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Rapport sur le projet de thèse
Equations aux dérivées partielles stochastiques et homogénéisation
présenté par Monsieur Mamadou Abdoul Diop

La thèse de Monsieur Diop est consacrée à l'étude de l'homogénéisation d'équations aux dérivées partielles (EDP) paraboliques avec coefficients aléatoires du second ordre avec terme non linéaire fortement oscillant. Cette thèse est composée de deux parties : la première partie traite l'homogénéisation dans le cadre markovien, et la deuxième partie étudie l'homogénéisation dans le cadre non-markovien. Nous allons examiner cette thèse en détail.

Dans la première partie, Monsieur Mamadou Abdoul Diop considère l'EDP :

$$\begin{aligned}\frac{\partial u^\epsilon}{\partial t}(t,x) &= \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\epsilon}, \xi_{t/\epsilon^\alpha}\right) \frac{\partial u^\epsilon}{\partial x_j}(t,x) + \frac{1}{\epsilon^{1\wedge\frac{\alpha}{2}}} g\left(\frac{x}{\epsilon}, \xi_{t/\epsilon^\alpha}, u^\epsilon(t,x)\right),\\ u^\epsilon(0,x) &= u_0(x),\end{aligned}$$

où $\{\xi_t, t \geq 0\}$ est un processus de diffusion stationnaire ergodique. Le cas d'une EDP linéaire a été étudié par Campillo, Kleptsyna et Piatnitski. Le cas avec terme non linéaire fortement oscillant de même type a été traité par Pardoux et Piatnitski dans le cas $\alpha = 2$.

L'auteur de cette thèse étend alors les résultats de Pardoux et Piatnitski aux cas $\alpha < 2$ et $\alpha > 2$. Pour le cas $\alpha < 2$, le résultat qu'il a obtenu est similaire au cas $\alpha = 2$: la famille des lois u^ϵ converge faiblement vers la loi d'une solution d'un problème de martingale. Pour le cas $\alpha > 2$, le résultat est différent : la famille des lois u^ϵ converge en probabilité vers la solution d'un problème de Cauchy (qui est deterministe). La méthode qu'il utilise est : d'abord, il obtient des estimations a priori ; ensuite, il montre la tension ; et enfin, il obtient le résultat par passage à la limite.

Dans la deuxième partie, Monsieur Mamadou Abdoul Diop considère l'EDP :

$$\begin{aligned}\frac{\partial u^\epsilon}{\partial t}(t,x) &= \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^\alpha}\right) \frac{\partial u^\epsilon}{\partial x_j}(t,x) + \frac{1}{\epsilon^{1\wedge\frac{\alpha}{2}}} g\left(\frac{x}{\epsilon}, \frac{t}{\epsilon^\alpha}, u^\epsilon(t,x)\right),\\ u^\epsilon(0,x) &= u_0(x),\end{aligned}$$

où les coefficients a et g sont périodiques en espace, aléatoires, stationnaires en temps.

Dans ce cadre non-markovien, l'auteur établit le résultat d'homogénéisation grâce à une méthode similaire à celle utilisée dans la première partie.

En conclusion, l'auteur étudie des problèmes intéressants d'homogénésation, et les résultats qu'il a obtenus sont nouveaux et originaux. En particulier, il a découvert un nouveau phénomène pour le cas $\alpha > 2$: la solution limite est déterministe. Les preuves sont techniques et montrent que l'auteur a bien maîtrisé l'outil du calcul stochastique et qu'il a aussi une bonne connaissance de la théorie des EDP. Pour toutes ces raison, je suis favorable à la soutenance de cette thèse.

Ying Hu.

Hu

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Report on the Ph. D. thesis entitled

"Equations aux derivees partielles stochastiques et homogeneisation"

by *Mamadou Abdoul Diop*

In the thesis the problem of homogenization of random non stationary parabolic operators with a large nonlinear potential is studied. It is assumed that the said operators have a periodic spatial microstructure whose characteristics are rapidly oscillating stationary random processes in time.

Two different cases of non diffusive scaling are addressed. Namely, the case when the oscillation in time is "faster" than that in spatial variables and the opposite case when the time oscillation is "slower" that the spatial one.

It is shown that in the former case, under certain mixing conditions, the corresponding Cauchy problem admits homogenization and its solution converges in probability to a solution of a deterministic semilinear operator.

In the latter case the limit equation is a stochastic partial differential equation. Here a solution of the original Cauchy problem converges in law in the energy functional space, while convergence in probability does not take place. This makes a big difference with classical homogenization results.

In the first part of the thesis the operators with Markov driving processes are considered. Here the author generalizes the results obtained earlier in [6] and [16] (see the bibliography in the thesis) respectively for linear and self similar operators.

In the second part the operators with non Markov coefficients are investigated. This part is our joint work with Mamadou Diop and Etienne Pardoux, and the contribution of Mamadou is important.

The results of the thesis are new and interesting. All the statements are accurately proved. The topic is interesting mathematically because the type of convergence and the effective models obtained are not the standard ones. It also promises interesting applications. The author demonstrates the ability to work successfully in such modern areas of analysis as random homogenization and asymptotic methods in stochastic analysis.

The thesis is well written, the presentation is clear.

Unfortunately, there are quite many misprints the in the text. This, however, does not reduce the mathematical quality of the work.

To conclude, on my point of view the thesis meets the Ph. D. thesis requirements and the author, Mamadou Diop, deserves Pd. D. degree.

Andrey Piatnitski

UNIVERSITÉ DE PROVENCE, AIX-MARSEILLE I

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THÈSE

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par

Mamadou Abdoul DIOP

le 06 octobre 2003

Titre :

Équations aux dérivées partielles stochastiques et homogénéisation

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.

A ma trés chère épouse Fatou Bintou

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Ma première rencontre avec Etienne Pardoux date d'Avril 2000, alors que je venais lui demander un sujet de thèse. J'ai vite été mis à l'aise par son enthousiasme et sa véritable disponibilité. Au cours de cette thèse, j'ai eu bien davantage l'occasion de profiter de sa personalité de mathématicien, d'abord de son esprit extraordinairement vif et brillant, de son intuition qui traverse les difficultés techniques, puis de l'étendue de sa culture mathématique. Sur le plan humain, j'ai profité chez Etienne, outre sa patience, de son optimisme constant. Il m'a appris que certainement, sans optimisme on ne peut pas démontrer de théorème. Le sien me paraît indestructible et je suis toujours sorti de son bureau plus assuré techniquement, j'y ai toujours retrouvé cette assurance de la valeur de l'activité mathématique, activité qui me semble chez lui naturelle. Il m'a fait partager toute sa compétence, son savoir et ses nombreuses idées avec confiance. C'est pourquoi je tiens aujourd'hui à le remercier de tout ce qu'il m'a appris, expliqué et fait partagé dans son bureau.

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J'ai eu également la chance de rencontrer Andrey Piatnitski au tout début de cette thèse, et de bénéficier de ses conseils et de ses encouragements. Ses travaux avec Etienne Pardoux m'ont ouvert la compréhension de l'homogénéisation des équations aux dérivées partielles stochastiques. J'ai retrouvé dans ces travaux le soin du détail et de la clarté de l'exposition : de tout cela je veux le remercier, aussi, bien-sûr que du travail de rapporter cette thèse qu'il a bien voulu accepter .

D'autre part, Mr HU Ying a bien voulu être rapporteur de ma thèse et membre de mon jury de thèse. Qu'il trouve ici mes sincères remerciements et reconnaissances.

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EDP -Contrôle- Modélisation dont je suis membre.

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J'en arrive maintenant à l'équipe de probabilités et statistique, que tous trouvent ici mes remerciements les plus sincères pour l'amitié et la bonne humeur tout au long de cette période. Je pense particuliérement à Amine, Christophe et Fabienne.
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Enfin, loin de la surface des choses, j'adresse à mon épouse Fatou Bintou un grand merci qui n'est destiné qu'à elle seule comme le murmure d'un fin silence entre nous.

Table des matières

Introduction

Ce travail comporte trois chapitres dont le premier est un chapitre d'exposition et les deux derniers sont des articles. Il traite de l'homogénéisation d'équations aux dérivées partielles (EDP dans toute la suite du texte) paraboliques du second ordre avec terme non linéaire fortement oscillant.

L'emploi de méthodes probabilistes pour l'étude des EDP est relativement classique depuis que l'on sait relier les solutions des EDP linéaires à des processus de diffusion à l'aide de la célèbre formule de Feynman-Kac. La modélisation des milieux physiques conduit souvent à des problèmes comportant de fortes hétérogénéités (grains, pores, fissures, canaux très fins, etc ...) dépendant d'un ou plusieurs paramétres, les rendant de ce fait difficilement exploitables d'un point de vue numérique. L'homogénéisation consiste en un ensemble de techniques d'analyse mathématique qui permettent d'obtenir des problèmes asymptotiques (en faisant tendre les tailles des hétérogénéités vers zéro) équivalents d'où les hétérogénéités ont été gommées. Les champs d'application en sont variés : de l'étude de la diffusion du pétrole dans une roche poreuse à celle des matériaux composites. Récemment, le développement du calcul stochastique a permis de porter un nouvel éclairage à cette théorie. Cette approche appelée probabiliste est ainsi différente de l'approche analytique, qui consiste à utiliser des développements asymptotiques. Elle tire profit des relations étroites, entre les opérateurs aux dérivées partielles et les processus stochastiques. Les premières utilisations de techniques stochastiques remontent essentiellement aux travaux de Freidlin en 1964, fondés sur l'étude de la convergence en loi des processus de diffusion sous-jacents. Le lecteur pourra trouver un aperçu de la diversité des problèmes et des modes de résolutions dans les ouvrages de Bensousssan, Lions, Papanicolaau [1] et de Jikov, Kozlov, Oleinik [9] .

Plus précisément, dans ce travail nous nous intéressons au comportement asymptotique $(\varepsilon \downarrow 0)$ de la solution u^ε des problèmes de Cauchy dans $\mathbb{R}^{\mathbf{n}}$ suivants :

a) des équations paraboliques du type :

$$(1)\quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t,x) = A^\varepsilon[u^\varepsilon(t,x)] + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)), \\ \qquad (t,x) \in (0,T)\times \mathbb{R}^n; \quad u^\varepsilon(0,x) = u_0(x), \end{cases}$$

avec $\alpha > 0$ fixé, $A^\varepsilon h(x) = \mathrm{div}(a(\frac{x}{\varepsilon}, y)\nabla h(x))$. Les coefficients $a(z,y)$ et $g(z,y,u)$ sont z-périodiques sur le tore. Le processus $\{\xi_t,\, t \geq 0\}$ est un processus de diffusion stationnaire ergodique, de générateur infinitésimal $\mathcal{L}$, dont on suppose que la mesure invariante admet une unique densité régulière $\rho(y)$. Les opérateurs A^1 et $\mathcal{L}$ sont supposés uniformément elliptiques.

b) des équations paraboliques du type :

$$(2)\quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t,x) = A^\varepsilon[u^\varepsilon(t,x)] + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, t/\varepsilon^\alpha, u^\varepsilon(t,x)), \\ \qquad (t,x) \in (0,T)\times \mathbb{R}^n; \quad u^\varepsilon(0,x) = u_0(x), \end{cases}$$

avec $\alpha > 0$ fixé, $A^\varepsilon h(x) = \mathrm{div}(a(\frac{x}{\varepsilon}, t)\nabla h(x))$. Les coefficients $a(z,s)$ et $g(z,s,u)$ sont périodiques en espace, aléatoires, stationnaires en temps et possèdent de bonnes propriétés de mélange.

Dans le chapitre 2, nous nous intéressons à l'équation (1). Nous obtenons un résultat d'homogénéisation dans le cadre markovien. Plus précisément sous certaines conditions de régularité et de centrage sur le terme non linéaire g nous obtenons d'une part pour $\alpha < 2$, la convergence faible de la loi $u^\varepsilon(t,x)$ vers la loi limite non dégénérée solution d'un problème de martingale. D'autre part pour $\alpha > 2$, la loi limite est dégénérée, i.e. concentrée sur la solution d'un problème de Cauchy. Mentionnons que le cas où g est linéaire a été traité par Campillo, Piatnitski, Klepstyna [6] et le cas où $\alpha = 2$ a été résolu par Pardoux, Piatnitski [16].
Dans le chapitre 3, nous traitons l'équation (2) dans le cadre non-markovien. Nous montrons en collaboration avec Etienne Pardoux et Andrey Piatnitski que les solutions pour $\alpha < 2$ et $\alpha > 2$ sont de même nature que celles de l'équation (1). Ces deux résultats s'appuient sur des techniques développées dans Viot [17], Bouc, Pardoux [5], Campillo, Kleptsyna, Piatnitski [6] et Pardoux, Piatnitski [16]. Les étapes pour la résolution des équations (1) et (2) sont :

1. Cadre fonctionnel

Il s'agit de déterminer le cadre fonctionnel de la résolution des équations (1) et (2). Pour résoudre les équations (1) et (2) nous considérons l'espace canonique

$$V_T := L^2(0,T; H^1(\mathbb{R}^n)) \cap C([0,T]; L^2(\mathbb{R}^n)).$$

Ceci résulte de la formule d'Itô. Plus précisément, grâce au caractère d'éllipticité de l'opérateur A^ε, la formule d'Itô nous donne,

$$(3) \quad \mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\right) \le \infty$$

$$(4) \quad \mathbf{E}\int_0^T \|\nabla_x u^\varepsilon(t)\|^2 dt \le \infty$$

où $\mathbf{E}$ désigne l'espérance mathématique.
Ceci entraîne que les trajectoires de la solution u^ε sont dans $L^2(0,T;H^1(\mathbb{R}^n))$.
Puisque les trajectoires sont continues, on considère alors V_T.

2. Tension

La deuxième étape consiste à considérer V_T muni de la topologie faible. Ceci résulte du critère de Prohorov et des équations (3) et (4) qui entraînent la tension de la famille u^ε d'éléments de V_T dans $\tilde{V}_T := L^2_w(0,T;H^1_w(\mathbb{R}^n)) \cap C([0,T];L^2_w(\mathbb{R}^n))$, où w signifie que l'espace est muni de la topologie faible.

3. Passage à la limite

Pour l'étude du passage à la limite nous introduisons des familles de correcteurs solutions d'équations de Poisson. Nous résolvons les problèmes de martingale associés aux équations (1) et (2) sur $\tilde{V}_T$.
Pour les équations paraboliques du type équation (1) le problème sera de montrer que la loi Q^0 de tout point d'accumulation $u \in \tilde{V}_T$ de la suite u^ε est solution du problème de martingale suivant que nous noterons (MP1) :

- Pour tout $\varphi \in C^0_\infty(\mathbb{R}^\mathbf{n})$, le processus

$$F_\varphi(t,u) := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (\hat{A}(u(s)),\varphi)ds, \ t \ge 0$$

est une martingale de carré intégrable par rapport à la filtration naturelle de u, de processus croissant,

$$<< F_\varphi(.,u) >> (t) = \int_0^t (R(u(s))\varphi,\varphi)ds$$

où

$$\hat{A}(v) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z\chi > \nabla_x v + < \bar{G}_u g > (v),$$

$$(R(u)\varphi,\varphi) = \int_{\mathbb{R}^d} \big((q(y)(\nabla_y\bar{G}(y,u),\varphi),(\nabla_y\bar{G}(y,u),\varphi))p(y)\big)\, dy.$$

Pour les équations paraboliques du type équation (2) le problème sera de montrer que la loi Q^0 de tout point d'accumulation $u \in \tilde{V}_T$ de la suite u^ε est solution du problème de martingale suivant que nous noterons (MP2) :

• Pour tout $\varphi \in C^0_\infty(\mathbb{R}^{\mathbf{n}})$, le processus

$$F_\varphi(t,u) := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (\hat{A}(u(s)),\varphi)ds, \ t \geq 0$$

est une martingale de carré intégrable par rapport à la filtration naturelle de u, de processus croissant,

$$<< F_\varphi(.,u) >> (t) = \int_0^t (R(u(s))\varphi,\varphi)ds$$

où

$$\hat{A}(v) = \mathrm{div}(\hat{\mathbf{a}}\nabla v) + \hat{\mathbf{g}}(v),$$

et de covariance $R(u(s))$, où

$$\begin{aligned}(R(u)\varphi,\varphi) &= 2\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\bar{G}(s,u(x))\varphi(x)\bar{g}(s,u(y))\varphi(y)dxdy \\ &= 2\mathbf{E}[(\bar{G}(s,u(.))\varphi)(\bar{g}(s,u(.)),\varphi)].\end{aligned}$$

Le problème des martingales devient donc l'étape essentielle de la recherche de solutions faibles des équations (1) et (2).
Nous présentons, à présent, un peu plus précisément les résultats obtenus.

Chapitre 1

Présentation des résultats

Nous exposons dans ce chapitre les résultats que nous avons obtenu au cours de cette thèse, lesquels sont détaillés dans les chapitres suivants. Nous présentons tout juste les résultats sans faire les preuves.

1.1 Homogénéisation dans le cadre markovien

Dans cette partie, on établit un résultat d'homogénéisation pour des EDP non linéaires, dans le cas de coefficients périodiques, avec une non linéarité fortement oscillante.
On considère pour chaque $\varepsilon > 0$, l'EDP parabolique non linéaire :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)), \tag{1.1}$$
$$(t,x) \in (0,T) \times \mathbb{R}^n \quad u^\varepsilon(0,x) = u_0(x).$$

avec $\alpha > 0$ fixé, $A^\varepsilon h(x) = \mathrm{div}(a(\frac{x}{\varepsilon}, y)\nabla h(x))$.
Les coefficients $a(z,y)$ et $g(z,y,u)$ sont z-périodiques sur le tore, $\{\xi_t,\, t \geq 0\}$ est un processus de diffusion stationnaire ergodique à valeurs dans $\mathbb{R}^{\mathbf{d}}$, solution de l'équation différentielle stochastique

$$d\xi_t = \sigma(\xi_t)dW_t + b(\xi_t)dt, \tag{1.2}$$

où $\sigma(y) = q^{1/2}(y)$, et $\{W_t\}$ est un mouvement brownien standard d-dimensionnel.
On note par $\mathcal{L}$, le générateur infinitésimal du processus $\{\xi_t,\, t \geq 0\}$ et on suppose que la mesure invariante admet une unique densité régulière $\rho(y)$.
Les opérateurs A^1 et $\mathcal{L}$ sont supposés uniformément elliptiques.
Le cas d'une EDP parabolique linéaire à coefficient d'ordre zéro fortement oscillant a

été étudié par Campillo, Kleptsyna, Piatnitski [6]. Le cas avec terme non linéaire fortement oscillant vient d'être traité par Pardoux, Piatnitski [16] dans le cas $\alpha = 2$, pour le premier type de coefficient précisé ci-dessus. Notre objectif ici est d'étendre aux cas $\alpha < 2$ et $\alpha > 2$ les résultats obtenus dans Pardoux, Piatnitski [16] en utilisant les techniques developpées dans Viot [17], Bouc, Pardoux [5], Campillo, Kleptsyna, Piatnitski [6] et Pardoux, Piatnitski [16]. Les notations et les hypothèses sont les suivantes :

1.1.1 Notations et Hypothèses

Notations

- $\overline{a(z,.)} = \int_{\mathbb{R}^d} a(z,y)p(y)dy.$
- $< a(\mathbf{I} + \nabla_z \chi) >= \mathbf{E} \int_{\mathbf{T^n}} a(z, \xi_{t/\varepsilon^\alpha})(\mathbf{I} + \nabla_z \chi(z, \xi_{t/\varepsilon^\alpha}))dz.$
- $< Eg >= \mathbf{E} \int_{\mathbf{T^n}} E(z)g(z,y,u)dz.$
- $< G_u g >= \mathbf{E} \int_{\mathbf{T^n}} G_u(\xi_{t/\varepsilon^\alpha}, u)g(z, \xi_{t/\varepsilon^\alpha}, u)dz.$
- $< N_u g >= \mathbf{E} \int_{\mathbf{T^n}} N_u(z,u)g(z, \xi_{t/\varepsilon^\alpha}, u)dz.$
- $< a\nabla_z N >= \mathbf{E} \int_{\mathbf{T^n}} a(z, \xi_{t/\varepsilon^\alpha})\nabla_z N(z,u)dz.$
- $\widehat{g}(z,u) = \int_{\mathbb{R}^d} g(z,y,u)p(y)dy.$
- On note $|.|$, $x.x'$ la norme et le produit scalaire dans $\mathbb{R}^n$.
- On note $\|.\|$, $(.,.)$ la norme et le produit scalaire dans $L^2(\mathbb{R}^n)$.
- $a^\varepsilon = a(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}).$
- $g(z,y,u) : g^\varepsilon = g(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}, u).$
- On notera par $L^2_p(\mathbf{T^n} \times \mathbb{R}^d)$ l'espace à poids muni de la norme :

$$\|f\|_p^2 = \int_{\mathbf{T^n}} \int_{\mathbb{R}^d} f(z,y)^2 p(y)dzdy.$$

- Nous introduisons les espaces suivants :

$$\overline{L}^2_p(\mathbf{T^n}\times\mathbb{R}^d)=\Big\{f\in L^2_p(\mathbf{T^n}\times\mathbb{R}^d);\ \int_{\mathbf{T^n}}\int_{\mathbb{R}^d}f(z,y)p(y)dzdy=0\Big\},$$

$$\overline{H}^1_p(\mathbf{T^n}\times\mathbb{R}^d)=\Big\{f\in\overline{L}^2_p(\mathbf{T^n}\times\mathbb{R}^d);\nabla_z f,\nabla_y f\in L^2_p(\mathbf{T^n}\times\mathbb{R}^d)\Big\}.$$

Hypothèses

H1 Les fonctions $a_{ij}(z,y)$ et $g(z,y,u)$ sont z-périodiques de période 1 dans chaque direction ; la matrice $\{a_{ij}(z,y)\}$ est uniformément définie positive :

$$0<\lambda\mathbf{I}\le a(z,y)\le\lambda^{-1}\mathbf{I};$$

le gradient de a_{ij} par rapport à y et z existe et est uniformément borné :

$$|\nabla_z a_{ij}(z,y)|+|\nabla_y a_{ij}(z,y)|\le c \tag{1.3}$$

H2 Pour tous $c,\ \mu_1>0$ nous avons les estimations suivantes

$$0<c\mathbf{I}\le q_{ij}(z,y)\le c^{-1}\mathbf{I},$$

$$|\nabla q_{ij}(y)|\le c,\quad |b(y)|+|\nabla b(y)|\le c(1+|y|)^{\mu_1},\quad \mu_1\ge 0$$

et il existe $M,C>0$, $\beta>-1$ tels que quelque soit $|y|>M$,

$$\frac{b(y)\cdot y}{|y|}\le -C|y|^{\beta}; \tag{1.4}$$

$b(y)\cdot y$ est le produit scalaire dans $\mathbb{R}^d$.
Sous ces différentes hypothèses, Pardoux et Veretennikov (voir [15]) ont montré que le processus $\{\xi_t\}$ possède une unique mesure de probabilité invariante $\pi(dy)=p(y)dy$ dont la densité à l'infini décroît plus vite que toute puissance négative de $|y|$.

H3 $g(z,y,u)$ satisfait les estimations suivantes

$$|\nabla_z g(z,y,u)|\le c(1+|u|), \tag{1.5}$$

$$|\nabla_y g(z,y,u)|\le c(1+|u|), \tag{1.6}$$

$$|g(z,y,u)|\le c|u|, \tag{1.7}$$

$$|g'_u(z,y,u)|\le c, \tag{1.8}$$

$$(1+|u|)|g''_{uu}(z,y,u)|\le c; \tag{1.9}$$

et g, g'_u, g''_{uu} sont continues ;

H4 Pour tout $u \in \mathbb{R}$ on a l'identité suivante

$$\int_{\mathbf{T^n}} \int_{\mathbb{R}^d} g(z,y,u)p(y)dzdy \ = \ 0. \tag{1.10}$$

Pour traiter le terme non linéaire $g(z,y,u)$ nous le décomposons comme suit

$$g(z,y,u) = \tilde{g}(z,y,u) + \bar{g}(y,u),$$

où

$$\bar{g}(y,u) = \int_{\mathbf{T^n}} g(z,y,u)dz,$$

tel que

$$\int_{\mathbf{T^n}} \tilde{g}(z,y,u)dz = 0, \ \forall y \in \mathbb{R}^d, \ u \in \mathbb{R}; \quad \int_{\mathbb{R}^d} \bar{g}(y,u)p(y)dy = 0, \ \forall u \in \mathbb{R}. \tag{1.11}$$

La première relation dans (1.11) implique l'existence d'une fonction vectorielle $\tilde{G}(z,y,u)$ telle que

$$\tilde{g}(z,y,u) = \mathrm{div}_z \tilde{G}(z,y,u). \tag{1.12}$$

Afin de construire une telle fonction $\tilde{G}(z,y,u)$, nous considérons l'équation aux dérivées partielles

$$\Delta v = \tilde{g}, \quad z \in \mathbf{T^n}.$$

En posant $\tilde{G}(z,y,u) = \nabla_z v(z,y,u)$, on obtient la representation désirée.
La fonction $\tilde{G}(z,y,u)$ satisfait les estimations (1.6), (1.7) et pour tout $u(x,t)$ nous avons

$$\mathrm{div}_x \tilde{G}(\frac{x}{\varepsilon}, y, u(t,x)) = \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon}, y, u(t,x)) + \tilde{G}'_u(\frac{x}{\varepsilon}, y, u(t,x))\nabla_x u(t,x). \tag{1.13}$$

D'après Pardoux,Veretennikov [15], sous les hypothèses **H2** et **H4** la seconde relation dans (1.11) implique que l'équation de Poisson

$$\mathcal{L}\bar{G}(y,u) + \bar{g}(y,u) = 0, \qquad \forall u \in \mathbb{R} \tag{1.14}$$

admet une solution dans l'espace $W^{2,p}_{loc}(\mathbb{R}^d)$. Cette solution $\bar{G}(\cdot,u)$ est à croissance polynômiale en $|y|$ et pour tout $u \in \mathbb{R}$, elle est unique à une constante additive près.
A présent nous présentons les deux principaux résultats de cette partie qui sont des résultats de convergence.

1.1.2 Résultats

Théorème 1.1.1. *Soit* $\alpha < 2$*, sous les hypothèses* **H1-H4***, la famille des lois* $\{u^\varepsilon\}$ *du problème (1.1) converge faiblement, quand* $\varepsilon \to 0$*, dans l'espace* $\tilde{V}_T$*, pour tout* $T > 0$*, vers la loi limite non dégénérée unique solution d'un problème de martingale de dérive* $\hat{A}(u(s))$*, où*

$$\hat{A}(u) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z \chi) > \nabla_x u + < \bar{G}_u g > (u),$$

et de covariance $R(u(s))$*, où*

$$(R(u)\varphi, \varphi) = \int_{\mathbb{R}^d} (q(y)(\nabla_y \bar{G}(y,u), \varphi), (\nabla_y \bar{G}(y,u), \varphi))p(y)\, dy.$$

Les fonctions $\chi^k \in \overline{H}^1_p(T^n \times \mathbb{R}^d)$ *et* $\bar{G}(.,u) \in W^{2,p}_{loc}(\mathbb{R}^d)$ *sont les solutions des équations :*

$$\mathcal{L}\bar{G}(y,u) + \bar{g}(y,u) = 0 \qquad \forall u \in \mathbb{R}, \tag{1.15}$$

$$A\chi^k(z,y) = -\sum_i \frac{\partial}{\partial z_i} a_{ik}(z,y), \tag{1.16}$$

$(z,y) \in \mathbf{T^n} \times \mathbb{R}^d, \quad 1 \le k \le n.$

Théorème 1.1.2. *Soit* $\alpha > 2$*, sous les hypothèses* **H1-H4***, la famille des lois* $\{u^\varepsilon\}$ *du problème(1.1) converge en probabilité dans l'espace* $\tilde{V}_T$*, pour tout* $T > 0$*, vers la solution du problème de Cauchy suivant :*

$$\frac{du(t,x)}{dt} = \hat{A}(u(t,x)) + \hat{B}(u(t,x)), \;\; u^\varepsilon(0,x) = u_0(x)$$

avec $(t,x) \in (0,T) \times \mathbb{R}^n$ *et*

$$\hat{A}(u) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z E) > \nabla_x u - \nabla_x \cdot < a \nabla_x N > (u),$$

$$\hat{B}(u) =< Eg > (u) + < N_u g > (u),$$

où les fonctions $E^k \in \bar{H}^1(\mathbf{T}^n)$*,* $F^k \in \overline{H}^1_\rho(\mathbf{T}^n \times \mathbb{R}^d)$*,* $N(.,u) \in W^{2,p}(\mathbf{T^n})$*,* $\Psi(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d)$ *sont les solutions des équations :*

$$\overline{A}E^k(z) = -\sum_i \overline{a^{ik}_{z_i}(z,.)}, \tag{1.17}$$

$$\mathcal{L}\chi^k(z,y) = -[AE^k(z) + \sum_i a^{ik}_{z_i}(z,y)], \tag{1.18}$$

$$\overline{A}N(z,u) = -\widehat{g}(z,u), \tag{1.19}$$

$$\mathcal{L}\Psi(z,y,u) = -[\tilde{g}(z,y,u) + AN(z,u)], \tag{1.20}$$

pour $z \in \mathbf{T}^n$ et $1 \le k \le n$, l'opérateur $\overline{A}$ est défini par

$$\overline{A}f(z) = \mathrm{div}(\overline{a(z,.)}\nabla f(z)).$$

1.2 Homogénéisation dans le cadre non-markovien

Dans cette partie nous étudions un problème d'homogénéisation pour l'EDP suivante :

$$\frac{\partial u^\varepsilon}{\partial t}(x,t) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha})\frac{\partial u^\varepsilon}{\partial x_j}(x,t) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x)), \qquad (1.21)$$
$$(t,x) \in (0,T)\times \mathbb{R}^n \quad u^\varepsilon(0,x) = u_0(x).$$

Les coefficients $a(z,s)$ et $g(z,s,u)$ sont périodiques en espace, aléatoires, stationnaires en temps et possèdent de bonnes propriétés de mélange. Notre objectif ici est d'étudier l'équation (1.21) dans les cas $\alpha < 2$ et $\alpha > 2$. Nous utiliserons largement dans cette partie les techniques décrites dans Viot [17], Bouc, Pardoux [5], Campillo, Kleptsyna, Piatnitski [6] et Pardoux, Piatnitski [16]. Par ailleurs, les notations, la démarche seront très semblables à ce que nous avons fait dans la première partie. A présent précisons les hypothèses sur les coefficients.

1.2.1 Hypothèses :

A1 Les coefficients $a_{ij}(z,s)$ et $g(z,s,u)$ sont z-périodiques de période 1 dans chaque direction.

A2 $a_{ij}(z,s)$ et $g(z,s,u)$ sont des processus aléatoires stationnaires à valeurs dans $\mathbf{C}(\mathbf{T^n})$ définis sur un espace de probabilité $(\Omega, \Im, \mathbf{P})$. Nous notons par $\Im_s$ la filtration correspondante, supposons que $\Im_s$ est continue et que $a_{ij}(s)$ et $g(s,u)$ sont presque sûrement continues.

A3 Nous avons les estimations suivantes

$$|a_{ij}(z,s)| \le C, \qquad (1.22)$$

$$|\nabla_z a_{ij}(z,s)| \le C, \qquad (1.23)$$

$$|g(z,s,u)| \le C|u|, \qquad (1.24)$$

$$|g'_u(z,s,u)| \le C, \qquad (1.25)$$

$$(1+|u|)|g''_{uu}(z,s,u)| \le C, \qquad (1.26)$$

où C est une constante générique.

A4 Pour tout $c > 0$,

$$\sum_{i,j} a_{ij}(z,s)\eta_i\eta_j \ge c|\eta|^2, \qquad \forall \eta \in \mathbb{R}^\mathbf{n}.$$

A5 La fonction qui à $t \longmapsto a(t,.)$ est p.s differentiable à valeurs dans $L^2(\mathbf{T^n})$ et

$$\|\frac{\partial}{\partial t}a(t,.)\|_{L^2(\mathbf{T^n})} \leq C.$$

A6

$$\forall u \in \mathbb{R}, \quad \mathbf{E}\int_{\mathbf{T}^n} g(z,t,u)dz = 0.$$

A7 $\phi(t)$ est le coefficient de mélange défini par

$$\phi(t) = \sup_{A,B\in\Im_s} |P(A|B) - P(A)|.$$

Nous supposons que

$$\int_0^\infty \phi(s)ds < \infty.$$

Sous ces différentes hypothèses le problème (1.21) admet une unique solution $\{u^\varepsilon, \ \ \varepsilon > 0\}$. Cette solution appartient à

$$V_T := L^2(0,T;H^1(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n)).$$

Nous décomposons $g(z,t,u)$ comme suit

$$g(z,t,u) = \tilde{g}(z,t,u) + \bar{g}(t,u),$$

où

$$\bar{g}(t,u) = \int_{\mathbf{T^n}} g(z,t,u)dz.$$

Il existe une fonction $\tilde{G}(z,t,u)$ telle que

$$\tilde{g}(z,t,u) = \mathrm{div}_z\tilde{G}(z,t,u) \tag{1.27}$$

où $\tilde{G}(z,t,u) = \nabla\Theta(z,t,u)$ et $\Theta(z,t,u)$ est une solution périodique de l'équation

$$\Delta_z\Theta(z,t,u) = \tilde{g}(z,t,u).$$

Introduisons le processus stationnaire suivant

$$\bar{G}(t,u) = \int_0^\infty \mathbf{E}^{\Im_t}[\bar{g}(s+t,u)]ds = \int_t^\infty \mathbf{E}^{\Im_t}[\bar{g}(s,u)]ds. \tag{1.28}$$

D'après [11], sous les hypothèse **A.5** et **A.6** , le processus $\bar{G}(t,u)$ est bien défini et nous avons les estimations suivantes

$$|\bar{G}(t,u)| \leq C|u|, \ |\bar{G}'_u(t,u)| \leq C, \ (1+|u|)|\bar{G}'_{uu}(t,u)| \leq C. \tag{1.29}$$

Avant de présenter les résultats obtenus dans cette partie, énoncons les résultats auxiliares suivants :

Lemme 1.2.1. *Pour tout* $u \in \mathbb{R}$*, le processus*

$$M_t = \bar{G}(t,u) + \int_0^t \bar{g}(s,u)ds$$

est une $\Im_t$*- martingale.*

Lemme 1.2.2. *Pour tout* $\varphi \in C_0^\infty(\mathbb{R}^{\mathbf{n}})$, $\varepsilon > 0$ *les fonctions*

$$M_t^u := \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}[(\bar{G}(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), u^\varepsilon(t)) - (\bar{G}(0, u^\varepsilon(0)), u^\varepsilon(0))] + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \int_0^t (\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s))ds$$

$$+ \quad \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})} \int_0^t (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s) u^\varepsilon(s) + 2\bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)))ds$$

$$- \quad \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})} \int_0^t (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)))ds$$

et

$$M_t^\varphi := \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}[(\bar{G}(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi) - (\bar{G}(0, u^\varepsilon(0)), \varphi)] + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \int_0^t (\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \varphi)ds$$

$$+ \quad \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})} \int_0^t (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)\varphi + \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla\varphi)ds$$

$$- \quad \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})} \int_0^t (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\varphi)ds$$

sont des $\{\Im_{\frac{t}{\varepsilon^\alpha}}\}$*- martingales.*

Nous présentons à présent les principaux résultats de cette partie qui sont également des résultats de convergence .

1.2.2 Résultats

Théorème 1.2.1. *Soit* $\alpha < 2$*, sous les hypothèses* **A1-A7***, la famille des lois* $\{u^\varepsilon\}$ *du problème (1.21) converge faiblement, quand* $\varepsilon \to 0$*, dans l'espace* $\tilde{V}_T$*, pour tout* $T > 0$*, vers la loi limite non dégénérée unique solution d'un problème de martingale de dérive* $\hat{A}(u(s))$*, où*

$$\hat{A}(u) = \mathrm{div}(\hat{\mathbf{a}}\nabla u) + \hat{\mathbf{g}}(u),$$

$$\hat{\mathbf{a}} = \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z\chi(z,s))dz,$$

$$\hat{\mathbf{g}}(u) = \mathbf{E}\int_{\mathbf{T^n}} \bar{G}'_u(s,u)g(z,s,u)dz,$$

et de covariance $R(u(s))$*, où*

$$\begin{aligned}(R(u)\varphi,\varphi) &= 2\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(\bar{G}(s,u(x))\varphi(x)\bar{g}(s,u(y))\varphi(y)dxdy \\ &= 2\mathbf{E}[(\bar{G}(s,u(.))\varphi)(\bar{g}(s,u(.)),\varphi)].\end{aligned}$$

Les fonctions $\chi^k(.,s) \in H^1(\mathbf{T^n})$*, sont les solutions des équations :*

$$A\chi^k(z,s) = -\sum_i \frac{\partial}{\partial z_i}a_{ik}(z,s), \;\; 1 \le k \le n. \tag{1.30}$$

Théorème 1.2.2. *Soit* $\alpha > 2$*, sous les hypothèses* **A1**-**A7***, la famille des lois* $\{u^\varepsilon\}$ *du problème(1.21) converge en probabilité dans l'espace* $\tilde{V}_T$*, pour tout* $T > 0$*, vers la solution du problème de Cauchy suivant :*

$$\frac{du(t,x)}{dt} = \operatorname{div}(\hat{\mathbf{a}}\nabla u(t,x)) + \operatorname{div}(\hat{\mathbf{b}}(u)), \;\; u^\varepsilon(0,x) = u_0(x)$$

avec $(t,x) \in (0,T) \times \mathbb{R}^n$ *et*

$$\hat{\mathbf{a}} = \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z\chi(z,s))dz,$$

$$\hat{\mathbf{b}}(u) = -\mathbf{E}\int_{\mathbf{T^n}} a(z,s)\nabla_z\tilde{H}(z,s,u)dz.$$

Les fonctions χ *et* $\tilde{H}$ *sont les solutions stationnaires des équations :*

$$\frac{\partial}{\partial s}\chi(z,s) + \operatorname{div}\left[a(z,s)\nabla\chi(z,s)\right] = -\operatorname{div}a(z,s), \qquad (z,s) \in \mathbf{T^n} \times (-\infty,+\infty), \tag{1.31}$$

$$\frac{\partial}{\partial s}\tilde{H}(z,s,u)(z,s) + \operatorname{div}\left[a(z,s)\nabla\tilde{H}(z,s,u)\right] = -\tilde{g}(z,s,u), \qquad (z,s) \in \mathbf{T^n} \times (-\infty,+\infty) \tag{1.32}$$

Cadre markovien

Chapitre 2

Averaging of a parabolic partial differential equation with random evolution

Résumé : Le chapitre est consacré à l'étude d'un problème de moyennisation pour des opérateurs paraboliques aléatoires sous forme divergence dans le cas d'un grand potentiel et avec des coefficients rapidement oscillants en temps aussi bien qu'en espace. On considére l'EDP parabolique non linéaire :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathbb{R}^n \quad u^\varepsilon(0,x) = u_0(x).$$

avec $\alpha > 0$ fixé, $A^\varepsilon h(x) = \operatorname{div}(a(\frac{x}{\varepsilon}, y)\nabla h(x))$.
On suppose que le milieu posséde une structure microscopique périodique alors que la dynamique temporelle est markovienne. Nous généralisons au cas $\alpha < 2$ et $\alpha > 2$, les résultats obtenus dans Pardoux, Piatnitski [16]. Nous montrons que l'équation limite est une équation aux dérivées partielles à coefficients constants, obtenue par passage à la limite en loi.

Abstract : In this paper, we study the homogenization problem of a semilinear parabolic second order partial differential equation of the type

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)),$$

with periodic coefficients rapidly oscillating both in space and time variables.We extend to $\alpha < 2$ and $\alpha > 2$, results of Pardoux, Piatnitski [16]. The structure of the limit problem depends crucially on the value of α.

2.1 Introduction

In this paper we study the limit as $\varepsilon \to 0$ of the solution u^ε of the second order semilinear parabolic PDE

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathrm{I\!R}^n \quad u^\varepsilon(0,x) = u_0(x).$$

The main assumptions are the periodicity (of period one in each direction) of a_{ij} and g with respect to their first variable, the fact that $\{\xi_t,\, t \geq 0\}$ is a d–dimensional ergodic diffusion process with a unique invariant measure π, and a centering condition for g :

$$\int_{\mathbf{T^n}} \int_{\mathrm{I\!R}^d} g(z,y,u) dz\, \pi(dy) = 0, \quad \forall u \in \mathrm{I\!R}.$$

Here and further below, $\mathbf{T^n}$ denotes the n–dimensional torus, $\mathbf{T^n} = \mathrm{I\!R}^n/\mathbf{Z}^n$. We shall identify periodic functions with functions defined on $\mathbf{T^n}$.

As in Pardoux, Piatnitski [16] the equation is a particular model of random homogenization, where the stochastic perturbation fluctuates as time evolves, in contradiction with the more traditional model where the coefficients are time invariant stationary random fields. The equation is nonlinear and the nonlinear term is highly oscillating. For the basic results on homogenization of periodic and random equations, we refer respectively to Bensoussan, Lions, Papanicolaou [1], and Jikov, Kozlov, Oleinik [9].

The homogenization of such equations, with g linear, has been studied quite widely by Campillo, Kleptsyna, Piatnitski [6]. Note also that the same problem, without the appearance of the process $\{\xi_t\}$, has been studied by Pardoux [14], and without the dependance upon x/ε, by Bouc, Pardoux [5]. Recently the same problem, in the case $\alpha = 2$ has been studied by Pardoux, Piatnitski [16]. The present work is very much inspired by the methods developed in [6] and [16]. The layout of the paper is as follows. In Section 2, we state the problem and the assumptions made throughout the paper concerning the diffusion process $\{\xi_t\}$, and recall the results from Pardoux, Piatnitski [16] about its associated Poisson equation. In Section 3 we prepare the ground for tightness for the family of functions u^ε by obtaining some a priori estimates. In section 4, the homogenization results for the parabolic PDE is proved. To this end we use the techniques developed in Viot [17], Bouc, Pardoux [5], Campillo, Kleptsyna, Piatnitski [6] and Pardoux, Piatnitski [16].

2.2 The setup and statement of main results

We consider the asymptotic behavior of the solution of the following Cauchy problem as $\varepsilon \to 0$

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathbb{R}^n; \quad u^\varepsilon(0,x) = u_0(x), \tag{2.1}$$

where $u_0 \in L^2(\mathbb{R}^n), \alpha > 0$ is a fixed parameter, $\{\xi_t, t > 0\}$ a stationary diffusion process with values in $\mathbb{R}^d$.
Let us introduce the following operators :
• the infinitesimal generator of the diffusion process $\{\xi_t\}$:

$$\mathcal{L} = \frac{1}{2} q_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + b_i(y) \frac{\partial}{\partial y_i}$$

• and

$$A^\varepsilon h(x) = \text{div}(a(\frac{x}{\varepsilon}, y) \nabla h(x))$$

A will denote A^ε for $\varepsilon = 1$.

2.2.1 Notations

• $\overline{a(z,.)} = \int_{\mathbb{R}^d} a(z,y)p(y)dy.$

• $< a(\mathbf{I} + \nabla_z \chi) >$ stands for $\mathbf{E} \int_{\mathbf{T^n}} a(z, \xi_{t/\varepsilon^\alpha})(\mathbf{I} + \nabla_z \chi(z, \xi_{t/\varepsilon^\alpha}))dz.$
The other uses of the notation $< . >$ in the paper can be made precise in a similar way. In order to avoid confusion, we shall use below in section 2.4 the notation $\{<< M >> (t); 0 \le t \le T\}$ to denote the increasing process associated to the continuous martingale $\{M_t; 0 \le t \le T\}$, i.e $t \to << M >> (t)$ is continuous and increasing, and $M_t^2 - << M >> (t)$ is a continuous martingale.

• $\widehat{g}(z,u) = \int_{\mathbb{R}^d} g(z,y,u)p(y)dy.$

• In $\mathbb{R}^n$; $x.x'$ will denote the scalar product and $|.|$ the corresponding norm.

• In the space $L^2(\mathbb{R}^n)$, $(.,.)$ will denote the inner product, and $\|.\|$ the norm.

• For a function or process $(t,x) \longmapsto u(t,x)$, $u(t)$ will denote the application

$x \longmapsto u(t,x)$. Hence $\|u(t)\| = \Big(\int_{\mathbb{R}^n} |u(t,x)|^2 dx \Big)^{\frac{1}{2}}$. This notation is also used for $u^\varepsilon(t,x)$ and for the gradient $\nabla u^\varepsilon(t,x)$. We use, as well, the contracted notation :

$$a^\varepsilon \quad = a(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}) \text{ and for a generic function } g(z,y,u) : g^\varepsilon = g(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}, u)$$

- $L^2_p(\mathbf{T^n} \times \mathbb{R}^d)$ denotes the weighted space with the norm :

$$\|f\|_p^2 = \int_{\mathbf{T^n}} \int_{\mathbb{R}^d} f(z,y)^2 p(y) dz dy.$$

- We introduce the spaces :

$$\overline{L}^2_p(\mathbf{T^n} \times \mathbb{R}^d) = \Big\{ f \in L^2_p(\mathbf{T^n} \times \mathbb{R}^d);\ \int_{\mathbf{T^n}} \int_{\mathbb{R}^d} f(z,y) p(y) dz dy = 0 \Big\},$$

$$\overline{H}^1_p(\mathbf{T^n} \times \mathbb{R}^d) = \Big\{ f \in \overline{L}^2_p(\mathbf{T^n} \times \mathbb{R}^d); \nabla_z f, \nabla_y f \in L^2_p(\mathbf{T^n} \times \mathbb{R}^d) \Big\}.$$

2.2.2 Hypotheses

In this section, we provide the precise assumptions on the coefficients of (2.1) and on the generator of the process ξ_t :

H1 The functions $a_{ij}(z,y)$ and $g(z,y,u)$ are periodic in z of period 1 in all the coordinate directions; the matrix $\{a_{ij}(z,y)\}$ is uniformly positive definite :

$$0 < \lambda \mathbf{I} \le a_{ij}(z,y) \le \lambda^{-1} \mathbf{I};$$

moreover, the gradient of a_{ij} both with respect to y and z exists and is uniformly bounded :

$$|\nabla_z a_{ij}(z,y)| + |\nabla_y a_{ij}(z,y)| \le c \tag{2.2}$$

H2 The following bounds hold for some c, $\mu_1 > 0$:

$$0 < c\mathbf{I} \le q_{ij}(z,y) \le c^{-1} \mathbf{I},$$

$$|\nabla q_{ij}(y)| \le c, \quad |b(y)| + |\nabla b(y)| \le c(1 + |y|)^{\mu_1},$$

and there exist $M, C > 0$, $\beta > -1$ such that whenever $|y| > M$,

$$\frac{b(y) \cdot y}{|y|} \le -C|y|^\beta; \tag{2.3}$$

here $b(y) \cdot y$ stands for the inner product in $\mathbb{R}^d$.
It follows from these assumptions that the process $\{\xi_t\}$ possesses a unique invariant probability measure $\pi(dy) = p(y)dy$ whose density decays at infinity faster than any negative power of $|y|$ (see [15]).

H3 $g(z,y,u)$ satisfies the estimates

$$|\nabla_z g(z,y,u)| \le c(1+|u|), \tag{2.4}$$

$$|\nabla_y g(z,y,u)| \le c(1+|u|), \tag{2.5}$$

$$|g(z,y,u)| \le c|u|, \tag{2.6}$$

$$|g'_u(z,y,u)| \le c, \tag{2.7}$$

$$(1+|u|)|g''_{uu}(z,y,u)| \le c; \tag{2.8}$$

and g, g'_u, g''_{uu} are jointly continuous ;

H4 The identity

$$\int_{\mathbf{T^n}} \int_{\mathbb{R}^d} g(z,y,u)p(y)dzdy \;=\; 0 \tag{2.9}$$

holds for any $u \in \mathbb{R}$.

By our assumptions the diffusion process $\{\xi_t\}$ is a solution of the stochastic equation

$$d\xi_t = \sigma(\xi_t)dW_t + b(\xi_t)dt, \tag{2.10}$$

where $\sigma(y) = q^{1/2}(y)$, and $\{W_t\}$ is a standard d-dimensional Wiener process.

It is convenient to decompose $g(z,y,u)$ as follows

$$g(z,y,u) = \tilde{g}(z,y,u) + \bar{g}(y,u),$$

where

$$\bar{g}(y,u) = \int_{\mathbf{T^n}} g(z,y,u)dz,$$

so that

$$\int_{\mathbf{T^n}} \tilde{g}(z,y,u)dz = 0,\ \forall y \in \mathbb{R}^d,\ u \in \mathbb{R}; \quad \int_{\mathbb{R}^d} \bar{g}(y,u)p(y)dy = 0,\ \forall u \in \mathbb{R}. \tag{2.11}$$

The first relation here implies in a standard way the existence of a vector function $\tilde{G}(z,y,u)$ such that

$$\tilde{g}(z,y,u) = \mathrm{div}_z \tilde{G}(z,y,u). \tag{2.12}$$

Indeed, we choose $\tilde{G} = \nabla v$, where for each $(y,u) \in \mathbb{R}^{d+1}$, $v(\cdot,y,u)$ solves the PDE $\Delta v = \tilde{g}$ on $\mathbf{T^n}$. Then the function $\tilde{G}(z,y,u)$ satisfies the estimates (2.6) and (2.7). For any $u(x,t)$ we have now

$$\mathrm{div}_x \tilde{G}(\frac{x}{\varepsilon}, y, u(t,x)) = \frac{1}{\varepsilon}\tilde{g}(\frac{x}{\varepsilon}, y, u(t,x)) + \tilde{G}'_u(\frac{x}{\varepsilon}, y, u(t,x))\nabla_x u(t,x). \tag{2.13}$$

According to [15], under assumptions **H2** and **H4** the second relation in (2.11) ensures the solvability of the Poisson equation

$$\mathcal{L}\bar{G}(y,u) + \bar{g}(y,u) = 0, \qquad \forall u \in \mathbb{R} \tag{2.14}$$

in the space $W^{2,p}_{loc}(\mathbb{R}^d)$. Moreover the solution $\bar{G}(\cdot,u)$ has polynomial growth in $|y|$ for all $u \in \mathbb{R}$. The solution is unique up to an additive constant, for definiteness we assume that it has zero mean w.r.t. the invariant measure $\pi(dy) = p(y)dy$.

2.2.3 Main results

Here we formulate the main results of the paper ; the proof will be given in the following section. It should be noted that for $\alpha \le 2$ we obtain the weak convergence of the law of $u^\varepsilon(t,x)$ towards the non trivial limit law which solves a proper martingale problem,while for $\alpha > 2$, the limit law is a Dirac measure concentrated on the solution of the Cauchy problem for the limit deterministic parabolic equation with constant coefficients.
We define $V_T := L^2(0,T;H^1(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n))$ and let $\tilde{V}_T$ denote the space V_T, equipped with the sup of the weak topology of $L^2(0,T;H^1(\mathbb{R}^n))$, and the topology of the space $C([0,T];L^2_w(\mathbb{R}^n))$, where $L^2_w(\mathbb{R}^n)$ denotes the corresponding L^2 space equipped with its weak topology.The space $\tilde{V}_T$ is a Lusin and completely regular space, see Viot[17]. We denote by $\tilde{\Im}$ its Borel σ-field. For any $\varepsilon > 0$, let Q^ε be the Radon probability measure on $(\tilde{V}_T, \tilde{\Im})$, which coincides with the law of $\{u^\varepsilon(t); 0 \le t \le T\}$. The asymptotic behavior of the solution $u^\varepsilon(t)$, as $\varepsilon \to 0$, depends on whether $\alpha < 2$ or $\alpha > 2$. The main results of the paper are summarized in the following theorems.

Theorem 2.2.1. *Let $\alpha < 2$, then under the Hypotheses* **H1**-**H4***, the family of laws of the solutions $\{u^\varepsilon\}$ to problem (2.1) converges weakly, as $\varepsilon \to 0$, in the space $\tilde{V}_T$, for all $T > 0$, to the unique solution of the martingale problem with the drift $\hat{A}(u(s))$, where*

$$\hat{A}(u) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z \chi) > \nabla_x u + < \bar{G}_u g > (u),$$

and the covariance $R(u(s))$, where

$$(R(u)\varphi,\varphi) = \int_{R^d} (q(y)(\nabla_y \bar{G}(y,u),\varphi),(\nabla_y \bar{G}(y,u),\varphi))p(y)\,dy,$$

where the functions $\chi^k \in \overline{H}^1_p(\mathbf{T^n} \times \mathbb{R}^d)$ and $\bar{G}(.,u) \in W^{2,p}_{loc}(\mathbb{R}^d)$ are the solutions of the equations :

$$\mathcal{L}\bar{G}(y,u) + \bar{g}(y,u) = 0 \qquad \forall u \in \mathbb{R}, \tag{2.15}$$

$$A\chi^k(z,y) = -\sum_i \frac{\partial}{\partial z_i} a_{ik}(z,y), \tag{2.16}$$

for $(z,y) \in \mathbf{T^n} \times \mathbb{R}^d, \quad 1 \le k \le n.$

Theorem 2.2.2. *Let* $\alpha > 2$*, then under the Hypotheses* **H1**-**H4***, the family of laws of the solutions* $\{u^\varepsilon\}$ *to problem (2.1) converges in probability in the space* $\tilde{V}_T$*, for all* $T > 0$*, to the solution of the following limit Cauchy problem :*

$$\frac{du(t,x)}{dt} = \hat{A}(u(t,x)) + \hat{B}(u(t,x)), \;\; u^\varepsilon(0,x) = u_0(x)$$

with $(t,x) \in (0,T) \times \mathbb{R}^n$ *and*

$$\hat{A}(u) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z E) > \nabla_x u - \nabla_x \cdot < a\nabla_x N > (u),$$

$$\hat{B}(u) =< Eg > (u)+ < N_u g > (u),$$

where the functions $E^k \in \bar{H}^1(\mathbf{T}^n)$, $F^k \in \overline{H}^1_\rho(\mathbf{T}^n \times \mathbb{R}^d)$, $N(.,u) \in W^{2,p}(\mathbf{T^n})$, $\Psi(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d)$ *are solutions of equations :*

$$\overline{A}E^k(z) = -\sum_i \overline{a^{ik}_{z_i}(z,.)}, \tag{2.17}$$

$$\mathcal{L}F^k(z,y) = -[AE^k(z) + \sum_i a^{ik}_{z_i}(z,y)], \tag{2.18}$$

$$\overline{A}N(z,u) = -\widehat{g}(z,u), \tag{2.19}$$

$$\mathcal{L}\Psi(z,y,u) = -[\tilde{g}(z,y,u) + AN(z,u)], \tag{2.20}$$

for $z \in \mathbf{T}^n$ and $1 \le k \le n$, where the operator $\overline{A}$ is defined by

$$\overline{A}f(z) = \mathrm{div}(\overline{a(z,.)}\nabla f(z)).$$

2.3 A priori estimates and tightness

In this section we obtain uniform a priori estimates for the solution u^ε and then use them in order to show tightness of the distributions of u^ε.

First, considering (2.11) and (2.13) one can rewrite the nonlinear term $g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x))$ on the right hand side of (2.1) in the form

$$\frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} g(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t,x)) = \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathrm{div}_x \tilde{G}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(x,t))$$
$$-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\tilde{G}'_u(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(x,t))\nabla_x u^\varepsilon(t,x) + \varepsilon^{-(1\wedge\frac{\alpha}{2})}\bar{g}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(x,t)). \tag{2.21}$$

For $u \in L^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^d$ denote

$$\Psi^\varepsilon(u,y) = \frac{1}{2}\|u\|^2 + \varepsilon^\rho(u, \bar{G}(y,u)),$$

where $\rho = \alpha - (1 \wedge \frac{\alpha}{2})$.
From Itô–Krylov's formula (see [15] for justifications), using (2.1), (2.21), we get

$$d\Psi^\varepsilon(u^\varepsilon(t), \xi_{t/\varepsilon^\alpha}) = (A^\varepsilon u^\varepsilon(t), u^\varepsilon(t))dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\nabla_x u^\varepsilon(t), \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)))dt$$

$$- \quad \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t))\nabla_x u^\varepsilon(t), u^\varepsilon(t))dt$$

$$+ \quad \varepsilon^\rho(A^\varepsilon u^\varepsilon(t), \bar{G}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)))dt + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)), \bar{G}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)))dt$$

$$+ \quad \varepsilon^\rho(A^\varepsilon u^\varepsilon(t), \bar{G}'_u(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t))u^\varepsilon(t))dt + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)), \bar{G}'_u(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t))u^\varepsilon(t))dt$$

$$+\varepsilon^{(\frac{\alpha}{2}-1)^+}(u^\varepsilon(t), \nabla_y \bar{G}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)))\sigma(\xi_{t/\varepsilon^\alpha})dW_t^\varepsilon, \tag{2.22}$$

where $W_t^\varepsilon = \varepsilon^{\frac{\alpha}{2}} W_{\frac{t}{\varepsilon^\alpha}}$ is a standard Wiener process.

We first prove the

Proposition 2.3.1. *Under our standing assumptions, if moreover $\beta > 0$ (where β is the exponent in (2.3)), there exists a constant C such that for all $\varepsilon_0 > 0$,*

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla_x u^\varepsilon(t)\|^2 dt\right) \le C, \forall \varepsilon \ge \varepsilon_0$$

Proof : It is not hard to see, using standard estimates, that for fixed $\varepsilon > 0$,

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla_x u^\varepsilon(t)\|^2 dt\right) < \infty.$$

Hence we can take the expectation in (2.22) integrated from 0 to t. Integrating by parts, one gets

$$\mathbf{E}\Psi^\varepsilon(u^\varepsilon(t),\xi_{t/\varepsilon^\alpha}) + \mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \nabla_x u^\varepsilon(s))ds$$

$$= \mathbf{E}\Psi^\varepsilon(u_0(x),\xi_0) - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t (\nabla_x u^\varepsilon(s), \tilde{G}(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s)))ds$$

$$-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s), u^\varepsilon(s))ds$$

$$-2\varepsilon^\rho\mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \bar{G}'_u(\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds$$

$$+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t (g(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s)), \bar{G}(\xi_{s/\varepsilon^\alpha},u^\varepsilon(s)) + \bar{G}'_u(\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s))ds$$

$$-\varepsilon^\rho\mathbf{E}\int_0^t (a(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), u^\varepsilon(s)\bar{G}''_{uu}(\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds. \tag{2.23}$$

According to Theorem 2 from [15], under condition **H3** the functions $\bar{G}(y,u)$, $\nabla_y\bar{G}(y,u)$, $\bar{G}'_u(y,u)$ and $\bar{G}''_{uu}(y,u)$ admit the following bounds

$$|\bar{G}(y,u)| \le c(1+|y|)^\mu|u|, \qquad |\nabla_y\bar{G}(y,u)| \le c(1+|y|)^\mu|u|,$$
$$|\bar{G}'_u(y,u)| \le c(1+|y|)^\mu,$$
$$(1+|u|)|\bar{G}''_{uu}(y,u)| \le c(1+|y|)^\mu,$$

where $\mu=\mu(\beta)$ is equal to 0 or strictly positive depending on whether $\beta>0$ or $\beta\le 0$, respectively.

The first two integrals on the r.h.s. of (2.23) can be estimated as follows

$$\begin{aligned}
\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Bigg|&\mathbf{E}\int_0^t (\nabla_x u^\varepsilon(s), \tilde{G}(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s)))ds \\
+\ &\mathbf{E}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s), u^\varepsilon(s))ds\Bigg| \\
\le\ &2c\mathbf{E}\int_0^t \|u^\varepsilon(s)\|\,\|\nabla_x u^\varepsilon(s)\|ds \\
\le\ &\frac{c}{\gamma}\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds + c\gamma\mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2 ds.
\end{aligned}$$

Since $\beta>0$ and thus $\mu=0$, then the two terms involving the factor ε^ρ in (2.23) are dominated by the corresponding terms on the l.h.s., and taking sufficiently small γ, we have

$$\mathbf{E}\|u^\varepsilon(t)\|^2 + \mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2 ds \le C_1 + C_2\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds. \tag{2.24}$$

An application of the Gronwall lemma now yields

$$\mathbf{E}\|u^\varepsilon(t)\|^2 + \mathbf{E}\int_0^t \|\nabla_x u^\varepsilon(s)\|^2 ds \le C, \qquad t \le T. \tag{2.25}$$

We then deduce from the Davis–Burkholder–Gundy inequality that

$$\mathbf{E}\sup_{0\le t\le T}\left|\int_0^t (u^\varepsilon(s), \nabla_y \bar{G}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))dW_s^\varepsilon\right| \le C\mathbf{E}\left[\left(\int_0^T \|u^\varepsilon(t)\|^4 dt\right)^{1/2}\right]$$

$$\le \frac{1}{4}\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\right) + C^2\mathbf{E}\int_0^T \|u^\varepsilon(t)\|^2 dt.$$

From (2.22) we obtain :

$$\Psi^\varepsilon(u^\varepsilon(t), \xi_{t/\varepsilon^\alpha}) + \int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \nabla_x u^\varepsilon(s))ds$$

$$= \Psi^\varepsilon(u_0(x), \xi_0) - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\int_0^t (\nabla_x u^\varepsilon(s), \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)))ds$$

$$-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s), u^\varepsilon(s))ds$$

$$-2\varepsilon^\rho\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \bar{G}'_u(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds$$

$$+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t (g(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)) + \bar{G}'_u(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s))ds$$

$$-\varepsilon^\rho\int_0^t (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), u^\varepsilon(s)\bar{G}''_{uu}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds$$

$$+\varepsilon^{(\frac{\alpha}{2}-1)^+}\int_0^t (u^\varepsilon(s), \nabla_y \bar{G}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)))\sigma(\xi_{s/\varepsilon^\alpha})dW_s^\varepsilon. \tag{2.26}$$

There exists $c > 0$ such that the following estimate hold

$$\varepsilon^\rho(\bar{G}(\xi_{\frac{t}{\varepsilon^\alpha}}, u^\varepsilon(t)), u^\varepsilon(t)) \quad \le \quad c\varepsilon^\rho|u|^2.$$

We choose ε_0 such that

$$\varepsilon_0^\rho c = \frac{1}{8}.$$

It then follows :

$$\frac{3}{8}\mathbf{E}\left(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2\right)+\bar{\gamma}\mathbf{E}\int_0^T\|\nabla_x u^\varepsilon(t)\|^2dt\leq C\|u^\varepsilon(0)\|^2+\overline{\gamma_1}\int_0^T\mathbf{E}\|u^\varepsilon(t)\|^2dt$$

Using (2.25), we deduce the proposition ♢

If $\beta\leq 0$ then the function $\bar{G}(y,u)$ admits polynomial growth in y and, as a result, the above method fails to work. In this case the expectation of $\|u^\varepsilon\|$ might explode, as $\varepsilon\to 0$, but we shall control the moments of a slightly different sequence $\{\tilde{u}^\varepsilon\}$.

Let $\theta>0$ be a constant to be chosen below. For each $\varepsilon>0$, let

$$\tau_\varepsilon=\inf\{0\leq t\leq T;\varepsilon^\rho(1+|\xi_{t/\varepsilon^\alpha}|)^\mu>\theta\},$$

and define $\{\tilde{u}^\varepsilon(t),\,0\leq t\leq T\}$ as the solution of the PDE

$$\frac{\partial\tilde{u}^\varepsilon}{\partial t}(t,x)=\frac{\partial}{\partial x_i}a_{ij}(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha})\frac{\partial\tilde{u}^\varepsilon}{\partial x_j}(t,x)+\mathbf{1}_{[0,\tau_\varepsilon]}(t)\frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}g(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},\tilde{u}^\varepsilon(t,x)),$$
$$(t,x)\in(0,T)\times\mathbb{R}^n;\quad \tilde{u}^\varepsilon(0,x)=u_0(x). \tag{2.27}$$

It follows from Corollary 1 in [15] that for all $\rho>0$, $\mu\in\mathbb{R}$

$$\varepsilon^\rho\sup_{0\leq t\leq T}\left(1+|\xi_{t/\varepsilon^\alpha}|\right)^\mu\to 0 \tag{2.28}$$

in probability, as $\varepsilon\to 0$. Hence, as $\varepsilon\to 0$,

$$\mathbf{P}(\tau_\varepsilon=T)\to 1,$$

and consequently

$$\mathbf{P}(u^\varepsilon(t)=\tilde{u}^\varepsilon(t),\,0\leq t\leq T)\to 1.$$

Hence tightness (resp. weak convergence to a limit u) of the sequence u^ε is equivalent to tightness (resp. weak convergence to the same limit u) of the sequence $\tilde{u}^\varepsilon$, for any topology.

We now prove the

Proposition 2.3.2. *Under our standing assumptions, if $\theta>0$ is small enough, then there exists another sequence of stopping times $\{T_\varepsilon,\,\varepsilon>0\}$, satisfying $\mathbf{P}(T_\varepsilon=T)\to 1$, as $\varepsilon\to 0$, and a constant C such that for all $\varepsilon>0$,*

$$\mathbf{E}\left(\sup_{0\leq t\leq T_\varepsilon}\|\tilde{u}^\varepsilon(t)\|^2+\int_0^{T_\varepsilon}\|\nabla_x\tilde{u}^\varepsilon(t)\|^2dt\right)\leq C,$$

and also

$$\mathbf{E}\left(\sup_{0\le t\le T_\varepsilon}\|\tilde u^\varepsilon(t)\|^4\right)\le C.$$

Proof : We repeat the argument of the previous Proposition, except that we develop by the Itô–Krylov formula the expression

$$\Psi^\varepsilon(t):=\frac{1}{2}\|\tilde u^\varepsilon(t)\|^2+\varepsilon^\rho\left(\bar G(\xi_{\frac{t\wedge\tau_\varepsilon}{\varepsilon^\alpha}},\tilde u^\varepsilon(t\wedge\tau_\varepsilon)),\tilde u^\varepsilon(t\wedge\tau_\varepsilon)\right).$$

We obtain the identity

$$\Psi^\varepsilon(t)+\int_0^t(a(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha})\nabla_x\tilde u^\varepsilon(s),\nabla_x\tilde u^\varepsilon(s))ds=\Psi^\varepsilon(0)+\varepsilon^{1-(1\wedge\frac{\alpha}{2})}A_\varepsilon(t\wedge\tau_\varepsilon)$$

$$+\quad\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}B_\varepsilon(t\wedge\tau_\varepsilon)+M_\varepsilon(t\wedge\tau_\varepsilon)+\varepsilon^\rho C_\varepsilon(t\wedge\tau_\varepsilon),$$

where

$$A_\varepsilon(t)=-\int_0^t(\nabla_x\tilde u^\varepsilon(s),\tilde G(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s)))ds-\int_0^t(\tilde G'_u(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s))\nabla_x\tilde u^\varepsilon(s),\tilde u^\varepsilon(s))ds,$$

$$B_\varepsilon(t)=\int_0^t(g(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s)),\bar G(\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s))+\bar G'_u(\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s))\tilde u^\varepsilon(s))ds,$$

$$M_\varepsilon(t)=\int_0^t(\tilde u^\varepsilon(s),\nabla_y\bar G^\varepsilon(s))\sigma(\xi_{s/\varepsilon^\alpha})dW^\varepsilon_s,$$

and

$$C_\varepsilon(t)=-\int_0^t(a(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha})\nabla_x\tilde u^\varepsilon(s),[2\bar G'_u(\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s))+\tilde u^\varepsilon(s)\bar G''_{uu}(\xi_{s/\varepsilon^\alpha},\tilde u^\varepsilon(s))]\nabla_x\tilde u^\varepsilon(s))ds.$$

We now choose θ. There exists $c>0$ such that the following estimate hold

$$\begin{aligned}\varepsilon^\rho C_\varepsilon(t\wedge\tau_\varepsilon)&\le\varepsilon^\rho c\int_0^{t\wedge\tau_\varepsilon}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu\|\nabla_x\tilde u^\varepsilon(s)\|^2ds\\&\le c\theta\int_0^t\|\nabla_x\tilde u^\varepsilon(s)\|^2ds.\end{aligned}$$

It then follows that

$$\begin{aligned}\frac{1}{2}\sup_{0\le s\le t}\|\tilde u^\varepsilon(s)\|^2&+\lambda\int_0^t\|\nabla\tilde u^\varepsilon(s)\|^2ds\\&\le\|u_0\|^2+\frac{c}{\gamma}\int_0^t\|\tilde u^\varepsilon(s)\|^2ds+(c\theta+c\gamma)\int_0^t\|\nabla\tilde u^\varepsilon(s)\|^2ds\\&\quad+c\int_0^t\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu\|\tilde u^\varepsilon(s)\|^2ds+\sup_{0\le s\le t}|M_\varepsilon(s)|.\end{aligned}$$

We choose θ and γ such that

$$\lambda > c\theta + c\gamma.$$

Hence, if ν is a stopping time such that $\nu \leq T$,

$$\begin{aligned}
\frac{1}{2}\mathbf{E}\sup_{0\leq s\leq\nu}\|\tilde{u}^\varepsilon(s)\|^2 \quad + \quad & \bar{\lambda}\mathbf{E}\int_0^\nu (a^\varepsilon(s)\nabla\tilde{u}^\varepsilon(s), \nabla\tilde{u}^\varepsilon(s))ds \\
\leq \quad & \|u_0\|^2 + c\mathbf{E}\int_0^\nu \|\tilde{u}^\varepsilon(s)\|^2 ds \\
& + c\mathbf{E}\int_0^\nu \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu\|\tilde{u}^\varepsilon(s)\|^2 ds \\
& + c\mathbf{E}\left[\left(\int_0^\nu (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu}\|\tilde{u}^\varepsilon(s)\|^4 ds\right)^{1/2}\right] \\
\leq \quad & \|u_0\|^2 + c\mathbf{E}\int_0^\nu \|\tilde{u}^\varepsilon(s)\|^2 ds \\
& + c\mathbf{E}\Big[\sup_{0\leq s\leq\nu}\|\tilde{u}^\varepsilon(s)\|^2\Big(\int_0^\nu \varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds \\
& + \sqrt{\int_0^\nu (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu} ds}\Big)\Big],
\end{aligned}$$

where $\rho' = (\alpha-2)^+$, $\bar{\lambda} > 0$.

As $\varepsilon \to 0$, $\int_0^t \varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds \to at$, $\int_0^t (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu} ds \to bt$ a.s. Note that a=0 if $\alpha > 2$, and $a > 0$ if $\alpha \leq 2$. We choose $r > 0$ such that $ar + \sqrt{br} < \frac{1}{4c}$. Let p be the smallest integer such that $pr \geq T$, and define the stopping times

$$\begin{aligned}
T_\varepsilon^1 &= \inf\{t \leq r; \int_0^t \varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds + \sqrt{\int_0^t (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu} ds} > \frac{1}{4c}\}, \\
T_\varepsilon^2 &= \inf\{r \leq t \leq 2r; \int_r^t \varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds + \sqrt{\int_r^t (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu} ds} > \frac{1}{4c}\}, \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \\
T_\varepsilon^p &= \inf\{(p-1)r \leq t \leq T; \int_{(p-1)r}^t \varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds + \sqrt{\int_{(p-1)r}^t (1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu} ds} > \frac{1}{4c}\}, \\
T_\varepsilon &= T_\varepsilon^1 \mathbf{1}_{\{T_\varepsilon^1 < r\}} + T_\varepsilon^2 \mathbf{1}_{\{T_\varepsilon^1 = r, T_\varepsilon^2 < 2r\}} + \cdots + T_\varepsilon^p \mathbf{1}_{\{T_\varepsilon^1 = r, T_\varepsilon^2 = 2r, \ldots, T_\varepsilon^{p-1} = (p-1)r\}}.
\end{aligned}$$

It follows from Birkhoff's ergodic theorem and the choice of r that

$$\mathbf{P}(T_\varepsilon = T) \to 1, \quad \text{as } \varepsilon \to 0.$$

We have, choosing $\nu = t \wedge T_\varepsilon^1$,

$$\frac{1}{4}\mathbf{E}\left(\sup_{s\le t\wedge T_\varepsilon^1}\|\tilde u^\varepsilon(s)\|^2\right)+\bar\lambda\mathbf{E}\int_0^{t\wedge T_\varepsilon^1}\|\nabla\tilde u^\varepsilon(s)\|^2ds$$
$$\le \|u_0\|^2 + c\mathbf{E}\int_0^{t\wedge T_\varepsilon^1}\|\tilde u^\varepsilon(s)\|^2ds,$$

hence

$$\mathbf{E}\left(\sup_{s\le T_\varepsilon^1}\|\tilde u^\varepsilon(s)\|^2\right)+\mathbf{E}\int_0^{T_\varepsilon^1}\|\nabla\tilde u^\varepsilon(s)\|^2ds\le C.$$

Moreover, for all $r < t \le 2r$, using the notation

$$\mathbf{E}(X;A) := \mathbf{E}(X\mathbf{1}_A),$$

$$\frac{1}{4}\mathbf{E}\left(\sup_{r\le s\le t\wedge T_\varepsilon^2}\|\tilde u^\varepsilon(s)\|^2;T_\varepsilon^1=r\right)+\mathbf{E}\left(\int_r^{t\wedge T_\varepsilon^2}\|\nabla\tilde u^\varepsilon(s)\|^2ds;T_\varepsilon^1=r\right)$$
$$\le \mathbf{E}\left(\|\tilde u^\varepsilon(r)\|^2;T_\varepsilon^1=r\right)+c\mathbf{E}\left(\int_r^{t\wedge T_\varepsilon^2}\|\tilde u^\varepsilon(s)\|^2ds;T_\varepsilon^1=r\right)$$
$$+c\mathbf{E}\left[\sup_{r\le s\le t\wedge T_\varepsilon^2}\|\tilde u^\varepsilon(s)\|^2\left(\int_r^t\varepsilon^{\rho'}(1+|\xi_{s/\varepsilon^\alpha}|)^\mu ds+\sqrt{\int_r^t(1+|\xi_{s/\varepsilon^\alpha}|)^{2\mu}ds}\right);T_\varepsilon^1=r\right],$$

and we deduce by the same arguments as above that

$$\mathbf{E}\left(\sup_{r\le t\le T_\varepsilon^2}\|\tilde u^\varepsilon(t)\|^2;T_\varepsilon^1=r\right)+\mathbf{E}\left(\int_r^{T_\varepsilon^2}\|\nabla\tilde u^\varepsilon(s)\|^2ds;T_\varepsilon^1=r\right)$$
$$\le C\mathbf{E}\left(\|\tilde u^\varepsilon(r)\|^2;T_\varepsilon^1=r\right).$$

Repeating the same argument with $T_\varepsilon^1, T_\varepsilon^2$ replaced by $T_\varepsilon^2, T_\varepsilon^3$, etc.., and combining all those estimates, we conclude that there exists a constant C such that

$$\mathbf{E}\left(\sup_{0\le t\le T_\varepsilon}\|\tilde u^\varepsilon(t)\|^2+\int_0^{T_\varepsilon}\|\nabla_x\tilde u^\varepsilon(s)\|^2ds\right)\le C.$$

The second result is proved quite similarly, with the same sequence of stopping times T_ε, starting with the quantity

$$\frac{1}{4}\|\tilde u^\varepsilon(t)\|^4+\varepsilon^\rho\left(\bar G(\xi_{\frac{t\wedge\tau_\varepsilon}{\varepsilon^\alpha}},\tilde u^\varepsilon(t\wedge\tau_\varepsilon)),\tilde u^\varepsilon(t\wedge\tau_\varepsilon)\right)\|\tilde u^\varepsilon(t\wedge\tau_\varepsilon)\|^2,$$

instead of $\Psi^\varepsilon(\tilde{u}^\varepsilon, t)$.

From now on, θ will be chosen as indicated in the above proof. Note that, for the same reasons as above, tightness (resp. convergence) of the sequence $\{\tilde{u}^\varepsilon(\cdot \wedge T_\varepsilon)\}$ is equivalent to tightness (resp. convergence) of the sequence $\{\tilde{u}^\varepsilon\}$ (resp. of the sequence $\{u^\varepsilon\}$).

We next establish the (here and in the rest of the paper $C_0^\infty(\mathbb{R}^n)$ denotes the class of mappings from $\mathbb{R}^n$ into $\mathbb{R}$, which are of class C^∞, and have compact support)

Proposition 2.3.3. *For any $\varphi \in C_0^\infty(\mathbb{R}^n)$, the collection of processes $\{(u^\varepsilon, \varphi), \varepsilon > 0\}$ is tight in $C([0,T])$.*

Proof : Fix $\varphi \in C_0^\infty(\mathbb{R}^n)$. We consider the random process

$$\Phi^{\varepsilon,\varphi}(t) = (u^\varepsilon(t), \varphi) + \varepsilon^\rho(\bar{G}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)), \varphi).$$

Applying the Itô–Krylov formula to develop $\Phi^{\varepsilon,\varphi}(t)$, we deduce that

$$(u^\varepsilon(t), \varphi) = (u_0, \varphi) + I^\varepsilon(t) + J_1^\varepsilon(t) + J_2^\varepsilon(t) - J_3^\varepsilon(t) + J_4^\varepsilon(t),$$

where

$$I^\varepsilon(t) = \varepsilon^{(\frac{\alpha}{2}-1)^+} \int_0^t (\varphi, \nabla_y \bar{G}(\xi_{r/\varepsilon^\alpha}, u^\varepsilon(s)))\sigma(\xi_{r/\varepsilon^\alpha}) dW_s^\varepsilon,$$

$$J_1^\varepsilon(t) = -\int_0^t \Big[(a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \nabla_x \varphi) + \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}'_u(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s), \varphi)\Big] ds,$$

$$J_2^\varepsilon(t) = \int_0^t \Big[\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\varphi)\Big] ds,$$

$$J_3^\varepsilon(t) = \int_0^t \Big[\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\nabla_x \varphi, \tilde{G}(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)))\Big] ds,$$

and

$$\begin{aligned} J_4^\varepsilon(t) &= \varepsilon^\rho \left[(\bar{G}(\xi_0, u^\varepsilon(0)), \varphi) - (\bar{G}(\xi_{t/\varepsilon^\alpha}, u^\varepsilon(t)), \varphi)\right] \\ &\quad - \varepsilon^\rho \int_0^t \big[(a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x \varphi, \bar{G}'_u(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s)) \\ &\quad + (a(\frac{\cdot}{\varepsilon}, \xi_{s/\varepsilon^\alpha})\nabla_x u^\varepsilon(s), \varphi \bar{G}''_{uu}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s))\nabla_x u^\varepsilon(s))\big] ds. \end{aligned}$$

We first note that

$$\begin{aligned} |J_4^\varepsilon(t)| &\le C\varepsilon^\rho \big[1 + (1 + |\xi_{t/\varepsilon^\alpha}|)^\mu \|u^\varepsilon(t)\| + (1 + |\xi_0|)^\mu \|u_0\| \\ &\quad + \int_0^t (1 + |\xi_{s/\varepsilon^\alpha}|)^\mu (\|\nabla_x u^\varepsilon(s)\| + \|\nabla_x u^\varepsilon(s)\|^2) ds\big] \\ &\le C\varepsilon^\rho \sup_{0\le s\le T} (1 + |\xi_{s/\varepsilon^\alpha}|)^\mu \left(\sup_{0\le s\le T} \|u^\varepsilon(s)\| + \int_0^T (1 + \|\nabla_x u^\varepsilon(s)\|^2) ds \right). \end{aligned}$$

It then follows from (2.28) and Proposition 2.3.2 that

$$\sup_{0\le s\le T} |J_4^\varepsilon(s)| \to 0 \text{ in probability, as } \varepsilon \to 0.$$

We note that for $0 \le s \le t \le T$,

$$|J_1^\varepsilon(t) - J_1^\varepsilon(s)|^2 \le c|t-s| \int_0^T \|\nabla_x u^\varepsilon(r)\|^2 dr.$$

But from Proposition 2.3.1, for any $\eta, \delta > 0$, one can choose $\theta > 0$ such that for all $\varepsilon > 0$,

$$\mathbf{P}\left(\int_0^T \|\nabla_x u^\varepsilon(r)\|^2 dr > \eta^2/c\theta\right) \le \delta,$$

hence for all $\varepsilon > 0$,

$$\mathbf{P}\left(\sup_{|t-s|\le\theta} |J_1^\varepsilon(t) - J_1^\varepsilon(s)| > \eta\right) \le \delta,$$

and the collection of continuous processes $\{J_1^\varepsilon,\, \varepsilon > 0\}$ is tight.

Set $J^\varepsilon(t) = J_2^\varepsilon(t) - J_3^\varepsilon(t)$ then we have

$$|J^\varepsilon(t) - J^\varepsilon(s)| \le c \sup_{0\le r\le T} \|u^\varepsilon(r)\| \int_s^t (1 + |\xi_{r/\varepsilon^\alpha}|)^\mu dr.$$

The tightness of the collection $\{J^\varepsilon,\, \varepsilon > 0\}$ now follows from Proposition 2.3.2 and the following estimate, for $p \ge 1$,

$$\mathbf{E}\left(\left|\int_s^t (1 + |\xi_{r/\varepsilon^\alpha}|)^\mu dr\right|^p\right) \le |t-s|^{p-1} C(T,p,\mu),$$

where $C(T,p,\mu) = \int_0^T \mathbf{E}\left[(1 + |\xi_{r/\varepsilon^\alpha}|)^{p\mu}\right] dr$ is finite and independent of ε, by the stationarity of ξ.

It remains to consider the stochastic integral term. For each $N > 0$, define $\tau_N^\varepsilon := \inf\{t > 0; \|u^\varepsilon(t)\| \ge N\}$, $u_N^\varepsilon := u^\varepsilon(t \wedge \tau_N^\varepsilon)$, and

$$I_N^\varepsilon(t) = \int_0^{t\wedge\tau_N^\varepsilon} (\varphi, \nabla_y \bar{G}(\xi_{s/\varepsilon^\alpha}, u^\varepsilon(s)))\sigma(\xi_{s/\varepsilon^\alpha}) dW_s^\varepsilon.$$

We have for $p > 2$

$$\begin{aligned}
\mathbf{E}\left(\sup_{t_0\le t\le t_0+\gamma} |I_N^\varepsilon(t) - I_N^\varepsilon(t_0)|^p\right) &\le CN^p \mathbf{E}\left(\left|\int_{t_0}^{t_0+\gamma} (1 + |\xi_{r/\varepsilon^\alpha}|)^{2\mu} dr\right|^{p/2}\right) \\
&\le C(T,p,\mu) N^p \gamma^{p/2-1},
\end{aligned}$$

hence for any $\delta > 0$ we first choose N large enough such that for all $\varepsilon > 0$

$$\mathbf{P}\left(\sup_{0\le t\le T} |I^\varepsilon(t) - I^\varepsilon_N(t)| > 0\right) \le \delta/2,$$

and note that

$$\begin{aligned}\mathbf{P}\left(\sup_{t_0\le t\le t_0+\gamma} |I^\varepsilon_N(t) - I^\varepsilon_N(t_0)| \ge \delta\right) &\le CN^p \frac{\gamma^{p/2-1}}{\delta^p} \\ &\le \gamma\delta/2,\end{aligned}$$

for $p = 6$ and $\gamma = \inf(1, \delta^7/2CN^6)$. The tightness of the collection $\{I^\varepsilon, \varepsilon > 0\}$ then follows from Theorem 8.3 in Billingsley [4]. The Proposition is established.

It follows from the results in Viot [17], Proposition 2.3.2 and Proposition 2.3.3 the

Proposition 2.3.4. *The collection $\{u^\varepsilon, \varepsilon > 0\}$ of elements of V_T is tight in $\tilde{V}_T$.*

2.4 Passage to the limit

The aim of this section is to pass to the limit, as $\varepsilon \to 0$, in the family of laws of $\{u^\varepsilon\}$ and to determine the limit problem. In view of the tightness result of the preceding section it is sufficient to find the limit distributions of the inner products (φ, u^ε) with $\varphi \in C_0^\infty(\mathbb{R}^n)$, see [17]. We study the cases $\alpha < 2$ and $\alpha > 2$ separately.We now prove Theorem 2.2.1 and Theorem 2.2.2.

Proof of Theorem 2.2.1 : We introduce the following equations

$$A\chi^k(z,y) = -\sum_i \frac{\partial}{\partial z_i} a_{ik}(z,y), \qquad k = 1, ..., n, \tag{2.29}$$

where the z-periodic functions $\chi^k \in \overline{H}^1_p(\mathbf{T^n} \times \mathbb{R}^d)$.

According to Hypotheses **H1** and **H2** the function on the right-hand side of (2.29) is uniformly bounded in $|y|$ and by Lemma 2.5 in Campillo,Kleptsyna, Piatnitski [6] the solutions χ^k admit a polynomial estimate. We assume for a while that $a(x,y)$ is three times differentiable in x and y, and that all its derivatives up to the third order admit polynomial estimates. Then, χ^k is differentiable and we have for some $m > 0$

$$|\chi^k(z,y)| + |\nabla_z \chi^k(z,y)| + |\nabla_{yy}\chi^k(z,y)| \le C(1 + |y|)^m,$$

For any arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$, we consider the real valued stochastic process $\{\Phi_1^\varepsilon(t),\ 0 \leq t \leq T\}$ defined as

$$\Phi_1^\varepsilon(t) \quad = \quad (\tilde{u}^\varepsilon(t), \varphi) + \varepsilon(\chi^\varepsilon(t)\tilde{u}^\varepsilon(t), \nabla_x \varphi) + \varepsilon^{\frac{\alpha}{2}}(\bar{G}(\xi^\varepsilon_{t\wedge\tau_\varepsilon}, \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)), \varphi),$$

where $\chi^\varepsilon(t)$ and ξ_t^ε stand for $\chi(\frac{\cdot}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}})$, and $\xi_{t/\varepsilon^\alpha}$ respectively. Let $\alpha_\varepsilon(t) := \mathbf{1}_{[0,\tau_\varepsilon]}(t)$.

By the Itô formula :

$$d\Phi_1^\varepsilon(t) = \Big\{(\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t), \varphi) + \varepsilon^{1-\alpha}(\mathcal{L}\chi^\varepsilon(t)\tilde{u}^\varepsilon(t), \nabla_x\varphi) + \varepsilon(\chi^\varepsilon(t)\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t), \nabla_x\varphi)$$

$$+ \quad \varepsilon^{-\frac{\alpha}{2}}\alpha_\varepsilon(t)(\mathcal{L}\bar{G}(\xi_t^\varepsilon, \tilde{u}^\varepsilon(t)), \varphi) + \varepsilon^{\frac{\alpha}{2}}\alpha_\varepsilon(t)(\bar{G}'_u(\xi_t^\varepsilon, \tilde{u}^\varepsilon(t))\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t), \varphi)\Big\}dt$$

$$+ \quad \alpha_\varepsilon(t)\varepsilon^{1-\frac{\alpha}{2}}(\nabla_y\chi^\varepsilon(t)\tilde{u}^\varepsilon(t)\sigma(\xi_t^\varepsilon), \nabla_x\varphi)dW_t^\varepsilon + \alpha_\varepsilon(t)(\nabla_y\bar{G}(\xi_t^\varepsilon, \tilde{u}^\varepsilon(t))\sigma(\xi_t^\varepsilon), \varphi)dW_t^\varepsilon.$$

Considering (2.29), after multiple integration by parts and simple rearrangements, we obtain

$$d\Phi_1^\varepsilon(t) = \Big\{(\tilde{u}^\varepsilon(t), a^\varepsilon\nabla_x\nabla_x\varphi) + \varepsilon^{-1}(\tilde{u}^\varepsilon(t), \nabla_z a^\varepsilon\nabla_x\varphi)$$

$$+ \quad \varepsilon^{-\frac{\alpha}{2}}\alpha_\varepsilon(t)(\bar{g}(\xi_t^\varepsilon, \tilde{u}^\varepsilon(t)), \varphi) + \varepsilon^{-\frac{\alpha}{2}}\alpha_\varepsilon(t)(\tilde{g}(\xi_t^\varepsilon, \tilde{u}^\varepsilon(t)), \varphi) + \varepsilon^{1-\alpha}(\mathcal{L}\chi^\varepsilon(t)\tilde{u}^\varepsilon(t), \nabla_x\varphi)$$

$$+ \quad \varepsilon^{-1}(\nabla_z(a^\varepsilon\nabla_z\chi^\varepsilon)(t)\tilde{u}^\varepsilon(t), \nabla_x\varphi) + (a^\varepsilon\nabla_z\chi^\varepsilon(t), \tilde{u}^\varepsilon(t)\nabla_x\nabla_x\varphi)$$

$$- \quad \varepsilon(a^\varepsilon\nabla_x\tilde{u}^\varepsilon(t), \chi^\varepsilon(t)\nabla_x\nabla_x\varphi) + \alpha_\varepsilon(t)\varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t)g^\varepsilon(t, \tilde{u}^\varepsilon(t)), \nabla_x\varphi)$$

$$+ \quad \varepsilon^{-\frac{\alpha}{2}}\alpha_\varepsilon(t)(\mathcal{L}\bar{G}^\varepsilon, \varphi) - \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}}(\bar{G}^\varepsilon_{uu}a^\varepsilon\nabla_x\tilde{u}^\varepsilon(t), \nabla_x\tilde{u}^\varepsilon(t)\varphi) - \varepsilon^{\frac{\alpha}{2}}(\bar{G}^\varepsilon_u a^\varepsilon\nabla_x\tilde{u}^\varepsilon(t), \nabla_x\varphi)$$

$$+ \quad (\bar{G}^\varepsilon_u g^\varepsilon(t), \varphi)\Big\}dt + \alpha_\varepsilon(t)\varepsilon^{1-\frac{\alpha}{2}}(\nabla_y\chi^\varepsilon(t)\tilde{u}^\varepsilon(t)\sigma(\xi_t^\varepsilon), \nabla_x\varphi)dW_t^\varepsilon + \alpha_\varepsilon(t)(\nabla_y\bar{G}^\varepsilon\sigma(\xi_t^\varepsilon), \varphi)dW_t^\varepsilon.$$

The functions χ^k satisfy the relation $\int_{\mathbf{T^n}}\chi^k(z,y)dz = 0,$ thus

$$\int_{\mathbf{T^n}}\mathcal{L}\chi^k(z,y)dz = \mathcal{L}\int_{\mathbf{T^n}}\chi^k(z,y)dz = 0,$$

and in the same way as in (2.12) we have

$$\mathcal{L}\chi^k(z,y) = \mathrm{div_z K^k(z,y)},$$

with continuous $K^k(z,y)$ of polynomial growth in $|y|$.
Taking into account (2.14),(2.13) after simple transformation we get

$$d\Phi_1^\varepsilon(t) = (\tilde u^\varepsilon(t), a^\varepsilon(\mathbf{I}+\nabla_z\chi^\varepsilon(t))\nabla_x\nabla_x\varphi)dt + \alpha_\varepsilon(t)(\bar G_u^\varepsilon g^\varepsilon,\varphi)dt$$

$$+ \quad \alpha_\varepsilon(t)(\nabla_y\bar G\sigma(\xi_t^\varepsilon),\varphi)dW_t^\varepsilon - \alpha_\varepsilon(t)\varepsilon^{2-\alpha}(K^\varepsilon(t),\nabla(\nabla_x\varphi\tilde u^\varepsilon(t)))dt$$

$$- \quad \varepsilon^{1-\frac{\alpha}{2}}(\nabla_x\varphi,\tilde G(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},u^\varepsilon(t)))dt - \varepsilon^{1-\frac{\alpha}{2}}(\tilde G_u'(\frac{\cdot}{\varepsilon},\xi_{t/\varepsilon^\alpha},u^\varepsilon(t))\nabla_x u^\varepsilon(t),\nabla_x\varphi)dt$$

$$- \quad \varepsilon(a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\chi^\varepsilon(t)\nabla_x\nabla_x\varphi)dt + \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t)g^\varepsilon(t,\tilde u^\varepsilon(t)),\nabla_x\varphi)dt$$

$$- \quad \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}}(\bar G_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\varphi)dt - \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}}(\bar G_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\tilde u^\varepsilon(t)\varphi)dt$$

$$+ \quad \alpha_\varepsilon(t)\varepsilon^{1-\frac{\alpha}{2}}(\nabla_y\chi^\varepsilon(t)\tilde u^\varepsilon(t)\sigma(\xi_t^\varepsilon),\nabla_x\varphi)dW_t^\varepsilon.$$

Now it is natural to rewrite the above expression as follows

$$(\tilde u^\varepsilon(t),\varphi) = (u_0,\varphi) + \int_0^t \Big\{(\tilde u^\varepsilon(s), < a(\mathbf{I}+\nabla_z\chi) > \nabla_x\nabla_x\varphi) + < \bar G_u' g > (\tilde u^\varepsilon(s)),\varphi)\Big\}ds$$

$$+\int_0^t (\nabla_y\bar G\sigma(\xi_s^\varepsilon),\varphi)dW_s^\varepsilon + R_1^\varepsilon(t) \tag{2.30}$$

where

$$R_1^\varepsilon(t) = \varepsilon(\chi^\varepsilon(0)u_0,\nabla_x\varphi) + \varepsilon^{\frac{\alpha}{2}}(\bar G(\xi_0^\varepsilon,u_0),\varphi) - \varepsilon(\chi^\varepsilon(t)\tilde u^\varepsilon(t),\nabla_x\varphi) - \varepsilon^{\frac{\alpha}{2}}(\bar G(\xi_{t\wedge\tau_\varepsilon}^\varepsilon,\tilde u^\varepsilon(t\wedge\tau_\varepsilon)),\varphi)$$

$$+ \quad \int_0^t (\tilde u^\varepsilon(s),[a^\varepsilon(\mathbf{I}+\nabla_z\chi^\varepsilon(s)) - < a(\mathbf{I}+\nabla_z\chi) >]\nabla_x\nabla_x\varphi)ds$$

$$+\int_0^t \{(\alpha_\varepsilon(s)\bar G_u^\varepsilon g^\varepsilon - < \bar G_u' g > (\tilde u^\varepsilon(s)),\varphi) + \alpha_\varepsilon(s)\varepsilon^{2-\alpha}(K^\varepsilon(s),\nabla(\nabla_x\varphi\tilde u^\varepsilon(s)))\}ds$$

$$+ \quad \int_0^t \Big\{ -\varepsilon^{1-\frac{\alpha}{2}}(\nabla_x\varphi,\tilde G(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))) - \varepsilon^{1-\frac{\alpha}{2}}(\tilde G_u'(\frac{\cdot}{\varepsilon},\xi_{s/\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s),\nabla_x\varphi)$$

$$- \quad \varepsilon(a^\varepsilon\nabla_x\tilde u^\varepsilon(s),\chi^\varepsilon(s)\nabla_x\nabla_x\varphi) + \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(s)g^\varepsilon(s,\tilde u^\varepsilon(s)),\nabla_x\varphi)$$

$$- \quad \alpha_\varepsilon(s)\varepsilon^{\frac{\alpha}{2}}(\bar G_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(s),\nabla_x\varphi) - \alpha_\varepsilon(s)\varepsilon^{\frac{\alpha}{2}}(\bar G_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(s),\nabla_x\tilde u^\varepsilon(s)\varphi)\Big\}ds$$

$$+ \quad \alpha_\varepsilon(s)\varepsilon^{1-\frac{\alpha}{2}}\int_0^t (\nabla_y\chi^\varepsilon(s)\tilde u^\varepsilon(s)\sigma(\xi_s^\varepsilon),\nabla_x\varphi)dW_s^\varepsilon.$$

We rewrite (2.30) as

$$\begin{aligned} F_\varphi(t, \tilde{u}^\varepsilon) &= \int_0^t \alpha_\varepsilon(s)(\nabla_y \bar{G}\sigma(\xi_s^\varepsilon), \varphi) dW_s^\varepsilon + R_1^\varepsilon(t) \\ &= M_\varphi^\varepsilon(t) + R_1^\varepsilon(t), \end{aligned}$$

where, for $u \in V_T$,

$$F_\varphi(t, u) := (u(t), \varphi) - (u_0, \varphi) - \int_0^t (u(s), < a(\mathbf{I} + \nabla_z \chi) > \nabla_x \nabla_x \varphi) ds$$
$$- \int_0^t (< \bar{G}_u g > (u(s)), \varphi) ds,$$

and the bracket of the local martingale M_φ^ε is given by

$$<< M_\varphi^\varepsilon >> (t) = \int_0^t \alpha_\varepsilon(s)(\nabla_y \bar{G}\sigma(\xi_s^\varepsilon), \varphi)^2 ds.$$

By **H3**, Proposition 7 in Pardoux, Piatnitski[16] , Proposition 2.3.1 and Burkholder-Davis-Gundy inequality , $R_1^\varepsilon(t \wedge T_\varepsilon)$ tends to zero uniformly in t, in $L^1(\Omega)$, as $\varepsilon \to 0$. Let $0 \le s < t$, and Θ_s^ε be any continuous (in the sense of the topology of $\tilde{V}_T$) and bounded functional of $\{\tilde{u}^\varepsilon(r),\, 0 \le r \le s\}$. We have that

$$\mathbf{E}\left[(F_\varphi(t \wedge T_\varepsilon, \tilde{u}^\varepsilon) - F_\varphi(s \wedge T_\varepsilon, \tilde{u}^\varepsilon))\, \Theta_s^\varepsilon\right] = \mathbf{E}\left[(R_1^\varepsilon(t \wedge T_\varepsilon) - R_1^\varepsilon(s \wedge T_\varepsilon))\, \Theta_s^\varepsilon\right],$$

$$\mathbf{E}\left[\left(M_\varphi^\varepsilon(t \wedge T_\varepsilon) - M_\varphi^\varepsilon(s \wedge T_\varepsilon)\right)^2 \Theta_s^\varepsilon\right] = \mathbf{E}\left[\left(< M_\varphi^\varepsilon > (t \wedge T_\varepsilon) - < M_\varphi^\varepsilon > (s \wedge T_\varepsilon)\right) \Theta_s^\varepsilon\right].$$

Let $u \in \tilde{V}_T$ be any accumulation point of the sequence $\tilde{u}^\varepsilon$, as $\varepsilon \to 0$. Taking the limit along the corresponding subsequence in the two last identities, using weak convergence and uniform integrability – see Proposition 2.3.2, we conclude with the help of Propositions 6, 8, and 9 in in Pardoux, Piatnitski [16] that

$$F_\varphi(t, u) := (u(t), \varphi) - (u_0, \varphi) - \int_0^t (u(s), < a(\mathbf{I} + \nabla_z \chi) > \nabla_x \nabla_x \varphi) ds$$
$$- \int_0^t (< \bar{G}_u g > (u(s)), \varphi) ds$$

is a square integrable martingale, if we equip $\tilde{V}_T$ with respect to the natural filtration of u, with the associated increasing process given by

$$\int_0^t (R(u(s))\varphi, \varphi) ds,$$

where

$$(R(u)\varphi,\varphi) = \int_{\mathbb{R}^d} \big((q(y)(\nabla_y \bar{G}(y,u),\varphi),(\nabla_y \bar{G}(y,u),\varphi))p(y)\big)\ dy.$$

We have shown that the law Q^0 of any accumulation point of the sequence u^ε solves the following martingale problem, which we shall denote problem (MP1). For all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$F_\varphi(t,u) := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (\hat{A}(u(s)),\varphi)ds,\ t \geq 0,$$

where

$$\hat{A}(v) = \nabla_x \cdot < a(\mathbf{I} + \nabla_z \chi > \nabla_x v + < \bar{G}_u g > (v),$$

is a martingale with the increasing process

$$< F_\varphi(\cdot,u) > (t) = \int_0^t (R(u(s))\varphi,\varphi)ds.$$

This completes the proof of theorem 2.2.1 in the smooth case. For a general matrix-valued function **a** satisfying the Hypotheses **H1** and **H2**, we approximate $a(x,y)$ with smooth functions, see e.g. Proof of Proposition 4.2 p.71 in [4]. For the uniqueness of the solution we follow the same scheme as in Pardoux, Piatnitski [16]. ♢

Proof of Theorem 2.2.2 : The approach used in this case is quite similar to that of the preceding case. We define the functions $N(.,u) \in W^{2,p}(\mathbf{T^n}) \cap C^2_\alpha(\mathbf{T^n})$, $\Psi(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d)$, to be solutions of the system of equations :

$$\overline{A}N(z,u) = -\widehat{g}(z,u), \tag{2.31}$$

$$\mathcal{L}\Psi(z,y,u) = -[\tilde{g}(z,y,u) + AN(z,u)], \tag{2.32}$$

and the functions $E^k \in \bar{H}^1(\mathbf{T}^n)$ and $F^k \in \overline{H}^1_\rho(\mathbf{T}^n \times \mathbb{R}^d), k = 1,...,n,$ to satisfy the system

$$\overline{A}E^k(z) = -\sum_i \overline{a^{ik}_{z_i}(z,.)}, \tag{2.33}$$

$$\mathcal{L}F^k(z,y) = -[AE^k(z) + \sum_i a^{ik}_{z_i}(z,.)], \tag{2.34}$$

$k = 1,...,n$. Having defined $\bar{G}(y,u)$, $E(z)$, $N(z,u)$, $F(z,y)$ and $\Psi(z,y,u)$, for any arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$,we consider the real valued stochastic process $\{\Phi_2^\varepsilon(t),\ 0 \leq t \leq T\}$ defined as

$$\begin{aligned}\Phi_2^\varepsilon(t) &= (\tilde{u}^\varepsilon(t),\varphi) + \varepsilon(E^\varepsilon \tilde{u}^\varepsilon(t),\nabla_x\varphi) + \varepsilon^{\alpha-1}(F^\varepsilon(t)\tilde{u}^\varepsilon(t),\nabla_x\varphi) + \varepsilon(N^\varepsilon(\frac{\cdot}{\varepsilon},\tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)),\varphi)\\ &+ \varepsilon^{\alpha-1}(\Psi^\varepsilon(t\wedge\tau_\varepsilon,\tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)),\varphi) + \varepsilon^{\alpha-1}(\bar{G}(\xi^\varepsilon_{t\wedge\tau_\varepsilon},\tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)),\varphi),\end{aligned}$$

where E^ε, $F^\varepsilon(t)$, $N^\varepsilon(\frac{\cdot}{\varepsilon}, u)$, $\Psi^\varepsilon(t,u)$ and ξ^ε_t stand for $E(\frac{\cdot}{\varepsilon})$, $F(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha})$, $N(\frac{\cdot}{\varepsilon}, u)$, $\Psi(\frac{\cdot}{\varepsilon}, \xi_{t/\varepsilon^\alpha}, u)$ and $\xi_{t/\varepsilon^\alpha}$ respectively. Let $\alpha_\varepsilon(t) := \mathbf{1}_{[0,\tau_\varepsilon]}(t)$. By the Itô formula,

$$
\begin{aligned}
d\Phi_2^\varepsilon(t) = \Big\{ &(\frac{\partial \tilde u^\varepsilon}{\partial t}(t), \varphi) + \varepsilon(E^\varepsilon \frac{\partial \tilde u^\varepsilon}{\partial t}(t), \nabla_x\varphi) + \varepsilon^{-1}(\mathcal{L}F^\varepsilon(t)\tilde u^\varepsilon(t), \nabla_x\varphi) \\
+ \;& \varepsilon^{\alpha-1}(F^\varepsilon(t)\frac{\partial \tilde u^\varepsilon}{\partial t}(t), \nabla_x\varphi) + \varepsilon(\frac{\partial N^\varepsilon}{\partial u}(t, \tilde u^\varepsilon(t))\frac{\partial \tilde u^\varepsilon}{\partial t}(t), \varphi) \\
+ \;& \varepsilon^{-1}\alpha_\varepsilon(t)(\mathcal{L}\Psi^\varepsilon(t, \tilde u^\varepsilon(t)), \varphi) + \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\frac{\partial \Psi^\varepsilon}{\partial u}(t, \tilde u^\varepsilon(t))\frac{\partial \tilde u^\varepsilon}{\partial t}(t), \varphi) \\
+ \;& \varepsilon^{-1}\alpha_\varepsilon(t)(\mathcal{L}\bar G(\xi^\varepsilon_t, \tilde u^\varepsilon(t)), \varphi) + \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\bar G'_u(\xi^\varepsilon_t, \tilde u^\varepsilon(t))\frac{\partial \tilde u^\varepsilon}{\partial t}(t), \varphi)\Big\} dt \\
+ \;& \varepsilon^{\frac{\alpha}{2}-1}(\tilde u^\varepsilon(t)\nabla_y F^\varepsilon(t)\sigma(\xi^\varepsilon_t), \nabla_x\varphi)dW_t + \varepsilon^{\frac{\alpha}{2}-1}(\nabla_y\Psi^\varepsilon(t, (\tilde u^\varepsilon(t))\sigma(\xi^\varepsilon_t), \varphi)dW_t \\
+ \;& \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}-1}(\nabla_y \bar G(\xi^\varepsilon_t, \tilde u^\varepsilon(t))\sigma(\xi^\varepsilon_t), \varphi)dW^\varepsilon_t.
\end{aligned}
$$

Considering (2.27), after multiple integration by parts and simple rearrangements, we obtain

$$
\begin{aligned}
d\Phi_2^\varepsilon(t) = \Big\{ &(\tilde u^\varepsilon(t), a^\varepsilon\nabla_x\nabla_x\varphi) + \varepsilon^{-1}(\tilde u^\varepsilon(t), \nabla_z a^\varepsilon \nabla_x\varphi) \\
+ \;& \varepsilon^{-1}\alpha_\varepsilon(t)(\bar g(\xi^\varepsilon_t, \tilde u^\varepsilon(t)), \varphi) + \varepsilon^{-1}\alpha_\varepsilon(t)(\tilde g(\xi^\varepsilon_t, \tilde u^\varepsilon(t)), \varphi) \\
+ \;& \varepsilon^{-1}(\nabla_z(a^\varepsilon\nabla_z E^\varepsilon)\tilde u^\varepsilon(t), \nabla_x\varphi) + (a^\varepsilon\nabla_z E^\varepsilon, \tilde u^\varepsilon(t)\nabla_x\nabla_x\varphi) \\
- \;& \varepsilon(a^\varepsilon\nabla_x\tilde u^\varepsilon(t), E^\varepsilon\nabla_x\nabla_x\varphi) + \alpha_\varepsilon(t)(E^\varepsilon g^\varepsilon(t, \tilde u^\varepsilon(t)), \nabla_x\varphi) \\
+ \;& \varepsilon^{\alpha-2}(\nabla_z(a^\varepsilon\nabla_z F^\varepsilon)(t)\tilde u^\varepsilon(t), \nabla_x\varphi) + \varepsilon^{\alpha-2}(a^\varepsilon\nabla_z F^\varepsilon(t), \tilde u^\varepsilon(t)\nabla_x\nabla_x\varphi) \\
- \;& \varepsilon^{\alpha-1}(a^\varepsilon\nabla_x\tilde u^\varepsilon(t), F^\varepsilon(t)\nabla_x\nabla_x\varphi) + \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(F^\varepsilon(t)g^\varepsilon(t, \tilde u^\varepsilon(t)), \nabla_x\varphi) \\
- \;& \alpha_\varepsilon(t)(\nabla_z N^\varepsilon_u(s, \tilde u^\varepsilon(t))a^\varepsilon\nabla_x\tilde u^\varepsilon(t), \varphi) - \varepsilon\alpha_\varepsilon(t)(N^\varepsilon_{uu}a^\varepsilon\nabla_x\tilde u^\varepsilon(t), \nabla_x\tilde u^\varepsilon(t)\varphi) \\
- \;& \varepsilon\alpha_\varepsilon(t)(N^\varepsilon_u a^\varepsilon\nabla_x\tilde u^\varepsilon(t), \nabla_x\varphi) + \alpha_\varepsilon(t)(N^\varepsilon_u g^\varepsilon, \varphi) + \varepsilon^{-1}\alpha_\varepsilon(t)(\mathcal{L}\Psi^\varepsilon(t, \tilde u^\varepsilon(t)), \varphi)
\end{aligned}
$$

$$
\begin{aligned}
- &\ \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(\nabla_z\Psi_u^\varepsilon(t,\tilde u^\varepsilon(t))a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\varphi) - \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\Psi_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\tilde u^\varepsilon(t)\varphi)\\
- &\ \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\Psi_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\varphi) + \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(\Psi_u^\varepsilon g^\varepsilon,\varphi) + \varepsilon^{-1}\alpha_\varepsilon(t)(\mathcal{L}\bar G^\varepsilon,\varphi)\\
- &\ \alpha_\varepsilon(t)\varepsilon^{\alpha-1}(\bar G_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\tilde u^\varepsilon(t)\varphi) - \varepsilon^{\alpha-1}(\bar G_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\varphi) + \varepsilon^{\alpha-2}(\bar G_u^\varepsilon g^\varepsilon,\varphi)\\
+ &\ \varepsilon^{-1}(\mathcal{L}F^\varepsilon(t)\tilde u^\varepsilon(t),\nabla_x\varphi)\Big\}dt + \varepsilon^{\frac{\alpha}{2}-1}(\tilde u^\varepsilon(t)\nabla_y F^\varepsilon(t)\sigma(\xi_t^\varepsilon),\nabla_x\varphi)dW_t^\varepsilon\\
+ &\ \varepsilon^{\frac{\alpha}{2}-1}(\nabla_y\Psi^\varepsilon(t,(\tilde u^\varepsilon(t))\sigma(\xi_t^\varepsilon),\varphi)dW_t^\varepsilon + \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}-1}(\nabla_y\bar G(\xi_t^\varepsilon,\tilde u^\varepsilon(t))\sigma(\xi_t^\varepsilon),\varphi)dW_t^\varepsilon.
\end{aligned}
$$

According to (2.14), (2.31),(2.32),(2.33),(2.34) and in view of the relation

$$(a^\varepsilon\nabla_z N_u^\varepsilon\nabla_x\tilde u^\varepsilon,\varphi) = -(a^\varepsilon\nabla_z N^\varepsilon,\nabla_x\varphi) - \varepsilon^{-1}(\nabla_z\cdot(a^\varepsilon\nabla_z N^\varepsilon),\varphi),$$

the following terms on the right-hand side are mutually cancelled :

$$\varepsilon^{-1}(\tilde u^\varepsilon(t),\nabla_z a^\varepsilon\nabla_x\varphi) + \varepsilon^{-1}(\nabla_z(a^\varepsilon\nabla_z E^\varepsilon)\tilde u^\varepsilon(t),\nabla_x\varphi) + \varepsilon^{-1}(\mathcal{L}F^\varepsilon(t)\tilde u^\varepsilon(t),\nabla_x\varphi) = 0,$$

$$\varepsilon^{-1}(\tilde g(\xi_t^\varepsilon,\tilde u^\varepsilon(t)),\varphi) + \varepsilon^{-1}(\nabla_z\cdot(a^\varepsilon\nabla_z N^\varepsilon),\varphi) + \varepsilon^{-1}(\mathcal{L}\Psi^\varepsilon(t,\tilde u^\varepsilon(t)),\varphi) = 0,$$

$$\mathcal{L}\bar G(y,u) + \bar g(y,u) = 0, \qquad \forall u\in\mathbb{R}.$$

Then the above expression can be simplified further as follows

$$
\begin{aligned}
d\Phi_2^\varepsilon(t) = &\Big\{(\tilde u^\varepsilon(t)),a^\varepsilon(\mathbf{I}+\nabla_z E^\varepsilon)\nabla_x\nabla_x\varphi)\\
+ &\ \alpha_\varepsilon(t)(E^\varepsilon g^\varepsilon(t,\tilde u^\varepsilon(t)),\nabla_x\varphi) + \alpha_\varepsilon(t)(a^\varepsilon\nabla_z N^\varepsilon,\nabla_x\varphi) + \alpha_\varepsilon(t)(N_u^\varepsilon g^\varepsilon,\varphi)\Big\}dt\\
+ &\ \Big\{\varepsilon^{\alpha-2}(a^\varepsilon\nabla_z F^\varepsilon(t),\nabla(\tilde u^\varepsilon(t)\nabla_x\varphi)) - \varepsilon(a^\varepsilon\nabla_x\tilde u^\varepsilon(t),E^\varepsilon\nabla_x\nabla_x\varphi)\\
- &\ \varepsilon^{\alpha-1}(a^\varepsilon\nabla_x\tilde u^\varepsilon(t),F^\varepsilon(t)\nabla_x\nabla_x\varphi) + \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(F^\varepsilon(t)g^\varepsilon(t,\tilde u^\varepsilon(t)),\nabla_x\varphi)\\
- &\ \varepsilon\alpha_\varepsilon(t)(N_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\tilde u^\varepsilon(t)\varphi) - \varepsilon\alpha_\varepsilon(t)(N_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\varphi)\\
- &\ \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(\nabla_z\Psi_u^\varepsilon(t,\tilde u^\varepsilon(t))a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\varphi) - \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\Psi_{uu}^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\tilde u^\varepsilon(t)\varphi)\\
- &\ \varepsilon^{\alpha-1}\alpha_\varepsilon(t)(\Psi_u^\varepsilon a^\varepsilon\nabla_x\tilde u^\varepsilon(t),\nabla_x\varphi) + \alpha_\varepsilon(t)\varepsilon^{\alpha-2}(\Psi_u^\varepsilon g^\varepsilon,\varphi) + \varepsilon^{\alpha-2}(a^\varepsilon\nabla_z F^\varepsilon(t),\tilde u^\varepsilon(t)\nabla_x\nabla_x\varphi)
\end{aligned}
$$

$$- \quad \alpha_\varepsilon(t)\varepsilon^{\alpha-1}(\bar{G}^\varepsilon_{uu} a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x \tilde{u}^\varepsilon(t)\varphi) - \varepsilon^{\alpha-1}(\bar{G}^\varepsilon_u a^\varepsilon \nabla_x \tilde{u}^\varepsilon(t), \nabla_x\varphi) + \varepsilon^{\alpha-2}(\bar{G}^\varepsilon_u g^\varepsilon, \varphi)\Big\}dt$$

$$+ \quad \varepsilon^{\frac{\alpha}{2}-1}(\tilde{u}^\varepsilon(t)\nabla_y F^\varepsilon(t)\sigma(\xi^\varepsilon_t), \nabla_x\varphi)dW^\varepsilon_t + \varepsilon^{\frac{\alpha}{2}-1}(\nabla_y \Psi^\varepsilon(t, (\tilde{u}^\varepsilon(t))\sigma(\xi^\varepsilon_t), \varphi)dW^\varepsilon_t$$

$$+ \quad \alpha_\varepsilon(t)\varepsilon^{\frac{\alpha}{2}-1}(\nabla_y \bar{G}(\xi^\varepsilon_t, \tilde{u}^\varepsilon(t))\sigma(\xi^\varepsilon_t), \varphi)dW^\varepsilon_t.$$

Now it is natural to rewrite $d\Phi^\varepsilon_2(t)$ as follows

$$(\tilde{u}^\varepsilon(t), \varphi) = (u_0, \varphi) + \int_0^t (\tilde{u}^\varepsilon(s), < a(\mathbf{I} + \nabla_z E) > \nabla_x \nabla_x \varphi)ds$$

$$+ \quad \int_0^t \Big\{(< a\nabla_x N > (\tilde{u}^\varepsilon(s)), \nabla_x\varphi) + (< Eg > (\tilde{u}^\varepsilon(s)), \nabla_x\varphi) + (< N_u g > (\tilde{u}^\varepsilon(s)), \varphi)\Big\}ds$$

$$+ \quad R^\varepsilon_2(t) + \varepsilon^{\frac{\alpha}{2}-1} M^\varepsilon(t),$$

where

$$R^\varepsilon_2(t) = \Big\{\varepsilon(E^\varepsilon u_0, \nabla_x\varphi) + \varepsilon^{\alpha-1}(F^\varepsilon(0)u_0, \nabla_x\varphi) + \varepsilon(N^\varepsilon(\frac{\cdot}{\varepsilon}, u_0), \varphi) + \varepsilon^{\alpha-1}(\Psi^\varepsilon(0, u_0), \varphi)$$

$$+ \quad \varepsilon^{\alpha-1}(\bar{G}(\xi^\varepsilon_0, u_0), \varphi) - \varepsilon(E^\varepsilon \tilde{u}^\varepsilon(t), \nabla_x\varphi) - \varepsilon^{\alpha-1}(F^\varepsilon(t)\tilde{u}^\varepsilon(t), \nabla_x\varphi)$$

$$- \quad \varepsilon(N^\varepsilon(t\wedge\tau_\varepsilon, \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)), \varphi) - \varepsilon^{\alpha-1}(\Psi^\varepsilon(t\wedge\tau_\varepsilon, \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)), \varphi)$$

$$- \quad \varepsilon^{\alpha-1}(\bar{G}(\xi^\varepsilon_{t\wedge\tau_\varepsilon}, \tilde{u}^\varepsilon(t\wedge\tau_\varepsilon)), \varphi)\Big\}$$

$$+ \quad \int_0^t \Big\{ -\varepsilon(a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), E^\varepsilon \nabla_x \nabla_x \varphi) + \varepsilon^{\alpha-2}(a^\varepsilon \nabla_z F^\varepsilon(s), \nabla(\tilde{u}^\varepsilon(s)\nabla_x\varphi))$$

$$+ \quad \varepsilon^{\alpha-2}(a^\varepsilon \nabla_z F^\varepsilon(s), \tilde{u}^\varepsilon(s)\nabla_x\nabla_x\varphi) - \varepsilon^{\alpha-1}(a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), F^\varepsilon(s)\nabla_x\nabla_x\varphi)$$

$$+ \quad \alpha_\varepsilon(s)\varepsilon^{\alpha-2}(F^\varepsilon(s)g^\varepsilon(s, \tilde{u}^\varepsilon(s)), \nabla_x\varphi) - \varepsilon\alpha_\varepsilon(s)(N^\varepsilon_{uu} a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x \tilde{u}^\varepsilon(s)\varphi)$$

$$- \quad \varepsilon\alpha_\varepsilon(s)(N^\varepsilon_u a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x\varphi) - \alpha_\varepsilon(s)\varepsilon^{\alpha-2}(\nabla_z \Psi^\varepsilon_u(s, \tilde{u}^\varepsilon(s))a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \varphi)$$

$$- \quad \varepsilon^{\alpha-1}\alpha_\varepsilon(s)(\Psi^\varepsilon_{uu} a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x \tilde{u}^\varepsilon(s)\varphi) - \varepsilon^{\alpha-1}\alpha_\varepsilon(s)(\Psi^\varepsilon_u a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x\varphi)$$

$$- \quad \alpha_\varepsilon(s)\varepsilon^{\alpha-1}(\bar{G}^\varepsilon_{uu} a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x \tilde{u}^\varepsilon(s)\varphi) - \varepsilon^{\alpha-1}(\bar{G}^\varepsilon_u a^\varepsilon \nabla_x \tilde{u}^\varepsilon(s), \nabla_x\varphi)$$

$$
\begin{aligned}
&+ \quad \varepsilon^{\alpha-2}(\bar{G}_u^\varepsilon g^\varepsilon, \varphi) + \alpha_\varepsilon(s)\varepsilon^{\alpha-2}(\Psi_u^\varepsilon g^\varepsilon, \varphi)\Big\}ds \\
&+ \quad \int_0^t (\tilde{u}^\varepsilon(s), [a^\varepsilon(\mathbf{I} + \nabla_z E^\varepsilon(s)) - < a(\mathbf{I} + \nabla_z E) >]\nabla_x \nabla_x \varphi)ds \\
&+ \quad \int_0^t \Big\{\alpha_\varepsilon(s)(E^\varepsilon(s)g^\varepsilon(s, \tilde{u}^\varepsilon(s)) - < Eg > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi) \\
&+ \quad \alpha_\varepsilon(s)(a^\varepsilon \nabla_z N^\varepsilon(s, \tilde{u}^\varepsilon(s)) - < a\nabla_z N > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi)\Big\}ds \\
&+ \quad \int_0^t \Big\{(\alpha_\varepsilon(s) N_u^\varepsilon(s, \tilde{u}^\varepsilon(s))g^\varepsilon(s, \tilde{u}^\varepsilon(s)) - < N_u g > (\tilde{u}^\varepsilon(s)), \varphi)\Big\}ds
\end{aligned}
$$

and $M^\varepsilon(t)$ is the stochastic term on the right hand side of the latter formula. By **H3** and Proposition 7 in Pardoux, Piatnitski [16], $R_2^\varepsilon(t \wedge T_\varepsilon)$ tends to zero uniformly in t in $L^1(\Omega)$, as $\varepsilon \to 0$.

We have also

$$\mathbf{E} \sup_{0\le t\le T} \Big| M^\varepsilon(t) \Big| \le C,$$

this limit relation follows from Proposition 2.3.1 and Burkholder-Davis-Gundy inequality.

Finally for any test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ the following limit relation holds :

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \ \mathbf{E} \ \sup_{t\le T} \Big| (\tilde{u}^\varepsilon(t), \varphi) - (u_0, \varphi) - \int_0^t (\tilde{u}^\varepsilon(s), < a(\mathbf{I} + \nabla_z E) > \nabla_x \nabla_x \varphi)ds \\
&- \int_0^t \Big\{(< a\nabla_x N > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi) + (< Eg > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi) + (< N_u g > (\tilde{u}^\varepsilon(s)), \varphi)\Big\}ds \Big| = 0.
\end{aligned}
$$

Let us introduce the bounded functional

$$
\begin{aligned}
&\Phi_\phi(u) = 1 \wedge \sup_{t\le T} \Big| (u(s), \varphi) - (u_0, \varphi) - \int_0^t (u(s), < a(\mathbf{I} + \nabla_z E) > \nabla_x \nabla_x \varphi)ds \\
&- \int_0^t \Big\{(< a\nabla_x N > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi) + (< Eg > (\tilde{u}^\varepsilon(s)), \nabla_x \varphi) + (< N_u g > (\tilde{u}^\varepsilon(s)), \varphi)\Big\}ds \Big|.
\end{aligned}
$$

From the above relation we get $\lim_{\varepsilon\downarrow 0} \mathbf{E}\Phi_\phi(u^\varepsilon) = 0$, so for any limiting point Q of the family of laws of u^ε in $\tilde{V}_T$ we obtain $\mathbf{E}^Q \Phi_\phi(u) = 0$.

The proof is complete. ♢

2.5 Appendix

In this appendix we study the kind of Poisson equations :

$$\mathcal{L}\Psi(z,.,u) = -\Phi(z,.,u), \quad \int_{\mathbb{R}^d} \Phi(z,y,u)p(y)dy = 0,$$

where

$$\Phi = [\tilde{g}(z,y,u) + AN(z,u)].$$

In particular we aim to study regularity of the function Ψ with respect to the variables z and u (see also [15] for related results).

Lemma 2.5.1. *Let us consider* $\Phi : \mathbf{T^n} \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ *such that for every* $p > 1$ *, the function* $(z,u) \mapsto \Phi(z,.,u)$ *is continuously differentiable from* $\mathbf{T^n} \times \mathbb{R}$ *into* $L^p(\mathbb{R}^d)$*. Then for every* $p > 1$ *the function* $(z,u) \mapsto \Psi(z,.,u)$ *is continuously differentiable from* $\mathbf{T^n} \times \mathbb{R}$ *into* $W^{2,p}_{loc}(\mathbb{R}^d)$ *and :*
$\forall p > 1, \forall i \in \{1,...,n\},\ \forall (z,u) \in \mathbf{T^n} \times \mathbb{R}^d,$

$$\begin{cases} \frac{\partial \Psi}{\partial z_i}(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d), \\ \frac{\partial \Psi}{\partial u}(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d), \\ \frac{\partial \Psi^2}{\partial uu}(z,.,u) \in W^{2,p}_{loc}(\mathbb{R}^d). \end{cases}$$

Morever, $\forall (z,u) \in \mathbf{T^n} \times \mathbb{R}$*,*

$$\begin{cases} \forall i \in \{1,...,n\} \ \mathcal{L}\frac{\partial \Psi}{\partial z_i}(z,.,u) = -\frac{\partial \Phi}{\partial z_i}(z,.,u), \\ \mathcal{L}\frac{\partial \Psi}{\partial u}(z,.,u) = -\frac{\partial \Phi}{\partial u}(z,.,u), \\ \mathcal{L}\frac{\partial \Psi^2}{\partial uu}(z,.,u) = -\frac{\partial \Phi^2}{\partial uu}(z,.,u). \end{cases}$$

Proof : We just detail the proof related to the partial differentiability with respect to the variables z. Following [15] we use extensively the representation

$$\Psi(z,y,u) = \int_0^\infty \mathbf{E_y}\Phi(z,Y_t,u)dt.$$

• Differentiability with respect to the variable z
Let us fix $i \in \{1,...,n\}$, $p > 1$ and $(z,u,\delta) \in \mathbf{T^n} \times \mathbb{R} \times \mathbb{R}$.We set $z + \delta = z + \delta e_i$. Then,

$$\mathcal{L}\left(\Psi(z+\delta,y,u) - \Psi(z,y,u)\right) + \left(\Phi(z+\delta,y,u) - \Phi(z,y,u)\right) = 0,$$

so that

$$\int_{\mathbb{R}^d} \left(\Phi(z+\delta,.,u) - \Phi(z,.,u)\right) p(y)dy = 0.$$

We deduce

$$\int_{\mathbb{R}^d} \frac{\partial \Phi}{\partial z_i}(z,y,u)p(y)dy = 0.$$

Moreover,we know that

$$\frac{\partial \Phi}{\partial z_i}(z,.,u) \in L^p(\mathbb{R}^d).$$

Therefore there exists $v_i(z,.,u) \in W^{2,p}(\mathbb{R}^d)$ such that

$$\mathcal{L}v_i(z,.,u) + \frac{\partial \Phi}{\partial z_i}(z,.,u) = 0.$$

Our aim is to prove that

$$\|\frac{1}{\delta}\left(\Psi(z+\delta,.,u) - \Psi(z,.,u)\right) - v_i(z,.,u)\|_{W^{2,p}} \to 0 \quad \text{as} \quad \delta \to 0.$$

We have

$$\int_0^M \int_{\mathbb{R}^d} \left(\frac{\Phi(z+\delta,y',u) - \Phi(z,y',u)}{\delta} - \frac{\partial \Phi}{\partial z_i}(z,y',u) \right) \left[p_t(y,dy') - \mu(dy')\right] dt$$

$$= I_1 + I_2,$$

where

$$I_1 = \int_0^M \int_{\mathbb{R}^d} \left(\frac{\Phi(z+\delta,y',u) - \Phi(z,y',u)}{\delta} - \frac{\partial \Phi}{\partial z_i}(z,y',u) \right) \left[p_t(y,dy')\right] dt,$$

$$I_2 = \int_M^\infty \int_{\mathbb{R}^d} \left(\frac{\Phi(z+\delta,y',u) - \Phi(z,y',u)}{\delta} - \frac{\partial \Phi}{\partial z_i}(z,y',u) \right) \left[p_t(y,dy') - \mu(dy')\right] dt.$$

With the help of the dominated convergence theorem we deduce $I_1 \to 0$ as $\delta \to 0$.

$$\begin{aligned} |I_2| &\leq \left| \int_M^\infty \int_{\mathbb{R}^d} \rho(z,y',u,\delta)[p_t(y,dy') - \mu(dy')]dt \right| \\ &\leq 2C \int_M^\infty var(p_t(y,.) - \mu) \\ &\leq \frac{C}{k}(1+|y|^m)\frac{1}{(1+M)^k} \to 0 \text{ when } M \to \infty. \end{aligned}$$

Finally we obtain

$$\lim_{\delta\to 0}\sup\left|\int_0^{\infty}\int_{\mathbb{R}^d}\left(\frac{\Phi(z+\delta,y^{'},u)-\Phi(z,y^{'},u)}{\delta}-\frac{\partial\Phi}{\partial z_i}(z,y^{'},u)\right)[p_t(y,dy^{'})-\mu(dy^{'})]dt\right|\to 0$$

from which we deduce the result.
For the differentiability of Φ with respect to u the proof is the same as above.

Cadre non-markovien

Chapitre 3

Homogenization in time stationary and in periodic space of random coefficients

Résumé : Dans le chapitre, on étudie un problème de moyennisation pour des opérateurs paraboliques, stationnaires avec des coefficients rapidement oscillants dans le cas d'un grand potentiel.
On considére l'EDP :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x)),$$

Sous l'hypothèse que les coefficients sont périodiques en espace, aléatoires, stationnaires en temps et qu'ils possèdent certaines propriétés de mélange, nous montrons que l'équation limite est une équation aux dérivées partielles à coefficients constants, obtenue par passage à la limite en loi.

Abstract : In this paper, we study the homogenization problem of a semilinear parabolic second order partial differential equation of the type

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(t,x) + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x)),$$

under the assumption that the coefficients are periodic in spatial variables and random stationary in time and that they possess certain mixing proprieties. We show that the structure of the limit problem depends crucially on the value of α.

3.1 Introduction

We study the homogenization problem for parabolic operators with a periodic spatial microstructure.We require that the characteristics of this microstructure constitute a stationary non-markov rapidly oscillating random process. In particular, we consider a model with a stationary driving process and we assume that the process possesses good mixing properties. Assuming that all the coefficients depend on time via a stationary non-markovian rapidly oscillating random process, we consider the corresponding Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t}(x,t) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(x,t) + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x)),$$
$$(t,x) \in (0,T) \times \mathbb{R}^n \quad u^\varepsilon(0,x) = u_0(x),$$

where ε is a small positive parameter and investigate the limit behavior of u^ε as $\varepsilon \to 0$.

Previously, parabolic equations of this type assuming that the process is markovian and ergodic were considered in Pardoux, Piatnitski [16], and in Diop, Pardoux [11]. In [16] and [11] it was shown that for divergence form operators the classical homogenization result holds true, that is the family of solutions to Cauchy problem for the original operators converges, as $\varepsilon \to 0$, to a solution of the corresponding homogenized Cauchy problem with constant nonrandom coefficients. The present work is very much inspired by the methods developped in [16] and [12]. In the paper we construct various correctors being usually solutions of auxiliary PDE problems, prove several a priori estimates and combine this technique with some ideas developed in [16], [12]. In the first section we start with the assumptions on the coefficients of the equation and then prove a number of auxiliary results. The following sections are devoted to the tightness and the passage to limit in cases $\alpha < 2$, and $\alpha > 2$ respectively. The structure of the limit depends crucially on α.

3.2 The Setup and statement of the main results

We study the asymptotic behaviors of solutions to the following Cauchy problem

$$\frac{\partial u^\varepsilon}{\partial t}(x,t) = \frac{\partial}{\partial x_i} a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}) \frac{\partial u^\varepsilon}{\partial x_j}(x,t) + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x)),$$
$$u^\varepsilon(0,x) = u_0(x) \in L^2(\mathbb{R}^n), \tag{3.1}$$

as $\varepsilon \downarrow 0$. The assumptions on the coefficients of the equation (3.1) are listed below.

3.2.1 Assumptions

A1 *Periodicity* . All the coefficients $a_{ij}(z,s)$ and $g(z,s,u)$ are periodic in z with period 1 in each coordinate direction.

A2 *Randomness.* $a_{ij}(.,s)$ and $g(.,s,u)$ are stationary random processes with values in $\mathbf{C}(\mathbf{T^n})$ defined on a probability space $(\Omega, \Im, \mathbf{P})$ for all $u \in \mathbb{R}$.
We denote by $\Im_s$ the corresponding filtration of σ-algebras, and suppose that $\Im_s$ is continuous and that the realizations of $a_{ij}(s)$ and $g(s,u)$ are almost surely (a.s) continuous.

A3 *Smoothness and growth conditions.* Uniformly in $s \in \mathbb{R}$ and $\omega \in \Omega$ the bounds hold

$$|a_{ij}(z,s)| \leq C, \tag{3.2}$$

$$|\nabla_z a_{ij}(z,s)| \leq C, \tag{3.3}$$

$$|g(z,s,u)| \leq C|u|, \tag{3.4}$$

$$|g'_u(z,s,u)| \leq C, \tag{3.5}$$

$$(1+|u|)|g''_{uu}(z,s,u)| \leq C, \tag{3.6}$$

here and afterwards C stands for a generic nonrandom constant.

A4 *Uniform ellipticity.* For some $c > 0$,

$$\sum_{i,j} a_{ij}(z,t)\eta_i\eta_j \geq c|\eta|^2, \qquad \forall \eta \in \mathbb{R}^{\mathbf{n}}.$$

A5 We assume that $t \longmapsto a(t,.)$ is a.s differentiable with values in $L^2(\mathbf{T^n})$ and morever there exist C such that

$$\|\frac{\partial}{\partial t} a(t,.)\|_{L^2(\mathbf{T^n})} \leq C.$$

A6 *Centering condition.* We assume that

$$\mathbf{E}\int_{\mathbf{T}^n} g(z,t,u)dz = 0, \ \forall u \in \mathbb{R}.$$

A7 *Mixing condition.* Let $\phi(t)$ be the uniform mixing coefficient defined by

$$\phi(t) = \sup_{A,B\in\Im_s} |P(A|B) - P(A)|.$$

We suppose that

$$\int_0^\infty \phi(s)ds < \infty.$$

Clearly, under these assumptions for each $\varepsilon > 0$ problem(3.1) has a unique solution u^ε and this solution is an element of $V_T := L^2(0,T;H^1(\mathbb{R}^n))\cap C([0,T];L^2(\mathbb{R}^n))$. Denote by Q^ε the law of u^ε in the space

$$\tilde{V}_T := L^2_w(0,T;H^1_w(\mathbb{R}^n)) \cap C([0,T];L^2_w(\mathbb{R}^n)),$$

where symbol w indicates that the corresponding space is equipped with its weak topology. Our first aim is to show that the family $\{Q^\varepsilon\}$ is tight in V.

It is convenient to decompose $g(z,t,u)$ as follows

$$g(z,t,u) = \tilde{g}(z,t,u) + \bar{g}(t,u),$$

where

$$\bar{g}(t,u) = \int_{\mathbf{T^n}} g(z,t,u)dz.$$

Then $\tilde{g}(z,t,u)$ admits the representation

$$\tilde{g}(z,t,u) = \mathrm{div}_z \tilde{G}(z,t,u), \tag{3.7}$$

where $\tilde{G}(z,t,u) = \nabla\Theta(z,t,u)$ and $\Theta(z,t,u)$ is a periodic solution of the equation

$$\Delta_z\Theta(z,t,u) = \tilde{g}(z,t,u).$$

The function $\tilde{G}(z,t,u)$ satisfies estimates (3.4) and (3.5) and for any $u(x,t)$ we have now

$$\begin{gathered}\frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} g(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t,x)) = \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathrm{div}_x\tilde{G}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)) \\ -\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\tilde{G}'_u(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(x,t))\nabla_x u^\varepsilon(t,x) + \varepsilon^{-(1\wedge\frac{\alpha}{2})}\bar{g}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)).\end{gathered} \tag{3.8}$$

We define χ^k, $k = 1,\cdots,n$ and $\tilde{H}$ as stationary solutions of

$$\frac{\partial}{\partial s}\chi^k(z,s) + \mathrm{div}\left[a(z,s)\nabla\chi^k(z,s)\right] = -\frac{\partial}{\partial z_i}a_{ik}(z,s), \qquad (z,s)\in\mathbf{T^n}\times(-\infty,+\infty), \tag{3.9}$$

$$\frac{\partial}{\partial s}\tilde{H}(z,s,u)(z,s) + \mathrm{div}\left[a(z,s)\nabla\tilde{H}(z,s,u)\right] = -\tilde{g}(z,s,u), \qquad (z,s)\in\mathbf{T^n}\times(-\infty,+\infty). \tag{3.10}$$

where u is a real parameter, see Lemma 3.5.2 below in Section 3.5. Next, we introduce the stationary process

$$\bar{G}(t,u) = \int_0^\infty \mathbf{E}^{\Im_t}[\bar{g}(s+t,u)]ds = \int_t^\infty \mathbf{E}^{\Im_t}[\bar{g}(s,u)]ds, \tag{3.11}$$

where $\mathbf{E}^{\Im_t}$ denotes conditional expectation given $\Im_t$ here and below. Using Proposition 7.2.6 in [12], the boundedness of $\bar{g}$ and the fact that $\mathbf{E}[\bar{g}(t,u)] = 0$, wich follows from the assumption **A.6** we deduce the estimate

$$\mathbf{E}^{\Im_t}[\bar{g}(s+t,u)] \leq 2C|u|\phi(s).$$

According to [12], under the assumptions **A.6** and **A.7** , the process $\bar{G}(t,u)$ is well-defined and, moreover,

$$|\bar{G}(t,u)| \leq C|u|, \ |\bar{G}_u^{'}(t,u)| \leq C, \ (1+|u|)|\bar{G}_{uu}^{'}(t,u)| \leq C. \tag{3.12}$$

3.2.2 Notations

- In $\mathbb{R}^n$; $x.x^{'}$ will denote the scalar product and $|.|$ the corresponding norm.

• In the space $L^2(\mathbb{R}^n)$, $(.,.)$ will denote the inner product, and $\|.\|$ the norm.
• For a function or process $(t,x) \longmapsto u(t,x), u(t)$ will denote the application $x \longmapsto u(t,x)$. Hence $\|u(t)\| = \left(\int_{\mathbb{R}^n} |u(t,x)|^2 dx\right)^{\frac{1}{2}}$. This notation is also used for $u^\varepsilon(t,x)$ and for the gradient $\nabla u^\varepsilon(t,x)$. We use, as well, the contracted notation :

$$a^\varepsilon \ = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}) \text{ and for a generic function } g(z,y,u) : g^\varepsilon = g(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u).$$

3.2.3 Auxiliary results

First all let us prove the following lemmas

Lemma 3.2.1. *For each $u \in \mathbb{R}$, the process*

$$M_t = \bar{G}(t,u) + \int_0^t \bar{g}(s,u)ds$$

is a martingale with respect to $\Im_t$.

Proof : This statement is a consequence of Proposition 2.7.6 from [12]. Let us verify that the conditions of the cited proposition are fulfiled. We should show that

$$\mathbf{P} - \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbf{E}^{\Im_t}[\bar{G}(t+\delta,u) - \bar{G}(t,u)] = -\bar{g}(t,u) \qquad \text{t a.s} \tag{3.13}$$

and

$$\frac{1}{\delta} \mathbf{E}^{\Im_t}[\bar{G}(t+\delta,u) - \bar{G}(t,u)] \text{ is uniformly integrable} \tag{3.14}$$

Define $A_\delta(t) = \frac{1}{\delta}\mathbf{E}^{\Im_t}[\bar{G}(t+\delta,u) - \bar{G}(t,u)]$.
From (3.11) we have the following relation

$$
\begin{aligned}
[A_\delta(t)] &= \frac{1}{\delta}\mathbf{E}^{\Im_t}\left[\left(\int_0^\infty \mathbf{E}^{\Im_{t+\delta}}\left[\bar{g}\left(s+t+\delta,u\right]\right)ds\right)\right] \qquad (3.15)\\
&\quad -\frac{1}{\delta}\int_0^\infty \mathbf{E}^{\Im_t}\left[\bar{g}\left(t+s,u\right)\right]ds\\
&= -\frac{1}{\delta}\int_0^\delta \mathbf{E}^{\Im_t}\left[\bar{g}\left(t+s,u\right)\right])ds. \qquad (3.16)
\end{aligned}
$$

Since $\bar{g}(t,u)$ is uniformly bounded for each u and has continuous trajectories, this quantity converges towards $-\bar{g}(t,u)$ for any t.
We next observe that $\{A_\delta(t) : 0 < \delta \leq 1\}$ is uniformly integrable for each $t > 0$, since $A_\delta(t) \leq C|u|$.

Lemma 3.2.2. *For any* $\varphi \in C_0^\infty(\mathbb{R}^{\mathbf{n}})$, $\varepsilon > 0$ *the functions*

$$
\begin{aligned}
M_t^u &:= \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}[(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),u^\varepsilon(t)) - (\bar{G}(0,u^\varepsilon(0)),u^\varepsilon(0))] + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}\int_0^t (\bar{g}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s))ds\\
&+ \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}\int_0^t (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s) + 2\bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla u^\varepsilon(s)))ds\\
&- \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t (g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)))ds
\end{aligned}
$$

and

$$
\begin{aligned}
M_t^\varphi &:= \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}[(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi) - (\bar{G}(0,u^\varepsilon(0)),\varphi)] + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}\int_0^t (\bar{g}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\varphi)ds\\
&+ \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}\int_0^t (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla u^\varepsilon(s)\varphi + \bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla\varphi)ds\\
&- \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t (g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\varphi)ds
\end{aligned}
$$

are $\{\Im_{\frac{t}{\varepsilon^\alpha}}\}$*- martingales.*

Proof : Fix $t_2 > t_1 \geq 0.$ For any partition $t_1 = s_0 < s_1 < s_2 < \cdots < s_n = t_2$ we have

$$
\begin{aligned}
&\mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[(\bar{G}(\frac{t_2}{\varepsilon^\alpha}, u^\varepsilon(t_2)), u^\varepsilon(t_2)) - (\bar{G}(\frac{t_1}{\varepsilon^\alpha}, u^\varepsilon(t_1)), u^\varepsilon(t_1))\right] \\
&= \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[\Big\{\sum_{k=0}^{n-1}(\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s_{k+1})), u^\varepsilon(s_{k+1})) - (\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k))\Big\}\right] \\
&= \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[\Big\{\sum_{k=0}^{n-1}(\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s_{k+1})), u^\varepsilon(s_{k+1})) - (\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k))\Big\}\right] \\
&+ \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[\Big\{\sum_{k=0}^{n-1}(\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k)) - (\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k))\Big\}\right] \\
&= \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\Big[\sum_{k=0}^{n-1}\int_{s_k}^{s_{k+1}} -(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s))ds\Big] \\
&+ \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\Big[\sum_{k=0}^{n-1}\int_{s_k}^{s_{k+1}} -(a^\varepsilon(s)\nabla u^\varepsilon(s), 2\bar{G}'_u(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)))ds\Big] \\
&+ \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[\sum_{k=0}^{n-1}\int_{s_k}^{s_{k+1}} \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}(g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)))ds\right] \\
&- \frac{1}{\varepsilon^\alpha}\mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\Big[\sum_{k=0}^{n-1}\int_{s_k}^{s_{k+1}} \Big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k)\Big)\, ds\Big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\Big[(\bar{G}(\frac{t_2}{\varepsilon^\alpha}, u^\varepsilon(t_2)), u^\varepsilon(t_2)) - (\bar{G}(\frac{t_1}{\varepsilon^\alpha}, u^\varepsilon(t_1)), u^\varepsilon(t_1)) \\
&+ \int_{t_1}^{t_2}(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s) + 2\bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)))ds \\
&- \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}\int_{t_1}^{t_2}(g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)))ds \\
&+ \frac{1}{\varepsilon^\alpha}\int_{t_1}^{t_2}(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s))ds\Big] \\
&= \mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\Big[\sum_{k=0}^{n-1}\int_{s_k}^{s_{k+1}} -(a^\varepsilon(s)\nabla u^\varepsilon(s), \nabla u^\varepsilon(s)u^\varepsilon(s)\mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}''_{uu}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))])ds
\end{aligned}
$$

$$+ \quad \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} (-2a^\varepsilon(s)\nabla u^\varepsilon(s), \nabla u^\varepsilon(s) u^\varepsilon(s) \mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}''_{uu}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))])ds$$

$$+ \quad \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \sum_{k=0}^{n-1} \int_{s_k}^{s_{k+1}} (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s) \mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}'_u(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))])ds$$

$$+ \quad \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \sum_{k=0}^{n-1} \int_{s_k}^{t_{k+1}} (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))])ds$$

$$- \quad \sum_{k=0}^{n-1} \frac{1}{\varepsilon^\alpha} \int_{s_k}^{s_{k+1}} [(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k)) - (\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s))]ds\Big].$$

We have the following estimates

$$\|\mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))]\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon)\frac{(s_{k+1}-s)}{\varepsilon^\alpha},$$

$$\|\mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}'_u(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))]\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon)\frac{(s_{k+1}-s)}{\varepsilon^\alpha},$$

$$\|\mathbf{E}^{\Im_{\frac{s}{\alpha}}}[\bar{G}''_{uu}(\frac{s_{k+1}}{\varepsilon^\alpha}, u^\varepsilon(s)) - \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))]\|_{L^2(\mathbb{R}^n)} \le C(\varepsilon)\frac{(s_{k+1}-s)}{\varepsilon^\alpha}.$$

Letting $\max |s_{k+1} - s_k| \to 0$, we derive from the above estimates

$$\mathbf{E}^{\Im_{\frac{t_1}{\alpha}}}\left[(\bar{G}(\frac{t_2}{\varepsilon^\alpha}, u^\varepsilon(t_2)), u^\varepsilon(t_2)) - (\bar{G}(\frac{t_1}{\varepsilon^\alpha}, u^\varepsilon(t_1)), u^\varepsilon(t_1))\right]$$

$$= \mathbf{E}^{\Im_{\frac{t_1}{\varepsilon^\alpha}}}[\int_{t_1}^{t_2} (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s) u^\varepsilon(s) + 2\bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)))ds,$$

$$+\frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \int_{t_1}^{t_2} (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)) u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)))ds$$

$$-\frac{1}{\varepsilon^\alpha}\mathbf{E}^{\Im_{\frac{t_1}{\varepsilon^\alpha}}}\left[\int_{t_1}^{t_2} (\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s))ds\right].$$

The argument used to show that M_t^u is a $\Im_{\frac{t}{\varepsilon^\alpha}}$–martingale now applies, almost word-forword, to show that M_t^φ is a $\Im_{\frac{t}{\varepsilon^\alpha}}$–martingale .

3.3 Main results

Here we formulate the main results of the paper ; the proof will be given in the following section.

It should be noted that for $\alpha \leq 2$ we obtain the weak convergence of the law of $u^\varepsilon(t,x)$ towards the non trivial limit law which solves a proper martingale problem,while for $\alpha > 2$, the limit law is a Dirac measure concentrated on the solution of the Cauchy problem for the limit deterministic parabolic equation with constant coefficients.

The space $\tilde{V}_T$ is a Lusin and completely regular space, see Viot [17]. We denote by $\tilde{\Im}$ its Borel σ-field. For any $\varepsilon > 0$, let Q^ε be the Radon probability measure on $(\tilde{V}_T, \tilde{\Im})$, which coincides with the law of $\{u^\varepsilon(t); 0 \leq t \leq T\}$. The asymptotic behavior of the solution $u^\varepsilon(t)$, as $\varepsilon \to 0$, depends on whether $\alpha < 2$, or $\alpha > 2$. The main results of the paper are summarized in the following theorems.

Theorem 3.3.1. *Let $\alpha < 2$, then under the assumptions* **A1**-**A7**, *for all $T > 0$ the family of laws of the solutions $\{u^\varepsilon\}$ to problem(3.1) converges weakly, as $\varepsilon \to 0$, in the space $\tilde{V}_T$, to the unique solution of the martingale problem with the drift $\hat{A}(u(s))$, where*

$$\hat{A}(u) = \mathrm{div}(\hat{\mathbf{a}}\nabla u) + \hat{\mathbf{g}}(u),$$

$$\hat{\mathbf{a}} = \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z \chi(z,s))dz,$$

$$\hat{\mathbf{g}}(u) = \mathbf{E}\int_{\mathbf{T^n}} \bar{G}'_u(s,u)g(z,s,u)dz,$$

and the covariance $R(u(s))$, where

$$\begin{aligned}(R(u)\varphi,\varphi) &= 2\mathbf{E}\int_{R^n}\int_{\mathbb{R}^n}(\bar{G}(s,u(x))\varphi(x)\bar{g}(s,u(y))\varphi(y)dxdy \\ &= 2\mathbf{E}[(\bar{G}(s,u(.))\varphi)(\bar{g}(s,u(.)),\varphi)],\end{aligned}$$

where the functions $\chi^k(.,s) \in H^1(\mathbf{T^n})$, are the solutions of the equations :

$$A\chi^k(z,s) = -\sum_i \frac{\partial}{\partial z_i} a_{ik}(z,s), \quad 1 \leq k \leq n. \tag{3.17}$$

Theorem 3.3.2. *Let $\alpha > 2$, then under the assumptions* **A1**-**A7**, *for all $T > 0$ the family of laws of the solutions $\{u^\varepsilon\}$ to problem(3.1) converges in probability in the space $\tilde{V}_T$, to the solution of the following limit Cauchy problem :*

$$\frac{du(t,x)}{dt} = \mathrm{div}(\hat{\mathbf{a}}\nabla u(t,x)) + \mathrm{div}\hat{\mathbf{b}}(u), \qquad u^\varepsilon(0,x) = u_0(x)$$

with $(t,x) \in (0,T) \times \mathbb{R}^n$ and

$$\hat{\mathbf{a}} = \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z \chi(z,s))dz,$$

$$\hat{\mathbf{b}}(u) = -\mathbf{E}\int_{\mathbf{T^n}} a(z,s)\nabla_z \tilde{H}(z,s,u)dz,$$

where the functions χ *and* $\tilde{H}$ *are stationary solutions of the equations :*

$$\frac{\partial}{\partial s}\chi(z,s) + \operatorname{div}\left[a(z,s)\nabla\chi(z,s)\right] = -\operatorname{div}a(z,s), \qquad (z,s)\in\mathbf{T^n}\times(-\infty,+\infty) \tag{3.18}$$

$$\frac{\partial}{\partial s}\tilde{H}(z,s,u)(z,s) + \operatorname{div}\left[a(z,s)\nabla\tilde{H}(z,s,u)\right] = -\tilde{g}(z,s,u), \qquad (z,s)\in\mathbf{T^n}\times(-\infty,+\infty). \tag{3.19}$$

We proceed with a priori estimates.

3.4 A priori estimates and tightness

In this section we obtain uniform a priori estimates for the solution u^ε and then use them in order to show tightness of the distributions of u^ε.

Denote $\bar{G}^\varepsilon(t) = \bar{G}(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t))$. By applying Itô's formula to the expression

$$\Phi^\varepsilon(u^\varepsilon(t),t) = \frac{1}{2}(u^\varepsilon(t),u^\varepsilon(t)) + \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t)),$$

where $\rho = \alpha - (1\wedge\frac{\alpha}{2})$, we get

$$\begin{aligned}
d\left[\Phi^\varepsilon(u^\varepsilon(t),\frac{t}{\varepsilon^\alpha})\right] &= (A^\varepsilon u^\varepsilon(t),u^\varepsilon(t))dt + \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}(g^\varepsilon(t),u^\varepsilon(t))dt \\
&+ \varepsilon^\rho(A^\varepsilon u^\varepsilon(t),\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)))dt + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)))dt \\
&+ \varepsilon^\rho(A^\varepsilon u^\varepsilon(t),\bar{G}'_u(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))u^\varepsilon(t))dt + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\bar{G}'_u(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))u^\varepsilon(t))dt \\
&- \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}(\bar{g}^\varepsilon(t),u^\varepsilon(t))dt + dM^u_t.
\end{aligned} \tag{3.20}$$

We first prove the

Proposition 3.4.1. *Under our standing assumptions, there exists a constant* C *such that*

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla_x u^\varepsilon(t)\|^2 dt\right) \le C.$$

Proof : Integrating by parts and using the representation (3.7), one gets

$$\begin{aligned}
&\mathbf{E}\Phi^\varepsilon(u^\varepsilon(t),\frac{t}{\varepsilon^\alpha})+\mathbf{E}\int_0^t(a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\nabla_x u^\varepsilon(s))ds\\
=\;& \mathbf{E}\left[\frac{1}{2}(u^\varepsilon(0),u^\varepsilon(0))+\varepsilon^\rho(\bar G^\varepsilon(0),u^\varepsilon(0))\right]-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t(\tilde G^\varepsilon(s),\nabla_x u^\varepsilon(s))ds\\
-\;& \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t(\tilde G_u^{\prime,\varepsilon}(s)\nabla_x u^\varepsilon(s),u^\varepsilon(s))ds\\
-\;& 2\varepsilon^\rho\mathbf{E}\int_0^t(a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\bar G_u^{\prime,\varepsilon}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds\\
+\;& \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\mathbf{E}\int_0^t(g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\bar G^\varepsilon(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))+\bar G_u^{\prime,\varepsilon}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s))ds\\
&-\varepsilon^\rho\mathbf{E}\int_0^t(a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),u^\varepsilon(s)\bar G_{uu}^{\prime\prime,\varepsilon}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds. \qquad (3.21)
\end{aligned}$$

The first two integrals on the r.h.s. of (3.21) can be estimated as follows

$$\begin{aligned}
&\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Big|\mathbf{E}\int_0^t(\tilde G^\varepsilon(s),\nabla_x u^\varepsilon(s))ds\\
&+\;\mathbf{E}\int_0^t(\tilde G_u^{\prime,\varepsilon}(s)\nabla_x u^\varepsilon(s),u^\varepsilon(s))ds\Big|\\
&\le\; 2c\mathbf{E}\int_0^t\|u^\varepsilon(s)\|\,\|\nabla_x u^\varepsilon(s)\|ds\\
&\le\; \frac{c}{\gamma}\mathbf{E}\int_0^t\|u^\varepsilon(s)\|^2ds+c\gamma\mathbf{E}\int_0^t\|\nabla_x u^\varepsilon(s)\|^2ds.
\end{aligned}$$

The two terms involving the factor ε^ρ in (3.21) are dominated by the corresponding terms on the l.h.s., and taking sufficiently small γ, we have

$$\mathbf{E}\|u^\varepsilon(t)\|^2+\mathbf{E}\int_0^t\|\nabla_x u^\varepsilon(s)\|^2ds\le C_1+C_2\mathbf{E}\int_0^t\|u^\varepsilon(s)\|^2ds. \qquad (3.22)$$

An application of the Gronwall lemma now yields

$$\mathbf{E}\|u^\varepsilon(t)\|^2+\mathbf{E}\int_0^t\|\nabla_x u^\varepsilon(s)\|^2ds\le C,\qquad t\le T. \qquad (3.23)$$

We then deduce from the Davis-Burkholder–Gundy inequality that

$$\mathbf{E}\sup_{0\le t\le T}|M^u_t| \le C\mathbf{E}\left[< M^u >_T^{1/2}\right]. \tag{3.24}$$

In order to estimate $< M^u >_T$, we apply Itô's formula to the expression

$$\tilde{\Psi}^\varepsilon(u^\varepsilon(t),t) = \left[\frac{1}{4}(u^\varepsilon(t),u^\varepsilon(t))^2 + \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))\right],$$

where $\rho = \alpha - (1\wedge\frac{\alpha}{2})$, we get,

$$d\left[\frac{1}{4}(u^\varepsilon(t),u^\varepsilon(t))^2 + \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))\right] = -(a^\varepsilon\nabla u^\varepsilon(t),\nabla u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt$$

$$- \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}^\varepsilon(t),\nabla u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t),u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt$$

$$- 2\varepsilon^\rho(\bar{G}'^{,\varepsilon}_u(t)a^\varepsilon\nabla u^\varepsilon(t),\nabla u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt + \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar{G}'^{,\varepsilon}_u(t)g^\varepsilon(t),u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt$$

$$+ \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar{G}^\varepsilon(t),g^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt - \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))(a^\varepsilon\nabla u^\varepsilon(t),\nabla u^\varepsilon(t))dt + dN^u_t$$

$$+ \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar{G}^\varepsilon(t),u^\varepsilon(t))(g^\varepsilon(t),u^\varepsilon(t))dt - \varepsilon^\rho(\bar{G}''^{,\varepsilon}_{uu}(t)a^\varepsilon\nabla u^\varepsilon(t),u^\varepsilon(t)\nabla u^\varepsilon(t))(u^\varepsilon(t),u^\varepsilon(t))dt,$$

where $N^u_t = \int_0^t(u^\varepsilon(s),u^\varepsilon(s))dM^u_s$ is a martingale. On the other hand, we deduce from (3.20) and Itô's formula that

$$d\left[\frac{1}{2}(u^\varepsilon(t),u^\varepsilon(t)) + \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))\right]^2 =$$

$$\left[(u^\varepsilon(t),u^\varepsilon(t)) + 2\varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))\right]\Big\{ - (a^\varepsilon\nabla u^\varepsilon(t),\nabla u^\varepsilon(t))dt$$

$$- \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}^\varepsilon(t),\nabla\varphi)dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde{G}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t),u^\varepsilon(t))dt$$

$$- 2\varepsilon^\rho(\bar{G}'^{,\varepsilon}_u(t)a^\varepsilon\nabla u^\varepsilon(t),\nabla u^\varepsilon(t))dt - \varepsilon^\rho(\bar{G}''^{,\varepsilon}_{uu}(t)a^\varepsilon\nabla u^\varepsilon(t),u^\varepsilon(t)\nabla u^\varepsilon(t))dt$$

$$+ \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar{G}'^{,\varepsilon}_u(t)g^\varepsilon(t),u^\varepsilon(t))dt + \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar{G}^\varepsilon(t),g^\varepsilon(t))dt + dM^u_t\Big\} + d< M^u >_t,$$

where $< M^u >_t$ stands for the quadratic characteristics of M^u. Comparing two previous expressions allow us to write down the formula for $< M^u >_t$:

$$
\begin{aligned}
< M^u >_t &= \varepsilon^{2\rho}(\bar{G}^\varepsilon(t), u^\varepsilon(t))^2 - \varepsilon^{2\rho}(\bar{G}^\varepsilon(0), u^\varepsilon(0))^2 \\
&+ 2\varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}\int_0^t (\bar{G}^\varepsilon(s), u^\varepsilon(s))(g^\varepsilon(s), u^\varepsilon(s))ds \\
&+ 2\varepsilon^{\rho+1-(1\wedge\frac{\alpha}{2})}\int_0^t \Big\{(\tilde{G}^\varepsilon(s), \nabla u^\varepsilon(s)) + (\tilde{G}_u^{\prime,\varepsilon}(s)\nabla u^\varepsilon(s), u^\varepsilon(s))\Big\}(\bar{G}^\varepsilon(s), u^\varepsilon(s))ds \\
&- 2\varepsilon^{2\rho-(1\wedge\frac{\alpha}{2})}\int_0^t \Big\{(\bar{G}_u^{\prime,\varepsilon}(s)g^\varepsilon(s), u^\varepsilon(s)) + (\bar{G}^\varepsilon(s), g^\varepsilon(s))\Big\}(\bar{G}^\varepsilon(s), u^\varepsilon(s))ds \\
&+ 4\varepsilon^{2\rho}\int_0^t (\bar{G}_u^{\prime,\varepsilon}(s)a^\varepsilon\nabla u^\varepsilon(s), \nabla u^\varepsilon(s))(\bar{G}^\varepsilon(s), u^\varepsilon(s))ds \\
&+ 2\varepsilon^{2\rho}\int_0^t (\bar{G}_{uu}^{\prime\prime,\varepsilon}(s)a^\varepsilon\nabla u^\varepsilon(s), u^\varepsilon(s)\nabla u^\varepsilon(s))(\bar{G}^\varepsilon(s), u^\varepsilon(s))ds \\
&- 2\varepsilon^{\rho}\int_0^t (\bar{G}^\varepsilon(s), u^\varepsilon(s))dM_s^u.
\end{aligned}
$$

Finally, we obtain the relation

$$
\begin{aligned}
< M^u >_T \leq C\Big[&\|u^\varepsilon(T)\|^4 + \|u^\varepsilon(0)\|^4 + \int_0^T \|u^\varepsilon(t)\|^4 dt \\
&+ \int_0^T \|u^\varepsilon(t)\|^2\|\nabla_x u^\varepsilon(t)\|^2 dt + 2\varepsilon^\rho\int_0^T (\bar{G}^\varepsilon(t), u^\varepsilon(t))dM_t^u\Big], \qquad (3.25)
\end{aligned}
$$

where C is a constant whose value could be changing from line to line. Taking the expectation in (3.25) one obtain :

$$
\begin{aligned}
\mathbf{E} < M^u >_T \leq C\Big[&\mathbf{E}\|u^\varepsilon(T)\|^4 + \mathbf{E}\|u^\varepsilon(0)\|^4 + \mathbf{E}\int_0^T \|u^\varepsilon(t)\|^4 dt \\
&+ \mathbf{E}\int_0^T \|u^\varepsilon(t)\|^2\|\nabla_x u^\varepsilon(t)\|^2 dt\Big]. \qquad (3.26)
\end{aligned}
$$

Let us verify that the terms on the r.h.s. of (3.26), are finite. To this end we come back to $\tilde{\Psi}^\varepsilon(u^\varepsilon(t), t)$, with similar arguments as those leading from (3.21) to (3.22) we deduce,

$$
\mathbf{E}\|u^\varepsilon(t)\|^4 + \mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2\|\nabla_x u^\varepsilon(s)\|^2 ds \leq C_3 + C_4\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^4 ds. \qquad (3.27)
$$

From (3.27), we derive, using Gronwall lemma :

$$\sup_{0\le t\le T} \mathbf{E}(\|u^\varepsilon(t)\|^4) \le k_1 \tag{3.28}$$

and reporting (3.28) in (3.27) :

$$\mathbf{E}\int_0^T \|u^\varepsilon(t)\|^2\|\nabla_x u^\varepsilon(t)\|^2 dt \le k_2.$$

It then follows that

$$\mathbf{E} < M^u >_T \le C.$$

Thanks to the Schwartz inequality

$$\mathbf{E} < M^u >_T^{\frac{1}{2}} \le \Big[\mathbf{E} < M^u >_T\Big]^{\frac{1}{2}}.$$

Collecting the results we have obtained it follows that :

$$\mathbf{E} < M^u >_T^{\frac{1}{2}} \le k_3. \tag{3.29}$$

Now comming back to (3.20) we obtain :

$$\begin{aligned}
&\left[\frac{1}{2}(u^\varepsilon(t),u^\varepsilon(t)) + \varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))\right] + \int_0^t (a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\nabla_x u^\varepsilon(s))ds \\
= \;& \left[\frac{1}{2}(u^\varepsilon(0),u^\varepsilon(0)) + \varepsilon^\rho(\bar{G}^\varepsilon(0),u^\varepsilon(0))\right] - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\int_0^t (\tilde{G}(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\nabla_x u^\varepsilon(s))ds \\
- \;& \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\int_0^t (\tilde{G}'_u(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s),u^\varepsilon(s))ds \\
- \;& 2\varepsilon^\rho\int_0^t (a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds \\
+ \;& \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t (g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\bar{G}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)) + \bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s))ds \\
- \;& \varepsilon^\rho\int_0^t (a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),u^\varepsilon(s)\bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))ds + M^u_t.
\end{aligned}$$

There exists $c>0$ such that the following estimate hold

$$\varepsilon^\rho(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),u^\varepsilon(t)) \quad\le\quad c\varepsilon^\rho|u|^2.$$

We choose ε_0 such that

$$\varepsilon_0^\rho c = \frac{1}{2}.$$

Making use of (3.4), (3.12), and standard inequalities it then follows :

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\right)+\gamma\mathbf{E}\int_0^T\|\nabla_x u^\varepsilon(t)\|^2dt \le k_4+\gamma_1\int_0^T\mathbf{E}\|u^\varepsilon(t)\|^2dt$$

$$+\mathbf{E}\left(\sup_{0\le t\le T}|M_t^u|\right). \tag{3.30}$$

Collecting the results we have obtained, we derive from the above inequality :

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\right)+\gamma\mathbf{E}\int_0^T\|\nabla_x u^\varepsilon(t)\|^2dt \quad\le\quad k_5.$$

Similarly it can be shown that

$$\mathbf{E}\left(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^4+\left(\int_0^T\|\nabla_x u^\varepsilon(t)\|^2dt\right)^2\right)\le C. \tag{3.31}$$

Indeed, it suffices to apply Itô's formula to the expression

$$\|u^\varepsilon(t)\|^2\left(\|u^\varepsilon(t)\|^2+\gamma\int_0^t\|\nabla_x u^\varepsilon(s)\|^2ds\right)$$

$$+\varepsilon^\rho(\bar{G}^\varepsilon(t),u^\varepsilon(t))\left(4\|u^\varepsilon(t)\|^2+2\gamma\int_0^t\|\nabla_x u^\varepsilon(s)\|^2ds\right),$$

and make the same rearrangements as above. $\diamondsuit$

We next establish the (here and in the rest of the paper $C_0^\infty(\mathbb{R}^n)$ denotes the class of mappings from $\mathbb{R}^n$ into $\mathbb{R}$, which are of class C^∞, and have compact support)

Proposition 3.4.2. *For any $\varphi\in C_0^\infty(\mathbb{R}^n)$, the collection of processes $\{(u^\varepsilon,\varphi),\ \varepsilon>0\}$ is tight in $C([0,T])$.*

Proof : Fix $\varphi\in C_0^\infty(\mathbb{R}^n)$. We consider the random process

$$\Phi^{\varepsilon,\varphi}(t)=(u^\varepsilon(t),\varphi)+\varepsilon^\rho(\bar{G}^\varepsilon(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi).$$

Applying the Itô's formula to develop $\Phi^{\varepsilon,\varphi}(t)$, we deduce that

$$
\begin{aligned}
d[(u^\varepsilon(t),\varphi)+\varepsilon^\rho(\bar G^\varepsilon(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi)] &= -(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi)dt \\
- &\ \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G^\varepsilon(t),\nabla\varphi)dt-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G_u^{\prime,\varepsilon}(t)\nabla u^\varepsilon(t),\varphi)dt \\
- &\ \varepsilon^\rho(\bar G_u^{\prime,\varepsilon}(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi)dt-\varepsilon^\rho(\bar G_u^{\prime\prime,\varepsilon}(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\varphi\nabla u^\varepsilon(t))dt \\
+ &\ \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(\bar G_u^{\prime,\varepsilon}(t)g^\varepsilon(t),\varphi)dt+dM_t^\varphi.
\end{aligned}
$$

Let

$$
I_1^\varepsilon(t)=-\int_0^t\Big[(a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\nabla_x\varphi)+\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G_u'(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s),\varphi)\Big]\,ds,
$$

$$
I_2^\varepsilon(t)=\int_0^t\Big[\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}(g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\bar G_u'(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\varphi)-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\nabla_x\varphi,\tilde G(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)))\Big]\,ds,
$$

and

$$
\begin{aligned}
I_3^\varepsilon(t)=\varepsilon^\rho&\left[(\bar G(t_0,u^\varepsilon(0)),\varphi)-(\bar G(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi)\right] \\
-\varepsilon^\rho&\int_0^t\big[(a(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha})\nabla_x\varphi,\bar G_u'(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s)) \\
+(a(\frac{\cdot}{\varepsilon},&\frac{s}{\varepsilon^\alpha})\nabla_x u^\varepsilon(s),\varphi\bar G_{uu}''(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s))\big]ds.
\end{aligned}
$$

We first note that

$$
\begin{aligned}
|I_3^\varepsilon(t)| &\le C\varepsilon^\rho\big[1+\|u^\varepsilon(t)\|+\|u_0\| \\
&\quad+\int_0^t(\|\nabla_x u^\varepsilon(s)\|+\|\nabla_x u^\varepsilon(s)\|^2)ds\big] \\
&\le C\varepsilon^\rho\left(\sup_{0\le s\le T}\|u^\varepsilon(s)\|+\int_0^T(1+\|\nabla_x u^\varepsilon(s)\|^2)ds\right).
\end{aligned}
$$

It then follows from Proposition 3.4.1 that

$$
\sup_{0\le s\le T}|I_3^\varepsilon(s)|\to 0 \text{ in probability, as } \varepsilon\to 0.
$$

We note that for $0\le s\le t\le T$,

$$
|I_1^\varepsilon(t)-I_1^\varepsilon(s)|\le c\sqrt{t-s}\|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))}.
$$

But from Proposition 3.4.1, for any $\eta, \delta > 0$, one can choose $\beta > 0$ such that for all $\varepsilon > 0$,

$$\mathbf{P}\left(\sup_{|t-s|<\beta} |I_1^\varepsilon(t) - I_1^\varepsilon(s)| > \eta\right) \le \delta,$$

and the collection of continuous processes $\{I_1^\varepsilon,\, \varepsilon > 0\}$ is tight.

We have

$$|I_2^\varepsilon(t) - I_2^\varepsilon(s)| \le c\sqrt{t-s}\|u^\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^n))}.$$

The tightness of the collection $\{I_2^\varepsilon,\, \varepsilon > 0\}$ now follows from Proposition 3.4.1.

It remains to consider the martingale term. In order to estimate the modulus of continuity of M_t^φ we calculate

$$\begin{aligned} d\left[(u^\varepsilon(t),\varphi)^2 + 2\varepsilon^\rho(\bar G^\varepsilon(t),\varphi)(u^\varepsilon(t),\varphi)\right] &= -2(a^\varepsilon\nabla u^\varepsilon(t),\nabla\varphi)(u^\varepsilon(t),\varphi)dt \\ - & \; 2\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G^\varepsilon(t),\nabla\varphi)(u^\varepsilon(t),\varphi)dt - 2\varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G_u^{',\varepsilon}(t)\nabla u^\varepsilon(t),\varphi)(u^\varepsilon(t),\varphi)dt \\ - & \; 2\varepsilon^\rho(\bar G_u^{',\varepsilon}(t)a^\varepsilon\nabla u^\varepsilon(t),\nabla\varphi)(u^\varepsilon(t),\varphi)dt - 2\varepsilon^\rho(\bar G_{uu}^{'',\varepsilon}(t)a^\varepsilon\nabla u^\varepsilon(t),\varphi\nabla u^\varepsilon(t)(u^\varepsilon(t),\varphi))dt \\ + & \; 2\varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar G_u^{',\varepsilon}(t)g^\varepsilon(t),\varphi)(u^\varepsilon(t),\varphi))dt - 2\varepsilon^\rho(\bar G^\varepsilon(t),\varphi)(a^\varepsilon\nabla u^\varepsilon(t),\nabla\varphi)dt \\ + & \; 2\varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar G^\varepsilon(t),\varphi)(g^\varepsilon(t),\varphi)dt + dN_t^\varphi, \end{aligned}$$

where $N_t^\varphi = 2\int_0^t (u^\varepsilon(s),\varphi)dM_s^\varphi$ is a martingale. On the other hand, by the Itô's formula we find

$$\begin{aligned} d\left[(u^\varepsilon(t),\varphi) + \varepsilon^\rho(\bar G^\varepsilon(t),\varphi)\right]^2 &= 2\left[(u^\varepsilon(t),\varphi) + \varepsilon^\rho(\bar G^\varepsilon(t),\varphi)\right]\Big\{ -(a^\varepsilon\nabla u^\varepsilon(t),\nabla\varphi)dt \\ - & \; \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G^\varepsilon(t),\nabla\varphi)dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}(\tilde G_u^{',\varepsilon}(t)\nabla u^\varepsilon(t),\varphi)dt \\ - & \; \varepsilon^\rho(\bar G_u^{',\varepsilon}(t)a^\varepsilon\nabla u^\varepsilon(t),\nabla\varphi)dt - \varepsilon^\rho(\bar G_{uu}^{'',\varepsilon}(t)a^\varepsilon\nabla u^\varepsilon(t),\varphi\nabla u^\varepsilon(t))dt \\ + & \; \varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}(\bar G_u^{',\varepsilon}(t)g^\varepsilon(t),\varphi)dt + dM_t^\varphi\Big\} + d < M^\varphi >_t, \end{aligned}$$

where $< M^\varphi >_t$ stands for the quadratic characteristics of M^φ. Comparing two previous expressions allow us to write down the formula for $< M^\varphi >_t$:

$$< M^\varphi >_t = \varepsilon^{2\rho}(\bar{G}^\varepsilon(t),\varphi)^2 - \varepsilon^{2\rho}(\bar{G}^\varepsilon(0),\varphi)^2 + 2\varepsilon^{\rho-(1\wedge\frac{\alpha}{2})}\int_0^t (\bar{G}^\varepsilon(s),\varphi)(g^\varepsilon(s),\varphi)ds$$

$$+ \quad 2\varepsilon^{\rho+1-(1\wedge\frac{\alpha}{2})}\int_0^t \Big\{(\tilde{G}^\varepsilon(s),\nabla\varphi) + (\tilde{G}_u^{\prime,\varepsilon}(s)\nabla u^\varepsilon(s),\varphi)\Big\}(\bar{G}^\varepsilon(s),\varphi)ds$$

$$- \quad 2\varepsilon^{2\rho-(1\wedge\frac{\alpha}{2})}\int_0^t (\bar{G}_u^{\prime,\varepsilon}(s)g^\varepsilon(s),\varphi)(\bar{G}^\varepsilon(s),\varphi)ds - 2\varepsilon^\rho\int_0^t (\bar{G}^\varepsilon(s),\varphi)dM_s^\varphi$$

$$+2\varepsilon^{2\rho}\int_0^t \Big\{(\bar{G}_u^{\prime,\varepsilon}(s)a^\varepsilon\nabla u^\varepsilon(s),\nabla\varphi) + (\bar{G}_{uu}^{\prime\prime,\varepsilon}(s)a^\varepsilon\nabla u^\varepsilon(s),\varphi\nabla u^\varepsilon(s))\Big\}(\bar{G}^\varepsilon(s),\varphi)ds. \quad (3.32)$$

From the Burkholder-Davis-Gundy inequality,

$$\mathbf{E}\Big(\sup_{t_1<s<t_2}\Big|M_s^\varphi - M_{t_1}^\varphi\Big|^4\Big) \le C\mathbf{E}\Big[\Big(< M^\varphi >_{t_2} - < M^\varphi >_{t_1}\Big)^2\Big]. \quad (3.33)$$

Using (3.12) and (3.31), we deduce from (3.33)

$$\mathbf{E}\Big(\sup_{t_1<s<t_2}\Big|M_s^\varphi - M_{t_1}^\varphi\Big|^4\Big)$$
$$\le C\Big(\varepsilon^{4\rho} + |t_2-t_1|^2\left\{\varepsilon^{2\rho-2(1\wedge\frac{\alpha}{2})} + \varepsilon^{2\rho+2-2(1\wedge\frac{\alpha}{2})} + \varepsilon^{4\rho-2(1\wedge\frac{\alpha}{2})} + \varepsilon^{4\rho}\right\}$$
$$+|t_2-t_1|\left\{\varepsilon^{4\rho} + \varepsilon^{2\rho+2-2(1\wedge\frac{\alpha}{2})}\right\} + |t_2-t_1|^2\Big).$$

and the tightness follows from Theorem 8.3 in [4].

Recall that $V_T := L^2(0,T;H^1(\mathbb{R}^n))\cap C([0,T];L^2(\mathbb{R}^n))$, and $\tilde{V_T} := L^2_w(0,T;H^1_w(\mathbb{R}^n))\cap C([0,T];L^2_w(\mathbb{R}^n))$. It follows from the results in Viot [17], Proposition 3.4.1 and Proposition 3.4.2 the

Proposition 3.4.3. *The collection $\{u^\varepsilon,\, \varepsilon > 0\}$ of elements of V_T is tight in $\tilde{V_T}$.*

3.5 Passage to the limit

The aim of this section is to pass to the limit, as $\varepsilon \to 0$, in the family of laws of $\{u^\varepsilon\}$ and to determine the limit problem. In view of the tightness result of the preceding section it is sufficient to find the limit distributions of the inner products (φ, u^ε) with $\varphi \in C_0^\infty(\mathbb{R}^n)$, see [17]. We study the cases $\alpha < 2$ and $\alpha > 2$ separately.

Proof of theorem 3.3.1 : We introduce the following equations

$$A\chi^k(z,s) = -\sum_i \frac{\partial}{\partial z_i} a_{ik}(z,s), \qquad k = 1, ..., n \tag{3.34}$$

where the z-periodic functions $\chi^k(.,s) \in H^1(\mathbf{T^n})$.
For any arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$, we consider the following expression $\{\Phi_3^\varepsilon(t),\ 0 \le t \le T\}$ defined as

$$\Phi_3^\varepsilon(t) \;=\; (u^\varepsilon(t), \varphi) + \varepsilon(\chi^\varepsilon(t)u^\varepsilon(t), \nabla_x\varphi) + \varepsilon^{\frac{\alpha}{2}}(\bar{G}^\varepsilon(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi),$$

where $\chi^\varepsilon(t)$ stand for $\chi(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha})$.

We have the following identities :

$$\begin{aligned}
d(u^\varepsilon(t), \varphi) &= -(a_\varepsilon \nabla u^\varepsilon(t), \nabla_x\varphi)dt + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(g^\varepsilon(t), \varphi)dt \\
&= (a_\varepsilon u^\varepsilon(t), \nabla_x\nabla_x\varphi)dt + \frac{1}{\varepsilon}(\nabla_z a_\varepsilon u^\varepsilon(t), \nabla_x\varphi)dt \\
&\quad + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\bar{g}^\varepsilon(t), \varphi)dt + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\tilde{g}^\varepsilon(t), \varphi)dt,
\end{aligned}$$

$$\begin{aligned}
\varepsilon d(\chi^\varepsilon(t)u^\varepsilon(t), \nabla_x\varphi) &= \varepsilon^{1-\alpha}(\frac{\partial\chi^\varepsilon(t)}{\partial t}u^\varepsilon(t), \nabla_x\varphi)dt + \varepsilon(\chi^\varepsilon(t)\frac{\partial u^\varepsilon}{\partial t}(t), \nabla_x\varphi)dt \\
&= \varepsilon^{1-\alpha}(\frac{\partial\chi^\varepsilon(t)}{\partial t}u^\varepsilon(t), \nabla_x\varphi)dt + \varepsilon(\chi^\varepsilon(t)A^\varepsilon(t)u^\varepsilon(t), \nabla_x\varphi)dt \\
&\quad + \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t)g^\varepsilon(t), \nabla_x\varphi)dt \\
&= \varepsilon^{1-\alpha}(\frac{\partial\chi^\varepsilon(t)}{\partial t}u^\varepsilon(t), \nabla_x\varphi)dt + \varepsilon^{-1}(\nabla_z(a^\varepsilon\nabla_z\chi^\varepsilon)(t)u^\varepsilon(t), \nabla_x\varphi)dt \\
&\quad + (a^\varepsilon\nabla_z\chi^\varepsilon(t), u^\varepsilon(t)\nabla_x\nabla_x\varphi)dt - \varepsilon(a^\varepsilon\nabla_x u^\varepsilon(t), \chi^\varepsilon(t)\nabla_x\nabla_x\varphi)dt \\
&\quad + \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t)g^\varepsilon(t, u^\varepsilon(t)), \nabla_x\varphi)dt,
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\frac{\alpha}{2}}d(\bar{G}^\varepsilon(t), \varphi) &= -\frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\bar{g}^\varepsilon(t), \varphi)dt + \varepsilon^{\frac{\alpha}{2}}(\bar{G}_u^{',\varepsilon}(t)A^\varepsilon(t)u^\varepsilon(t), \varphi)dt \\
&+ \quad (\bar{G}_u^{',\varepsilon}(t)g^\varepsilon(t), \varphi)dt + dM_t^\varphi.
\end{aligned}$$

After simple rearrangements, we obtain

$$
\begin{aligned}
d\Phi_3^\varepsilon(t) &= (u^\varepsilon(t), a^\varepsilon \nabla_x \nabla_x \varphi)dt + \varepsilon^{-1}(u^\varepsilon(t), \nabla_z a^\varepsilon \nabla_x \varphi)dt \\
&+ \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\bar{g}^\varepsilon(t), \varphi)dt + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\tilde{g}^\varepsilon(t), \varphi)dt + \varepsilon^{1-\alpha}(\frac{\partial \chi^\varepsilon(t)}{\partial t} u^\varepsilon(t), \nabla_x \varphi)dt \\
&+ \varepsilon^{-1}(\nabla_z(a^\varepsilon \nabla_z \chi^\varepsilon)(t) u^\varepsilon(t), \nabla_x \varphi)dt + (a^\varepsilon \nabla_z \chi^\varepsilon(t), u^\varepsilon(t) \nabla_x \nabla_x \varphi)dt \\
&- \varepsilon(a^\varepsilon \nabla_x u^\varepsilon(t), \chi^\varepsilon(t) \nabla_x \nabla_x \varphi)dt + \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t) g^\varepsilon(t, u^\varepsilon(t)), \nabla_x \varphi)dt \\
&- \varepsilon^{\frac{\alpha}{2}}(\bar{G}_{uu}^{\prime\prime,\varepsilon}(t) a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi)dt - \varepsilon^{\frac{\alpha}{2}}(\bar{G}_u^{\prime,\varepsilon}(t) a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi)dt \\
&+ (\bar{G}_u^{\prime,\varepsilon}(t) g^\varepsilon(t), \varphi)dt - \frac{1}{\varepsilon^{\frac{\alpha}{2}}}(\bar{g}^\varepsilon(t), \varphi)dt + dM_t^\varphi.
\end{aligned}
$$

The functions χ^k satisfy the relation $\int_{\mathbf{T^n}} \chi^k(z,s)dz = 0,$ thus

$$
\int_{\mathbf{T^n}} \frac{\partial \chi^k}{\partial s}(z,s)dz = \frac{\partial}{\partial s} \int_{\mathbf{T^n}} \chi^k(z,s)dz = 0, \tag{3.35}
$$

and in the same way as in (3.7) there exists $E^k(z,s) \in H^2(\mathbf{T^n})$ such that

$$
\frac{\partial \chi^k}{\partial s}(z,s) = \mathrm{div_z E^k(z,s)}. \tag{3.36}
$$

Taking into account the relations (3.7), (3.34) and (3.36) after simple transformations we get

$$
\begin{aligned}
d\Phi_1^\varepsilon(t) &= (u^\varepsilon(t), a^\varepsilon(\mathbf{I} + \nabla_z \chi^\varepsilon(t)) \nabla_x \nabla_x \varphi)dt + (\bar{G}_u^{\prime,\varepsilon}(t) g^\varepsilon(t), \varphi)dt \\
&+ dM_t^\varphi - \varepsilon^{2-\alpha}(E^\varepsilon(t), \nabla(\nabla_x \varphi u^\varepsilon(t)))dt - \varepsilon^{1-\frac{\alpha}{2}}(\nabla_x \varphi, \tilde{G}(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)))dt \\
&- \varepsilon^{1-\frac{\alpha}{2}}(\tilde{G}_u^\prime(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)) \nabla_x u^\varepsilon(t), \nabla_x \varphi)dt - \varepsilon(a^\varepsilon \nabla_x u^\varepsilon(t), \chi^\varepsilon(t) \nabla_x \nabla_x \varphi)dt \\
&+ \varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(t) g^\varepsilon(t), \nabla_x \varphi)dt - \varepsilon^{\frac{\alpha}{2}}(\bar{G}_u^{\prime,\varepsilon}(t) a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x \varphi)dt \\
&- \varepsilon^{\frac{\alpha}{2}}(\bar{G}_{uu}^{\prime\prime,\varepsilon}(t) a^\varepsilon \nabla_x u^\varepsilon(t), \nabla_x u^\varepsilon(t) \varphi)dt.
\end{aligned}
$$

Now it is natural to rewrite the above expression as follows

$$(u^\varepsilon(t),\varphi)=(u_0,\varphi)+\int_0^t\Big\{(u^\varepsilon(s),\hat{\mathbf{a}}\nabla_x\nabla_x\varphi)+(\hat{\mathbf{g}}(u^\varepsilon(s))(u^\varepsilon(s)),\varphi)\Big\}ds+M_t^\varphi+\tilde{R}_3^\varepsilon(t), \tag{3.37}$$

where

$$\begin{aligned}
\tilde{R}_3^\varepsilon(t)&=\varepsilon(\chi^\varepsilon(0)u_0,\nabla_x\varphi)+\varepsilon^{\frac{\alpha}{2}}(\bar{G}^\varepsilon(0),\varphi)-\varepsilon(\chi^\varepsilon(t)u^\varepsilon(t),\nabla_x\varphi)-\varepsilon^{\frac{\alpha}{2}}(\bar{G}^\varepsilon(t),\varphi)\\
&+\int_0^t(u^\varepsilon(s),[a^\varepsilon(\mathbf{I}+\nabla_z\chi^\varepsilon(s))-\hat{\mathbf{a}}]\nabla_x\nabla_x\varphi)ds\\
&+\int_0^t\Big\{(\bar{G}_u^{\prime,\varepsilon}(s)g^\varepsilon-\hat{\mathbf{g}}(u^\varepsilon(s))(u^\varepsilon(s)),\varphi)-\varepsilon^{2-\alpha}(E^\varepsilon(s),\nabla(\nabla_x\varphi u^\varepsilon(s)))\Big\}ds\\
&+\int_0^t\Big\{-\varepsilon^{1-\frac{\alpha}{2}}(\nabla_x\varphi,\tilde{G}(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)))-\varepsilon^{1-\frac{\alpha}{2}}(\tilde{G}_u'(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla_x u^\varepsilon(s),\nabla_x\varphi)\\
&-\varepsilon(a^\varepsilon\nabla_x u^\varepsilon(s),\chi^\varepsilon(s)\nabla_x\nabla_x\varphi)+\varepsilon^{1-\frac{\alpha}{2}}(\chi^\varepsilon(s)g^\varepsilon(s,u^\varepsilon(s)),\nabla_x\varphi)\\
&-\varepsilon^{\frac{\alpha}{2}}(\bar{G}_u^{\prime,\varepsilon}(s)a^\varepsilon\nabla_x u^\varepsilon(s),\nabla_x\varphi)-\varepsilon^{\frac{\alpha}{2}}(\bar{G}_{uu}^{\prime\prime,\varepsilon}(s)a^\varepsilon\nabla_x u^\varepsilon(s),\nabla_x u^\varepsilon(s)\varphi)\Big\}ds.
\end{aligned}$$

We rewrite (3.37) as

$$F_\varphi(t,u^\varepsilon)\;=\;M_t^\varphi+\tilde{R}_3^\varepsilon(t),$$

where, for $u\in V_T$,

$$F_\varphi(t,u):=(u(t),\varphi)-(u_0,\varphi)-\int_0^t(u(s),\hat{\mathbf{a}}\nabla_x\nabla_x\varphi)ds-\int_0^t(\hat{\mathbf{g}}(u)(u(s)),\varphi)ds,$$

and the quadratic variation of M_t^φ has been calculated in (3.32). By **A3**, Proposition 7 in Pardoux, Piatnitski [16], and Burkholder-Davis-Gundy inequality , $\tilde{R}_1^\varepsilon(t)$ tends to zero uniformly in t, in $L^1(\Omega)$, as $\varepsilon\to 0$. Let $0\le s<t$, and Θ_s^ε be any continuous (in the sense of the topology of $\tilde{V}_T$) and bounded functional of $\{u^\varepsilon(r),\,0\le r\le s\}$. We have that

$$\begin{aligned}
\mathbf{E}\left[\left(F_\varphi(t,u^\varepsilon)-F_\varphi(s,u^\varepsilon)\right)\Theta_s^\varepsilon\right]&=\mathbf{E}\left[\left(\tilde{R}_1^\varepsilon(t)-\tilde{R}_1^\varepsilon(t)\right)\Theta_s^\varepsilon\right],\\
\mathbf{E}\left[\left(M_\varphi^\varepsilon(t)-M_\varphi^\varepsilon(s)\right)^2\Theta_s^\varepsilon\right]&=\mathbf{E}\left[\left(<M_\varphi^\varepsilon>(t)-<M_\varphi^\varepsilon>(s)\right)\Theta_s^\varepsilon\right].
\end{aligned}$$

Let $u\in\tilde{V}_T$ be any accumulation point of the sequence u^ε, as $\varepsilon\to 0$. Taking the limit along the corresponding subsequence in the two last identities, we conclude with the help of Propositions 6, 8, and 9 in in Pardoux, Piatnitski [16] that

$$F_\varphi(t,u) := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (u(s), \hat{\mathbf{a}}\nabla_x\nabla_x\varphi)ds - \int_0^t (\hat{\mathbf{g}}(u)(u(s)),\varphi)ds,$$

is a square integrable martingale, if we equip $\tilde{V_T}$ with respect to the natural filtration, of u with the associated increasing process given by

$$\int_0^t (R(u(s))\varphi,\varphi)ds,$$

where

$$\begin{aligned}(R(u)\varphi,\varphi) &= 2\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\bar{G}(s,u(x))\varphi(x)\bar{g}(s,u(y))\varphi(y)dxdy \\ &= 2\mathbf{E}[(\bar{G}(s,u(.))\varphi)(\bar{g}(s,u(.)),\varphi)].\end{aligned}$$

We have shown that the law Q^0 of any accumulation point u of the sequence u^ε solves the following martingale problem, which we shall denote problem (MP2). For all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$F_\varphi(t,u) := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (\hat{A}(u(s)),\varphi)ds,\ t \geq 0,$$

where

$$\hat{A}(v) = \nabla_x \hat{\mathbf{a}} \nabla_x v + \hat{\mathbf{g}}(v),$$

is a martingale with the increasing process

$$<< F_\varphi(\cdot,u) >> (t) = \int_0^t (R(u(s))\varphi,\varphi)ds.$$

We finally establish the

Lemma 3.5.1. *The martingale problem (MP2) has a unique solution.*

Proof : The usual argument of Yamada-Watanabe establishes that the uniqueness of the solution of the martingale problem is a consequence of pathwise uniqueness for a corresponding SDE(see[17]) for the adaptation of that argument to SPDEs).
We define a linear space K of stationary random processes as follows. Let W denote the collection of processes $\{\bar{g}(t,u);\quad t \geq 0\}$, where u various in $\mathbb{R}$, and Span(W) be the linear space generated by W. The space Span(W) may be endowed with the bilinear form :

$$< \bar{g}_u, \bar{g}_v >:= \int_0^\infty \mathbf{E}\left[(\bar{g}(0,u)\bar{g}(t,v) + \bar{g}(t,u)\bar{g}(0,v)\right] dt.$$

Let $\mathcal{N} = \{h \in Span(W) :< h, h >= 0\}$ and let K the closure of $Span(W)/\mathcal{N}$ under $< \cdot, \cdot >$. For each $w(x) \in L^2(\mathbb{R}^\mathbf{n})$, denote by $C(w)$ the operator : $L^2(\mathbb{R}^\mathbf{n}) \mapsto \mathbf{K}$ given by

$$[C(w)\varphi](t) := \int_{\mathbb{R}^\mathbf{n}} \bar{g}(t, w(x))\varphi(x)dx.$$

Consider the following SPDE in $L^2(\mathbb{R}^\mathbf{n})$

$$du(t) = \hat{A}(u(t))dt + C(u(t))dB_t, \quad u(0) = u_0, \tag{3.38}$$

where B_t is a cylindrical Brownian motion over K,i.e for any $h \in K$, $\{< h, B_t >, \ t \geq 0\}$ is a real valued Brownian motion with covariance $t\|h\|^2$. By [8] this equation does have a solution in the space $V := \bigcup_{T>0} V_T$, such that for all $T > 0$,

$$\mathbf{E}\left(\sup_{0\leq t\leq T} \|u(t)\|^2 \int_0^T \|\nabla_x u(t)\ \|^2 dt\right) < \infty. \tag{3.39}$$

We introduce the following notation for the coefficients of the operator $\hat{A}(u)$:

$$\hat{A}(u) = \nabla_x.\bar{a}\nabla_x u + \bar{F}^0(u)$$

with $\bar{a} = \bar{a}_{ij}$.

We now establish uniqueness. Assume that there are two solutions $w_1(t)$ and $w_2(t)$ with the same initial condition. It is not hard to see that they both satisfy the estimate (3.39). Apply Itô's formula to the expression $\|w_1 - w_2\|^2$, where $\| \cdot \|$ stands for the norm in $L^2(\mathbb{R}^\mathbf{n})$. This gives

$$\|w_1 - w_2\|^2 + 2\int_0^t \bar{a}\nabla_x(w_1(s) - w_2(s)).\nabla_x(w_1(s) - w_2(s))ds =$$

$$+2\int_0^t \big(F^0(w_1(s)) - F^0(w_2(s)), w_1(s) - w_2(s)\big)\, ds + \mathcal{M}_t$$

$$+\int_0^t \Big[\int_0^\infty \mathbf{E}\left(\bar{g}(0, w_1(s)) - \bar{g}(0, w_2(s)), \bar{g}(r, w_1(s)) - \bar{g}(r, w_2(s))\right) dr\Big] ds,$$

where $\mathcal{M}_t$ is a martingale. Taking the expectation on both sides of this identity and considering the Lipschitz properties in w of all the functions involved, we obtain after simple rearrangements

$$\begin{aligned}\mathbf{E}\|w_1(t) - w_2(t)\|^2 \ &+\ 2\mathbf{E}\int_0^t (\bar{a} - \gamma\mathbf{I})\nabla_x(w_1(s) - w_2(s)).\nabla_x(w_1(s) - w_2(s))ds \\ &\leq\ C(\gamma)\mathbf{E}\int_0^t \|w_1(s) - w_2(s)\|^2 ds.\end{aligned}$$

For sufficiently small γ by the Gronwall lemma then implies

$$\mathbf{E}\|w_1(t) - w_2(t)\|^2 = 0$$

for any $t \geq 0$. This completes the proof. ♢

Proof of theorem 3.3.2 : We follow the same scheme as above and we introduce, in addition to $\bar{G}^\varepsilon(t)$, two more correctors. Let $\chi = \chi(z,s)$ and $\tilde{H}(z,s,u)$ be stationary solutions of the following equations

$$\frac{\partial}{\partial s}\chi(z,s) + \operatorname{div}\left[a(z,s)\nabla\chi(z,s)\right] = -\operatorname{div} a(z,s), \qquad (z,s) \in \mathbf{T^n} \times (-\infty,+\infty), \tag{3.40}$$

$$\frac{\partial}{\partial s}\tilde{H}(z,s,u)(z,s) + \operatorname{div}\left[a(z,s)\nabla\tilde{H}(z,s,u)\right] = -\tilde{g}(z,s,u), \qquad (z,s) \in \mathbf{T^n} \times (-\infty,+\infty) \tag{3.41}$$

Lemma 3.5.2. *The equations (3.40) and (3.41) have stationary solutions. Under the normalizations*

$$\int_{\mathbf{T^n}} \chi(z,s)dz = 0, \quad \int_{\mathbf{T^n}} \tilde{H}(z,s,u)dz = 0,$$

the solutions are unique and ergodic, and the following bounds hold

$$\|\chi\|_{L^\infty} + \int_t^{t+1} \|\chi(.,s)\|^2_{H^1(\mathbf{T^n})}ds \leq C,$$

$$\|\tilde{H}\|_{L^\infty} + \Big(\int_t^{t+1} \|\tilde{H}(.,s,u)\|^2_{H^1(\mathbf{T^n})}ds\Big)^{1/2} \leq C|u|,$$

$$\|\tilde{H}'_u\|_{L^\infty} + \Big(\int_t^{t+1} \|\tilde{H}'_u(.,s,u)\|^2_{H^1(\mathbf{T^n})}ds\Big)^{1/2} \leq C,$$

$$\|\tilde{H}''_{uu}\|_{L^\infty} + \Big(\int_t^{t+1} \|\tilde{H}''_{uu}(.,s,u)\|^2_{H^1(\mathbf{T^n})}ds\Big)^{1/2} \leq C/(1+|u|).$$

Proof : See[10].

For any arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$, we consider $\{\Phi_4^\varepsilon(t), 0 \leq t \leq T\}$ defined as

$$\Phi_4^\varepsilon(t) = (u^\varepsilon(t),\varphi) + \varepsilon^{\alpha-1}(\chi^\varepsilon(t)u^\varepsilon(t),\nabla_x\varphi) + \varepsilon^{\alpha-1}(\tilde{H}^\varepsilon(t),\varphi) + \varepsilon^{\alpha-1}(\bar{G}^\varepsilon(t),\varphi),$$

where $\chi^\varepsilon(t)$, $\tilde{H}^\varepsilon(t)$ stand for $\chi(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha})$, $\tilde{H}^\varepsilon(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t,x))$.
We have the following identities :

$$\begin{aligned}
d(u^\varepsilon(t),\varphi) &= -(a^\varepsilon \nabla u^\varepsilon(t), \nabla_x\varphi)dt + \frac{1}{\varepsilon}(g^\varepsilon(t),\varphi)dt \\
&= (a^\varepsilon u^\varepsilon(t), \nabla_x\nabla_x\varphi)dt + \frac{1}{\varepsilon}(\nabla_z a^\varepsilon u^\varepsilon(t), \nabla_x\varphi)dt \\
&\quad + \frac{1}{\varepsilon}(\bar{g}^\varepsilon(t),\varphi)dt + \frac{1}{\varepsilon}(\tilde{g}^\varepsilon(t),\varphi)dt,
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\alpha-1} d(\chi^\varepsilon(t)u^\varepsilon(t), \nabla_x\varphi) &= -\frac{1}{\varepsilon}(u^\varepsilon(t)A^\varepsilon\chi^\varepsilon(t), \nabla_x\varphi)dt - \frac{1}{\varepsilon}(u^\varepsilon(t)\mathrm{div}_z a^\varepsilon, \nabla_x\varphi)dt \\
&\quad + \varepsilon^{\alpha-1}(\chi^\varepsilon(t)A^\varepsilon u^\varepsilon(t), \nabla_x\varphi)dt + \varepsilon^{\alpha-2}(\chi^\varepsilon(t)g^\varepsilon(t), \nabla_x\varphi)dt \\
&= (\nabla\chi^\varepsilon(t)a^\varepsilon u^\varepsilon(t), \nabla\nabla_x\varphi)dt - \frac{1}{\varepsilon}(u^\varepsilon(t)\mathrm{div}_z a^\varepsilon, \nabla_x\varphi)dt \\
&+ \varepsilon^{\alpha-2}(\chi^\varepsilon(t)g^\varepsilon(t), \nabla_x\varphi)dt - \varepsilon^{\alpha-1}(\chi^\varepsilon(t)a^\varepsilon\nabla u^\varepsilon(t), \nabla_x\nabla_x\varphi)dt \\
&\quad + \varepsilon^{\alpha-1}(\chi^\varepsilon(t)h^\varepsilon(t), \nabla_x\varphi)dt,
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\alpha-1} d(\tilde{H}^\varepsilon(t),\varphi) &= -\frac{1}{\varepsilon}(\mathrm{div}_z(a^\varepsilon\nabla_z\tilde{H}^\varepsilon(t)),\varphi)dt - \frac{1}{\varepsilon}(\tilde{g}^\varepsilon(t),\varphi)dt \\
&\quad + \varepsilon^{\alpha-1}(\tilde{H}_u^{',\varepsilon}(t)A^\varepsilon(t)u^\varepsilon(t),\varphi)dt + \varepsilon^{\alpha-2}(\tilde{H}_u^{',\varepsilon}(t)g^\varepsilon(t),\varphi)dt \\
&= (a^\varepsilon\nabla_z\tilde{H}^\varepsilon(t), \nabla_x\varphi)dt - \frac{1}{\varepsilon}(\tilde{g}^\varepsilon(t),\varphi)dt + \varepsilon^{\alpha-2}(\tilde{H}_u^{',\varepsilon}(t)g^\varepsilon(t),\varphi)dt \\
&\quad - \varepsilon^{\alpha-1}(\tilde{H}_u^{',\varepsilon}(t)a^\varepsilon(t)\nabla u^\varepsilon(t), \nabla_x\varphi)dt \\
&\quad - \varepsilon^{\alpha-1}(\tilde{H}_{uu}^{'',\varepsilon}(t)a^\varepsilon(t)\nabla u^\varepsilon(t), \nabla u^\varepsilon(t)\varphi)dt,
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\alpha-1} d(\bar{G}^\varepsilon(t),\varphi) &= -\frac{1}{\varepsilon}(\bar{g}^\varepsilon(t),\varphi)dt + \varepsilon^{\alpha-1}(\bar{G}_u^{',\varepsilon}(t)A^\varepsilon(t)u^\varepsilon(t),\varphi)dt \\
&\quad + \varepsilon^{\frac{\alpha}{2}-1}(\bar{G}_u^{',\varepsilon}(t)g^\varepsilon(t),\varphi)dt + \varepsilon^{\frac{\alpha}{2}-1}d\tilde{M}_t^\varphi,
\end{aligned}$$

where

$$\tilde{M}_t^{\varphi} := \varepsilon^{\frac{\alpha}{2}}[(\bar{G}(\frac{t}{\varepsilon^{\alpha}}, u^{\varepsilon}(t)), \varphi) - (\bar{G}(0, u^{\varepsilon}(0)), \varphi)] + \frac{1}{\varepsilon^{\frac{\alpha}{2}}} \int_0^t (\bar{g}(\frac{s}{\varepsilon^{\alpha}}, u^{\varepsilon}(s)), \varphi) ds$$

$$+\varepsilon^{\frac{\alpha}{2}} \int_0^t (a^{\varepsilon}(s) \nabla u^{\varepsilon}(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^{\alpha}}, u^{\varepsilon}(s)) \nabla u^{\varepsilon}(s) \varphi + \bar{G}'_u(\frac{s}{\varepsilon^{\alpha}}, u^{\varepsilon}(s)) \nabla \varphi) ds$$

$$- \int_0^t (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^{\alpha}}, u^{\varepsilon}(s)), \bar{G}'_u(\frac{s}{\varepsilon^{\alpha}}, u^{\varepsilon}(s)) \varphi) ds$$

is a $\{\Im_{\frac{t}{\varepsilon^{\alpha}}}\}$- martingale.

We obtain after simple transformations

$$\begin{aligned} d\Phi_4^{\varepsilon}(t) &= ((I + \nabla \chi^{\varepsilon}(t)) a^{\varepsilon} u^{\varepsilon}(t), \nabla \nabla_x \varphi) dt + (a^{\varepsilon} \nabla_z \tilde{G}^{\varepsilon}(t), \nabla_x \varphi) dt \\ &-\varepsilon^{\alpha-1} \Big\{ (\chi^{\varepsilon}(t) a^{\varepsilon} \nabla u^{\varepsilon}(t), \nabla \nabla_x \varphi) + (\tilde{G}'^{,\varepsilon}_u(t) a^{\varepsilon}(t) \nabla u^{\varepsilon}(t), \nabla_x \varphi) \Big\} dt \\ &-\varepsilon^{\alpha-1} \Big\{ (\tilde{H}''^{,\varepsilon}_{uu}(t) a^{\varepsilon}(t) \nabla u^{\varepsilon}(t), \nabla u^{\varepsilon}(t) \varphi) + (\bar{G}'^{,\varepsilon}_u(t) A^{\varepsilon}(t) u^{\varepsilon}(t), \varphi) \Big\} dt \\ &+\varepsilon^{\alpha-2} \Big\{ (\chi^{\varepsilon}(t) g^{\varepsilon}(t), \nabla_x \varphi) + (\tilde{H}'^{,\varepsilon}_u(t) g^{\varepsilon}(t), \varphi) - (\bar{G}'^{,\varepsilon}_u(t) g^{\varepsilon}(t), \varphi) \Big\} dt \\ &+\varepsilon^{\frac{\alpha}{2}-1} d\tilde{M}_t^{\varphi} + \varepsilon^{\frac{\alpha}{2}-1} (\bar{G}'^{,\varepsilon}_u(t) g^{\varepsilon}(t), \varphi) dt. \end{aligned}$$

We rewrite $d\Phi_4^{\varepsilon}(t)$ as follows

$$\begin{aligned} (u^{\varepsilon}(t), \varphi) &= (u_0, \varphi) + \int_0^t \Big\{ (u^{\varepsilon}(s), \hat{\mathbf{a}} \nabla_x \nabla_x \varphi) + (\hat{\mathbf{b}}(u^{\varepsilon}(s))(u^{\varepsilon}(s)), \nabla_x \varphi) \Big\} ds \\ &+ \varepsilon^{\frac{\alpha}{2}-1} \tilde{M}_t^{\varphi} + {R_4}^{\varepsilon}(t), \end{aligned}$$

where

$$\mathbf{E} \sup_{o \le t \le T} |\tilde{M}_t^{\varphi}| \le C.$$

By **A3** and Proposition 7 in Pardoux, Piatnitski [16], $R_4^{\varepsilon}(t)$ tends to zero uniformly in t, in $L^1(\Omega)$, as $\varepsilon \to 0$.
Finally for any test function $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ the following limit relation holds :

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} \sup_{t \le T} \Big| (u^{\varepsilon}(t), \varphi) - (u_0, \varphi) - \int_0^t (u^{\varepsilon}(s), \hat{\mathbf{a}} \nabla_x \nabla_x \varphi) ds - \int_0^t (\hat{\mathbf{b}}(u^{\varepsilon}(s))(u^{\varepsilon}(s)), \nabla_x \varphi) ds \Big| = 0.$$

The proof is complete. ♢

Bibliographie

[1] A.Bensoussan, J.-L. Lions, G.Papanicolaou, *Asymptotic Analysis of Periodic Structures*, North-Holland Publ. Comp., Amsterdam, 1978.

[2] M. CESSENAT, G. LEDANOIS, P .L. LIONS, E. PARDOUX, ET R. SENTIS, *Méthodes probabilistes pour les équations de la physique.* Collection du C.E.A sous la direction de R. Dautray, Série synthèses, Eyrolles, 1989.

[3] A.Friedman, *Stochastic Differential Equations and Applications.* Academic Press, New-york, 1975.

[4] P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968.

[5] R.Bouc, E.Pardoux, *Asymptotic Analysis of PDEs with wide-band noise disturbance ; expansion of the moments.* J. of Stochastic Analysis and Appl., **2**(4), p.369-422, 1984.

[6] F.Campillo, M.Kleptsyna, A.Piatnitski, *Averaging of random parabolic operators with large potential*, Stoch. Proc. Appl. **93**, 57–85, 2001.

[7] M. Freidlin, *The Dirichlet problem for an equation with periodic coefficients depending on a small parameter.*, Teor. Veroyanost. I. Primenen 9, 133-139, 1964.

[8] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, 1992.

[9] V.Jikov, S.Kozlov, O.Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.

[10] M.Kleptsyna, A.Piatnitski, *Averaging of non-selfadjoint parabolic equations with random evolution*, INRIA, Report 3951, 2000

[11] M.A.Diop, E.Pardoux, *Averaging of parabolic partial differential equation with random evolution*, To appear in Ascona Proceedings, 2002

[12] S.N.Ethier,T.G.Kurtz, *Markov Processes. Characterization and convergence.*,John Wiley & Sons, New York, 1986.

[13] R.Liptser, A.Shiryaev, *Theory of Martingales*, Kluwer Academic Publishers, 1989.

[14] E.Pardoux, *Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients : a probabilistic approach* J. of Funct. Analysis, **167**, p.498-520, 1999.

[15] E.Pardoux, A.Veretennikov, *On Poisson equation and diffusion approximation. I.* Ann. of Probability, **29**, 1061-1085,2001.

[16] E.Pardoux, A.Piatnitski, *Homogenization of a nonlinear random parabolic differential equation.* Stoch. Proc. Appl. **104**,1-27,2003.

[17] M.Viot, Solutions Faibles d'Equations Dérivées Stochastiques Non Linéaires. Thèse, Université Paris VI, 1976.