

where ω, k and ρ are chosen in such a way that $\omega(k\rho - \omega) \ge 0$. When $m \longrightarrow 1$, Eq. (11) reduces to

$$u(x, t) = \pm 2\sqrt{\omega(k\rho - \omega)}\tanh(kx + \omega t + \eta_0), \tag{12a}$$

$$v(x, t) = 2(k\rho - \omega)\tanh(kx + \omega t + \eta_0). \tag{12b}$$

From this Eq. (12), it is clear that while drawing u versus v, one will obtain a straight line.

2.2. Jacobi elliptic cosine function method

After the sine elliptic function expansion, we can also set an ansatz constituted of expansion of elliptic cosine function as follow

$$u(\eta) = \sum_{j=0}^{N} q_j \mathrm{cn}^j(\eta). \tag{13}$$

As for the previous analysis, we restrict our attention to the first order, that is

$$u(\eta) = q_0 + q_1\mathrm{cn}(\eta). \tag{14}$$

Replacing Eq. (14) into Eq. (7), one obtains

$$q_0 = 0, \tag{15a}$$

$$q_1 = \pm 2m\sqrt{\omega(\omega - k\rho)}, \tag{15b}$$

$$a_0 = (2m^2 - 1)(\omega - k\rho), \tag{15c}$$

solution being explicitly expressed as

$$u(\eta) = \pm 2m\sqrt{\omega(\omega - k\rho)}\mathrm{cn}(\eta), \tag{16a}$$

$$v(\eta) = (\omega - k\rho)\eta - 2(\omega - k\rho)E(\eta, m). \tag{16b}$$

For $m \longrightarrow 1$, Eq. (16) reduces to

$$u(x, t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{sech}(kx + \omega t + \eta_0), \tag{17a}$$

$$v(x, t) = (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). \tag{17b}$$

2.3. Jacobi elliptic function of the third kind method

Similarly to the previous analysis, we set the following ansatz that is an expansion of the elliptic function of the third kind as follows

$$u(\eta) = \sum_{j=0}^{N} r_j \mathrm{dn}^j(\eta). \tag{18}$$

Restriction made to the first order, u gives

$$u(\eta) = r_0 + r_1 \mathrm{dn}(\eta). \tag{19}$$

By replacing restriction into Eq. (7), we obtain as a result

$$r_0 = 0, \tag{20a}$$

$$r_1 = \pm 2\sqrt{\omega(\omega - k\rho)}, \tag{20b}$$

$$a_0 = (2 - m^2)(\omega - k\rho), \tag{20c}$$

which help writing u and v in their explicit form

$$u(\eta) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{dn}(\eta), \tag{21a}$$

$$v(\eta) = (2 - m^2)(\omega - k\rho)\eta - 2(\omega - k\rho)E(\eta, m). \tag{21b}$$

For $m \longrightarrow 1$, Eq. (21) reduces to

$$u(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{sech}(kx + \omega t + \eta_0), \tag{22a}$$

$$v(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). \tag{22b}$$

2.4. Jacobi elliptic function cs(η) method

We now proceed to the expansion of the $cs(\eta)$ elliptic function as follows

$$u(\eta) = \sum_{j=0}^{N} \theta_j \mathrm{cs}^j(\eta), \tag{23}$$

where $cs(\eta) = cn(\eta)/sn(\eta)$. Restriction to the first order allow us to write

$$u(\eta) = \theta_0 + \theta_1 \mathrm{cs}(\eta). \tag{24}$$

As for the previous cases, while inserting the previous Eq. (24) into Eq. (7) one obtains

$$\theta_0 = 0, \tag{25a}$$

$$\theta_1 = \pm 2\sqrt{\omega(\omega - k\rho)}, \tag{25b}$$

$$a_0 = (2 - m^2)(\omega - k\rho), \tag{25c}$$

and with the help of this parameters, the solutions are expressed as follows

$$u(\eta) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{cs}(\eta), \tag{26a}$$

$$v(\eta) = (2 - m^2)(\omega - k\rho)\eta - 2(\omega - k\rho)\int \mathrm{cs}^2(\eta)d\eta. \tag{26b}$$

When $m \longrightarrow 1$, Eq. (26) reduces to

$$u(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{csch}(kx + \omega t + \eta_0), \tag{27a}$$

$$v(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) + 2\frac{\omega - k\rho}{\tanh(kx + \omega t + \eta_0)}. \tag{27b}$$

2.5. Rational function solution

To determine algebraic solution to the system of our interest, we set $a_0 = 0$. Multiplying Eq. (7) with u_η one obtains

$$(\omega - k\rho)u_\eta u_{\eta\eta} = -\frac{1}{2\omega}u_\eta u^3, \tag{28a}$$

$$\omega v_{\eta\eta} = -u u_\eta, \tag{28b}$$

and by simple integration we obtain

$$u(x,t) = \pm 2\frac{\sqrt{\omega(k\rho - \omega)}}{kx + \omega t + \eta_0}, \tag{29a}$$

$$v(x,t) = 2\frac{\omega(k\rho - \omega)}{kx + \omega t + \eta_0} + C_0, \tag{29b}$$

which, when depicting u versus v with $C_0 = 0$, one will obtain a straight line as for limiting case obtained in Eq. (12). All the solutions obtained above are new and deserve many attentions in investigating the influence of inhomogeneous exchange effect on the dynamics of magnetic wave in ferrites. It is also important to point out the fact that solutions given by Eqs. (12) and (29) are particular because they are due to the presence of inhomogeneities, while the other solutions can exist even in the absence of inhomogeneities.

3. Influence of the inhomogeneous exchange effects on the dynamics of magnetic waves in ferrites

As pointed out before, the expansion of Jacobi elliptic functions has allow to construct a series of nonlinear wave solution and periodic solution to the system (1). We now aim at using some of these solutions to describe the propagation of magnetic waves in inhomogeneous ferrites. The solutions that particularly retain our attention are

$$u(x,t) = \pm 2\sqrt{\omega(\omega - k\rho)}\mathrm{sech}(kx + \omega t + \eta_0), \tag{30a}$$

$$v(x,t) = (\omega - k\rho)(kx + \omega t + \eta_0) - 2(\omega - k\rho)\tanh(kx + \omega t + \eta_0). \tag{30b}$$

These solutions are different from the one obtained in Ref. [26], where solutions are given under the form

$$u(x,t) = \pm 2\sqrt{\omega/k}\mathrm{sech}(kx + \omega t + \eta_0), \tag{31a}$$

$$v(x,t) = x - (2/k)\tanh(kx + \omega t + \eta_0). \tag{31b}$$

Its comes clearly that to the expression given in Eq. (31) one must add to x a time-like parameter. There is an information that does not appears directly through this method, that is the dispersion relation. Such a relation is important to choose appropriately the parameters k standing for wave number, and the parameter ω standing for frequency. But in Ref. [26], such a relation has been given that from the Hirota's bilinear scheme. From the results obtained above, it is possible to derive this dispersion relation. Indeed, let us recall the system of our interest:

$$(\omega - k\rho)u_{\eta\eta} = a_0 u - \frac{1}{2\omega}u^3,$$
$$\omega v_{\eta\eta} + u u_\eta = 0.$$

By simple integration, the second of the above system can be rewritten as follows

$$v_\eta = -\frac{1}{2\omega}u^2 + a_0, \tag{32}$$

where a_0 is the constant of integration. Using the solutions given in Eq. (22), we easily obtain the relation

$$\omega - k\rho = a_0. \tag{33}$$

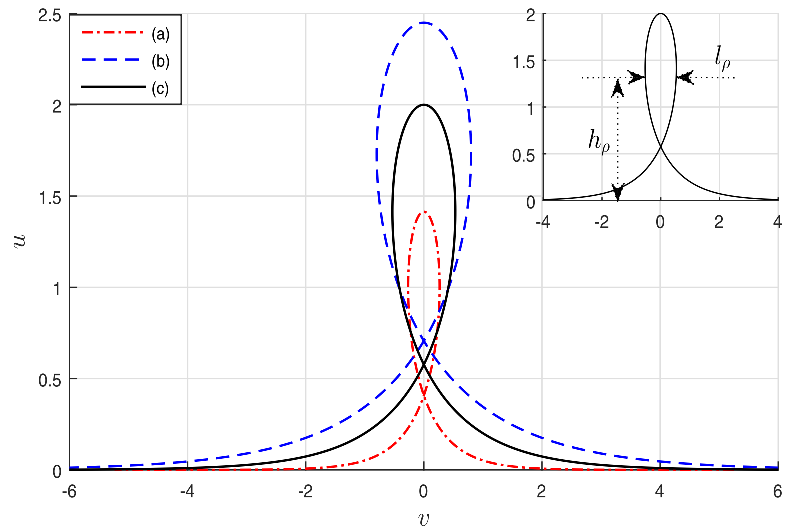

Fig. 1. One-soliton solution u to the system (1), with the wave number $k = 0.5$, (a) $\rho = 1$ (dashed-dotted line), (b) $\rho = -1$ (dashed line) and (c) $\rho = 0$ (solid line) at times $t = 0$, depicted versus the magnetization v. One observes that, inhomogeneous term contribute by increasing the amplitude of the wave when ρ takes negative values, or contribute by slowing down this amplitude when ρ is taken positive. One also observe that the parameter ρ also has a contribution on the maximum width of the loop soliton l_ρ, that increases or decreases as the amplitude of the wave increases or decreases. The hight h_ρ at which this maximum width of the wave occurs is also subjected to changes in the same way as maximum width.

Considering in particular that $a_0 = 1/k$, Eq. (33) becomes

$$k\omega - k^2\rho = 1, \tag{34}$$

which is the dispersion relation that has been obtained in Ref. [26]. We can choose arbitrarily without lost of generality $\omega = 1$, to go further with our analysis. Therefore, using this value of ω, solution provided in Eq. (22) can be rewritten as follows

$$u(x,t) = \pm 2\sqrt{1-k\rho}\,\mathrm{sech}(kx + t + \eta_0), \tag{35a}$$

$$v(x,t) = (1-k\rho)(kx + t + \eta_0) - 2(1-k\rho)\tanh(kx + t + \eta_0). \tag{35b}$$

Deriving this solution with respect to η, one obtains

$$u_\eta = \pm\sqrt{1-k\rho}\sinh(\eta)\mathrm{sech}^2(\eta), \tag{36a}$$

$$v_\eta = (1-k\rho)(1-2\mathrm{sech}^2(\eta)). \tag{36b}$$

From this system Eq. (36), it comes that u_η is positive definite for $\eta \in]-\infty, 0]$ and negative definite for $\eta \in [0, +\infty[$. It also comes that $v_\eta = 0$ admits two solutions η_1 and η_2. Then, v_η is positive definite for $\eta \in]-\infty, \eta_1] \cup [\eta_2, +\infty[$ and negative definite for $\eta \in]\eta_1, \eta_2[$. It is then easy to conclude from the above analysis that v admits two extrema and changes direction twice, while in the same time u admits one extremum and changes direction ones. Therefore, by depicting u versus v, one obtains a loop soliton as presented in Fig. 1.

In Eq. (35), one observes clearly that the wave amplitude depends on the inhomogeneous parameter ρ. To complete our analysis on the impact of the inhomogeneous term on the dynamics of the wave, we define the parameter l_ρ as the maximum width of the wave and h_ρ the hight at which it occurs (see Fig. 1). Then by direct calculus, these parameters can be expressed as follows

$$l_\varrho = 2(1-k\rho)\left(\sqrt{3} - \frac{1}{2}\mathrm{arctanh}\left(\frac{\sqrt{3}}{2}\right)\right), \tag{37a}$$

$$h_\varrho = \sqrt{2(1-k\rho)}, \tag{37b}$$

which allow the following discussion:

(i) By setting $\rho = 0$ one obtains the solution of the system free of inhomogeneous term. Such a solution has been provided in Refs. [15,17,18,20,24]. under some tractable methods including the Hirota's billinearization method, inverse scattering transform, expansion functions, just to name a few. This case then constitutes the reference case that serves to evaluate the impact of the inhomogeneous parameter on the dynamics of the wave.

(ii) Choosing ρ to be negative ($\rho < 0$), the amplitude of the wave increases. The maximum width l_ρ of the wave also increases along with the height h_ρ at which this maximum width occurs. The inhomogeneous exchange effects act here by amplifying the pulse in ferrites.

(iii) While selecting ρ to be positive valued ($\rho > 0$), the amplitude of the wave decreases along with l_ρ and h_ρ. This means that, inhomogeneous exchange effects contribute by reducing the pulse profile in ferrite.

As illustration, we depict in Fig. 1 the loop-like soliton solution of the system (1) for different values of the parameter ρ where all the conclusions that have emerged in the above items, are observed.

In fact, inhomogeneities known as deformations in a system, can be due to external fields or to the presence of defects, voids and gaps in the materials, and has a significant effect on the magnetization dynamics of the ferromagnet. The soliton responsible for the localization is deformed by the presence of inhomogeneities and in particular its structure. The presence of inhomogeneity delivers dissipation into the system which supports soliton excitation, but can also inject energy into it.

4. Conclusion

Throughout this work, we have paid attention to a nonlinear system describing the propagation of short-waves in ferrite, in which inhomogeneous exchange effects have been taken into account. This system, namely (1), has been investigated under the Hirota's bilinear method [26] and multi-soliton solutions have been provided, which are of loop types. The present work was devoted first of all in constructing new solutions to the same system (1). Indeed, the Jacobi elliptic functions method has been used and, as a result, number of new exact solutions have been derived which are periodic functions. Their limiting cases have been provided where the modulus of the Jacobi's elliptic functions tend to one ($m \longrightarrow 1$). These limiting case solutions appeared to be localized solutions. One solution has particularly retain our attention, which is the one given in Eqs. (17) and (22) which, compare to the result obtained in Ref. [26]. possess an addition term of time in the v solution. The second step of this work was to investigate the influence of inhomogeneous terms in ferrites. To achieve such a task, we have used the solution given in Eqs. (17) and (22). From this equation, we have evaluated the maximum width of the loop wave and the height at which it occurs. With their explicit expressions at hand, it was possible to conclude on the influence of inhomogeneous exchange on the dynamics of the wave in ferrite. It appears that for ρ negative, the amplitude of the wave increases along with the maximum width l_ρ and the height h_ρ at which it occurs. When ρ is taken positive, the opposite phenomenon is observed. It is then clear that, under the light cone of the analysis made here and the solutions obtained, there are two types of inhomogeneities in ferrites that are represented with the values taken by ρ. One cannot exclude the fact that in ferrite, the two types of inhomogeneities can appear periodically or can be randomly distributed. In the periodical appearance of inhomogeneities, one may observe a wave that moves while breathing due to the compensation between inhomogeneities such that the mean wave can move as if there were no inhomogeneous exchange effects in the considered material. For the case were the inhomogeneities appear randomly, the wave moves while breathing irregularly in function of the type of inhomogeneities encounter.

Further investigations on the contribution of such inhomogeneous exchange effects are needed to understand more deeply the behavior of the wave in magnetic insulators. Jacobi elliptic function are shown to be efficient method that can be handle to look for alternative solution to nonlinear evolution systems. It will be used in forthcoming papers to investigate solutions of nonlinear evolution equations.

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Research articles

On the dynamics of magnetic wave in ferrites: Influence of damping and inhomogeneous exchange effects

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ABSTRACT

We investigate throughout this paper the contributions of the damping effects and inhomogeneous exchange on the propagation of magnetic waves in ferrites. Indeed, we focus our attention on a nonlinear evolution system derived by Kuetche and coworkers (2015 J. Magn. Magn. Mater. **374** 1), that takes into account the contributions that we aim at studying. We first proceed by some analytical analysis of the contribution of damping effects on the solutions to the system of our interest and after, the contribution of the inhomogeneous exchange, to unearth analytical expression of a solution that well approximates the above mentioned global system. It appears as a result that contribution of damping reduces the amplitude of waves during its motion. We unearth two types of inhomogeneities that effects act on the waves in different manner by increasing or decreasing the width of the soliton in ferrites. For ferrites made of poly-crystals in which involves the two types of inhomogeneities appearing periodically, wave moving therein will breath as it moves and there must be a case in which the two types of inhomogeneities can be combined in such a way to reduce their effects on the dynamics of the wave. We confirm the validity of the analytical analysis carried out by proceeding to some numerical simulations.

1. Introduction

Waves can propagate in wide variety of media including fluids [1], stretched ropes [2], optical fibers [3–6] magnetic materials [7,8] etc. Propagation of waves in such media has attracted deep attention over years, in view to explain some huge phenomena that occur in our daily lives such as rogue waves in oceans [9,10], or because of technological benefits. In these Investigations, researchers, interested in the dynamics of waves in particular medium, often construct nonlinear equations that model the dynamic of wave moving therein. It then appears in the literature a wide number of evolution equations that describe propagation of waves in various media. As example, one may enumerate the Korteweg-de Vries (KdV) equation [11] that models the dynamic of water waves, the vector counterpart of the KdV-equation that describe evolution of waves in multi-layer fluids [12,13], the nonlinear Schrödinger equation [14] that models dynamic of waves in optical fibers, just to name a few. The real need that comes out of these equations are their solutions that are more expressive than the equations themselves. Therefore, the question of integrability is posed along with explicit expressions of solutions. To carry out such a task, there are techniques available in the literature that can be handled namely: The inverse scattering transform (IST) method [14–19] that is useful when Lax pairs to nonlinear evolution equation is provided. Beside the IST method, the Hirota's bilinearization [20–24] which stands as a more direct method to investigate solutions to nonlinear evolution equations when the bilinear equation associated to nonlinear systems are provided. To find such bilinear equation associated to nonlinear systems, one often make use of the Painlevé analysis [25,26] that is also a technique that help verifying if a system can be integrated or not before providing analytical solutions.

Having solutions at hand, understanding of the behavior of waves in considered media is possible and, some investigations may also be carried out theoretically before going back in laboratories to test their efficiency. In the case where, exact solution is difficult to be constructed, one may used the aforementioned methods to find solutions that approximates very well the solution of the system under

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investigation and numerical method can be used to confirm results obtained analytically. As far as we are concerned with a nonlinear system describing the propagation of short waves in magnetic insulators, Kraenkel, Manna and Merle [27] have derived a system known as the KMM-system, that describes propagation of ultra short light pulses in these media. Integrability of such a system has been fully investigated from prolongations structure analysis [28], Hirota's bilinearization [29,30], auxiliary equation method [31,34], Riccati method [32], Jacobi elliptic function method [33] and inverse scattering transform method [35,36]. Pursuing in the same analysis as Kraenkel, Manna and Merle the investigation of the dynamics of waves in ferrites, Nguepjouo and coworker [37] have derived a system that takes into account the damping effects from the Landau-Lifshitz-Gilbert equation [38,39]. Since this system has not been proven integrable, phase portraits analysis has been carried out to discuss the influence of such damping effects on the wave. Going further Kuetche and coauthors [40] have proposed a nonlinear system, that takes into account inhomogeneous exchange along with damping effects. They have investigated phase portraits analysis, to predict the effects of the combined effects on the dynamic of waves in ferrites. For the case of system with the term standing for inhomogeneous exchange process, loop wave has been constructed from Hirota's bilinear method [41] where the magnetic field has been constructed versus external magnetization. The case of magnetic field versus position has not been investigated. The comparison with the solutions to the KMM-system has not been investigated explicitly to know precisely the contribution of damping and inhomogeneous exchange of the magnetic field in ferrites. These questions constitute the main aim of our investigations. Thus, the present paper is organized as follows: We present in Section 2 the solution of the KMM-system that is free of damping and inhomogeneous exchange effects. We pursue in the same section by providing and approximated solution to the system taking into account only damping effects, and also in this section we discuss solution to the system in which involve only inhomogeneous exchange effects. We then deduce and approximated analytical solution of the system taking into account the combined effects of damping and inhomogeneous exchange. In Section 3 we proceed to some numerical solution to complete the analytical analysis made in Section 4. We discuss in Section 5 the results obtained and we end this work with a brief conclusion and perspectives.

2. Analytical investigation of the solutions to KMM-system taking into account inhomogeneous exchange and damping effects

2.1. The KMM-system

While investigating the propagation of short-waves in saturated ferromagnetic materials with zero-conductivity in the presence of an external field, Kraenkel and coauthors [27] have constructed the following nonlinear evolution system

$$B_{xt} = BC_x, \tag{1a}$$

$$C_{xt} = -BB_x, \tag{1b}$$

from the Maxwell's equations, where B is standing for the magnetic field, when C is standing for the external magnetization. The subscripts x and t denoting partial derivatives according to space-like and time-like variables respectively. Integrability properties of such a system has been investigated intensively under some framework including prolongation structure analysis, Hirota's bilinearization [29], and inverse scattering transform [35]. As a result, multi-soliton solutions have been predicted and constructed, the expression of the one soliton solution being given as:

$$B = -\frac{c}{\iota\eta^2}\frac{\exp(-2\eta h)}{T}, \tag{2a}$$

$$C = h - \frac{2}{\eta T}, \tag{2b}$$

$$\varepsilon_+ = \frac{1}{\eta T}, \tag{2c}$$

where ε_+ is an explicit function coming from a variable change connected to x as: $x = h - \varepsilon_+$, details of such a result being provided in Ref. [35] where the full inverse scattering method has been helpful in constructing such a result. The functions c and T are given as follows

$$c = c_0\exp(-t/2\eta), \tag{3a}$$

$$T = 1 - \frac{c^2}{4\eta^2}\exp(-4\eta h). \tag{3b}$$

The envelop moving left at the constant velocity $\mathfrak{v} = 1/(4\eta^2)$. This solution has been shown to be of loop-shape when depicting B versus C [29,35] and of spike shape when depicting B versus x [35]. The solution Eq. (2) will serve to discuss the contribution of additional terms that occur in the system taking into account inhomogeneous exchange and damping effects.

2.2. The KMM-system with damping effects

In the windings of Kraenkel, Manna and Merle ideas, Nguepjouo and coworkers [37] have derived, while considering the Landau-Lifschitz-Gilbert damping, the system

$$B_{xt} = BC_x - sB_x, \tag{4a}$$

$$C_{xt} = -BB_x, \tag{4b}$$

where s is a constant parameter that translates the damping effects on the dynamic of waves in ferrites. The initial value problem of this equation has also been investigated in details by the same authors and, as conclusion, it has appeared that the energy of the system decreases as time evolves. But till now, analytical expression of the associated solution has not been provided. It has not even been proven yet that this equation is integrable or not. Paying attention of such a question of integrability, we propose to investigate solution to a nonlinear system from which, through and approximation one obtains the solutions to the system Eq. (3). The system we are talking about is expressed as follows:

$$B_{xt} = BC_x - sB_x, \tag{5a}$$

$$C_{xt} = -\alpha^2 BB_x, \qquad \alpha = \exp(st), \tag{5b}$$

with $s \neq 0$. α can be developed in Taylor series near the initial point as follows:

$$\exp(st) = 1 + st + \frac{(st)^2}{2} + \cdots\cdots \tag{6}$$

This equation Eq. (5) has the advantage to possess associated Lax-pairs:

$$y_x = \iota\lambda\begin{pmatrix} C_x & \alpha B_x \\ \alpha B_x & -C_x \end{pmatrix} y, \tag{7a}$$

$$y_t = \begin{pmatrix} 1/4\iota\lambda & -\alpha B/2 \\ \alpha B/2 & -1/4\iota\lambda \end{pmatrix} y. \tag{7b}$$

Following the inverse scattering transform method, while considering $\alpha^2 \simeq 1$, one easily obtains a solution to the system Eq. (4). This approximated solution is given as [36]

$$B = \frac{e^{-st}}{\eta}\operatorname{sech}(\varphi + \varphi_0), \tag{8a}$$

$$C = h - \frac{1}{\eta}(\tanh(\varphi + \varphi_0) + 1), \tag{8b}$$

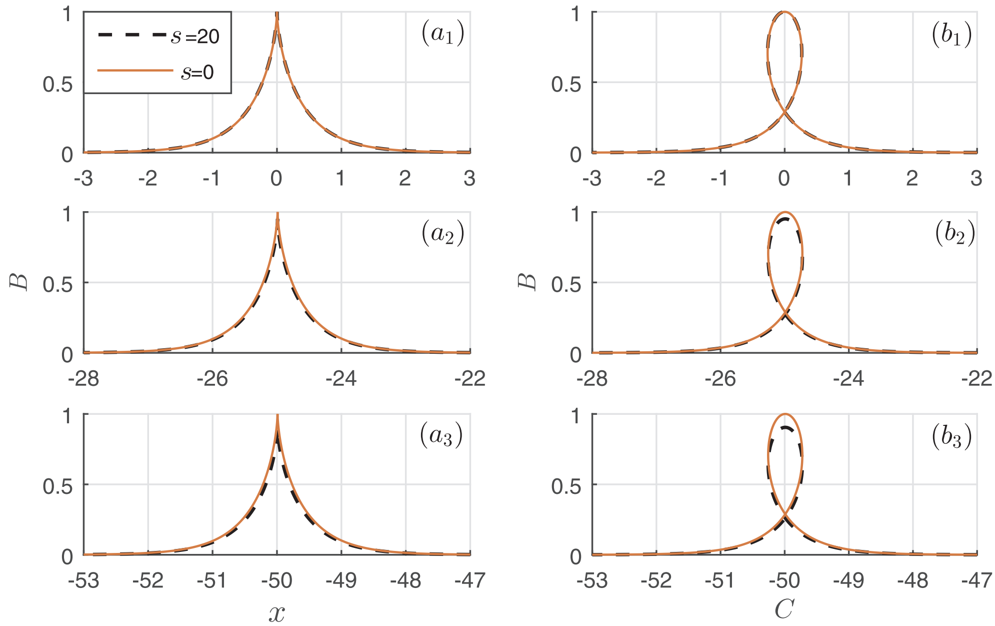

Fig. 1. Propagation of one-soliton solution B to the system Eq. (4) for $s = 0$ (solid line) and $s = 20$ (dashed line), with the eigenvalue $\eta = 1$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). The solid line ($s = 0$) is the wave solution of the KMM-system Eq. (1) while the dashed line (s = 20) is standing for the approximated solution of Eq. (4). It is observed on this figure that damping acts principally on the amplitude of the wave while its influence on the width is very low.

$$\varepsilon_+ = \frac{1}{2\eta}(\tanh(\varphi + \varphi_0) + 1), \tag{8c}$$

$$\varphi = 2\eta\left(h + \frac{t}{4\eta^2}\right), \tag{8d}$$

$$\varphi_0 = -\frac{1}{2\eta}\ln\left(\frac{\imath c_0}{2\eta}\right), \tag{8e}$$

which moves at the velocity $\mathfrak{v} = 1/(4\eta^2)$. Observing solutions Eq. (8) and the one given in Eq. (2), it appears that the damping effects contributes on the dynamic of wave by decreasing its amplitude as wave evolve. The shapes of the waves remaining the same as for the case of wave solution to the KMM-system. Influence of such a parameter is depicted in Fig. 1 where it is well observe that wave decreases as time evolve. The impact of damping effects stands to act principally on the amplitude of the wave. It also has an impact on the width of the wave but, this impact on the width is too weak that it can be neglected, as shown in Fig. 1, due to ultra fast process assumption occurring in ferrite during re-magnetization. Since influence of damping effects has been investigated, we then further discuss the influence of the inhomogeneous exchange effects.

2.3. Inhomogeneous exchange influence on the propagation of waves in ferrites

Known as deformations, inhomogeneity in general represents variations of the physical interactions due to spatial distortion of the crystal lattice [42,43]. It also represents the symmetry breaking or disorder in crystal systems [44]. Inhomogeneities can occur in materials due to external fields, presence of defects, vacuum or gaps. On the other hand, inhomogeneity can also be the result of spin interactions. Indeed, knowing that the magnetization is treated in terms of spin waves and the Heisenberg model allows to directly treat a set of spins relative interaction depends on the distance between the nearest neighbors [45]. The usual picture of the two-magnon model is that inhomogeneities introduce weak interactions between the spin-wave modes that allow the energy of the uniform precession to leak into a number of other modes, providing an effective damping of the uniform mode. Alternatively, the effect of the inhomogeneity may be regarded as a mixing of the eigenmodes of the uniform film in a way that distributes the ferromagnetic resonance intensity over a number of eigenmodes, resulting in an ferromagnetic resonance peak composed of a number of overlapping resonances [46,47]. In certain cases, due to presence of imperfection, intra-sublattice interaction becomes comparable with intersublattice interaction. In such a situation disorder and frustration takes place in the spin subsystem. Similarly, inhomogeneity can also be simulated by the deliberate introduction of imperfections (impurities or organic complexes) in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion [42,43]. These defects cause distortions in atomic shells and induce deformation of the materials. Inhomogeneities affect significantly the dynamics of magnetization of ferrites [43]. In order to investigate the inhomogeneous exchange effects on the dynamic of magnetic waves in ferrites, Kuetche and coworkers [40] have consider the Landau-Lifschitz-Gilbert damping at higher order and have derived the following system

$$B_{xt} = BC_x - sB_x + \varrho B_{xx}, \tag{9a}$$

$$C_{xt} = -BB_x, \tag{9b}$$

where B_{xx} stands to translate the inhomogeneous exchange within ferrites. We first consider the case where the damping effects are neglected (s = 0) to investigate the contribution of this inhomogeneous exchange on the propagating wave only. Looking for solutions to such a nonlinear system, Kuetche [41] has derived via Hirota's bilinear method soliton solutions and has depicted B versus C. We aim in this work at constructing in addition to B versus C, B versus x. Indeed, we consider that solution to Eq. (9) can be written as follows

$$B = k\,\mathrm{sech}\gamma, \tag{10a}$$

$$C = h - p\tanh\gamma, \tag{10b}$$

$$x = h - r\tanh\gamma, \tag{10c}$$

$$\gamma = nh + mt, \tag{10d}$$

the constant parameters k, n, m, p and r are to be determine. Introducing Eq. (10) into Eq. (9) we obtain the following system

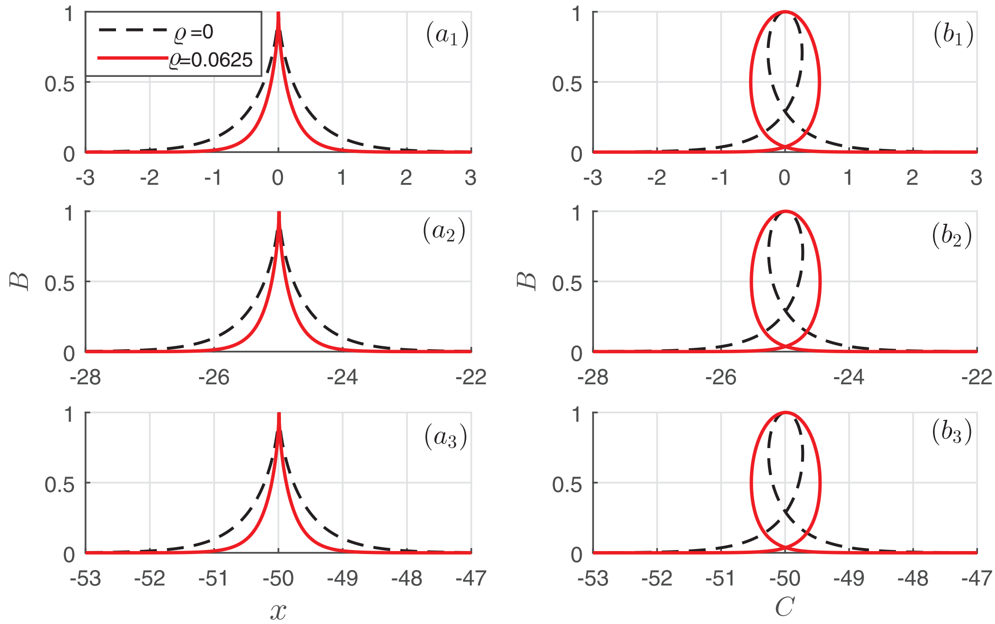

Fig. 2. Propagation of one-soliton solution B to the system Eq. (9) for $\varrho = 0$ (dashed line) and $\varrho = 0.0625$ solid line, with the eigenvalue $\eta = 1$, $s = 0$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). The dashed line ($\varrho = 0$) is the wave solution of the KMM-system Eq. (1) while the continuous line ($\varrho = 0.0625$) is standing for exact one soliton solution of Eq. (9) for ($s = 0$). It is observed on this figure that inhomogeneous effects act not on the amplitude, but on the width of the waves. For spike soliton the width of the wave decreases as observed in panels (a_1), (a_2) and (a_3) while the width of the loop soliton increases as observed on panels (b_1) (b_2) and (b_3).

$$n^2\varrho - mn + 1 = 0, \tag{11a}$$

$$np - 2nm = 0, \tag{11b}$$

$$2mp - k^2 = 0, \tag{11c}$$

The first of the above equation Eq. (11a) corresponds directly to the dispersion relation provided in Ref. [41]. Solving this system, we obtain

$$k = \pm 2m, \tag{12a}$$

$$p = 2m, \tag{12b}$$

$$n = \frac{m \pm \sqrt{m^2 - 4\varrho}}{2\varrho}, \tag{12c}$$

$$r = \frac{2\varrho}{m \pm \sqrt{m^2 - 4\varrho}}, \tag{12d}$$

with $m^2 - 4\varrho \geqslant 0$. Considering propagation of waves in ferrites without any inhomogeneous exchange processes, one has to set $\varrho = 0$ before solving Eq. (9). As a result, one recovers the solutions obtained for the case of KMM-system.

2.3.1. Case of $m^2 - 4\varrho = 0$

Explicit expression of the analytical solution of the system Eq. (9) is written as

$$B = \frac{1}{\eta}\mathrm{sech}\gamma, \tag{13a}$$

$$C = h - \frac{1}{\eta}\tanh\gamma, \tag{13b}$$

$$x = h - \frac{1}{4\eta}\tanh\gamma, \tag{13c}$$

$$\gamma = 4\eta h + \frac{t}{2\eta}, \tag{13d}$$

$$\varrho = \left(\frac{1}{4\eta}\right)^2, \tag{13e}$$

where we have replaced m by $1/2\eta$ for convenience and have chosen $\varrho = \left(\frac{1}{4\eta}\right)^2$ for simplification in the analysis. Eq. (13) stands for exact one-soliton solution to the system Eq. (9) that allows us to depict B vs C as in Ref. [41] and B versus x. To discuss the contribution of the parameter ϱ on the dynamic of the waves, we evaluate bandwidth at half height of the wave, the maximum width of the loop wave and the height at which this maximum width occurs. Indeed we define the quantities L_0 and L_ϱ as the maximum width of loop solitons, H_0 and H_ϱ the height at which it occurs and the quantities l_0 and l_ϱ the width at half maximum height of the wave when $\varrho = 0$ and when $\varrho \neq 0$. It appears that

$$L_\varrho = \frac{1}{\eta}\left(\sqrt{3} - \frac{1}{2}\mathrm{arctanh}\left(\frac{\sqrt{3}}{2}\right)\right), \tag{14a}$$

$$L_0 = \frac{1}{\eta}\left(\sqrt{2} - \mathrm{arctanh}\left(\frac{\sqrt{2}}{2}\right)\right), \tag{14b}$$

$$l_\varrho = \frac{1}{4\eta}(2\ln(2 + \sqrt{3}) - \sqrt{3}), \tag{14c}$$

$$l_0 = \frac{1}{2\eta}(2\ln(2 + \sqrt{3}) - \sqrt{3}), \tag{14d}$$

$$H_\varrho = \frac{1}{2\eta}, \tag{14e}$$

$$H_0 = \frac{\sqrt{2}}{2\eta}, \tag{14f}$$

From these expressions it is shown clearly that $\Delta = L_\varrho - L_0 > 0$ and $\delta = l_\varrho - l_0 < 0$. The influence of inhomogeneous exchange within ferrites begin to give some ideas since for the same amplitude, the additional dispersive term increase the maximum width of the loop soliton while the bandwidth at half maximum height of the spike wave decreases as presented in Figs. 2 and 3, comforted with expressions given in Eq. (14). To investigate more deeply the influence of inhomogeneous exchange effects on the dynamics of magnetic waves in ferrites, further investigations are needed while considering the case where $m^2 - 4\varrho > 0$.

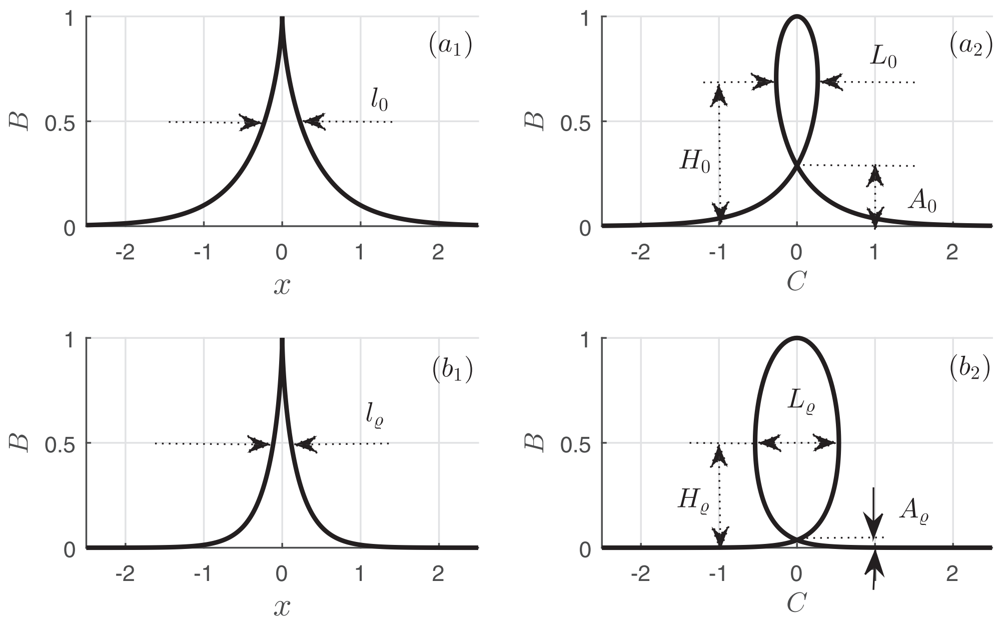

Fig. 3. Propagation of one-soliton solution B to the system Eq. (9) for $s = 0$, with the eigenvalue $\eta = 1$, $\varrho = 0.0625$ at times $t = 0$, depicted versus x and versus C (panels (a_1), (a_2)) and the one-soliton solution B to the KMM-system Eq. (1) versus x and versus C (panels (b_1), (b_2)). The parameters l_0 and l_ϱ stand for the width at half maximum height, L_0 and L_ϱ stand for the maximum width of the loop solitons, H_0 and H_ϱ represent the magnitude at which the width of the loops solitons are maximal and A_0 and A_ϱ represent the magnitude at which the cross over occurs.

Investigating solutions to the system Eq. (9), since it is not proven integrable, it is possible to find a solution that, under some approximation, verify Eq. (9). On the regards of the above solutions provided for $s = 0, \varrho \neq 0$ given in Eq. (13) and the solution obtained for $s \neq 0, \varrho = 0$, we deduce an approximated solution to Eq. (9), written as follow

$$B = \frac{e^{-st}}{\eta}\mathrm{sech}\gamma, \tag{15a}$$

$$C = h - \frac{1}{\eta}\tanh\gamma, \tag{15b}$$

$$x = h - \frac{1}{4\eta}\tanh\gamma, \tag{15c}$$

$$\gamma = 4\eta h + \frac{t}{2\eta}, \tag{15d}$$

$$\varrho = \left(\frac{1}{4\eta}\right)^2, \tag{15e}$$

provided st is very small that $e^{-2st} \simeq 1$. This consideration is in accordance with the ultra fast processes in ferrites which practically can be refer to data inputs which undergo some fast re-magnetization process within magnetic memory devices [41].

2.3.2. Case of $m^2 - 4\varrho > 0$

In this case, while considering the system free of damping effects, Eq. (13) takes the following expression

$$B = \frac{1}{\eta}\mathrm{sech}\gamma, \tag{16a}$$

$$C = h - \frac{1}{\eta}\tanh\gamma, \tag{16b}$$

$$x = h - \frac{4\eta\varrho}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}\tanh\gamma, \tag{16c}$$

$$\gamma = \frac{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}{4\eta\varrho}h + \frac{t}{2\eta}, \tag{16d}$$

which alow construction of magnetic waves with arbitrarily chosen non zero ϱ. Similarly to the previous analysis, at $t = 0$ we evaluate H_ϱ, l_ϱ and L_ϱ to be expressed as follows

$$H_\varrho = \sqrt{\frac{4\varrho}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}}, \tag{17a}$$

$$l_\varrho = \frac{4\eta\varrho}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}(2\ln(2 + \sqrt{3}) - \sqrt{3}), \tag{17b}$$

$$L_\varrho = \frac{2}{\eta}\sqrt{\frac{1 - 4\eta^2\varrho + \epsilon\sqrt{1 - 16\eta^2\varrho}}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}} - \frac{8\eta\varrho}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}\mathrm{arctanh}\left(\sqrt{\frac{1 - 4\eta^2\varrho + \epsilon\sqrt{1 - 16\eta^2\varrho}}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}}\right), \tag{17c}$$

where $\epsilon = \pm 1$. According to the parameter ϵ, we observe different behavior for the same value of ϱ. Then, the parameter ϵ characterizes two types of inhomogeneities that can occur in ferrite having different effects on the parameter of the wave. Such information is illustrated in Fig. 4 and confirmed in Fig. 5. When considering the case with $s \geqslant 0$, one obtains

$$B = \frac{e^{-st}}{\eta}\mathrm{sech}\gamma, \tag{18a}$$

$$C = h - \frac{1}{\eta}\tanh\gamma, \tag{18b}$$

$$x = h - \frac{4\eta\varrho}{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}\tanh\gamma, \tag{18c}$$

$$\gamma = \frac{1 + \epsilon\sqrt{1 - 16\eta^2\varrho}}{4\eta\varrho}h + \frac{t}{2\eta}, \tag{18d}$$

which approaches the solution of the system Eq. (9) under the ultra fast

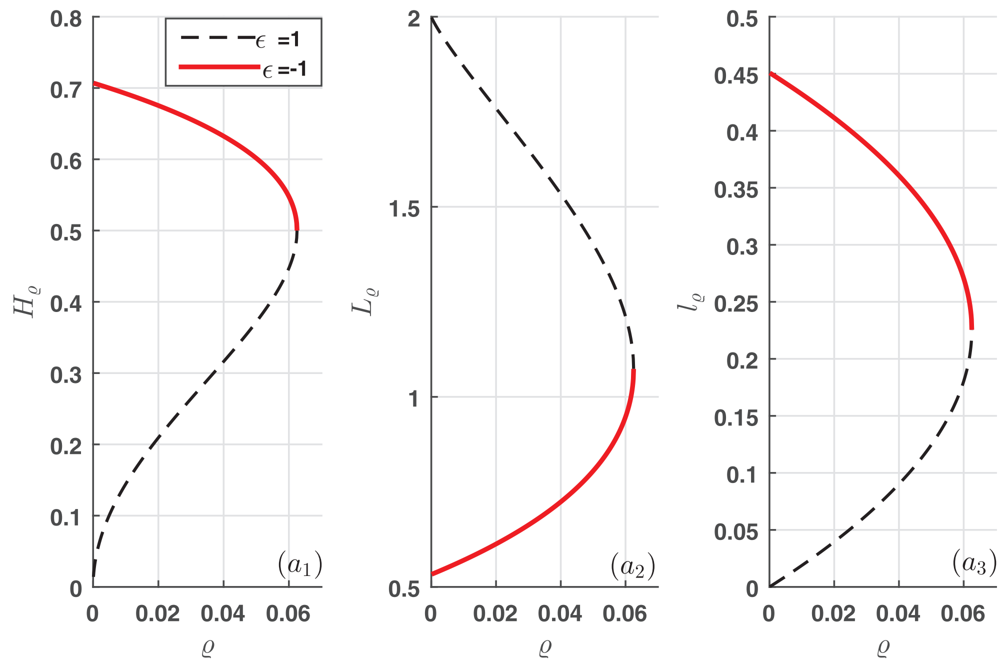

Fig. 4. Evolution of height H_ϱ at which maximum width of loop occurs versus ϱ (panel a_1), evolution of the width L_ϱ versus ϱ (panel a_2) and variation of width at half maximum l_ϱ versus ϱ (panel a_3) For different values of ϵ.

process assumption. Under this approximation, the values of L_ϱ, l_ϱ and H_ϱ do not vary significantly. Such a result is in accordance with the one obtained in Ref.[48] for YIG film. It appears that the damping effects are not only characteristic of the properties of the material, but must have something to do also with some parameters of the wave itself Ref. [43]. To bring further information about the contribution of damping effects and inhomogeneous exchange processes on the dynamics of waves in ferrites we provide the following equation

$$\begin{aligned} E=&\frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{16\eta^4\varrho}\frac{\sqrt{1-\mathrm{sech}^2\gamma}}{1-\mathrm{sech}^2\gamma}(2s\eta+\sqrt{1-\mathrm{sech}^2\gamma})\mathrm{sech}^2\gamma\\ &-\frac{1}{4\eta^3}\frac{\mathrm{sech}^2\gamma}{1-\mathrm{sech}^2\gamma}\left(\eta-\frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}\mathrm{sech}^2\gamma\right)\\ &+\frac{1}{2\eta^2}\left(\frac{1+\epsilon\sqrt{1-16\eta^2\varrho}}{4\eta\varrho}\right)^2(1-\mathrm{sech}^2\gamma)\mathrm{sech}^2\gamma, \end{aligned} \tag{19}$$

which stands for the energy density of the moving wave given in Eq. (18). In all of the cases $\epsilon = 1$ or $\epsilon = -1$ this energy density E decreases as x increases and, when ϱ increases, energy density decreases for $\epsilon = 1$ and increases for $\epsilon = -1$. Presence of inhomogeneities cause dissipation in the system exciting the soliton or can also bring in energy. We depict

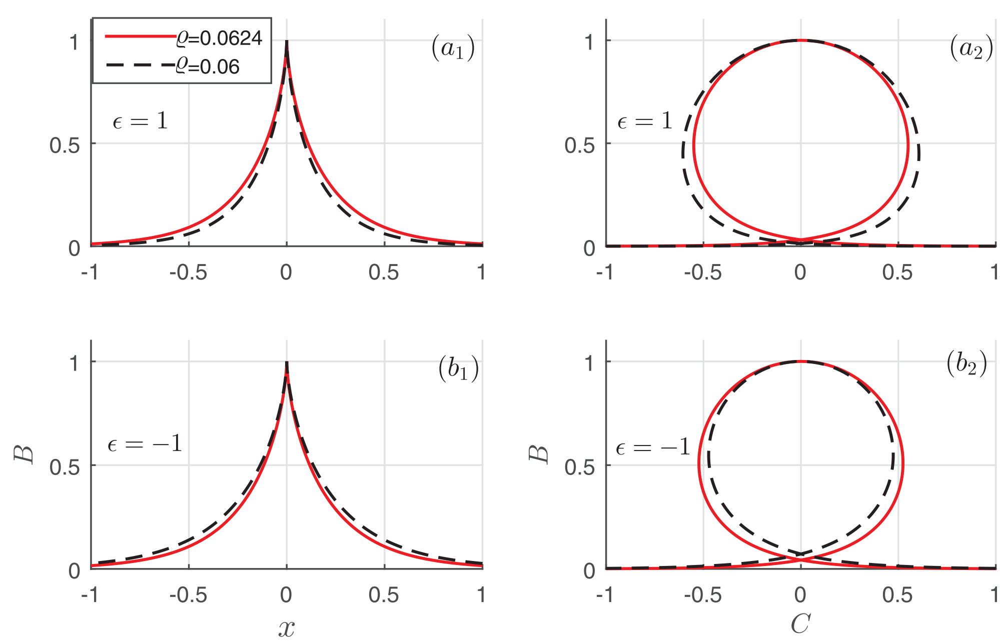

Fig. 5. One-soliton solution of the system Eq. (9). For the case $\epsilon = +1$ (panels a_1 and a_2) one observe that H_ϱ and L_ϱ decrease as ϱ increase for the loop (panel a_2) while the bandwidth increase as ϱ increase for the spike (panels a_1). When $\epsilon = -1$ the previous process reverse (panels b_1 and b_2).

in Fig. 4 the evolution of height H_ϱ versus ϱ alongside with the evolution of the width of the spike l_ϱ versus ϱ. This figures show that according to the parameter ϵ these quantities behave in the same manner, increasing when $\epsilon = +1$ and decreasing when $\epsilon = -1$. This result is coherent with the result obtained in Eq. (16) and depicted in Fig. 4 for different values of ϱ. For $\epsilon = +1$ it appears clearly that L_ϱ decreases as ϱ increases and for $\epsilon = -1$ L_ϱ increases as ϱ increases. It was not evident to provide analytical expression of A_ϱ. Meanwhile, it appears in Fig. 5 that A_ϱ increases as ϱ increases for $\epsilon = 1$ and A_ϱ decreases as ϱ increases for $\epsilon = -1$.

Now that we have investigated analytically the solution of the system Eq. (9), and have constructed its solution without damping effects and, have also constructed approximated solution while considering zero inhomogeneous exchange effects under the ultra-fast process assumption, it is thus important to verify if this solution can be confirmed numerically.

3. Numerical simulations

Investigating solutions to nonlinear evolution equations, one often derive when possible exact analytical solutions when equation is proven integrable. When on the contrary it appears difficult to solve exactly the nonlinear system under interest, it sometimes appears that approximated solutions can be derived and, in this case, further investigations are necessary. As far as we are concerned with inhomogeneous exchange within ferrite, we found it difficult to solve exactly the nonlinear system Eq. (9), but we have been able to propose a solution that approximates the solutions to the aforementioned system under a condition that translate ultra fast processes in ferrites. We now pursue with numerical investigations to confirm the results that have come out from analytical investigations. Indeed, we use the finite difference method with initial conditions given as:

$$B(x, 0) = \frac{1}{\eta}\mathrm{sech}(2\eta h + \varphi_0), \tag{20a}$$

$$C(x, 0) = h - \frac{1}{\eta}\tanh(2\eta h + \varphi_0), \tag{20b}$$

$$x(h, 0) = h - \frac{1}{2\eta}\tanh(2\eta h + \varphi_0). \tag{20c}$$

We present in Fig. 7 the evolution of loop and spike soliton, solutions of Eq. (9). On this figure, it appears that numerical simulation and analytical results are matching very well. Such a result confirms the investigations carried out in the previous section. Then, going forward in the windings of the previous analysis, we discuss numerically the case where the damping effect is taken into account ($s \neq 0$). We depict in Fig. 7 the solution of the system Eq. (9) in which one observe that once more, analytical and numerical simulations are matching. It appears that the combined effects of damping an inhomogeneous exchange contributions on the dynamics of the wave are: decreasing the amplitude of the wave, increase the width of the loop wave and decrease the bandwidth at half maximum amplitude of the wave.

The numerical simulations then confirm the analytical result.

4. Discussion of the results

We have investigated the influence of inhomogeneous exchange and damping effects on the dynamics of wave in ferrites. Indeed, we have constructed analytical solution of the system Eq. (9) that model the dynamics of such waves in which involve parameters s and ϱ that model damping and inhomogeneous exchange respectively. As a result, the following conclusions have risen out.

(i) For the case where $\varrho = 0$, we observe that for $s > 0$, the damping effects act on the wave which decrease as time evolve. The bandwidth not changing significantly in the case of ultra fast process assumption as observed in Fig. 1. Such a result has been predicted and obtained in Ref.[48] and references therein.

(ii) When considering the inhomogeneous exchange only ($\varrho \neq 0$), it appears that two cases came out according to a parameter ϵ that takes two different values which are $\epsilon = +1$ and $\epsilon = -1$. The parameters that help characterizing these inhomogeneous exchange effects being the maximum width of the loop wave L_ϱ, the height H_ϱ at which this maximum width occurs and the width l_ϱ at half maximum as provided in Fig. 3.

* When $\epsilon = +1$ it has appeared that the maximum width L_ϱ of the loop soliton decreases as ϱ increases and the height H_ϱ at which its occurs increases as ϱ increases. As illustration we have depicted in Fig. 4 (dashed line) and in Fig. 5 (panels (a_1) and (a_2)) the evolution of H_ϱ and L_ϱ versus ϱ. For the spike soliton solution that are originated from the depiction of B versus x, one has observed reverse processes where the width at half maximum l_ϱ increase as ϱ increases as depicted Fig. 4 (panels (a_3)).

* When $\epsilon = -1$ One observe a reverse process that is, L_ϱ increases while H_ϱ decreases as ϱ increases, while l_ϱ decreases as ϱ increases as observed in Figs. 4 and 5. In both cases ($\epsilon = +1$ and $\epsilon = -1$), energy densities have been provided in Eq. (19) and depicted in Fig. 6 for two different values of ϱ, which complete and confirm the previous results. All the behaviors described here are simply due to the type of inhomogeneity present in the ferrite.

(iii) When considering the combined effects of damping an inhomogeneous exchange process within ferrites, the evolution of waves are subjected to loss of energy with time, characterized by the decreasing of the wave amplitude and, according to the value taken by ϵ the bandwidth of the wave increase or decrease with ϱ.

(iv) Further investigations have been carried out to confirm the above analytical results. Indeed we have proceeded to some numerical simulations. We have followed the finite difference method and, as a result, it has come out that the analytical and numerical simulations are matching very well. We have depicted in Fig. 7 the analytical and numerical results where the combined effects of damping and inhomogeneous exchange have been taken into account.

In ferrites made of polycrystal, inhomogeneous exchange process and damping effects must be taken into account when investigating the propagation of wave in such media. The system Eq. (9) thus respond to this demand. Investigating the solution of such equation, it appears that two different types of inhomogeneities occur that increase or decrease the width of the wave according to the parameter that characterizes the inhomogeneity. One can also think about ferrites that combine the two different types of inhomogeneities that come periodically. Therefore combined effects will occur that is, width of the wave will increase and decrease alternatively and one will observe a sort of wave that breath as time evolves. Another scenario can occur that is, the combined types of inhomogeneities in the same material may help in reducing the effects of inhomogeneous exchange within ferrite which will result to a stable profile that is near to the result of KMM-system, that does not takes into consideration these combined effects.

5. Conclusion

Throughout this work, we have investigated propagation of waves in ferrites subjected to influence of damping and inhomogeneous exchange effects. Indeed, we have considered the KMM-system that appears to describe propagation of wave in ferrites free of damping and inhomogeneous exchanges. We have recalled the analytical solutions obtained for the previous equation and used the associated one-soliton for comparison purpose. Aiming at investigating the influence of damping effects in the dynamic of waves in such a medium we have derived a solution that, under some approximation (approximation that

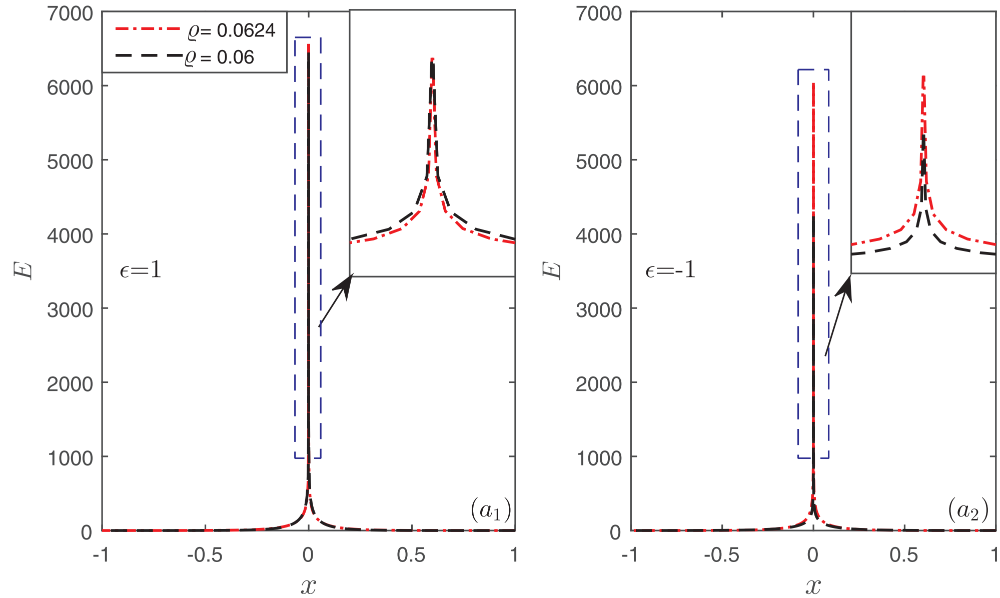

Fig. 6. Energy densities Eq. (19) depicted for the case $\epsilon = +1$ (panels a_1) where one observes that energy density increases when ϱ increases. For the case $\epsilon = -1$ (panels a_2) a reverse process is observed where energy density decrease when ϱ increases. Such a result complete and confirms the one presented in Fig. 4.

approves the ultra-fast process assumption) verify the system Eq. (4). As a result, it appears that damping has as effect to decrease the amplitude of the wave as it evolves. Pursuing our analysis, we have discussed the case of inhomogeneous exchange effects and have proposed analytical one soliton solution to the system Eq. (9) where damping effects are neglected ($s = 0$). It appeared as a result that, for a leading case verified by ϱ, inhomogeneous exchange parameter acts on the wave by reducing it bandwidth for the case of B versus x, or increasing the maximum width of the loop soliton for B depicted versus C. The amplitude of the wave and its shape are conserved in this case. In regards to the previous solutions, we have proposed an approximated solution to the system Eq. (9) that takes into account the combined effects of damping and inhomogeneous exchange. Going over the leading case verified by ϱ, we have investigated the influence of the inhomogeneous exchange process in ferrites of different homogeneities. Indeed, we constructed solution of Eq. (9) without damping, that is expressed explicitly as function of ϱ. Then, through this solution, we have characterized the wave moving in such medium while evaluating the maximum width of the loop soliton L_ϱ, the height H_ϱ at which it occurs and, for the spike soliton we have evaluated the width at half maximum of the wave l_ϱ. It has appeared as a result that, according to a parameter ϵ, l_ϱ, the height H_ϱ decrease as ϱ increases when in the same time L_ϱ increases ($\epsilon = -1$). A reverse process has been obtained for the case ($\epsilon = +1$). We have illustrated the different situations in Figs. 4 and 5. To complete these analysis, we have provided the energy densities for both cases of ϵ in Eq. (19) and depicted these energies in Fig. 6 where for two different values of ϱ one observe in each case how energy is influenced by inhomogeneity.

From the results obtained throughout this work, it has appeared two

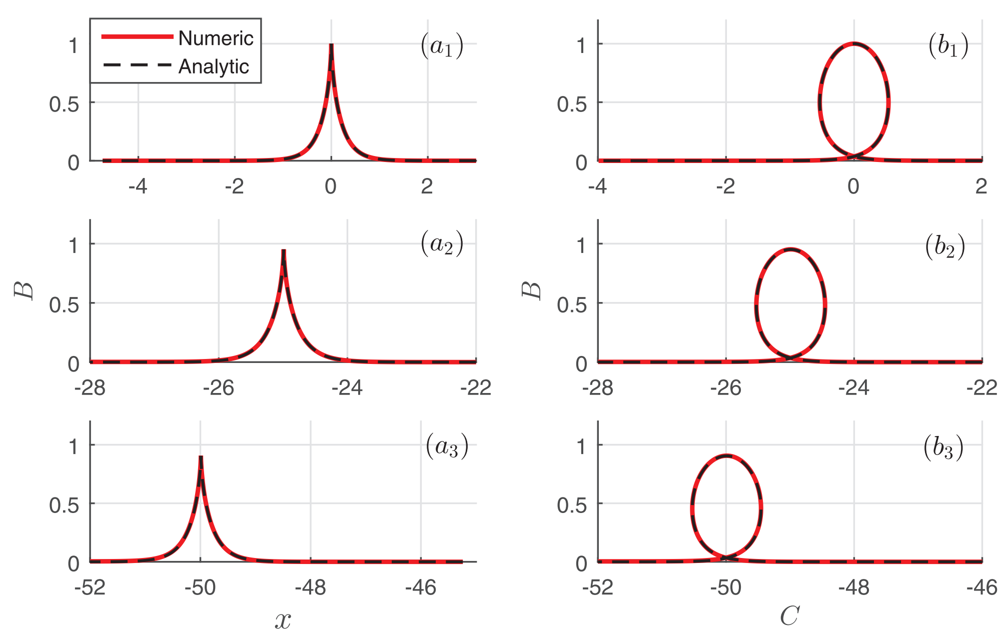

Fig. 7. Propagation of one-soliton solution B to the system Eq. (9) for $s = 20$, with the eigenvalue $\eta = 1$, $\varrho = 0.0625$ at times $t = 0$, $t = 0.0025$ and $t = 0.005$ depicted versus x (panels (a_1), (a_2) and (a_3)) and versus the Magnetization induction C (panels (b_1) (b_2) and (b_3)). One observes that, for the combine effects of damping and inhomogeneous exchange, analytical and numerical solutions match.

types of inhomogeneities that can be recognized by their effects on the waves. These effects consisting of increasing or decreasing the width of the waves as it moves. Each effect will then be observed in ferrites made of one crystal with particular inhomogeneity. But one may think about poly crystals in which the two different types of inhomogeneities are structured periodically, then one will observe increasing and decreasing of width of the wave leading to a sort of wave that breath as it evolve. One may also think about reducing the effects of inhomogeneous exchange within ferrite by introducing the two types of inhomogeneities so that the effect of these two types of inhomogeneities will compensate each order. In this case, the mean wave will evolve as if there is no inhomogeneity in the medium as described by the KMM-system.

To further investigate the analytical results obtained, we have proceeded to some numerical simulations. We have followed indeed the finite difference method. Conclusion that appeared at the end of such analysis is that: numerical and analytical results are matching very well. Illustration is given in Fig. 7, where we have depicted analytical solution Eq. (18) and numerical solution obtained under the initial conditions provided in Eq. (20). This result sustains the analytical results obtained in Section 2. Now that numerical simulations have been carried out along with analytical results that are not in contradiction, the further investigation should consist of experimental investigations. Such question remains open.

We have obtained in this work solutions that are loop-shaped for the case of magnetic field depicted against the magnetization. We have not heard as far as we are concerned with such a loop-soliton that it has been obtained experimentally. It seems then judicious to look for alternative model equations which solution may be single-valued. Such systems may help understanding very deeply the dynamic of waves in magnetic materials, and one will have as advantage that single valued solutions have been obtained experimentally. Such an approach has recently been carried out in Ref. [43] where a modified KdV equation has been shown to describe propagation of short-waves in ferrites. Breathers as solutions have been constructed. Similar investigations can be made under some different conditions and one may obtain novel equation that, we are sure, will be helpful in describing the propagation of single-valued short waves in inhomogeneous ferrites. There are in the literature so many equations of physical implication that have not been proven integrable and there are poor knowledge on the analytical description of the dynamics of their wave solutions. One of such example is the vector short pulse equation describing the propagation of ultra short waves in optical fibers [49]. Such systems deserve deep investigation both analytically and numerically. These problems will constitute the matter of future investigations.

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