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DÉPARTEMENT DE MATHÉMATIQUES
DEPARTMENT OF MATHEMATICS

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ON AXIOMS AND VALIDITY DOMAINS IN GAME THEORY

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ATTESTATION DE CORRECTION DE LA THÈSE DE DOCTORAT/PhD

Nous soussignés, Pr. **AYISSI Raoul Domingo,** Pr. **MOYOUWOU Issofa,** Pr. **TAKAM SOH Patrice;** membres du jury de la thèse de Doctorat/PhD présenté par **Monsieur SAFOKEM Adin**, Matricule **11Y680**, Thèse intitulé: **«ON AXIOMS AND VALIDITY DOMAINS IN GAME THEORY»** et soutenu en vue de l'obtention du diplôme de **DOCTORAT/PhD en Mathématiques**, attestons que toutes les corrections demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

En foi de quoi la présente attestation lui est délivrée pour servir et valoir ce que de droit.

Rapporteur	**Président**	**Examinateur**
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⋆⋆ Declaration ⋆⋆

I hereby declare that this submission is my own work and to the best of my knowledge, it contains no materials previously published or written by another person, no material which to a substantial extent has been accepted for the award of any other degree or diploma at the University of Yaounde I or at any other educational institution except where due acknowledgment is made in this thesis. Any contribution made to the present research by others, with whom I have worked at the University of Yaounde I or elsewhere, is explicitly cited in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in style, presentation, and linguistic expression is acknowledged.

Adin SAFOKEM.

⋆ ⋆ Dedication ⋆⋆

I dedicate this thesis to
my late parents NDADJIO André and FEUPE Pauline.

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Gratitude is the memory of the heart.

French Proverb.

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⋆⋆ Résumé ⋆⋆

Les travaux de cette thèse portent sur le problème de l'axiomatisation des solutions ponctuelles (aussi appelées valeurs) aux jeux coopératifs à utilités transférables (JCUTs) issus des travaux de Von Neumann and Morgenstern (1944). Plusieurs valeurs sur les JCUTs existent dont, entre autres, la valeur de Shapley (1953), le nucleolus de Schmeidler (1969), la valeur de Owen (1977) et la valeur de solidarité de Nowak and Radzik (1994). Pour motiver le choix d'une valeur, l'axiomatisation s'est avérée indispensable. Nous enrichissons la littérature existante à partir de deux approches.

L'approche *supra-domaine* consiste pour une valeur axiomatisée sur une famille $\mathcal{F}$ de jeux à rechercher de nouvelles axiomatisations sur des familles $\mathcal{F}' \supseteq \mathcal{F}$. C'est par exemple le cas du passage d'un électorat fixe à un électorat variable. Ici, notre attention porte sur l'indice de Deegan and Packel (1978) et l'indice de Holler-Packel introduit par Holler (1982). Nous contribuons à l'analyse de ces deux indices en proposant de nouvelles axiomatisations dans le cadre d'un électorat variable resté inexploré. À la suite de Fishburn (1970) sur les électorats variables, de nouveaux axiomes sont introduits relativement à l'arrivée d'un nouveau joueur et à une fusion.

L'approche *intra-domaine* consiste pour une axiomatisation d'une valeur sur une famille $\mathcal{F}$ de jeux à rechercher les domaines $\mathcal{F}' \subseteq \mathcal{F}$ sur lesquels restreindre les axiomes initiaux caractérise de nouveau la valeur considérée. Ici, nous portons notre attention sur la valeur de Shapley (1953). Nous introduisons la notion de classe de symétrie non triviale d'un JCUT et montrons que les axiomes de Shapley (1953) caractérisent de nouveau une seule valeur sur toute famille $\mathcal{F}$ dont les générateurs ont chacun au plus une classe de symétrie non triviale. Nos résultats généralisent celui de Neyman (1989) aussi bien que celui de Peleg and Sudhölter (2007). Le passage d'une famille $\mathcal{F}$ à une sous-famille $\mathcal{F}'$ peut nécessiter la reformulation de certains axiomes. C'est le cas de l'axiomatisation de la valeur de Shapley par Van den Brink (2001) étendue aux jeux de vote par l'auteur lui-même et à tout cône convexe contenant tous les JCUTs à l'unanimité par Casajus (2011). Ces deux résultats occultent le cas des jeux de vote monotones (ou jeux simples). Nous levons cette impasse due à la formulation de l'axiome d'équité et obtenons de nouvelles axiomatisations sur les jeux simples.

Mots clés: Jeux coopératifs à utilités transférables, jeux simples, valeur de Shapley, indice de Holler-Packel, indice de Deegan-Packel, électorat variable, axiomatisation.

⋆ ⋆ Abstract ⋆⋆

In this thesis, we deal with the problem of axiomatizing single-valued solutions (also called values) for transferable utility games (TU-games) that has been initiated by Von Neumann and Morgenstern (1944). Several values for TU-games exist and include, among others, the Shapley value by Shapley (1953), the nucleolus by Schmeidler (1969), the Owen value by Owen (1977) and the solidarity value by Nowak and Radzik (1994). To motivate the choice of a value among several others, the operation of axiomatization has become an essential step. We enrich the existing literature from two approaches. Given an existing characterization of a value on a family $\mathcal{F}$ of TU-games, the *supra-domain* approach consists in searching for new axiomatizations of the value in consideration over some families $\mathcal{F}' \supseteq \mathcal{F}$. This is for instance the case when one moves from a fixed electorate to a variable electorate. Here, our attention is focused on the Deegan-Packel index by Deegan and Packel (1978) and the Public Good Index introduced by Holler (1982). We contribute to the analysis of these two power indices by proposing new axiomatizations in the framework of a variable electorate that was not yet explored. Following Fishburn (1970) on the variable electorates, new axioms are introduced related to the arrival of a new player and to the merging of players.

Given an axiomatization of a value over a set $\mathcal{F}$ of TU-games, the *intra-domain* approach consists in searching for domains $\mathcal{F}' \subseteq \mathcal{F}$ on which restricting the initial axioms still characterizes the value in consideration. We focus our attention here on the Shapley value; see Shapley (1953). We introduce the notion of non-trivial symmetry class of a TU-game and show that the axioms of Shapley (1953) again characterize a single value on any set $\mathcal{F}$ of TU-games whose generators each has at most one non-trivial symmetry class. Our results generalize those by Neyman (1989) and Peleg and Sudhölter (2007). Moving from a set $\mathcal{F}$ of TU-games to a subset $\mathcal{F}'$ may require an appropriate reformulation of some axioms. This is the case of the axiomatization of the Shapley value by Van den Brink (2001) which was extended to voting games by the author himself and to any convex cone containing all unanimity games by Casajus (2011). These two results escape the case of monotone voting games (or simple games). We note that this impasse is due to the formulation of the fairness axiom and obtain new axiomatizations of the Shapley value on simple games.

Keywords: Cooperative games with transferable utilities, simple games, Shapley value, Public Good Index, Deegan-Packel index, variable electorate, axiomatization.

⋆ ⋆ Introduction ⋆⋆

Since the seminal work by Von Neumann and Morgenstern (1944), investigations in game theory have become borderless since they cover economic models as well as many applications to various fields such as politics, security, networks, computer sciences, etc. It generally involves modeling and analyzing situations (called games) where agents (hereafter called players) with conflicting interests interact in such a way that the well-being or the worth of each of them depends not only on his actions but also on those of others. Mathematical formalism and logic are therefore assets for modeling and analyzing the actions of agents and the possible causal links between individual strategies and possible outcomes in a game. In this thesis, we focus our attention on cooperative games with transferable utilities (TU-games) where players can form coalitions through binding cooperative agreements to achieve collective payoffs and redistribute these achievements among members of actual coalitions in a conceivable way. More specifically, our concerns are related to the properties of single-valued solutions (also called values) each of which associates any TU-game on a given set of players with a payoff vector whose coordinates indicate individual payments in the game.

There exists a rich panel of values for TU-games. A non exhaustive list includes for example the Shapley value by Shapley (1953), the nucleolus by Schmeidler (1969), the Owen value by Owen (1977) and the solidarity value by Nowak and Radzik (1994). To handle this multitude of values for TU-games, axiomatization has become essential. An axiomatization of a value on a family $\mathcal{F}$ for TU-games is in fact any operation which consists in providing a minimal list of desirable properties that the value in consideration is the only one to satisfy among all values defined on $\mathcal{F}$. Such an operation consists, according to Ferrières (2016), of two steps: one should first introduce and motivate some appealing properties (called axioms); and then proves that combining those axioms necessarily emerges to a unique value. Our contribution is related to this strand of the literature and is based on two main approaches we subdivide into two parts.

The first part of our investigations is a *supra-domain* study of two well-known voting power indices. A supra-domain approach consists, given an axiomatization of a value over a set $\mathcal{F}$ of TU-games, in searching for new axiomatizations of the value under consideration over some families $\mathcal{F}' \supseteq \mathcal{F}$. This is for instance the case when one moves from a fixed set of players to a variable set of players as it is the case from fixed electorate with no possible abstention to a variable electorate with eventually the possibility of some voters abstaining. Typical contributions within the variable electorate include Driessen and Funaki (1997a) or Van den Brink and Funaki (2009) who provided axiomatic characterizations of certain solution concepts. Our concerns here can be scrutinized from two distinct corners, both related to the abundant literature of voting power indices.

For a brief overview, the reader is referred to Andjiga et al. (2003) for an impregnation reading on power indices; or for a selected list of power indices and their features to Shapley and Shubik (1954); Banzhaf (1965); Coleman (1971); Deegan and Packel (1978) or Holler (1982). Power indices are of practical importance; see for examples, Bilbao (1998) who used power indices to measure individual voting power in the Council of Ministers of the European Union, Laruelle and Widgrén (1998) who presented an explanation of some voting behavior in terms of voting power among members of the European Union or Alonso-Meijide et al. (2011b) who quantified voting power in the Portuguese Parliament. Concretely, the intuition behind a simple game is the following: each player is a voter who either votes for the adoption or the rejection of a proposal; each coalition is either winning or losing; the grand coalition is winning and any superset of a winning coalition is winning. Values on simple games are called (voting) power indices and thus viewed as tools for evaluating individual voting power. Given a simple game, a power index thus assigns to each player a real number that can be interpreted as his decisiveness in the game or simply his ability to influence the final outcome (rejection or adoption).

Our first attempt within the supra-domain approach consists in providing new axiomatizations of the Holler-Packel index also known as the Public Good Index. Holler and Packel (1983) have provided an axiomatization of the Public Good Index in the case of fixed electorates in terms of (E), (S), (NP) and an axiom we refer to as the Holler-Packel mergeability condition (HPM). To the best of our knowledge, no such axiomatization result exists in the variable electorate setting. To address this issue, we bring into consideration some new axioms such as supplementation consistency (SC); supplementation invariance (SI) or the membership equivalence property (MEP) among others. These axioms are in relation with the arrival of a new player or the merging of some players (who decide to let a single player act on behalf of others) and their respective intuitions are sketched below.

The merging operation occurs when two or more players are merged into a single player. When players are independent in some sense, merging should not impact on the power of partners. Our axiom of Non Profitable Merging (NPM) of independent players requires that in such situation, the power of the representative player is equal to the sum of the powers of merged players. Alternatively, our axiom of Independence of External Merging (IEM) of independent players requires that in such situation, the power of other players should remain unchanged. Similar requirements to Axiom (NPM) can be found in Knudsen and Østerdal (2012) or in Lehrer (1988) under the name 2-efficiency when the merging operation is restricted to two players.

Supplementing a player is observable when a new player gets in a simple game without breaking any of the existing minimal winning coalitions. For such events, moving from one simple game to another, the number of minimal winning coalitions is unchanged. Our Supplementation Consistency (SC) axiom requires that when we move from a simple game to another by a supplementation operation, the power of all players, other than the new comer, in the supplemented game should be proportional to their power in the initial game (their respective voting powers are affected in the same way). Such condition were used by Moulin (2002) in the context of cost allocation problems to characterize the proportional rule.

The Supplementation Invariance axiom that is newly introduced in this thesis states that the power of a player, say i, should not change from a game to another whenever the new arrival player k did not, during a supplementation operation, join any minimal winning coalition containing player i. Similar but different axioms were used by Sen (1977) and Basu (1983) in the framework of Social Choice Theory to exhibit particular social welfare functions; and by Béal et al. (2015) in the framework of TU-games to characterize four solutions of TU-games.

Two simple games are said to be equivalent for a player i when moving from one of the two simple games to the other, only one minimal winning coalition of the first game is affected by allowing only the substitution of some players other than i (the size of the affected coalition as well as the membership of i remain unchanged). The membership equivalence property (MEP) we introduce here is based on equivalent games and simply requires that when two simple games are equivalent for a player, that player should enjoy the same power in both games.

As results, we obtain five axiomatizations of the Public Good Index thanks to the newly introduced axioms we just describe. In our second attempt within the supra-domain approach, we apply the analysis developed with the Public Good Index to establish new axiomatizations of the Deegan-Packel index. The linking aspect of both attempts is that the two power indices we consider depend only on the set of minimal winning coalitions in a simple games. Moreover, exactly as the axiomatizations of these two power indices by Holler and Packel (1983) and Deegan and Packel (1978) differ only on how merging two games is treated; we also present two new axiomatizations that also differ only on how a supplementation of a simple game is handled. Some of the results presented here are extended to simple games with a priori unions; (see Alonso-Meijide et al. (2010a) or Alonso-Meijide et al. (2010b)).

The second part of our investigations is an *intra-domain* study of the Shapley value. An intra-domain approach that consists, given an axiomatization of a value over a set $\mathcal{F}$ of TU-games, in searching for domains $\mathcal{F}' \subseteq \mathcal{F}$ on which restricting the initial axioms still characterizes the value in consideration. We refer to such a sub-domain $\mathcal{F}'$ as a valid domain of the initial axiomatization. As in the case of the supra-domain approach, our objective on the intra-domain approach is twofold.

Our first objective within the intra-domain approach is the analysis of valid domains of a pioneering and renowned axiomatization result due to Shapley (1953) who proved that, on the set of all TU-games on a given finite and nonempty set of players, the Shapley value is the only value that satisfies efficiency (E), symmetry (S), additivity (A) and the null player property (NP). Of course, many other characterizations of the Shapley value have been established so far using alternative axioms; for a selected list of papers, see Young (1985) ; Chun (1989); Feltkamp (1995); Van den Brink and Van der Laan (1998); Van den Brink (2001); Laruelle and Valenciano (2001); Hamiache (2001) or de Clippel (2018). There are also many books and book chapters that further reflect the notoriety of the Shapley value and its extensions to a wide variety of game classes; see for examples, Roth (1988); Aumann (1990); Winter (2002) or Algaba et al. (2019).

Despite this impressive expansion of contributions on the Shapley value, we were only able to identify two publications related to valid domains of the (E)+(S)+(A)+(NP) characterization mentioned above; hereafter, such a domain will simply be called a Shapley valid domain. The first type of Shapley valid domain is due to Neyman (1989) who showed that the (E)+(S)+(A)+(NP) characterization of the Shapley value remains valid on any additive subgroup generated by a given TU-game and its subgames. Another type of Shapley valid domain is provided by Peleg and Sudhölter (2007) and consists in any convex cone (that is any nonempty set of TU-games stable under linear combinations with non negative coefficients) containing all unanimity TU-games. We introduce the notion of non-trivial symmetry class of a TU-game and show that the axioms of Shapley (1953) again characterize a single value on any set $\mathcal{F}$ of TU-games whose generators each has at most one non-trivial symmetry class. Our results on conically consistent sets of TU-games generalize those of Neyman (1989) and Peleg and Sudhölter (2007).

Still on the intra-domain approach, our second objective is to show that the set of simple games on a given nonempty set of players can be seen as a valid domain of a characterization of the Shapley value due to Van den Brink (2001). Using a fairness axiom, here called Van den Brink fairness (VDB-F), the author proved that the Shapley value is the only value, on the set of all TU-games on any finite and nonempty set of players, that meets (E), (NP) and (VDB-F). Before we continue, it is worth mentioning that the Axiom (VDB-F) is the requirement that if from any TU-game to another, one simply adds TU-game in which two players are symmetric, then the payoffs of these two players must change by the same amount. The author himself proved that his (E)+(NP)+(VDB-F) characterization of the Shapley value is still valid on the set of all voting games (these are $\{0, 1\}$-valued TU-games) on any finite and nonempty set of players. In a follow up paper, Casajus (2011) remarkably proved the same result on any cone of TU-games that contains all unanimity games. Simple games are monotonous voting games and are very often the model encountered in voting bodies where the voting rule is a yes-no voting such as the simple (also known as qualified) majority voting. It is thus of real interest to check whether the (E), (NP) and (VDB-F) still characterize the restriction of the Shapley value on simple games.

Exceptionally, the restriction of the Shapley value on the domain of simple games is known as the Shapley-Shubik (voting power) index after Shapley and Shubik (1954) made it a suitable measure of voting power. Unfortunately, the set of simple games is not a valid domain of the (E)+(NP)+(VDB-F) characterization of the Shapley value. We note that this is only due to the formulation of the Axiom (VDB-F). Clearly, the sum of two simple games is no longer a simple game. In order to get out of this impasse, we note that (VDB-F) is, over the set of all TU-games over any finite and nonempty set of players, the requirement that if from one TU-game to another, two players play symmetric roles in the changes that are observed, their respective payoffs should change by the same amount. Now, instead of summing up two simple games, we focus our attention on the contribution of each player to the changes that occur from one simple game to another. The new axiom is simply denoted by (F) and we prove on the one hand that (F) and (VDB-F) are equivalent in terms of axioms on the set of all TU-games; and on the other hand that, the Shapley-Shubik index is the only voting power index that satisfies (E), (NP) and (F).

In the sequel, the presentation is structured around five chapters that include some preliminaries on simple games, results within the supra-domain approach and results within the intra-domain approach. More precisely, Part I is subdivided into three chapters. Chapter 1 of Part I presents basic concepts on simple games and power indices as well as new concepts such as merging, supplementation and equivalence on simple games together with some associated axioms. In Chapter 2 of Part I, we provide new axiomatizations of the Public Good Index on simple games with a variable electorate. The same work is done in Chapter 3 of Part I for the Deegan-Packel index. Part II is an intra-domain study of the Shapley value and comprises two chapters. In Chapter 4 of Part II, we provide our analysis of Shapley valid domains and then prove that Neyman's domains and the Peleg-Sudhölter's domains are particular instances of ours. The fairness condition (VDB-F) is revisited in Chapter 5 of Part II and this leads us to a new axiomatization of the Shapley-Shubik index using a restatement of the fairness condition.

Part I

A supra-domain study of two voting power indices

Chapter One

Preliminaries on simple games

In this chapter, we present basic concepts related to simple games. Simple games are tools used to model yes-no voting: these are decision-making processes where proposals are made to the voting body and each voter provides his opinion by saying Yes or No without abstention. The final outcome is the Adoption or the Rejection of that proposal. One of the main problem on this family of games is the power measurement. Power indices are methods for numerical evaluation of players' voting power in simple (voting) games. But a question still persists over decades: *what is a "good" power index?* Axiomatizing power indices provides a partial and practical answer to this question as axioms that characterize a given power index highlight most of its key features.

When the electorate is fixed, classical axioms of *symmetry*, *efficiency* and the *null player property* are intra-game axioms: these are axioms that are applied on a single game. Besides, inter-game axioms are used to indicate how a power index changes when we move from a game to another; this is the case of the *Transfer* (axiom) from Dubey (1975), the *Deegan-Packel mergeability* from Deegan and Packel (1978) and the *Holler-Packel mergeability* from Holler and Packel (1983).

The aim of this chapter is to present formal definitions of basic concepts on simple games that will be used in subsequent chapters. This involves classical concepts as well as the ones we newly introduce. The whole chapter is organized as follows: Section 1.1 formally describes simple games and power indices. For an illustrative example, we consider the voting rule of the Senegalese Parliament that resulted from the parliamentary election held on 31 July 2022. Section 1.2 introduces the merging and the supplementation operations together with the notion of equivalent games. The chapter ends with the statement of some axioms and some preliminary results that highlight some immediate properties deduced from the axioms presented.

1.1 Simple games and power indices

We denote by $\mathbb{N}$ the set of all non negative integers and by $\mathbb{N}^*$ the set of all positive integers.

1.1.1 Simple games

In this section, we consider an infinite set $\mathcal{P}$ of potential players indexed by positive integers; that is $\mathcal{P} = \mathbb{N}^*$. Each finite and nonempty subset S of $\mathcal{P}$ is called a coalition and $|S|$ or s denotes the cardinality of S, that is the number of players in S.

Definition 1.1.1. *A simple game is a pair $G = (N, v)$, where N is a finite nonempty subset of $\mathcal{P}$, and v is a $\{0,1\}$-valued map defined on the powerset of N such that:*

(i) $v(\emptyset) = 0$;

(ii) $v(N) = 1$;

(iii) *for all* $S, T \subseteq N$, $S \subseteq T$ *implies* $v(S) \leq v(T)$.

We set, unless otherwise stated, $N = \{1, 2, \ldots, n\}$ *with* $|N| = n \geq 2$.

The set N is the electorate associated to the simple game (N, v) and elements of N are called voters (or players). The set of all coalitions of N will be denoted by 2^N. Firstly, condition (i) simply means that when all voters are against a proposal, that proposal should be rejected. Secondly, condition (ii) is the disposition that when all voters are for the adoption of a proposal, that proposal should be collectively adopted. Finally, condition (iii) is the statement that if S can ensure the adoption of a proposal and T contains all members of S, then T too can ensure the adoption of that proposal; that is, v is *monotonic.*

Notation 1.1.1. *The set of all simple games with the same electorate N is denoted by $\mathcal{G}_N$ and the set of all possible simple games by $\mathcal{G}$; that is*

$$\mathcal{G} = \bigcup_{N \in \mathbb{E}} \mathcal{G}_N$$

where $\mathbb{E}$ is the set of all finite and nonempty subsets of the set $\mathcal{P}$ of all potential voters.

Notation 1.1.2. *Let N be a given electorate and L a coalition, we denote by $\mathbb{1}_L$ the indicator function of L; that is for all $i \in N$,*

$$\mathbb{1}_L(i) = \begin{cases} 1 \text{ if } i \in L \\ 0 \text{ otherwise.} \end{cases}$$

By definition, coalitions in a simple game are partitioned into two groups as in the following definition.

Definition 1.1.2. *Given $G = (N, v) \in \mathcal{G}$,*

- *a coalition S is called* winning *if $v(S) = 1$, and* losing *otherwise;*
- *a winning coalition S is called* minimal *if $v(S \setminus \{i\}) = 0$ for all player $i \in S$.*

The set of all winning coalitions in G is denoted by $\mathcal{W}(G)$, the set all minimal winning coalitions by $\mathcal{M}(G)$ and the set of all minimal winning coalitions that contain a player $i \in N$ by $\mathcal{M}_i(G)$.

Each coalition is either winning or losing in a simple game. Thus, a simple game $G = (N, v)$ is completely described by its set N of voters and its set $\mathcal{W}(G)$ of winning coalitions. Therefore, the simple game G will also be denoted by $G = (N, \mathcal{W}(G))$ or simply by $G = (N, \mathcal{M}(G))$ as sometimes used by Felsenthal and Machover (1998) and Peleg and Sudhölter (2007).

Example 1.1.1. *Here are some examples of simple games given a set N of voters.*

1) **Unanimity simple game**: $G = (N, \mathcal{W}(G))$ *with* $\mathcal{W}(G) = \{N\}$.

2) **Absolute majority**: $G = (N, \mathcal{W}(G))$ *where* $\mathcal{W}(G) = \{S \in 2^N : |S| > \frac{1}{2}|N|\}$.

3) **Qualified majority with quota** q: $G = (N, \mathcal{W}(G))$ *where* $\mathcal{W}(G) = \{S \in 2^N : |S| \geq q|N|\}$ *with* $q \in]\frac{1}{2}; 1]$.

4) **Weighted voting game**: $G = (N, \mathcal{W}(G))$ *where* $\mathcal{W}(G) = \{S \in 2^N : \sum_{i \in S} w_i \geq q\}$ *where* q, w_1, w_2, ..., w_n *are some positive real numbers such that* $\sum_{i \in N} w_i > q > 0$.

5) **The decision rule of the Parliament of the Basque Country resulted from the elections held on April 14^{th}, 2005 (see Alonso-Meijide et al. (2010b))** *can be seen as a simple game* $G = (N, \mathcal{W}(G))$ *where* $N = \{1, 2, 3, 4, 5, 6, 7\}$ *and* $\mathcal{M}(G)$ *consists of the following coalitions* $\{1, 2\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 3, 7\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$ *and* $\{2, 4, 5, 6, 7\}$.

Depending on how the complement of a winning coalitions is treated, here are two classes of simple games.

Definition 1.1.3. *A simple game $G = (N, \mathcal{W}(G))$ is,*

- *proper if* $\forall S \subseteq N : S \in \mathcal{W}(G) \Rightarrow N \backslash S \notin \mathcal{W}(G)$;
- *strong if* $\forall S \subseteq N : S \notin \mathcal{W}(G) \Rightarrow N \backslash S \in \mathcal{W}(G)$.

Being a proper simple game simply means that the complement of any winning coalition is losing. In the same way, a simple game is strong if the complement of any losing coalition is winning.

Definition 1.1.4. *Given a simple game $G = (N, v)$,*

- *a null player in G is any player i such that* $v(S \cup \{i\}) = v(S)$ *for all* $S \subseteq N \setminus \{i\}$;

- a *dictator* in G is any player i such that $v(S \cup \{i\}) = 1$ for all $S \subseteq N \setminus \{i\}$;
- given a coalition S, a player $i \in S$ is *decisive* in S if $v(S) = 1$, but $v(S \setminus \{i\}) = 0$. In this case, player i is said to be *complementary* to $S \setminus \{i\}$.

We denote by $\mathcal{D}_i(G)$ the set of all winning coalitions in which player i is decisive in G.

Note that, a player i in a simple game G is decisive in a coalition S if $S \in \mathcal{W}(G)$ and $S\backslash\{i\} \notin \mathcal{W}(G)$. Moreover, player i is a null player and player j a dictator in a simple game G if

$$\mathcal{D}_i(G) = \mathcal{M}_i(G) = \emptyset \text{ and } \mathcal{M}_j(G) = \{\{j\}\}.$$

In other words, a null player in G is any player who is decisive nowhere in G and a dictator is any player whose vote for the adoption of a proposal commits everyone.

Notation 1.1.3. *The set of all null players in a simple game G is denoted by $N^0(G)$.*

Before we continue, here are some specific simple games that will be useful in the sequel.

Definition 1.1.5. *Given a finite and nonempty subset S of $\mathcal{P}$, the singleton game associated to S denoted by G_S is the simple game $G_S = (S, \mathcal{W}(G_S))$ with*

$$\mathcal{M}(G_S) = \{\{i\} : i \in S\}.$$

A singleton game associated to a coalition S is a simple game in which players in S and only those players are each a dictator.

Next, given a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $j \in \mathcal{P}\backslash N$, we denote by $G^{i\leftrightarrow j}$ the simple game obtained from G by only replacing i by j. Formally, $G^{i\leftrightarrow j} = (N^{i\leftrightarrow j}, \mathcal{W}(G^{i\leftrightarrow j}))$ with $N^{i\leftrightarrow j} = (N \cup \{j\})\backslash\{i\}$ and

$$\mathcal{W}(G^{i\leftrightarrow j}) = \{S \in \mathcal{W}(G) : i \notin S\} \cup \{(S \cup \{j\})\backslash\{i\} : S \in \mathcal{W}(G) \text{ and } i \in S\}.$$

Note that given a coalition $T \subseteq \mathcal{P}\backslash \{i, j\}$,

$$G_{T\cup\{i\}} = \left(G_{T\cup\{j\}}\right)^{j\leftrightarrow i}.$$

Furthermore, given a positive integer p and $\{i_1, i_2, ..., i_p\} \subseteq \mathcal{P}\backslash N$, we denote by $G\,[i_1, i_2, ..., i_p]$ the simple game obtained from G by successively introducing players $i_1, i_2, ..., i_{p-1}$ and i_p as null players in such a way that the player set in $G\,[i_1, i_2, ..., i_p]$ is $N \cup \{i_1, i_2, ..., i_p\}$ and the set of minimal winning coalitions is still $\mathcal{M}\,(G)$. Note that player i_1 gets in first, player i_2 second and so on. Moreover, it follows by definition that,

$$G\,[i_1, i_2, ..., i_p] = G\,[j_1, j_2, ..., j_p] \text{ whenever } \{i_1, i_2, ..., i_p\} = \{j_1, j_2, ..., j_p\}\,.$$

That is, the entry order does not matter in defining $G\,[i_1, i_2, ..., i_p]$.

The following propositions underline the link between winning coalitions and minimal winning coalitions.

Proposition 1.1.1 (Safokem et al. (2021)). *Given a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition $S \subseteq N$. If $S \in \mathcal{W}(G)$, then $K \subseteq S$ for some $K \in \mathcal{M}(G)$.*

Proof. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition $S \subseteq N$. Suppose that $S \in \mathcal{W}(G)$. Pose $S_0 = S$ and $T_0 = \{i \in S : S \setminus \{i\} \in \mathcal{W}(G)\}$. If $T_0 = \emptyset$, then for all $i \in S, S \setminus \{i\} \notin \mathcal{W}(G)$. Since $S \in \mathcal{W}(G)$, then $S \in \mathcal{M}(G)$. Otherwise, consider a player $i_1 \in T_0$. It follows that $S \setminus \{i_1\} \in \mathcal{W}(G)$. Pose $S_1 = S \setminus \{i_1\}$ and $T_1 = \{i \in S_1 : S_1 \setminus \{i\} \in \mathcal{W}(G)\}$. We have $|S_1| < |S|$. Since S is finite, by iterating this procedure, one constructs three finite sequences $(i_l)_{1\leq l \leq p}$, $(S_l)_{0\leq l\leq p}$ and $(T_l)_{0\leq l\leq p}$ such that $p \leq |S|$ and for all $l \in \{1, ..., p\}, S_l = S \setminus \{i_1, ..., i_l\} \in \mathcal{W}(G)$, $T_l = \{i \in S_l : S_l \setminus \{i\} \in \mathcal{W}(G)\}$, $i_l \in T_{l-1}$ and $T_p = \emptyset$. Hence $S_p \setminus \{i\} \notin \mathcal{W}(G)$ for all $i \in S_p$. Thus $S_p = S \setminus \{i_1, i_2, ..., i_p\} \in \mathcal{M}(G)$ and $S_p \subseteq S$. ■

Proposition 1.1.2 (Safokem et al. (2021)). *Consider a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $S \subseteq N$.*

If $S \in \mathcal{D}_i(G)$, then there exists $K \subseteq S$ such that $K \in \mathcal{M}_i(G)$.

Proof. Suppose that $S \in \mathcal{D}_i(G)$ in the simple game $G = (N, \mathcal{W}(G))$. Then $S \in \mathcal{W}(G)$ and it follows by Proposition 1.1.1 that there exists $T \in \mathcal{M}(G)$ such that $T \subseteq S$. Suppose that $i \notin T$. Then $T \subseteq S\backslash\{i\}$ and it follows by monotonicity that $S\backslash\{i\} \in \mathcal{W}(G)$. A contradiction arises since $S \in \mathcal{D}_i(G)$. It holds that $T \in \mathcal{M}_i(G)$ and $T \subseteq S$. ■

Let $G = (N, v)$ be a simple game. If $\pi : N \to N$ is a *permutation* of the player set N (that is a bijection from N to N), then the simple game $G_\pi = (N, v_\pi)$ is defined by $v_\pi(S) = v(S_\pi)$ for all $S \subseteq N$ where

$$S_\pi = \{\pi^{-1}(i) : i \in S\}$$

and for all $i, j \in N$,

$$\pi^{-1}(i) = j \text{ if and only if } \pi(j) = i.$$

The mapping π^{-1} refers to the inverse of π. In particular, the transposition of two players i and j is the bijection $\tau_{i,j} : N \to N$ defined by

$$\tau_{i,j}(i) = j,\ \tau_{i,j}(j) = i \text{ and } \tau_{i,j}(k) = k \text{ for all } k \in N \setminus \{i, j\}.$$

Definition 1.1.6. *Two players i and j in the game $G = (N, \mathcal{W}(G))$ are called symmetric if*

$$\forall S \subseteq N\backslash\{i,j\}, S \cup \{i\} \in \mathcal{W}(G) \Longleftrightarrow S \cup \{j\} \in \mathcal{W}(G).$$

That is $\mathcal{W}\left(G_{\tau_{i,j}}\right) = \mathcal{W}(G)$.

In words, two symmetric players act in the same way when one replaces the other.

Example 1.1.2. *In the simple game of the Parliament of the Basque Country in Example 1.1.1, players 4 and 5 as well as players 6 and 7 are symmetric.*

Dubey (1975), Deegan and Packel (1978) and Holler and Packel (1983) introduced some operators on simple games that we recall below.

Definition 1.1.7. *Given two simple games $G_1 = (N, \mathcal{W}(G_1))$ and $G_2 = (N, \mathcal{W}(G_2))$ with the same electorate N,*

(i) The merging game *obtained from G_1 and G_2 is denoted by $G_1 \oplus G_2 = (N, \mathcal{W}(G_1 \oplus G_2))$ and is such that*

$$\mathcal{W}(G_1 \oplus G_2) = \mathcal{W}(G_1) \cup \mathcal{W}(G_2).$$

(ii) G_1 and G_2 are mergeable *if*

$$\forall (S_1, S_2) \in \mathcal{M}(G_1) \times \mathcal{M}(G_2), S_1 \not\subseteq S_2 \text{ and } S_2 \not\subseteq S_1.$$

Saying that two simple games G_1 and G_2 are mergeable simply means that in G_1 and G_2 each minimal winning coalition from one game is losing in the other. Furthermore, note that:

- for all $i \in N, \mathcal{M}_i(G_1 \oplus G_2) = \mathcal{M}_i(G_1) \cup \mathcal{M}_i(G_2)$;
- when the simple games G_1 and G_2 are mergeable, it follows that for all $i \in N$, $\mathcal{M}_i(G_1) \cap \mathcal{M}_i(G_2) = \emptyset$ and $|\mathcal{M}_i(G_1 \oplus G_2)| = |\mathcal{M}_i(G_1)| + |\mathcal{M}_i(G_2)|$.

1.1.2 Review of some power indices

In this section, we provide a short view on tools for quantitative measurement of voting power. For qualitative approaches of measuring voting power, we refer to Isbell (1958); Tomiyama (1987); Diffo Lambo and Moulen (2002) or Tchantcho et al. (2008).

Definition 1.1.8. *A* power index *φ is a map defined on $\mathcal{G}$ such that $\varphi(G) \in \mathbb{R}^N$ for every $G = (N, \mathcal{W}(G)) \in \mathcal{G}$.*

The share $\varphi_i(G)$ measures the ability of voter i to turn a losing coalition, as he/she gets in, into a winning one; or conversely, to make a winning coalition losing, as he/she moves out. Intuitively, the larger $\varphi_i(G)$, the greater the power of voter i. For many authors, the definition of a power index includes the disposition that players' shares are non negative numbers; see Kóczy (2009) or Kong and Peters (2021). But in general, the positivity fulfillment is deduced from some reasonable properties of power indices. Many power indices exist in the literature, what follows is the definition of some classical power indices.

The Shapley-Shubik index was introduced by Shapley and Shubik (1954) and it is the restriction of the Shapley value to the class of simple games. For this power index, all possible orderings of voters are considered with equal probabilities and each voter is associated with the total number of being the swing voter (each time he/she is the first to turn the ascending coalition from losing to winning). The voting power of a voter by the Shapley-Shubik index is simply the average number of times he/she is the swing voter. Formally,

Definition 1.1.9. *The Shapley-Shubik index is the map SS defined on $\mathcal{G}$ such that*

$$\forall G = (N, \mathcal{W}(G)) \in \mathcal{G}, \forall i \in N, \ \mathrm{SS}_i(G) = \sum_{S \in \mathcal{D}_i(G)} \frac{(s-1)!(n-s)!}{n!}$$

where $s = |S|$ and $n = |N|$.

The Banzhaf power index appears in Banzhaf (1965), although Penrose (1946) defines a measure which is the half of the Banzhaf's power index. For this index, all coalitions that contain a given voter are considered with equal probabilities and he/she is associated with the total number of times he/she is decisive. The voting power of a voter by the Banzhaf power index is simply the average number of times he/she is decisive in the coalitions he/she belongs to. Formally,

Definition 1.1.10. *Given a simple game $G = (N, \mathcal{W}(G))$ and a player $i \in N$,*

- *the Banzhaf (or absolute Banzhaf or Banzhaf-Penrose) index of voter i is given by*

$$B_i(G) = \frac{|\mathcal{D}_i(G)|}{2^{n-1}};$$

- *The normalized Banzhaf (or relative Banzhaf or Banzhaf-Coleman) index of voter i, is given by*

$$\tilde{B}_i(G) = \frac{|\mathcal{D}_i(G)|}{\sum_{j \in N} |\mathcal{D}_j(G)|}.$$

The Johnston index is due to Johnston (1978). Given a simple game G, a coalition S and a voter i, let $d(S)$ denotes the total number of decisive voters in S. For the Johnston index, all coalitions containing i are considered and i receives a Johnston reward of $1/d(T)$ each time he is decisive in a coalition T and nothing otherwise. The voting power of i by the Johnston index is the sum of all his Johnston rewards over the coalitions to which it belongs. Formally,

Definition 1.1.11. *Given a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a player $i \in N$,*

- *the non-normalized Johnston index of voter i is given by*

$$J_i(G) = \sum_{S \in \mathcal{D}_i(G)} \frac{1}{d(S)};$$

- The *Johnston index* of voter i is given by

$$\tilde{J}_i(G) = \frac{J_i(G)}{\sum_{j\in N} J_j(G)}.$$

The Deegan-Packel index was introduced by Deegan and Packel (1978). For this power index, all minimal winning coalitions are considered and each voter receives a Deegan-Packel reward of $1/|S|$ each time he is a member of a minimal winning coalition S and nothing otherwise. The voting power of a voter by the Deegan-Packel index is his average Deegan-Packel reward over all minimal winning coalitions. Formally,

Definition 1.1.12. *Given a simple game* $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ *and a player* $i \in N$, *the Deegan-Packel index of voter* i *is given by*

$$\mathrm{DP}_i(G) = \frac{1}{|\mathcal{M}(G)|} \sum_{S\in\mathcal{M}_i(G)} \frac{1}{|S|}. \tag{1.1}$$

The Holler-Packel index was first introduced and used in Holler (1978) to measure the voting power of parties in the Finnish Parliament; it was explicitly defined in Holler (1982). For this power index, each voter is associated with a Holler-Packel reward equal to the total number of minimal winning coalitions he belongs to. The voting power of a voter by the Holler-Packel index is his relative Holler-Packel reward assuming that individual voting powers sum to 1 and are each proportional to the corresponding Holler-Packel reward. Formally,

Definition 1.1.13. *The Holler-Packel index (also known as the Public Good Index) is the map HP define on* $\mathcal{G}$ *such that*

$$\forall G = (N, \mathcal{W}(G)) \in \mathcal{G}, \forall i \in N, \mathrm{HP}_i(G) = \frac{|\mathcal{M}_i(G)|}{\sum_{j\in N} |\mathcal{M}_j(G)|}. \tag{1.2}$$

For more literature on power indices, see Andjiga et al. (2003).

1.1.3 An illustration example

In this section, we model the Senegalese Parliament as a simple game where the data origin from the results of parliamentary elections held in Senegal on 31 July 2022. We also compute individual voting powers for each of the preceding power indices.

The situation

The Senegalese National Assembly is a 6-party Parliament with 165 seats distributed as described in Table 1.1 below:

Party	Simplified designation	Total number of seats (weight)
United in Hope	1	82
Liberate the People	2	56
Wallu Sénégal	3	24
The Servants – MPR	4	1
AAR Sénégal	5	1
Bokk Gis Gis	6	1

Table 1.1: Senegalese National Assembly: parties and distribution of seats.

Case of absolute majority voting

We consider here the case of a standard bill whose adoption requires an absolute majority of favorable votes by parliamentarians. Our aim is the evaluation of the ratio of power enjoyed by each party in the Senegalese National Assembly. We assume that votes are coordinated within each party and a party then acts as a single voter whose weight is the total number of seats the party records. The voting rule is modeled as a weighted simple game the quota of which is exactly 83 and the weights is as in Table 1.1. For the sake of simplicity, we identify each party with a positive integer as in the table. The set of voters is then identified with $N = \{1, 2, 3, 4, 5, 6\}$ and the corresponding set of minimal winning coalitions is

$$\mathcal{M}(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3, 4, 5, 6\}\}.$$

The computation of voting power of Senegalese parties

The set of minimal winning coalitions of the weighted simple game in consideration reveals a preliminary qualitative information: except, the leading party, all other parties are symmetric. This is of course a contrast with the distribution of seats observed. The Liberate the People party with 56 seats and the Wallu Sénégal party with 24 seats both have substitutable roles with the three small sized parties with one seat each!

Numerical results are as in Table 1.2 below.

Power index (φ)	$\varphi_1(G)$	$\varphi_2(G)$	$\varphi_3(G)$	$\varphi_4(G)$	$\varphi_5(G)$	$\varphi_6(G)$
Shapley-Shubik (SS)	$\frac{10}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$
Banzhaf-Coleman ($\tilde{B}$)	$\frac{30}{40}$	$\frac{2}{40}$	$\frac{2}{40}$	$\frac{2}{40}$	$\frac{2}{40}$	$\frac{2}{40}$
Johnston ($\tilde{J}$)	$\frac{275}{310}$	$\frac{7}{310}$	$\frac{7}{310}$	$\frac{7}{310}$	$\frac{7}{310}$	$\frac{7}{310}$
Deegan-Packel (DP)	$\frac{25}{60}$	$\frac{7}{60}$	$\frac{7}{60}$	$\frac{7}{60}$	$\frac{7}{60}$	$\frac{7}{60}$
Holler-Packel (HP)	$\frac{5}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$

Table 1.2: Computation of the power indices of parties in the Senegalese Parliament

Each of the five power indices gives equal voting power to the five parties other than the dominant party. This is consistent with the fact that these parties all play symmetric roles in case of an absolute majority voting. Moreover, the dominant party's voting power is maximal under the Johnston index and minimal under the Holler-Packel index. Of course, this is due to the distinct scenarios on which the five power indices are based. Once more, the change of the distribution of individual voting power from one power index to another requires additional analyzes such as axiomatization results that would motivate the choice of a power index.

1.2 New operations and axioms of power indices

In this section, we present new operations on simple games namely merging and supplementation. Concepts of equivalent simple games from the view point of a voter is also introduced. This leads us to some classical axioms and new axioms for power indices.

1.2.1 Merging of independent players

Players in a coalition T can enter a simple game $G = (N, \mathcal{W}(G))$ with a *prior agreement* to act as a single player $i_T \in \mathcal{P}\backslash N$. In this case, players in T are merged into i_T, their representative in the new simple game. Before a formal definition, we associated, given a coalition T and a voter $i_T \in \mathcal{P}\backslash N$, any coalition S with the coalition denoted by S_T and defined by

$$S_T = \begin{cases} (S \setminus \{i_T\}) \cup T & \text{if } i_T \in S \\ S & \text{otherwise.} \end{cases}$$

The presence of i_T in a coalition S is seen as that of all the members of T.

Definition 1.2.1. *Given a simple game $G = (N, v)$, a coalition T and a player $i_T \in \mathcal{P}\backslash N$, the merged game obtained from G by merging members of T into i_T is denoted by $G^T = (N^T, v^T)$ and defined by:*

(i) $N^T = (N \setminus T) \cup \{i_T\}$;

(ii) $v^T(S) = v(S_T),\ \forall S \subseteq N^T$.

The move from G to G^T is called a merging operation while the move from G^T to G is called a splitting operation.

We will sometimes replace G^T by $G^{T\to k}$ to specify that i_T is identified with a given player $k \in \mathcal{P}\backslash N$. Note that in Definition 1.2.1, S_T is a subset of N for each $S \subseteq N^T$. Similarly, for each $S \subseteq N$, we denote by S^T the subset of N^T defined by

$$S^T = \begin{cases} (S \setminus T) \cup \{i_T\} & \text{if } S \cap T \neq \emptyset \\ S & \text{otherwise.} \end{cases}$$

Now, the membership of any voter in T is seen as that of i_T.

The merging operation is well-known in the literature although authors sometimes used distinct notation options, see Knudsen and Østerdal (2012) or Slavov and Evans (2017). It is also known under the name of *amalgamation* as in Lehrer (1988) or in Haviv (1995).

Example 1.2.1. Consider the simple game $G = (N, \mathcal{W}(G))$ such that $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$. Let $T = \{1, 5\}$. Here, $N^T = \{i_T, 2, 3, 4\}$ and $\mathcal{M}(G^T) = \{\{i_T, 2\}, \{i_T, 3, 4\}\}$. If we set $i_T = 6$, then $N^{T\to 6} = \{6, 2, 3, 4\}$ and $\mathcal{M}(G^{T\to 6}) = \{\{6, 2\}, \{6, 3, 4\}\}$. Similarly, for $K = \{4, 5\}$, $N^K = \{1, 2, 3, i_K\}$ and $\mathcal{M}(G^K) = \{\{1, 2\}, \{2, 3, i_K\}, \{1, 3, i_K\}\}$. Still for illustration with $T = \{1, 5\}$, note that $\{1, 4\}^T = \{i_T, 4\}$, $\{1, 4, 5\}^T = \{i_T, 4\}$ and $\{2, 3, 4\}^T = \{2, 3, 4\}$, $\{i_T, 3\}_T = \{1, 3, 5\}$ and $\{2, 3, 4\}_T = \{2, 3, 4\}$.

Remark 1.2.1. *Note that the merging operation in the sense of Deegan and Packel (1978) and Holler and Packel (1983) is related to merging two simple games with the same electorate meanwhile our merging operation is related to merging players in one simple game; thus in our case the electorate is variable.*

We now define two particular configurations of players: disconnected players and independent players.

Definition 1.2.2. *Two players i and j in the simple game $G = (N, \mathcal{W}(G))$ are disconnected if $|S \cap \{i, j\}| \leq 1$ for all $S \in \mathcal{M}(G)$.*

Definition 1.2.2 simply means that $\mathcal{M}_i(G) \cap \mathcal{M}_j(G) = \emptyset$. That is, two players are disconnected in a simple game if no minimal winning coalition contains both players at the same time.

Definition 1.2.3. *Two players i and j in a simple game $G = (N, \mathcal{W}(G))$ are independent if i and j are disconnected and for all losing coalitions $S \subseteq N \setminus \{i, j\}$:*

$$\begin{pmatrix} S \cup \{i\} \in \mathcal{D}_i(G) \\ \text{and} \\ S \cup \{j\} \in \mathcal{D}_j(G) \end{pmatrix} \Rightarrow \text{for some } S', S'' \subseteq S, \begin{pmatrix} S' \cup \{i\} \in \mathcal{D}_i(G),\ S' \cup \{j\} \notin \mathcal{D}_j(G) \\ \\ S'' \cup \{j\} \in \mathcal{D}_j(G) \text{ and } S'' \cup \{i\} \notin \mathcal{D}_i(G) \end{pmatrix}.$$

Furthermore, we say that a coalition T consists of independent players in G if any two players in T are independent in G.

Intuitively, two players are independent in a simple game if each time they are both complementary to the same losing coalition S, each of the two players is complementary to some subset of S while the other is not.

Example 1.2.2. *In Example 1.2.1 where $G = (N, \mathcal{W}(G))$ is such that $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$, players 4 and 5 are disconnected in the simple game G and there is no losing coalition S such that $S \cup \{4\}$ and $S \cup \{5\}$ are winning. Thus, 4 and 5 are independent in G. Now, players 1 and 5 are disconnected in G; but* *for $S = \{2,3\}$, S is losing, $S \cup \{1\}$ and $S \cup \{5\}$ are winning, but $S' \cup \{5\}$ is losing for all $S' \subsetneq S$. Thus, 1 and 5 are not independent in G. Also observe that players 1 and 3 are not disconnected in the simple game G.*

It is worth mentioning that if i is a null player in a simple game G, then for any other player j, i and j are independent. In the general case, we have the following result.

Proposition 1.2.1 (Safokem et al. (2021)). *Given a simple game $G = (N, \mathcal{W}(G))$, two disconnected players i and j are independent if and only if for all $S \in \mathcal{M}_i(G)$ and for all $S' \in \mathcal{M}_j(G)$, $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$.*

Proof. Consider two disconnected players i and j in a simple game $G = (N, \mathcal{W}(G))$. First suppose that players i and j are independent. Consider $S \in \mathcal{M}_i(G)$ and $S' \in \mathcal{M}_j(G)$. To prove that $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$, suppose the contrary. Without loss of generality (w.l.o.g), suppose that $S \setminus (S' \cup \{i\}) = \emptyset$; that is, $S \subseteq S' \cup \{i\}$. Since i and j are disconnected, $j \notin S$. Therefore $S \setminus \{i\} \subseteq S' \setminus \{j\}$. Let $K = S' \setminus \{j\}$. It follows that K is losing, $K \cup \{i\} \in \mathcal{D}_i(G)$ and $K \cup \{j\} \in \mathcal{D}_j(G)$. Since i and j are independent, there exists $S'' \subseteq K$ such that $S'' \cup \{j\} \in \mathcal{D}_j(G)$ and $S'' \cup \{i\} \notin \mathcal{D}_i(G)$. Note that $K \cup \{i\}$ is winning while $S'' \cup \{i\}$ is losing. We deduce that S'' is a proper subset of K and that $S'' \cup \{j\}$ is a proper winning subset of S'; that is $S'' \cup \{j\} \subseteq S'$ and $S'' \cup \{j\} \neq S'$. A contradiction arises since $S' \in \mathcal{M}_j(G)$.

Now suppose that for all $S \in \mathcal{M}_i(G)$ and for all $S' \in \mathcal{M}_j(G)$, $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$. Consider a losing coalition $S \subseteq N \setminus \{i,j\}$ such that both $S \cup \{i\} \in \mathcal{D}_i(G)$ and $S \cup \{j\} \in \mathcal{D}_j(G)$. Then, by Proposition 1.1.2, there exist $S', S'' \subseteq S$ such that $S' \cup \{i\} \in \mathcal{M}_i(G)$ and $S'' \cup \{j\} \in \mathcal{M}_j(G)$. To prove that $S' \cup \{j\} \notin \mathcal{D}_j(G)$ and $S'' \cup \{i\} \notin \mathcal{D}_i(G)$, suppose the contrary. W.l.o.g., suppose that $S' \cup \{j\} \in \mathcal{D}_j(G)$. Then by Proposition 1.1.2, there exists $L \subseteq S'$ such that $L \cup \{j\} \in \mathcal{M}_j(G)$. It holds that $S' \cup \{i\} \in \mathcal{M}_i(G)$, and $L \cup \{j\} \in \mathcal{M}_j(G)$ with $L \setminus S' = \emptyset$. A contradiction arises since by assumption we should have $L \setminus S' = (L \cup \{j\}) \setminus (S' \cup \{i,j\}) \neq \emptyset$. ■

The next result tells us how merging independent players impacts the structure of a simple game.

Proposition 1.2.2 (Safokem et al. (2021)). *Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two players. If T is a coalition of independent players, then*

(a) *For all $S, R \in \mathcal{M}(G)$, $S \neq R$ implies $S^T \neq R^T$;*

(b) *$\mathcal{M}(G^T) = \{S^T : S \in \mathcal{M}(G)\}$;*

(c) *$|\mathcal{M}_j(G^T)| = |\mathcal{M}_j(G)|$ for all $j \in N \setminus T$;*

(d) $|\mathcal{M}_{i_T}(G^T)| = \sum_{i\in T} |\mathcal{M}_i(G)|$.

Proof.

Suppose that T is a coalition of independent players with $|T| \geq 2$.

(a). Consider $S, R \in \mathcal{M}(G)$ such that $S \neq R$. First suppose that $S \cap T = R \cap T = \emptyset$. By definition, $S^T = S \neq R = R^T$. Now suppose that $S \cap T \neq \emptyset$ and $R \cap T = \emptyset$. Then $i_T \in S^T$ and $i_T \notin R^T$. Therefore $S^T \neq R^T$. Similarly, $S^T \neq R^T$ if $S \cap T = \emptyset$ and $R \cap T \neq \emptyset$. Finally suppose that $S \cap T \neq \emptyset$ and $R \cap T \neq \emptyset$. Since T contains only pairs of independent (disconnected) players, then $S \cap T = \{i\}$ and $R \cap T = \{j\}$ for some $i, j \in T$. If $i = j$, then $S^T \setminus R^T = S \setminus R \neq \emptyset$ since $R \in \mathcal{M}(G)$. Otherwise $i \neq j$. Players i and j are independent, $S \in \mathcal{M}_i(G)$ and $R \in \mathcal{M}_j(G)$, thus by Proposition 1.2.1, $S \setminus (R \cup \{i\}) \neq \emptyset$. Since $S \setminus (R \cup \{i\}) \subseteq S^T \setminus R^T$, it follows that $S^T \neq R^T$. In each possible case, $S^T \neq R^T$.

(b). We first prove that $\mathcal{M}(G^T) \supseteq \{S^T : S \in \mathcal{M}(G)\}$. Let $S \in \mathcal{M}(G)$. If $S \cap T = \emptyset$, then $S^T = S \in \mathcal{M}(G^T)$ since $(S \setminus \{i\})_T = S \setminus \{i\}$ is losing in both G and G^T for all $i \in S$. Otherwise, $|S \cap T| = 1$ since $S \in \mathcal{M}(G)$ and T contains only pairs of disconnected players. Set $S \cap T = \{i\}$. First note that $(S^T \setminus \{i_T\})_T = S \setminus \{i\} \notin \mathcal{W}(G)$ and thus $S^T \setminus \{i_T\} \notin \mathcal{W}(G^T)$. Now consider $j \in S^T \setminus \{i_T\}$ and suppose that $S^T \setminus \{j\} \in \mathcal{W}(G^T)$. By definition, $(S^T \setminus \{j\})_T = (S \setminus \{i, j\}) \cup T \in \mathcal{W}(G)$, it follows that there exists $A \subseteq (S \setminus \{i, j\}) \cup T$ such that $A \in \mathcal{M}(G)$. Set $A = A' \cup A''$ with $A' \subseteq S \setminus \{i, j\}$ and $A'' \subseteq T$. Note that $A'' \neq \emptyset$ since $A' \subseteq S \setminus \{i, j\} \notin \mathcal{W}(G)$. Moreover $A'' = \{k\}$ for some $k \in T$ since T contains only pairs of disconnected players. Recalling that $S \setminus \{j\} \notin \mathcal{W}(G)$, it follows that $k \neq i$. Note that $\{i, k\} \subseteq T$ such that $S = (S \setminus \{i\}) \cup \{i\} \in \mathcal{D}_i(G)$ and $A \subseteq (S \setminus \{i\}) \cup \{k\} \in \mathcal{D}_k(G)$. Since i and k are independent, therefore there exists $K \subseteq S \setminus \{i\}$ such that $K \cup \{i\} \in \mathcal{D}_i(G)$ and $K \cup \{k\} \notin \mathcal{D}_k(G)$. Hence $K \subsetneq S \setminus \{i\}$ and thus $K \cup \{i\} \subsetneq S$. A contradiction arises since $K \cup \{i\} \in \mathcal{W}(G)$ and $S \in \mathcal{M}(G)$. This proves that $S^T \setminus \{j\} \notin \mathcal{W}(G^T)$ for all $j \in S^T \setminus \{i_T\}$. Therefore $S^T \in \mathcal{M}(G^T)$. We conclude that $\mathcal{M}(G^T) \supseteq \{S^T : S \in \mathcal{M}(G)\}$.

Now, we prove that $\mathcal{M}(G^T) \subseteq \{S^T : S \in \mathcal{M}(G)\}$. Let $R \in \mathcal{M}(G^T)$. If $i_T \notin R$, then $R_T = R \in \mathcal{W}(G^T)$ and for all $i \in R$, $(R \setminus \{i\})_T = R \setminus \{i\} \notin \mathcal{W}(G^T)$. By definition of G^T, $R \in \mathcal{W}(G)$ and for all $i \in R$, $R \setminus \{i\} \notin \mathcal{W}(G)$. Thus, $R \in \mathcal{M}(G)$ and $R = R^T \in \{S^T : S \in \mathcal{M}(G)\}$. Otherwise, $i_T \in R$. Two possible cases arise. First suppose that $R = \{i_T\}$. Then $R_T = T \in \mathcal{W}(G)$. This implies that T contains in the simple game G some minimal winning coalition K. Therefore $R = K^T \in \{S^T : S \in \mathcal{M}(G)\}$. Now suppose that $R \neq \{i_T\}$. Since $R \in \mathcal{M}(G^T)$, we have $R_T = (R \setminus \{i_T\}) \cup T \in \mathcal{W}(G)$ while $R \setminus \{i_T\} \notin \mathcal{W}(G)$. It follows that there exists some nonempty subset L of T such that $(R \setminus \{i_T\}) \cup L \in \mathcal{D}_i(G)$ for all $i \in L$. Such a coalition L can be obtained from $(R \setminus \{i_T\}) \cup T$ by removing, one by one, some members of T. Note that $[(R \setminus \{i_T\}) \cup L]^T = R$. To prove that $R \in \{S^T : S \in \mathcal{M}(G)\}$, we only have to prove that $(R \setminus \{i_T\}) \cup L$ is necessary a minimal winning coalition in the simple game G. Suppose on the contrary that this is not the case. Then, there exists $j \in R \setminus \{i_T\}$ such that $(R \setminus \{i_T, j\}) \cup L$ is still a winning coalition in the simple game G. By the definition of G^T, $[(R \setminus \{i_T, j\}) \cup L]^T = R \setminus \{j\}$ is a proper subset of R which is a winning coalition in G^T. A contradiction arises since $R \in \mathcal{M}(G^T)$. Therefore $(R \setminus \{i_T\}) \cup L$ is a minimal winning coalition in the simple game G. In each possible case, $R \in \{S^T : S \in \mathcal{M}(G)\}$. We conclude that $\mathcal{M}(G^T) \subseteq \{S^T : S \in \mathcal{M}(G)\}$. Finally, $\mathcal{M}(G^T) = \{S^T : S \in \mathcal{M}(G)\}$.

(c). Given $j \in N \setminus T$, note that for all coalitions $S \subseteq N$, $j \in S^T$ if and only if $j \in S$. Therefore, Part (b) implies that $\mathcal{M}_j(G^T) = \{S^T : S \in \mathcal{M}(G)$ and $j \in S^T\} = \{S^T : S \in \mathcal{M}_j(G)\}$; moreover, the above operator is onto by definition. By Part (a), the operator $S \longmapsto S^T$ is injective in $\mathcal{M}(G)$. Therefore $|\mathcal{M}_j(G^T)| = |\{S^T : S \in \mathcal{M}_j(G)\}| = |\mathcal{M}_j(G)|$.

(d). By Part (b), $\mathcal{M}_{i_T}(G^T) = \{S^T : S \in \mathcal{M}(G)$ and $T \cap S \neq \emptyset\}$. Recall that T contains only pairs of independent (disconnected) players, then, $|S \cap T| \leq 1$ for all $S \in \mathcal{M}(G)$. It follows that $\mathcal{M}_{i_T}(G^T) = \{S^T : S \in \mathcal{M}(G)$ and $i \in S$ for some $i \in T\} = \cup_{i \in T}\{S^T : S \in \mathcal{M}_i(G)\}$. Moreover $\mathcal{M}_i(G) \cap \mathcal{M}_j(G) = \emptyset$ for pairs $\{i, j\}$ of distinct players in T. Taking into account that the operator $S \longmapsto S^T$ is injective in $\mathcal{M}(G)$, one finally gets $|\mathcal{M}_{i_T}(G^T)| = \sum_{i \in T} |\mathcal{M}_i(G)|$. ■

Remark 1.2.2. *Note that the results in Proposition 1.2.2 fail if T is not a coalition of independent players; even if T contains pairs of disconnected players as we can see in the simple game $G = (N, \mathcal{W}(G))$ in Example 1.2.1 defined such that $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G)=\{\{1,2\},\{2,3,5\},\{1,3,4\}\}$. By considering $T = \{1, 5\}$, $N^T = \{i_T, 2, 3, 4\}$ and $\mathcal{M}(G^T) = \{\{i_T, 2\}, \{i_T, 3, 4\}\}$. Players 1 and 5 are disconnected, $\{2, 3, 5\} \in \mathcal{M}(G)$ but $\{2, 3, 5\}^T = \{i_T, 2, 3\} \notin \mathcal{M}(G^T)$; $|\mathcal{M}_2(G^T)| \neq |\mathcal{M}_2(G)|$ and $|\mathcal{M}_{i_T}(G^T)| \neq |\mathcal{M}_1(G)| + |\mathcal{M}_5(G)|$.*

1.2.2 Supplementation of a simple game

The scenario we now consider, for a given simple game $G = (N, \mathcal{W}(G))$, is the arrival of a new player $k \notin N$.

Definition 1.2.4. *Given $k \in \mathcal{P} \setminus N$, a k-supplementation of a simple game $G = (N, \mathcal{W}(G))$ is any simple game $G' = (N', \mathcal{W}(G'))$ such that $N' = N \cup \{k\}$ and for all coalitions $S \subseteq N$:*

$$S \in \mathcal{M}(G) \Longleftrightarrow (S \in \mathcal{M}(G') \text{ or } S \cup \{k\} \in \mathcal{M}(G')).$$

From G to a k-supplementation G' of G, the arrival of player k is such that, for each minimal winning coalition S in G, either S or $S \cup \{k\}$ remains a minimal winning coalition in G'. For each winning coalition S in G, we say that k becomes *supplementary* to S from G to G' when S is losing in G', while $S \cup \{k\}$ is winning in G'. For cost allocation problems, a similar definition was considered by Hougaard and Moulin (2014) in their definition of an item that is supplementary to an agent needs. From G to a k-supplementation of G, the arrival of player k changes the set of minimal winning coalition of G exactly as the introduction of a supplementary resource to the needs of an agent, say i, reshapes the collection of agent i's minimal serving sets.

Example 1.2.3. *Let $G = (N, \mathcal{W}(G))$ be the simple game defined in Example 1.2.1 with $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$. Then G is a 3-supplementation of $G_1 = (N_1, \mathcal{W}(G_1))$ where $N_1 = \{1, 2, 4, 5\}$ and $\mathcal{M}(G_1) = \{\{1, 2\}, \{2, 5\}, \{1, 4\}\}$. From G_1 to G, player 3 is supplementary not only to the minimal winning coalitions $\{2, 5\}$ and $\{1, 4\}$ but also to some non minimal winning coalitions such as $\{2, 4, 5\}$. Similarly, the simple game G is a 5-supplementation of $G_2 = (N_2, \mathcal{W}(G_2))$ with $N_2 = \{1, 2, 3, 4\}$ and $\mathcal{M}(G_2) = \{\{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}$. Conversely, G is not a 1-supplementation of $G_3 = (N_3, \mathcal{W}(G_3))$ where $N_3 = \{2, 3, 4, 5\}$ and $\mathcal{M}(G_3) = \{\{2\}, \{3, 4\}\}$. Since $\{2, 3, 5\} \in \mathcal{M}(G)$ but $\{2, 3, 5\} \notin \mathcal{M}(G_3)$ and $1 \notin \{2, 3, 5\}$.*

The next proposition shows how a supplementation impacts on the set of minimal winning coalitions of a simple game.

Proposition 1.2.3 (Safokem et al. (2021))**.** *Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a simple game $G' = (N', \mathcal{W}(G'))$ such that $N' = N \cup \{k\}$.*

Then G' is a k-supplementation of G if and only if there exists a subset E of $\mathcal{M}(G)$ such that

$$\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}.$$

Proof.

Let $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$ and $G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ such that $N' = N \cup \{k\}$.

Necessity. Suppose that G' is a k-supplementation of G. Let $E = \mathcal{M}(G) \setminus \mathcal{W}(G')$. We prove that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. We first show that $\mathcal{M}(G') \supseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. To see this, first consider a coalition $R \subseteq N'$ such that $R = S \cup \{k\}$ for some $S \in E$. By definition of E, $R \setminus \{k\} = S \notin \mathcal{W}(G')$, then $S \notin \mathcal{M}(G')$. Since $S \in E \subseteq \mathcal{M}(G)$, it follows by assumption on G' that $S \cup \{k\} \in \mathcal{M}(G')$. That is $R \in \mathcal{M}(G')$. Now, consider a coalition $R \subseteq N'$ such that $R \in \mathcal{M}(G) \setminus E = \mathcal{M}(G) \cap \mathcal{W}(G')$. Then $R \in \mathcal{M}(G)$ and $R \cup \{k\} \notin \mathcal{M}(G')$ since $R \in \mathcal{W}(G')$. It follows by assumption on G' that $R \in \mathcal{M}(G')$.

We now prove that $\mathcal{M}(G') \subseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. For this purpose, consider $R \in \mathcal{M}(G')$. If $k \notin R$, then by definition of G', $R \in \mathcal{M}(G)$ and therefore $R \in \mathcal{M}(G) \cap \mathcal{M}(G') \subseteq \mathcal{M}(G) \cap \mathcal{W}(G') = \mathcal{M}(G) \setminus E$. Now suppose that $k \in R$, that is $R = S \cup \{k\}$ for some $S \subseteq N$. It follows that $S \in \mathcal{M}(G)$ since G' is a k-supplementation of G by assumption. Moreover, $R \setminus \{k\} = S \notin \mathcal{W}(G')$ Therefore $S \in \mathcal{M}(G) \setminus \mathcal{W}(G') = E$. We conclude that $\mathcal{M}(G') \subseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$.

In summary, $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$.

Sufficiency. Assume that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$ for some subset E of $\mathcal{M}(G)$. We prove that G' is a k-supplementation of G. Consider $S \subseteq N$.

First suppose that $S \in \mathcal{M}(G)$. If $S \notin E$, then $S \in \mathcal{M}(G) \setminus E$ and this implies that $S \in \mathcal{M}(G')$. If $S \in E$, then it follows by assumption that $S \cup \{k\} \in \mathcal{M}(G')$. We have prove that $S \in \mathcal{M}(G)$ implies $S \in \mathcal{M}(G')$ or $S \cup \{k\} \in \mathcal{M}(G')$. Now suppose that $S \in \mathcal{M}(G')$ or $S \cup \{k\} \in \mathcal{M}(G')$. If $S \in \mathcal{M}(G')$ then $S \in \mathcal{M}(G) \setminus E$ since $k \notin S$. Thus $S \in \mathcal{M}(G)$. If $S \cup \{k\} \in \mathcal{M}(G')$ then $S \in E \subseteq \mathcal{M}(G)$. Therefore in both cases, $S \in \mathcal{M}(G)$. In summary, G' is a k-supplementation of G. ■

From Proposition 1.2.3, we obtain the following result that tells us how the supplementation operation impacts on the cardinalities of minimal winning coalitions containing a given player.

Proposition 1.2.4 (Safokem et al. (2021)). *Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$ and a k-supplementation $G' = (N', \mathcal{W}(G'))$ of G. Then*

(a) $|\mathcal{M}_i(G')| = |\mathcal{M}_i(G)|$ *for all* $i \in N$;

(b) $|\mathcal{M}_k(G')| = |\mathcal{M}(G) \setminus \mathcal{W}(G')|$.

Proof. Suppose that G' is a k-supplementation of G. It follows from Proposition 1.2.3 that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$ for some $E \subseteq \mathcal{M}(G)$.

(a). Given $i \in N$, $\mathcal{M}_i(G') = \{S \in \mathcal{M}(G') : i \in S\} = \{S \cup \{k\} : i \in S \in E\} \cup \{S : i \in S \in \mathcal{M}(G) \setminus E\}$. That is $\mathcal{M}_i(G') = \{S \cup \{k\} : S \in E \cap \mathcal{M}_i(G)\} \cup \{S : S \in \mathcal{M}_i(G) \setminus E\}$. Since $E \cap \mathcal{M}_i(G)$ and $\mathcal{M}_i(G) \setminus E$ are disjoint sets, it follows that $|\mathcal{M}_i(G')| = |E \cap \mathcal{M}_i(G)| + |\mathcal{M}_i(G) \setminus E| = |\mathcal{M}_i(G)|$.

(b). By assumption, $k \notin N$. Therefore $\mathcal{M}_k(G') = \{S \in \mathcal{M}(G') : k \in S\} = \{S \cup \{k\} : S \in E\}$. Therefore, $|\mathcal{M}_k(G')| = |E|$. Now, note that for all $S \in E$, we have $S \in \mathcal{M}(G)$ since $E \subseteq \mathcal{M}(G)$ and $S \notin \mathcal{W}(G')$ since $S \cup \{k\} \in \mathcal{M}(G')$. Therefore, $E \subseteq \mathcal{M}(G) \setminus \mathcal{W}(G')$. Conversely, all $S \in \mathcal{M}(G) \setminus \mathcal{W}(G')$ are such that $S \in \mathcal{M}(G)$ and $S \notin \mathcal{M}(G')$. By assumption on $\mathcal{M}(G')$, $S \notin \mathcal{M}(G) \setminus E$. That is $S \in E$. Therefore $\mathcal{M}(G) \setminus \mathcal{W}(G') \subseteq E$. We conclude that $E = \mathcal{M}(G) \setminus \mathcal{W}(G')$. This proves that $|\mathcal{M}_k(G')| = |\mathcal{M}(G) \setminus \mathcal{W}(G')|$. ■

1.2.3 Equivalent games

We now present a notion of equivalent games from a player's perspective. The importance of a player in a collective decision process depends not only on all the situations where his opinion counts, but also on all the situations in which decisions can be made without him. It is therefore necessary for each player to make a round up about: on the one hand, the situations where he is decisive (internal decisiveness parameter) and on the other hand, the situations where decision can be made without him (external decisiveness parameter), taking into account each time the number of his partners. Two games are thus equivalent for a player's viewpoint if they keep the above parameters unchanged. For example, in a parliamentary system, one can imagine that the formation of a government can only be initiated by a minimal winning coalition. A player's influence parameters are therefore the minimal winning coalitions that contain him and their sizes. Formally,

Definition 1.2.5. *Given two simple games $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}$, and a player $i \in N \cap N'$.*

The games G and G' are said to be $\mathcal{M}_i$-equivalent, if there exists $(S,T) \in \mathcal{M}(G) \times \mathcal{M}(G')$ such that $\mathcal{M}(G') = (\mathcal{M}(G)\backslash\{S\})\cup\{T\}$ with $|S| = |T|$ and $(i \in S\cap T$ or $i \notin S\cup T)$.

In this case, we write $G\Delta_i G'$ or more precisely $G\Delta_i^{S,T}G'$ if coalitions S and T involved in the definition are specified.

Note that for a given player $i \in \mathcal{P}$, Δ_i is a binary relation on $\mathcal{G}$ putting together games that are somewhat similar for player i, in the sense that moving from one of those two games to the other, only one minimal winning coalition of the first game is replaced by a minimal winning coalition of the second with the same cardinality and the particularity that the membership of i is unaffected (he belongs to both coalitions or to none of them).

Example 1.2.4. *Consider the simple games $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G'))$ and $G'' = (N'', \mathcal{W}(G''))$ such that $N = \{1,2,3,4,5\}, N' = \{1,2,3,4,6\}, N'' = \{1,2,3,4,5,7,8,9\}$ and $\mathcal{M}(G) = \{\{1,2\},\{2,3,5\},\{1,3,4\}\}, \mathcal{M}(G') = \{\{1,2\},\{2,4,6\},\{1,3,4\}\}$, $\mathcal{M}(G'') = \{\{1,2\},\{2,3,4\},\{1,3,4\}\}$. Let $S = \{2,3,5\}, T = \{2,4,6\}, K = \{2,3,4\}$. It follows that*

$$G\Delta_2^{S,T}G', G\Delta_1^{S,T}G', G\Delta_2^{S,K}G'', G\Delta_3^{S,K}G'', G'\Delta_4^{T,K}G'' \text{ and } G'\Delta_2^{T,K}G''.$$

Notation 1.2.1. *Given $i \in \mathcal{P}$, we denote by $\mathcal{G}_i$ the set of all simple games $G = (N, v)$ such that $i \in N$. Hence, $\mathcal{G} = \cup_{i\in\mathcal{P}}\mathcal{G}_i$.*

The next proposition exhibits some algebraic properties[1].

Proposition 1.2.5. *For all $i \in \mathcal{P}$, the binary relation Δ_i is reflexive and symmetric on $\mathcal{G}_i$, but fails to be transitive.*

Proof.

[1]A binary relation R on a set X (that is any subset of the Cartesian product $X \times X$) is *reflexive* if $(x,x) \in R$ for all $x \in X$; *symmetric* if for all $x,y \in X$, $(x,y) \in R \Longrightarrow (y,x) \in R$; and *transitive* if for all $x,y,z \in X, ((x,y) \in R$ and $(y,z) \in R) \Longrightarrow (x,z) \in X$. It is usual to write xRy instead of $(x,y) \in R$.

Consider a potential player $i \in \mathcal{P}$.

(i) **Reflexivity**: Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_i$. Since $\mathcal{M}(G) \neq \emptyset$, consider $S \in \mathcal{M}(G)$.

If $i \in S$, then $i \in S \cap S$ and $\mathcal{M}(G) = (\mathcal{M}(G) \backslash \{S\}) \cup \{S\}$, it follows that $G\Delta_i^{S,S}G$.

Else $i \notin S$, then $i \notin S \cup S$ and $\mathcal{M}(G) = (\mathcal{M}(G) \backslash \{S\}) \cup \{S\}$, it follows that $G\Delta_i^{S,S}G$.

In both cases $G\Delta_i G$. Therefore Δ_i is reflexive on $\mathcal{G}_i$.

(ii) **Symmetry**: Consider two simple games $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}_i$ such that $G\Delta_i G'$. Then there exists $(S, T) \in \mathcal{M}(G) \times \mathcal{M}(G')$ with $|S| = |T|$ and ($i \in S \cap T$ or $i \notin S \cup T$) such that $\mathcal{M}(G') = (\mathcal{M}(G) \backslash \{S\}) \cup \{T\}$. One obtains $\mathcal{M}(G) = (\mathcal{M}(G') \backslash \{T\}) \cup \{S\}$. It follows that $G'\Delta_i^{T,S}G$. This proves that Δ_i is symmetric on $\mathcal{G}_i$.

(iii) **Transitivity**: To see that the relation Δ_i is not transitive on $\mathcal{G}_i$, consider the following simple games $G = (N, \mathcal{W}(G))$, $G' = (N', \mathcal{W}(G'))$ and $G'' = (N'', \mathcal{W}(G''))$ with $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$; $N' = \{1, 2, 3, 4, 6\}$ and $\mathcal{M}(G') = \{\{1, 2\}, \{2, 4, 6\}, \{1, 3, 4\}\}$; $N'' = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathcal{M}(G'') = \{\{2, 3\}, \{2, 4, 6\}, \{1, 3, 4\}\}$. Let $S = \{2, 3, 5\}, T = \{2, 4, 6\}, T' = \{1, 2\}, K = \{2, 3\}$. It follows that $G\Delta_2^{S,T}G', G'\Delta_2^{T',K}G''$ but $\rceil(G\Delta_2 G'')$ since $|\mathcal{M}(G) \backslash \mathcal{M}(G'')| = 2 > 1$. Clearly, $G\Delta_2 G'$ and $G'\Delta_2 G''$ but $\rceil(G\Delta_2 G'')$. We conclude that Δ_i is not transitive on $\mathcal{G}_i$.

■

The following remark is useful in the sequel. It is straightforward from the definition of two equivalent simple games.

Remark 1.2.3. If $G\Delta_i^{S,T}G'$, then $\mathcal{M}(G') = (\mathcal{M}(G) \backslash \{S\}) \cup \{T\}$ with $|S| = |T|$ and:

1. $\mathcal{M}_i(G') = (\mathcal{M}_i(G) \backslash \{S\}) \cup \{T\}$ if $i \in S \cap T$;
2. $\mathcal{M}_i(G') = \mathcal{M}_i(G)$ otherwise.

1.2.4 Axioms of power indices

Among classical axioms for power indices, we have the following when one considers a power index φ:

Axiom 1. ***Null Player* (NP)**: *For all $G \in \mathcal{G}$, $\varphi_i(G) = 0$ whenever i is a null player in G.*

Axiom (NP) simply means that a player who is never decisive in a simple game necessarily enjoys a null voting power.

Axiom 2. ***Efficiency* (E)**: *For all* $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $\sum_{i \in N} \varphi_i(G) = 1$.

A power index that meets (E) is said to be *normalized*. Note that for any power index φ such that individual shares in all simple games always sum to non-zero values, one gets an efficient power index $\overline{\varphi}$ defined for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by:

$$\overline{\varphi}_i(G) = \frac{\varphi_i(G)}{\sum_{j \in N} \varphi_j(G)}.$$

The power index $\overline{\varphi}$ is called the normalized version of φ. The Banzhaf-Coleman index (see Banzhaf (1965) or Coleman (1971)) is the normalized version of the Banzhaf-Penrose index; see Banzhaf (1965) or Penrose (1946).

Axiom 3. ***Anonymity* (AN)**: *For all* $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, *for all permutation* $\pi : N \longrightarrow N$, $\varphi_{\pi(i)}(G_\pi) = \varphi_i(G)$ *for all* $i \in N$.

Axiom (AN) says that individual voting powers should not depend on the players' labels. The three preceding axioms are classical axioms of a power index very often used; see for example Allingham (1975).

Axiom 4. ***Deegan-Packel Mergeability* (DPM)**: *For all* $G_1 = (N, \mathcal{W}(G_1)) \in \mathcal{G}_N$, *for all* $G_2 = (N, \mathcal{W}(G_2)) \in \mathcal{G}_N$ *such that* G_1 *and* G_2 *are mergeable,*

$$\varphi_i(G_1 \oplus G_2) = \frac{|\mathcal{M}(G_1)|\varphi_i(G_1) + |\mathcal{M}(G_2)|\varphi_i(G_2)}{|\mathcal{M}(G_1 \oplus G_2)|}, \ \forall i \in N.$$

This axiom is due to Deegan and Packel (1978) and says that the voting power of a player in a merged game obtained from two mergeable simple games is a weighted mean of voting powers of the component games, with the number of minimal winning coalitions in each component game being its weight.

Axiom 5. ***Holler-Packel Mergeability* (HPM)**: *For all* $G_1 = (N, \mathcal{W}(G_1)) \in \mathcal{G}_N$, *for all* $G_2 = (N, \mathcal{W}(G_2)) \in \mathcal{G}_N$ *such that* G_1 *and* G_2 *are mergeable,*

$$\varphi_i(G_1 \oplus G_2) = \frac{\lambda(G_1)\varphi_i(G_1) + \lambda(G_2)\varphi_i(G_2)}{\lambda(G_1) + \lambda(G_2)}, \ \forall i \in N$$

where for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$,

$$\lambda(G) = \sum_{i \in N} |\mathcal{M}_i(G)| = \sum_{i \in N} \sum_{S \in \mathcal{M}_i(G)} 1 = \sum_{S \in \mathcal{M}(G)} \sum_{i \in S} 1 = \sum_{S \in \mathcal{M}(G)} |S|.$$

Axiom (HPM) is due to Holler and Packel (1983) and tells us that the voting power of a player in a game obtained from two mergeable simple games is a weighted mean of voting powers of the component games, the weight of each component game being equal to the sum of the sizes of all its minimal winning coalitions.

The next three axioms are due to Safokem et al. (2021).

Axiom 6. ***Non Profitable Merging (NPM) of independent players***: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players,*

$$\varphi_{i_T}(G^T) = \sum_{i \in T} \varphi_i(G) \tag{1.3}$$

whenever T contains only independent players.

Axiom (NPM) is a weak condition of the lack of incentive for independent players to merge in a simple game; for similar requirements, see Knudsen and Østerdal (2012). When the merging equality (1.3) is considered only for coalitions of size 2 (without any restriction), one obtains the 2-efficiency condition of Lehrer (1988)[2].

The following remark highlights the differences between Axiom (NPM) and the well-known Deegan-Packel mergeability and Holler-Packel mergeability axioms.

Remark 1.2.4. *It can be checked that the power index that associates each simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ with the n-tuple $(1/n; 1/n; \cdots; 1/n)$ satisfies the Deegan-Packel mergeability axiom as well as the Holler-Packel mergeability axiom but not (NPM). And the Deegan-Packel index (see Deegan and Packel (1978)) satisfies Axiom (NPM) but fails to meet the Holler-Packel mergeability Axiom; also, the Holler-Packel index (see Holler (1982)) satisfies Axiom (NPM) but fails to meet the Deegan-Packel mergeability axiom. Furthermore, the Holler-Packel and the Deegan-Packel mergeability conditions apply to a fix electorate while (NPM) is designed for variable electorates.*

The next result provides a relationship between (NPM) and (NP).

Proposition 1.2.6 (Safokem et al. (2021)). *All power indices that satisfy (NPM) also satisfy (NP).*

Proof. Suppose that φ is a power index that satisfies (NPM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a null player $k \in N$ in G. Denote by G_0 the simple game obtained when k leaves the game G while the set of minimal winning coalitions is unchanged; that is $G = G_0[k]$. To prove that $\varphi_k(G) = 0$, we consider $\{k_1, k_2, k_3, k_4, k_5, k_6\} \subset \mathcal{P} \backslash N$ and the following simple games :

$$G_1 = G_0[k_1, k_2, k_3, k_4],\ G_2 = G_0[k_1, k_2, k_5], G_3 = G_0[k_5, k_6] \text{ and } G_4 = G_0[k_1, k_2, k_3, k_6].$$

Note that any player from $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ who is involved in a games G_j for $j = 1, 2, 3, 4$ is a null player. Moreover, in terms of the merging operation, the following holds

$$G = G_1^{\{k_1,k_2,k_3,k_4\} \to k} = G_2^{\{k_1,k_2,k_5\} \to k}, G_2 = G_1^{\{k_3,k_4\} \to k_5} \text{ and } G_3 = G_2^{\{k_1,k_2\} \to k_6}. \tag{1.4}$$

[2] The author proves that the Banzhaf value for TU-games (that is an extension of the Banzhaf-Penrose index on the class of TU-games) is uniquely determined by the 2-efficiency condition among all values that coincide with the Shapley value on all 2-player games. It follows that the Banzhaf power index (normalized or not) satisfies (1.3) for all coalitions T of size 2.

Furthermore, applying (NPM) on each of the above mentioned merging operations leads to

$$\begin{aligned}\varphi_{k_1}(G_1)+\varphi_{k_2}(G_1) &= \varphi_k(G)-\varphi_{k_3}(G_1)-\varphi_{k_4}(G_1) \text{ since by (1.4), } G=G_1^{\{k_1,k_2,k_3,k_4\}\to k}\\ &= \varphi_k(G)-\varphi_{k_5}(G_2) \text{ since by (1.4), } G_2=G_1^{\{k_3,k_4\}\to k_5}\\ &= \varphi_{k_1}(G_2)+\varphi_{k_2}(G_2) \text{ since by (1.4), } G=G_2^{\{k_1,k_2,k_5\}\to k}\\ &= \varphi_{k_6}(G_3) \text{ since by (1.4), } G_3=G_2^{\{k_1,k_2\}\to k_6}.\end{aligned}$$

In a similar way, $\varphi_i(G_1)+\varphi_j(G_1)=\varphi_{k_6}(G_3)$ for all pairs $\{i,j\}\subseteq\{k_1,k_2,k_3,k_4\}$. This proves that $\varphi_i(G_1)+\varphi_j(G_1)$ does not depend on the pair $\{i,j\}\subseteq\{k_1,k_2,k_3,k_4\}$. Since $\varphi_{k_1}(G_1)+\varphi_{k_2}(G_1)+\varphi_{k_3}(G_1)+\varphi_{k_4}(G_1)=\varphi_k(G)$, it follows that $\varphi_i(G_1)=\frac{1}{4}\varphi_k(G)$ for all $i\in\{k_1,k_2,k_3,k_4\}$ and $\varphi_{k_6}(G_3)=\frac{1}{2}\varphi_k(G)$. In the same way, $\varphi_i(G_4)=\frac{1}{4}\varphi_k(G)$ for all $i\in\{k_1,k_2,k_3,k_6\}$.

Noting that $G_3=G_4^{\{k_1,k_2,k_3\}\to k_5}$, it follows by (NPM) that $\varphi_{k_5}(G_3)=\frac{3}{4}\varphi_k(G)$. Moreover $G=G_3^{\{k_5,k_6\}\to k}$. Thus $\varphi_{k_6}(G_3)=\varphi_k(G)-\varphi_{k_5}(G_3)=\frac{1}{4}\varphi_k(G)$. Hence $\frac{1}{2}\varphi_k(G)=\frac{1}{4}\varphi_k(G)$ and therefore $\varphi_k(G)=0$.

■

Axiom 7. ***Independence of External Merging (IEM) of independent players*** : *For all $G=(N,\mathcal{W}(G))\in\mathcal{G}$, for all coalitions $T\subseteq N$ of at least two players, $\varphi_i(G^T)=\varphi_i(G)$ for all $i\in N\setminus T$ whenever T is a coalition of independent players.*

When a merging operation involves independent players from one game to another, (IEM) is the requirement that the shares of other players remain unchanged. It is shown in the next proposition that (IEM) and (E) imply (NPM).

Proposition 1.2.7. *If a power index φ satisfies (E) and (IEM), then φ satisfies (NPM).*

Proof. Suppose that a power index φ satisfies (E) and (IEM). Consider $G=(N,\mathcal{W}(G))\in\mathcal{G}$ and a coalition $T\subseteq N$ of independent players. We have:

$$\begin{aligned}\varphi_{i_T}(G^T) &= 1-\sum_{i\in N\setminus T}\varphi_i(G^T) \text{ by (E)}\\ &= 1-\sum_{i\in N\setminus T}\varphi_i(G) \text{ by (IEM)}\\ &= \sum_{i\in T}\varphi_i(G) \text{ by (E).}\end{aligned}$$

Therefore φ satisfies (NPM).

■

In the following axiom, the members of a coalition T of independent players in a simple game G have to choose their representative in T. This is the case in an annexation described by Aziz et al. (2011) when a voter in T takes the voting weight of other members of T. The notation $G^{T \to t}$ is extended below so that the possibility for t to be a player from T is now included.

Axiom 8. ***Independence of Internal Merging (IIM) independent players***: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players, for $t \in T$, $\varphi_i(G^{T \to t}) = \varphi_i(G)$ for all $i \in N \setminus T$ whenever T is a coalition of independent players.*

When the members of a coalition T of independent players are offered the possibility to merge into a player in T, (IIM) requires that this should not impact the shares of players out of T.

Proposition 1.2.8. *All power indices on $\mathcal{G}$ that satisfy (IIM) also satisfy (IEM).*

Proof. Suppose that a power index φ on $\mathcal{G}$ satisfies (IIM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition T of independent players with $|T| \geq 2$ and a player $i_T \in \mathcal{P} \backslash N$. Set $G_1 = G[i_T]$. By noting that $|N| \geq |T| \geq 2$, consider for all $i \in N$, some $j \in N \backslash \{i\}$. Since $G = G_1^{\{j, i_T\} \to j}$, then by (IIM), it follows that $\varphi_i(G) = \varphi_i(G_1)$. Now, note that $G^T = (G_1)^{T \cup \{i_T\} \to i_T}$. Therefore, (IIM) implies that for all $i \in N \backslash T$, $\varphi_i(G^T) = \varphi_i(G_1)$ and therefore, $\varphi_i(G^T) = \varphi_i(G)$. This proves that φ satisfies (IEM).

■

Here is a new axiom relied on equivalent games that we have introduced.

Axiom 9. ***Membership Equivalence Property* (MEP)**: *For all $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}$, for all player $i \in N \cap N', \varphi_i(G) = \varphi_i(G')$ whenever $G \Delta_i G'$.*

This axiom requires that a power index should give a player, say i, the same power in two simple games that are $\mathcal{M}_i$-equivalent, in other words, the $\mathcal{M}_i$-equivalence of simple games should guarantee to player the same power in both games.

We establish that any power index that satisfies (MEP) and (E) also satisfies (NP).

Proposition 1.2.9. *If a power index φ on $\mathcal{G}$ satisfies (MEP) and (E), then φ also satisfies (NP).*

Proof. Consider a power index φ on $\mathcal{G}$ that satisfies (MEP) and (E). Let $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ be a simple game and $i \in N^0(G)$ be a null player in G. Define the simple game $G_0 = (N_0, \mathcal{W}(G_0))$ with $N_0 = N \backslash \{i\}$ and $\mathcal{M}(G_0) = \mathcal{M}(G)$. Set $S \in \mathcal{M}(G)$ and consider $j \in N_0$.

If $j \in S$, then $j \in S \cap S$ and $\mathcal{M}(G_0) = (\mathcal{M}(G)\backslash\{S\}) \cup \{S\}$, that is $G\Delta_j G_0$.

If $j \notin S$, then $j \notin S \cup S$ and $\mathcal{M}(G_0) = (\mathcal{M}(G)\backslash\{S\}) \cup \{S\}$, that is $G\Delta_j G_0$.

It follows by (MEP) that $\varphi_j(G_0) = \varphi_j(G)$. Then $\sum_{j\in N_0} \varphi_j(G_0) = \sum_{j\in N_0} \varphi_j(G)$. Axiom (E) implies that $1 = 1 - \varphi_i(G)$, that is $\varphi_i(G) = 0$. Thus, φ satisfies (NP).

■

We now establish that (MEP) is stronger than (IEM).

Proposition 1.2.10. *If a power index φ on $\mathcal{G}$ satisfies (MEP), then φ satisfies (IEM).*

Proof. Suppose that φ is a power index on $\mathcal{G}$ that satisfies (MEP). Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $i_T \in \mathcal{P}\backslash N$ and $T \subseteq N$ such that members of T are independents players. Also consider a player $i \in N\backslash T$, we show that $\varphi_i(G^T) = \varphi_i(G)$.

Set $T = \{i_1, i_2, ..., i_t\}$, where $t = |T|$. Write:

$\mathcal{M}_{i_k}(G) = \{S_{k,1}, S_{k,2}, ..., S_{k,n_k}\}$ for all $k \in \{1, 2, ..., t\}$.

Since members of T are independent players, they are also disconnected. That is for $k, k' \in \{1, ..., t\}$ with $k \neq k'$, for all $l \in \{1, ..., n_k\}$ and $l' \in \{1, ..., n_{k'}\}$, one have $i_k \notin S_{k',l'}$ and $i_{k'} \notin S_{k,l}$.

For all $k \in \{1, 2, ..., t\}$ and $l \in \{1, ..., n_k\}$, set $T_{k,l} = (S_{k,l}\backslash\{i_k\}) \cup \{i_T\}$. Define the following games:

$G_{1,1} = (N_{1,1}, \mathcal{W}(G_{1,1}))$ with $N_{1,1} = N \cup \{i_T\}$ and $\mathcal{M}(G_{1,1}) = (\mathcal{M}(G)\backslash\{S_{1,1}\}) \cup \{T_{1,1}\}$.

$G_{1,2} = (N_{1,2}, \mathcal{W}(G_{1,2}))$ with $N_{1,2} = N_{1,1} = N \cup \{i_T\}$ and $\mathcal{M}(G_{1,2})=(\mathcal{M}(G_{1,1})\backslash\{S_{1,2}\})\cup\{T_{1,2}\}$.

$\vdots$

$G_{1,n_1} = (N_{1,n_1}, \mathcal{W}(G_{1,n_1}))$ with $N_{1,n_1} = (N\backslash\{i_1\}) \cup \{i_T\}$ and $\mathcal{M}(G_{1,n_1})=(\mathcal{M}(G_{1,n_1-1})\backslash\{S_{1,n_1}\})\cup\{T_{1,n_1}\}$.

$G_{2,1} = (N_{2,1}, \mathcal{W}(G_{2,1}))$ with $N_{2,1} = (N\backslash\{i_1\}) \cup \{i_T\}$ and $\mathcal{M}(G_{2,1})=(\mathcal{M}(G_{1,n_1})\backslash\{S_{2,1}\})\cup\{T_{2,1}\}$.

$G_{2,2} = (N_{2,2}, \mathcal{W}(G_{2,2}))$ with $N_{2,2} = N_{2,1} = (N\backslash\{i_1\})\cup\{i_T\}$ and $\mathcal{M}(G_{2,2})=(\mathcal{M}(G_{2,1})\backslash\{S_{2,2}\})\cup\{T_{2,2}\}$.

$\vdots$

$G_{2,n_2} = (N_{2,n_2}, \mathcal{W}(G_{2,n_2}))$ with $N_{2,n_2} = (N\backslash\{i_1, i_2\}) \cup \{i_T\}$ and $\mathcal{M}(G_{2,n_2})=(\mathcal{M}(G_{2,n_2-1})\backslash\{S_{2,n_2}\})\cup\{T_{2,n_2}\}$.

$\vdots$

$G_{t,1} = (N_{t,1}, \mathcal{W}(G_{t,1}))$ with $N_{t,1} = (N\backslash\{i_1, i_2, ..., i_{t-1}\}) \cup \{i_T\}$ and $\mathcal{M}(G_{t,1})=(\mathcal{M}(G_{t-1,n_{t-1}})\backslash\{S_{t,1}\})\cup\{T_{t,1}\}$.

$G_{t,2} = (N_{t,2}, \mathcal{W}(G_{t,2}))$ with $N_{t,2} = N_{t,1} = (N\backslash\{i_1, i_2, ..., i_{t-1}\}) \cup \{i_T\}$ and $\mathcal{M}(G_{t,2})=(\mathcal{M}(G_{t,1})\backslash\{S_{t,2}\})\cup\{T_{t,2}\}$.

$\vdots$

$G_{t,n_t} = (N_{t,n_t}, \mathcal{W}(G_{t,n_t}))$ with $N_{t,n_t} = (N\backslash\{i_1, i_2, ..., i_{t-1}, i_t\}) \cup \{i_T\} = (N\backslash T) \cup \{i_T\}$ and $\mathcal{M}(G_{t,n_t})=(\mathcal{M}(G_{t,n_t-1})\backslash \{S_{t,n_t}\}) \cup \{T_{t,n_t}\}$. Note that $G_{t,n_t} = G^T$.

$\forall k \in \{1, ..., t\}$ and $l \in \{2, ..., n_k\}$, one have $\mathcal{M}(G_{k,l}) = (\mathcal{M}(G_{k,l-1})\backslash\{S_{k,l}\}) \cup \{T_{k,l}\}, |T_{k,l}| = |(S_{k,l}\backslash\{i_k\}) \cup \{i_T\}| = |S_{k,l}|, i \in N_{k,l-1} \cap N_{k,l}$ and ($i \in S_{k,l} \cap T_{k,l}$ or $i \notin S_{k,l} \cup T_{k,l}$) since $i \notin T$. That is $G_{k,l}\Delta_i G_{k,l-1}$.

Since φ satisfies (MEP), one obtains $\varphi_i(G_{k,l}) = \varphi_i(G_{k,l-1})$.

Similarly, one show that $G\Delta_i G_{1,1}$ and $G_{k+1,1}\Delta_i G_{k,n_k}$ for all $k \in \{1, ..., t-1\}$.

By (MEP), one obtains $\varphi_i(G) = \varphi_i(G_{1,1})$ and $\varphi_i(G_{k+1,1}) = \varphi_i(G_{k,n_k})$ for all $k \in \{1, ..., t-1\}$.

Finally, one gets $\varphi_i(G) = \varphi_i(G_{1,1}) = \varphi_i(G_{t,n_t}) = \varphi_i(G^T)$.

Therefore, φ satisfies (IEM).

■

Remark 1.2.5. *Using Proposition 1.2.7 and Proposition 1.2.10, we observe that (MEP) and (E) together implies Axiom (NPM). But (MEP) alone is not sufficient to imply (NPM) as we can see with the power index Υ defined below for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by*

$$\Upsilon_i(G) = \begin{cases} \frac{1}{|\mathcal{M}_i(G)|} & \text{if } i \notin N^0(G) \\ 1 & \text{otherwise} \end{cases}$$

To see this, first consider two simple games $G = (N, \mathcal{W}(G))$, $G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ and a player $i \in N \cap N'$ such that $G\Delta_i G'$. Note that $|\mathcal{M}_i(G')| = |\mathcal{M}_i(G)|$. If $i \in N^0(G)$, then $|\mathcal{M}_i(G)| = 0 = |\mathcal{M}_i(G')|$. That is $i \in (N')^0(G)$. It follows that $\Upsilon_i(G) = 1 = \Upsilon_i(G')$. If $i \notin N^0(G)$, then $|\mathcal{M}_i(G)| \neq 0$ and $|\mathcal{M}_i(G')| \neq 0$ since $|\mathcal{M}_i(G)| = |\mathcal{M}_i(G')|$. That is $i \notin (N')^0(G)$. It follows that $\Upsilon_i(G) = \frac{1}{|\mathcal{M}_i(G)|} = \frac{1}{|\mathcal{M}_i(G')|} = \Upsilon_i(G')$. Thus, Υ satisfies (MEP).

Now, consider the simple game $G = (N, \mathcal{W}(G))$ such that $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$. Since player 5 is a null player in G, then merging players 4 and 5 into 6 in G is a proper merging operation and it leads us to the simple game $G^T = (N^T, \mathcal{W}(G^T))$ where $T = \{4, 5\}$, $N^T = \{1, 2, 3, 6\}$ and $\mathcal{M}(G^T) = \{\{1, 2\}, \{1, 3\}, \{1, 6\}, \{2, 3, 6\}\}$. We obtain $\Upsilon(G) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$ and $\Upsilon(G^T) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. It then appears that $\Upsilon_6(G^T) = \frac{1}{2} \neq \Upsilon_4(G) + \Upsilon_5(G)$. That is Υ does not satisfies (NPM).

In what follows, it is shown that any power index that meet (MEP) and (E) assigns a specific distribution of voting power to each singleton simple game.

Proposition 1.2.11. *If a power index φ on $\mathcal{G}$ satisfies (MEP) and (E), then for all coalition S, $\varphi_i(G_S) = \dfrac{1}{|S|}$ for all voter $i \in S$.*

Proof.

Suppose that φ is a power index on $\mathcal{G}$ that satisfies (MEP) and (E). Consider a singleton game $G_S = (S, \mathcal{W}(G_S))$. By Proposition 1.2.9, φ satisfies (NP). If $|S| = 1$, then $S = \{i\}$ for some $i \in \mathcal{P}$; it follows by Axiom (E) that $\varphi_i(G_S) = 1 = \dfrac{1}{|S|}$.

Otherwise $|S| \geq 2$. In this case, consider three players $i, j \in S$ and $k \in \mathcal{P} \backslash S$. Define the simple games $G_1 = (N_1, \mathcal{W}(G_1))$ and $G_2 = (N_2, \mathcal{W}(G_2))$ by $N_1 = (S\backslash\{i\}) \cup \{k\}, N_2 = (S\backslash\{j\}) \cup \{k\}, \mathcal{M}(G_1) = (\mathcal{M}(G)\backslash\{\{i\}\}) \cup \{\{k\}\}$ and $\mathcal{M}(G_2) = (\mathcal{M}(G)\backslash\{\{j\}\}) \cup \{\{k\}\}$. In the one hand, we have for all $l \in S\backslash\{i\}, G_1 \Delta_l G_S$, it follows by Axiom (MEP) that $\varphi_l(G_1) = \varphi_l(G_S)$. Applying Axiom (E) to the simple games G_1 and G_S lead us to $1 - \varphi_k(G_1) = 1 - \varphi_i(G_S)$. That is

$$\varphi_k(G_1) = \varphi_i(G_S). \tag{1.5}$$

In the other hand, we have for all $l \in S\backslash\{j\}, G_2 \Delta_l G_S$, it follows by Axiom (MEP) that $\varphi_l(G_2) = \varphi_l(G_S)$. Applying Axiom (E) to the simple games G_2 and G_S lead us to $1 - \varphi_k(G_2) = 1 - \varphi_j(G_S)$. That is

$$\varphi_k(G_2) = \varphi_j(G_S). \tag{1.6}$$

Note that $G_1 \Delta_k G_2$, it then follows by (MEP) that

$$\varphi_k(G_1) = \varphi_k(G_2). \tag{1.7}$$

Equations (1.5), (1.6) and (1.7) imply that $\varphi_i(G_S) = \varphi_j(G_S)$.

Applying Axiom (E) to the singleton game G_S leads us to $\varphi_i(G_S) = \dfrac{1}{|S|}$. ■

As mentioned right at the beginning of this chapter, we have presented basic concepts and some preliminary results on simple games and power indices. We are now ready to combine some of those axioms to provide some new axiomatizations of the Public Good Index. This is the subject of the next chapter.

Chapter Two

Axiomatizations of the Public Good Index for simple games with a variable electorate

In this chapter, we focus our attention on the Holler-Packel index, also known as the Public Good Index. It is worth mentioning that the Public Good Index was introduced by Holler (1978) and was explicitly interpreted as a power index by Holler (1982). Its first axiomatization was provided by Holler and Packel (1983) on the domain of simple games with a fixed electorate as an (AN)+(E)+(NP)+(HPM) characterization. Our main objective is to provide new characterizations of the Public Good Index on the domain of simple games with a variable electorate which was not yet explored. We mainly present some characterizations using efficiency and inter-game axioms.

The current chapter is organized as follows: Section 2.1 provides preliminary results on Public Good Index. We recall the first axiomatization of the Public Good Index due to Holler and Packel (1983) and then introduce an axiom that relies to the supplementation operation: the Supplementation Consistency (SC). We then show that the Public Good Index satisfies (SC) as well as other new properties. We mainly state and prove several characterization results in Section 2.2 built on (SC), on (NPM) and some of its variants. The independence of the axioms we use is also confirmed for each of the results provided. Finally, in section 2.3, we provide an axiomatization of the Public Good Index using a weak version of Axiom (NPM) or Axiom (MEP) based on the concept of equivalent games presented in Chapter 1. Extensions to coalitional versions of the Public Good Index are also considered.

2.1 Preliminaries

This section deals with preliminaries on the Public Good Index. We essentially present a result due to Holler and Packel (1983) which is the first axiomatic description of the Public Good Index. Useful references are also provided. In order to ease the proofs of some characterizations that appear later in the chapter, some preliminary properties of the Public

Good Index are established.

2.1.1 The Holler-Packel axiomatization of the Public Good Index

As already announced, the Public Good Index was axiomatized by Holler and Packel (1983) using Axiom (HPM) as the main novelty in the analysis of voting powers. The independence of the axioms the authors used was later established by Napel (2001).

Theorem 2.1.1 (Holler and Packel (1983)).

Given a nonempty and finite subset N of $\mathcal{P}$, the Public Good Index is the unique power index that simultaneously satisfies (NP), (E), (AN) and (HPM) on $\mathcal{G}_N$. ⋆

Other axiomatizations of the Public Good Index with a fixed electorate include Haradau and Napel (2007) who built a *potential function* to characterize the Public Good Index; and Alonso-Meijide et al. (2008) who provided an alternative characterization by substituting a monotonicity axiom to the Holler-Packel Mergeability axiom. Related contributions also include, for example, Holler and Li (1995) for the non-normalized version, Freixas and Kurz (2016) for some type of monotonic power indices obtained as convex combinations of the Public Good Index and the Banzhaf index, Alonso-Meijide et al. (2015) for an extension to simple games with externalities, Courtin and Tchantcho (2020) for extensions to $(j, 2)$ simple games and Kurz (2021) for an extension to (j, k) simple games.

We move to the variable electorate setting.

2.1.2 Axiom of Supplementation Consistency

The next axiom is a way to handle changes due to a supplementation that may reasonably affect the shares of voters by a power index.

Axiom 10. ***Supplementation Consistency* (SC)**: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \setminus N$, for all k-supplementations G' of G, for all $i \in N$, $\varphi_i(G') = \varphi_i(G)\lambda_{G'}$ for some constant $\lambda_{G'}$.*

In (SC), it is stated that, from a simple game G to a k-supplementation G' of G, changes, if any, on each player's share is proportional to his/her power in G. More specifically, when the power index is normalized, the constant $\lambda_{G'}$ becomes more explicit as shown below:

Proposition 2.1.1 (**Safokem et al. (2021)**). *If a power index φ satisfies (E) and (SC), then for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \setminus N$, for all k-supplementations G' of G, for all $i \in N$,*

$$\varphi_i(G') = (1 - \varphi_k(G'))\varphi_i(G). \tag{2.1}$$

Proof. Suppose that a power index φ satisfies (E) and (SC). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a k-supplementation G' of G. Then, there exists a constant $\lambda_{G'}$ such that $\varphi_i(G') = \varphi_i(G)\lambda_{G'}$ for all $i \in N$. Then by (E), $1 - \varphi_k(G') = \sum_{i\in N} \varphi_i(G') = \lambda_{G'} \left(\sum_{i\in N} \varphi_i(G)\right) = \lambda_{G'}$. ■

Equation (2.1) effectively matches the consistency requirement by Hougaard and Moulin (2014) as an agent is removed (or introduced) in a cost allocation problem.

The next result is a property of power indices that meet (SC) and (E).

Proposition 2.1.2 (**Safokem et al. (2021)**)**.** *Consider a simple game $G = (N, \mathcal{W}(G))$, a player $k \in \mathcal{P} \setminus N$, a k-supplementation $G'$of G and two power indices φ and ψ that both satisfy (E) and (SC) .*

Then $(\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1)$ implies $\psi(G) = \varphi(G)$.

Proof. Suppose that two power indices φ and ψ satisfy (E) and (SC). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a k-supplementation G' of G such that $\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1$. Then, by Proposition 2.1.1, it follows that for all $i \in N$, $\psi_i(G') = (1 - \psi_k(G'))\psi_i(G)$ and $\varphi_i(G') = (1 - \varphi_k(G'))\varphi_i(G)$. Since $\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1$, then $\psi_i(G) = \varphi_i(G)$ for all $i \in N$. That is $\psi(G) = \varphi(G)$. ■

2.1.3 Preliminary results

It is shown that the Public Good Index meets (NP), (E), (NPM) and (SC). As observed by Holler and Packel (1983), the following result is straightforward:

Proposition 2.1.3. *The Public Good Index HP satisfies (NP) and (E).*

Thanks to Proposition 1.2.4, we are able to establish that the Public Good Index satisfies (SC) as stated in the next result.

Proposition 2.1.4. *The Public Good Index satisfies Axiom (SC).*

Proof. Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$ and a k-supplementation $G' = (N', \mathcal{W}(G'))$ of G. Given $i \in N$, it follows from Proposition 1.2.4 that

$$\begin{aligned}
\mathrm{HP}_i(G') &= \frac{|\mathcal{M}_i(G')|}{\sum_{j\in N'} |\mathcal{M}_j(G')|} \\
&= \frac{|\mathcal{M}_i(G)|}{|\mathcal{M}_k(G')| + \sum_{j\in N} |\mathcal{M}_j(G)|} \\
&= \frac{\sum_{j\in N} |\mathcal{M}_j(G)|}{|\mathcal{M}_k(G')| + \sum_{j\in N} |\mathcal{M}_j(G)|} \times \frac{|\mathcal{M}_i(G)|}{\sum_{j\in N} |\mathcal{M}_j(G)|} = (1 - \mathrm{HP}_k(G'))\,\mathrm{HP}_i(G).
\end{aligned}$$

Therefore HP satisfies Axiom (SC).

■

Using Proposition 1.2.2, we prove that the Public Good Index satisfies (NPM).

Proposition 2.1.5. *The Public Good Index satisfies (NPM).*

Proof. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of independent players such that $|T| \geq 2$. Parts (c) and (d) in Proposition 1.2.2 imply that

$$\sum_{j \in N^T} |\mathcal{M}_j(G^T)| = \sum_{j \in N} |\mathcal{M}_j(G)| \text{ and } |\mathcal{M}_{i_T}(G^T)| = \sum_{i \in T} |\mathcal{M}_i(G)|.$$

Furthermore,

$$\mathrm{HP}_{i_T}(G^T) = \frac{|\mathcal{M}_{i_T}(G^T)|}{\sum_{j \in N^T} |\mathcal{M}_j(G^T)|} = \frac{\sum_{i \in T} |\mathcal{M}_i(G)|}{\sum_{j \in N} |\mathcal{M}_j(G)|} = \sum_{i \in T} \mathrm{HP}_i(G).$$

Therefore HP satisfies (NPM).

■

Remark 2.1.1. *By Proposition 1.2.6, any power index φ that satisfies (NPM) also satisfies (NP). Moreover, if φ satisfies both (E), (SC) and (NPM), then $\varphi_i(G) = \varphi_i(G_0)$ for all $i \in N_0$ whenever $G = G_0[i_1, ..., i_p]$ for all nonempty and finite subset $N_0 \subseteq \mathcal{P}$.*

Proposition 2.1.6 (Safokem et al. (2021))**.** *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC).*

Then for all simple games $G = (N, \mathcal{W}(G))$, for all $i \in N$ and for all $j \in \mathcal{P} \backslash N$, $\varphi_j\left(G^{i \leftrightarrow j}\right) = \varphi_i\left(G\right)$.

Proof. Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Consider a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $j \in \mathcal{P} \backslash N$. Let $k \in \mathcal{P} \backslash (N \cup \{j\})$. By Proposition 1.2.6, $\varphi_k(G[k]) = 0$ and therefore, $\varphi_i(G[k]) = \varphi_i(G)$ by (E) and (SC). Moreover, $G^{i \leftrightarrow j}$ is obtained from $G\left[k\right]$ by merging i and k into j. Therefore by (NPM), $\varphi_j\left(G^{i \leftrightarrow j}\right) = \varphi_k(G[k]) + \varphi_i(G[k]) = \varphi_i\left(G\right)$.

■

Proposition 2.1.6 tells us that, if from one simple game to another, only one player in the initial simple game is replaced by another player, then the new player simply inherits the replaced player's share. However, nothing is said about the shares of other players in the new simple game.

Remark 2.1.2. *Suppose that φ is a power index that satisfies* (E), (NPM) *and* (SC). *We denote by $\varphi_{T,i}$ the share of player i in the simple game $G_{T\cup\{i\}}$ for a given coalition $T \subseteq \mathcal{P}$ and a given player $i \in \mathcal{P}\backslash T$. Note that for any other player $j \in \mathcal{P}\backslash(T \cup \{i\})$, $G_{T\cup\{i\}} = \left(G_{T\cup\{j\}}\right)^{j\leftrightarrow i}$. Therefore, it follows from Proposition* 2.1.6 *that*

$$\varphi_{T,i} = \varphi_i\left(G_{T\cup\{i\}}\right) = \varphi_j\left(G_{T\cup\{j\}}\right) = \varphi_{T,j}.$$

This shows that $\varphi_{T,i}$ only depends on T but not on the player i we choose from $\mathcal{P}\backslash T$. That is why, from now on, we simply denote by φ_T the share, with respect to φ, of an arbitrary player $i \in \mathcal{P}\backslash T$ in the simple game $G_{T\cup\{i\}}$. By efficiency, it follows that for all coalitions $S \subseteq \mathcal{P}$ of at least two players, the collection $(\varphi_T)_{T\subseteq S/|T|=|S|-1}$ satisfies the following equation (E_S) :

$$\sum_{T\subseteq S/|T|=|S|-1} \varphi_T = 1. \tag{2.2}$$

Recall that the set $\mathcal{P}$ of potential players is infinite. So, there is an infinite number of equations similar to (2.2). As we will show in the next section, those equations are sufficient to determine the collection $(\varphi_T)_{\emptyset\neq T\subseteq\mathcal{P}}$ for any power index that satisfies (E), (NPM) and (SC). For illustration, we show, in the example below, how to determine φ_T when T is a singleton.

Example 2.1.1. *Suppose that $|S| = 2$ and set $S = \{i,j\}$. We have to determine $\varphi_{\{i\}} = \varphi_j\left(G_{\{i,j\}}\right)$ and $\varphi_{\{j\}} = \varphi_i\left(G_{\{i,j\}}\right)$ assuming that φ is a power index on $\mathcal{G}$ that satisfies* (E), (NPM) *and* (SC). *We first bring into consideration a new player, say $k \in \mathcal{P}\backslash\{i,j\}$. As stated in* (2.2), *we have*

$$E_{\{i,j\}} : \varphi_{\{i\}} + \varphi_{\{j\}} = 1,\ E_{\{i,k\}} : \varphi_{\{i\}} + \varphi_{\{k\}} = 1,\ E_{\{j,k\}} : \varphi_{\{j\}} + \varphi_{\{k\}} = 1.$$

Equivalently

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi_{\{i\}} \\ \varphi_{\{j\}} \\ \varphi_{\{k\}} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since the corresponding matrix is invertible, we get

$$\begin{pmatrix} \varphi_{\{i\}} \\ \varphi_{\{j\}} \\ \varphi_{\{k\}} \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Instead of inverting the matrix, it is sufficient to note that $\frac{1}{2}\left(E_{\{i,j\}}\right) + \frac{1}{2}\left(E_{\{i,k\}}\right) - \frac{1}{2}\left(E_{\{j,k\}}\right)$ would have immediately lead us to $\varphi_{\{i\}} = \frac{1}{2}$. In the general case, we provide in the next section appropriate combinations to obtain φ_T.

2.2 Axiomatizations of the Public Good Index

We state and prove some axiomatizations of the Public Good Index using our merging and supplementation axioms.

2.2.1 Main result

It is shown in the precedent section that the Public Good Index satisfies (NP), (E), (NPM) and (SC). It is now shown that it is the unique power index on $\mathcal{G}$ that meets (E), (NPM) and (SC). Before we state and prove this result, the following lemmas are useful to ease the presentation of the main steps of the proof.

In the following lemma, it is shown that power indices that satisfy (E), (NPM) and (SC) all coincide with HP on singleton simple games.

Lemma 2.2.1. *Let φ be a power index on $\mathcal{G}$ that satisfies* (E), (NPM) *and* (SC).

Then for all coalitions $C \subseteq \mathcal{P}$, $\varphi(G_C) = \mathrm{HP}(G_C)$.

Proof. To ease the proof, we introduce, for all integers $p \geq 2$, the sequence $(c_m)_{1\leq m\leq p}$ defined by

$$c_m = \frac{(-1)^{m-1}}{\binom{p-1}{m-1}}.$$

where $\binom{p-1}{m-1}$ is the binomial coefficient. Note that it can be easily checked that for $0\leq m<p$,

$$(p-m)\, c_{m+1} + m c_m = 0. \tag{2.3}$$

Now, let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Consider a coalition $C \subseteq \mathcal{P}$ and set $|C| = p$. If $p = 1$, then $C = \{k\}$ and $\mathcal{M}(G_C) = \{\{k\}\}$ for some $k \in \mathcal{P}$. By efficiency, $\varphi_k(G_C) = 1 = \mathrm{HP}_k(G_C)$. Clearly, $\varphi(G_C) = \mathrm{HP}(G_C)$. Now suppose that $p \geq 2$. As we announced earlier, we consider a coalition $C' \subseteq \mathcal{P}\backslash C$ of $p-1$ players. Set $N = C \cup C'$ and denote by $\mathcal{H}$ the set of all simple games G_S such that $S \subseteq N$ and $|S| = p$. Note that there are exactly $\binom{2p-1}{p}$ simple games G_S in $\mathcal{H}$ that lead to $\binom{2p-1}{p}$ equation $(E_S)_{S\subseteq N,|S|=p}$, with exactly $\binom{2p-1}{p-1}$ variables $(X_T)_{T\subseteq N,|T|=p-1}$.

Since $\mathrm{HP}_i(G_C) = \frac{1}{p}$ for all $i \in C$, we have to prove that $\varphi_i(G_C) = \frac{1}{p}$ for all $i \in C$. That is, $X_{C\backslash\{i\}} = \frac{1}{p}$ for all $i \in C$. Consider $i \in C$ and set $K = C\backslash\{i\}$. In (2.2) (see Remark 2.1.2, Page 36), we multiply by $c_{p-|S\cap K|}$ the left-hand-side and the right-hand-side of each equation (E_S) such that $S \subseteq N$ and $|S| = p$. By summing over all left-hand-sides and over all right-hand-sides, we obtain

$$\sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \sum_{T\subseteq S,|T|=p-1} X_T = \sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \times 1 \tag{2.4}$$

On the one hand, the right-hand-side of (2.4), say $\sum_R$, is simplified as follows:

$$\sum\nolimits_R = \sum_{S\subseteq N, |S|=p} c_{p-|S\cap K|} \times 1 = \sum_{k=0}^{p-1} \sum_{S\subseteq N, |S|=p, |S\cap K|=k} c_{p-k}$$

Each coalition S such that $S \subseteq N$, $|S| = p$ and $|S \cap K| = k$ consists in k players from K and $p-k$ players from $N\backslash K$. Since $|K| = p-1$ and $|N\backslash K| = p$, there are exactly $\binom{p-1}{k}\binom{p}{p-k}$ such coalitions. Noting that $0 \leq |S \cap K| \leq p-1$, it follows that

$$\sum\nolimits_R = \sum_{k=0}^{p-1} \binom{p}{p-k}\binom{p-1}{k} \frac{(-1)^{p-1-k}}{\binom{p-1}{p-1-k}}$$

Since $\binom{p-1}{p-1-k} = \binom{p-1}{k}$ and $\binom{p}{p-k} = \binom{p}{k}$, it follows that

$$\sum\nolimits_R = (-1)^{p-1} \sum_{k=0}^{p-1} (-1)^k \binom{p}{k} = (-1)^{p-1}\left(0 - (-1)^p\right) = 1. \tag{2.5}$$

On the other hand, the left-hand-side of (2.4), say $\sum_L$, is simplified as follows:

$$\sum\nolimits_L = \sum_{S\subseteq N, |S|=p} c_{p-|S\cap K|} \sum_{T\subseteq S, |T|=p-1} X_T = \sum_{T\subseteq N, |T|=p-1} X_T \sum_{S\subseteq N, |S|=p, T\subseteq S} c_{p-|S\cap K|}.$$

Given $T \subseteq N$ such that $|T| = p-1$, each coalition $S \subseteq N$ such that $|S| = p$ and $T \subseteq S$ can be rewritten as $S = T \cup \{l\}$ for some $l \in N\backslash T$. Furthermore, $N\backslash T = (K\backslash T) \cup (N\backslash (K \cup T))$. Thus,

$$\sum\nolimits_L = \sum_{T\subseteq N, |T|=p-1} X_T \left(\sum_{l\in K\backslash T, S=T\cup\{l\}} c_{p-|S\cap K|} + \sum_{l\in N\backslash (T\cup K), S=T\cup\{l\}} c_{p-|S\cap K|} \right).$$

Consider a coalition $T \subseteq N$ such that $|T| = p-1$. First suppose that $T = K$. Then no coalition S exists such that $S = T \cup \{l\}$ for some $l \in K\backslash T$, since $K\backslash T = \emptyset$. And there are exactly p coalitions S such that $S = T \cup \{l\}$ for some $l \in N\backslash (T \cup K) = N\backslash K$. Now suppose that $T \neq K$. For all $l \in K\backslash T$ and $S = T \cup \{l\}$, $|S \cap K| = |T \cap K| + 1 \leq p-1$. And for all $l \in N\backslash (T \cup K)$ and $S = T \cup \{l\}$, $|S \cap K| = |T \cap K|$. Therefore,

$$\sum\nolimits_L = pX_K + \sum_{T\subseteq N, T\neq K, |T|=p-1} \left[|K\backslash T|\, c_{p-1-|T\cap K|} + |N\backslash (T \cup K)|\, c_{p-|T\cap K|} \right] X_T.$$

Since $|K| = p-1$ and $|N| = 2p-1$, it follows that for all $T \subseteq N$ such that $T \neq K$ and $|T| = p-1$, we have $|K\backslash T| = p-1-|T \cap K|$ and $|N\backslash (T \cup K)| = 2p-1-|T|-|K\backslash T| = p - |K\backslash T| = |T \cap K| + 1$. Therefore,

$$\sum\nolimits_L = pX_K + \sum_{T\subseteq N, T\neq K, |T|=p-1} \left[(p-1-|T\cap K|)\, c_{p-1-|T\cap K|} + (|T\cap K|+1)\, c_{p-|T\cap K|} \right] X_T.$$

For all $T \subseteq N$ such that $T \neq K$ and $|T| = p-1$, note that, the relation (2.3), taking $m = p-1-|T \cap K|$, implies that

$$(p-1-|T\cap K|)\, c_{p-1-|T\cap K|} + (|T\cap K|+1)\, c_{p-|T\cap K|} = 0 \text{ and } \sum\nolimits_L = pX_K. \tag{2.6}$$

We conclude from (2.4), (2.5) and (2.6) that $X_K = \frac{1}{p}$ for all coalitions $K \subseteq \mathcal{P}$. Thus, $\varphi(G_C) = \mathrm{HP}(G_C)$.

■

Remark 2.2.1. *Note that to prove Lemma 2.2.1, we mainly use the fact that the power index φ satisfies Axiom (E) and for all simple games $G = (N, \mathcal{W}(G))$, for all $i \in N$ and for all $j \in \mathcal{P} \backslash N$, $\varphi_j(G^{i\leftrightarrow j}) = \varphi_i(G)$.*

In the next lemma, it is shown that all power indices that meet (E), (NPM) and (SC) necessarily coincide on all simple games in which every voter is decisive in at most one coalition.

Lemma 2.2.2. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC).*

Then $\varphi(G) = \mathrm{HP}(G)$ for all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Proof. Suppose that φ is a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). We denote by $\mathcal{G}_n$ the set of all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ with n players such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. We prove by induction on $n = |N| \geq 1$ the assertion $\mathcal{A}(n)$ that for all simple games $G \in \mathcal{G}_n$, $\varphi(G) = \mathrm{HP}(G)$.

Initialization step.

In the initialization step, we consider $n \in \{1, 2\}$.

Suppose that $n = 1$ and let $G = (N, \mathcal{W}(G)) \in \mathcal{G}_1$. Then $N = \{i\}$ for some $i \in \mathcal{P}$. By efficiency, $\varphi(G) = \mathrm{HP}(G)$.

Suppose that $n = 2$ and let $G = (N, \mathcal{W}(G)) \in \mathcal{G}_2$. Then $\mathcal{M}(G) = \{\{i\}\}$; or $\mathcal{M}(G) = \{\{j\}\}$; or $\mathcal{M}(G) = \{\{i\}, \{j\}\}$; or $\mathcal{M}(G) = \{\{i, j\}\}$ with $N = \{i, j\}$ for some $i, j \in \mathcal{P}$. First suppose that $\mathcal{M}(G) = \{\{i\}\}$ with $N = \{i, j\}$. Then by efficiency and Proposition 1.2.6, $\varphi_i(G) = 1 = \mathrm{HP}_i(G)$ and $\varphi_j(G) = 0 = \mathrm{HP}_j(G)$ since j is a null player in G. That is $\varphi(G) = \mathrm{HP}(G)$. Similarly, if $\mathcal{M}(G) = \{\{j\}\}$, then $\varphi_j(G) = 1 = \mathrm{HP}_j(G)$ and $\varphi_i(G) = 0 = \mathrm{HP}_i(G)$. That is $\varphi(G) = \mathrm{HP}(G)$. Now suppose that $\mathcal{M}(G) = \{\{i\}, \{j\}\}$ with $N = \{i, j\}$. Then $G = G_{\{i,j\}}$ and $\varphi(G) = \mathrm{HP}(G)$ by Lemma 2.2.1. Finally, suppose that $\mathcal{M}(G) = \{\{i, j\}\}$ with $N = \{i, j\}$. Consider a player $k \in \mathcal{P} \backslash \{i, j\}$. Let $G_1 = (N_1, \mathcal{W}(G_1))$ and $G_2 = (N_2, \mathcal{W}(G_2))$ be the simple games defined by $N_1 = \{i, k\}$, $N_2 = \{j, k\}$, $\mathcal{M}(G_1) = \{\{i, k\}\}$ and $\mathcal{M}(G_2) = \{\{j, k\}\}$. Note that $G_1 = G^{j\leftrightarrow k}$, $G_2 = G^{i\leftrightarrow k}$ and $G_2 = G_1^{i\leftrightarrow j}$. By Proposition 2.1.6, it follows that

$$\varphi_i(G) = \varphi_k(G_2), \ \varphi_j(G) = \varphi_k(G_1) \ \text{ and } \varphi_i(G_1) = \varphi_j(G_2).$$

It then follows by efficiency with respect to G, G_1 and G_2 that

$$\varphi_i(G) + \varphi_j(G) = 1, \ \varphi_i(G_1) + \varphi_j(G) = 1 \text{ and } \varphi_i(G_1) + \varphi_i(G) = 1.$$

Solving this three equations leads to $\varphi_i(G) = \varphi_j(G) = \varphi_i(G_1) = \frac{1}{2}$. Since $\text{HP}_i(G) = \text{HP}_j(G) = \frac{1}{2}$, it holds that $\varphi(G) = \text{HP}(G)$.

Induction step.

For the induction step, suppose that $\mathcal{A}(n)$ holds for some integer $n \geq 2$. We prove that $\mathcal{A}(n+1)$ necessarily holds. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{n+1}$. Set $C = \{i \in N : |\mathcal{M}_i(G)| = 1\}$ and $N^0(G) = \{i_1, i_2, ..., i_{n_0}\}$ with $n_0 = |N^0(G)|$ that is the number of null players in the simple game G. Note that $N = C \cup N^0(G)$. First suppose that $|S| = 1$ for all $S \in \mathcal{M}(G)$. Then $G = G_C[i_1, i_2, ..., i_{n_0}]$ and $\varphi_i(G) = 0 = \text{HP}_i(G)$ for all $i \in N^0(G)$ by Proposition 1.2.6. And for all $i \in C$, one have in the one hand $\varphi_i(G) = \varphi_i(G_C)$ and $\text{HP}_i(G) = \text{HP}_i(G_C)$ by Remark 2.1.1, and in the other hand $\varphi_i(G_C) = \text{HP}_i(G_C)$ by Lemma 2.2.1; therefore $\varphi_i(G) = \text{HP}_i(G)$. Then $\varphi(G) = \text{HP}(G)$. Now suppose that there exists some $S \in \mathcal{M}(G)$ such that $|S| \geq 2$. Consider three distincts players i, j and k in N such that $i, j \in S$ and $k \in C$. Let $S_i = S \setminus \{i\}$ and $S_j = S \setminus \{j\}$. Define the simple games $G_1 = (N_1, \mathcal{W}(G_1))$ and $G_2 = (N_2, \mathcal{W}(G_2))$ by $N_1 = N \setminus \{i\}$, $N_2 = N \setminus \{j\}$, $\mathcal{M}(G_1) = [\mathcal{M}(G) \setminus \{S\}] \cup \{S_i\}$ and $\mathcal{M}(G_2) = [\mathcal{M}(G) \setminus \{S\}] \cup \{S_j\}$. Since $G \in \mathcal{G}_{n+1}$, coalitions in $\mathcal{M}(G)$ are disjoints. Thus, no coalition in $\mathcal{M}(G) \setminus \{S\}$ contains S_i or S_j. This guarantees that the simple games G_1 and G_2 are well-defined. By the induction assumption, $\varphi_t(G_1) = \frac{1}{n_1} = \text{HP}_t(G_1)$ for all $t \in C \setminus \{i\}$ and $\varphi_t(G_2) = \frac{1}{n_2} = \text{HP}_t(G_2)$ for all $t \in C \setminus \{j\}$. Note that $n_1 = |N_1 \setminus N^0(G_1)| = |N \setminus (N^0(G) \cup \{i\})| = |N \setminus (N^0(G) \cup \{j\})| = |N_2 \setminus N^0(G_2)| = n_2$. Moreover, G is an i-supplementation of G_1 as well as a j-supplementation of G_2. Therefore, moving from G_1 to G, (SC) implies that

$$\varphi_j(G) = (1 - \varphi_i(G))\, \varphi_j(G_1) = \frac{1 - \varphi_i(G)}{n_1} \text{ and } \varphi_t(G) = (1 - \varphi_i(G))\, \varphi_t(G_1) = \frac{1 - \varphi_i(G)}{n_1}.$$

Similarly, from G_2 to G, (SC) implies that

$$\varphi_i(G) = (1 - \varphi_j(G))\, \varphi_i(G_2) = \frac{1 - \varphi_j(G)}{n_1} \text{ and } \varphi_t(G) = (1 - \varphi_j(G))\, \varphi_t(G_2) = \frac{1 - \varphi_j(G)}{n_1}.$$

We then deduce that

$$\varphi_t(G) = \frac{1 - \varphi_i(G)}{n_1} = \frac{1 - \varphi_j(G)}{n_1} \text{ with } \varphi_i(G) = \frac{1 - \varphi_j(G)}{n_1}.$$

Therefore

$$\varphi_i(G) = \varphi_j(G) = \frac{1}{n_1 + 1}.$$

Recalling that G is an i-supplementation of G_1, we deduce by (SC) that for all $t \in C \setminus \{i\}$,

$$\varphi_t(G) = (1 - \varphi_i(G))\, \varphi_t(G_1) = \frac{1 - \varphi_i(G)}{n_1} = \frac{1}{n_1 + 1}.$$

This proves that $\varphi_t(G) = \frac{1}{n_1+1} = \text{HP}_t(G)$ for all $t \in C$ where $n_1 + 1 = |\{t \in N : |\mathcal{M}_t(G)| = 1\}| = |N \setminus N^0(G)|$. Also, one have $\varphi_t(G) = 0 = \text{HP}_t(G)$ for all $t \in N^0(G)$ by Proposition 1.2.6. Finally, $\varphi(G) = \text{HP}(G)$. This proves that $\mathcal{A}(n+1)$ holds. Therefore, we conclude that $\mathcal{A}(n)$ holds for all integers $n \geq 2$.

■

We are now ready to state and prove the following:

Theorem 2.2.1 (Safokem et al. (2021)).

A power index φ on $\mathcal{G}$ satisfies (E), (NPM) and (SC) if and only if $\varphi = \text{HP}$. ⋆

Proof.

Necessity. By Propositions 2.1.3, 2.1.4 and 2.1.5, HP necessary satisfies (E), (SC) and (NPM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Given a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, we denote by $E(G)$ the set of all players i in N such that $|\mathcal{M}_i(G)| \geq 2$ and by $\mathcal{G}_{(m)}$ the set of all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ in which $E(G)$ contains exactly m players. We prove by induction on integer $m \geq 0$ the assertion $\mathcal{A}'(m)$ that for all simple games $G \in \mathcal{G}_{(m)}$, $\varphi(G) = \text{HP}(G)$.

For the initialization step, suppose that $m = 0$. For all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(0)}$, $E(G) = \emptyset$. That is $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Therefore, $\varphi(G) = \text{HP}(G)$ by Lemma 2.2.2. This prove that $\mathcal{A}'(0)$ holds.

For the induction step, suppose that $\mathcal{A}'(m)$ holds for some integer $m \geq 0$. We prove that $\mathcal{A}'(m+1)$ necessarily holds. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(m+1)}$. Then $|E(G)| = m + 1 \geq 1$. Thus, there exists some player $i \in E(G)$. Let $p = |\mathcal{M}(G)|$ and $q = |\mathcal{M}_i(G)|$. We write $\mathcal{M}_i(G) = \{S_1, S_2, ..., S_q\}$ and $\mathcal{M}(G) = \{S_1, S_2, ..., S_q, S_{q+1}, ..., S_p\}$. Consider $p+q$ distincts players $j_1, j_2, ..., j_p, i_1, i_2, ..., i_q \in \mathcal{P} \setminus N$. We define the simple games:

- $G_1 = (N_1, \mathcal{W}(G_1))$ with $N_1 = (N \setminus \{i\}) \cup \{j_1, j_2, ..., j_p\} \cup \{i_1, i_2, ..., i_q\}$ and

$$\mathcal{M}(G_1) = \{T_1 \cup \{i_1, j_1\}, ..., T_q \cup \{i_q, j_q\}, S_{q+1} \cup \{j_{q+1}\}, ..., S_p \cup \{j_p\}\}$$

  where $T_t = S_t \setminus \{i\}$ for all $t \in \{1, 2, ..., q\}$.

- $G_2 = (N_2, \mathcal{W}(G_2))$ with $N_2 = N \cup \{j_1, j_2, ..., j_p\}$ and

$$\begin{aligned} \mathcal{M}(G_2) &= \{T_1 \cup \{i, j_1\}, ..., T_q \cup \{i, j_q\}, S_{q+1} \cup \{j_{q+1}\}, ..., S_p \cup \{j_p\}\} \\ &= \{S_1 \cup \{j_1\}, ..., S_q \cup \{j_q\}, S_{q+1} \cup \{j_{q+1}\}, ..., S_p \cup \{j_p\}\}. \end{aligned}$$

- $G_3 = (N_3, \mathcal{W}(G_3))$ with $N_3 = N_2 \setminus \{i\}$ and

$$\mathcal{M}(G_3) = \{T_1 \cup \{j_1\}, ..., T_q \cup \{j_q\}, S_{q+1} \cup \{j_{q+1}\}, ..., S_p \cup \{j_p\}\}.$$

Note that G_1 is obtained from G by adding j_t to S_t for $t \in \{1, ..., p\}$ and replacing player i in each S_t by i_t for $t \in \{1, ..., q\}$. Since each new player belongs to exactly one minimal winning coalition in G_1, then $E(G_1) = E(G) \setminus \{i\}$. It follows that $|E(G_1)| = m$.

Therefore by the induction assumption, $\varphi(G_1) = \mathrm{HP}(G_1)$. Also note that moving from G_1 to G_2 consists in merging players $i_1, i_2, ..., i_q$ into i. Since $\{i_1, i_2, ..., i_q\}$ is a coalition of independent players in the simple game G_1, we then deduce that

$$\begin{aligned}\varphi_i(G_2) &= \sum_{t=1}^{q} \varphi_{i_t}(G_1) \text{ since } \varphi \text{ satisfies (NPM)} \\ &= \sum_{t=1}^{q} \mathrm{HP}_{i_t}(G_1) \text{ since } \varphi(G_1) = \mathrm{HP}(G_1) \\ &= \mathrm{HP}_i(G_2) \text{ since HP satisfies (NPM).}\end{aligned}$$

To continue, also note that $E(G_3) = E(G) \setminus \{i\}$, it follows that $|E(G_3)| = m$. Therefore by the induction assumption, $\varphi(G_3) = \mathrm{HP}(G_3)$. By observing that G_2 is an i-supplementation of G_3, it follows that for all $k \in N_2 \setminus \{i\}$,

$$\begin{aligned}\varphi_k(G_2) &= (1 - \varphi_i(G_2))\, \varphi_k(G_3) \text{ since } \varphi \text{ satisfies (SC)} \\ &= (1 - \mathrm{HP}_i(G_2))\, \mathrm{HP}_k(G_3) \text{ since } \varphi_i(G_2) = \mathrm{HP}_i(G_2) \text{ and } \varphi(G_3) = \mathrm{HP}(G_3) \\ &= \mathrm{HP}_k(G_2) \text{ since HP satisfies (SC).}\end{aligned}$$

This proves that $\varphi(G_2) = \mathrm{HP}(G_2)$. Finally, we define the simple games $(G'_t)_{0 \leq t \leq p}$ by $G'_0 = G$ and for all $t \in \{1, 2, ..., p\}$, $G'_t = (N'_t, \mathcal{W}(G'_t))$ with

$$N'_t = N \cup \{j_1, j_2, ..., j_t\} \text{ and } \mathcal{M}(G'_t) = \{S_1 \cup \{j_1\}, ..., S_t \cup \{j_t\}, S_{t+1}, ..., S_p\}.$$

Note that $G'_p = G_2$ and that G'_t is an j_t-supplementation of G'_{t-1} for all $t \in \{1, 2, ..., p\}$. Moreover $i \in N'_t$ and $\mathcal{M}_i(G'_t) \neq \emptyset$ for all $t \in \{1, 2, ..., p\}$. By the definition of HP, it follows that $\mathrm{HP}_i(G'_t) > 0$ for $t \in \{1, 2, ..., p\}$. This proves that, for all $k \in N'_t \setminus \{i\}$, $\mathrm{HP}_k(G'_t) < 1$. Since $\varphi(G_2) = \mathrm{HP}(G_2)$ and G'_p is an j_p-supplementation of G'_{p-1}, we deduce from Proposition 2.1.2 that $\varphi(G'_{p-1}) = \mathrm{HP}(G'_{p-1})$ with $\mathrm{HP}_{j_{p-1}}(G'_{p-1}) < 1$. By iterating this procedure for $t = p$, $p-1, ..., 1$, it holds that $\varphi(G'_0) = \mathrm{HP}(G'_0)$. Since $G'_0 = G$, we get $\varphi(G) = \mathrm{HP}(G)$. This proves that $\mathcal{A}'(m+1)$ holds. In conclusion, $\mathcal{A}'(m)$ holds for all integers $m \geq 0$.

■

2.2.2 Independence of axioms

Theorem 2.2.1 provides an axiomatic characterization of the Public Good Index on the domain of simple games with a variable electorate. We now prove that those axioms are independent (none of the three axioms is redundant).

Proposition 2.2.1. *The three axioms used in Theorem 2.2.1 are logically independent on $\mathcal{G}$.*

Proof.

(i) **Non redundancy of Non Profitable Merging of independent players**

Consider the power index φ defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by:

$$\varphi_i(G) = \begin{cases} \dfrac{1}{|N \backslash N^0(G)|} & \text{if } i \in N \backslash N^0(G) \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that φ satisfies (E) and (SC). To see that φ fails to meet (NPM), let $G = (N, \mathcal{W}(G))$ with $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 3, 5\}, \{2, 3, 4\}\}$. Set $T = \{1, 2\}$. Then T is a pair of independent players. Moreover $N^T = \{3, 4, 5, i_T\}, \varphi_1(G) + \varphi_2(G) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \neq \varphi_{i_T}(G^T) = \frac{1}{4}$. Hence φ does not satisfy (NPM). It follows that (NPM) is not redundant.

(ii) **Non redundancy of Efficiency**

The power index 2 HP (where HP is the Public Good Index) satisfies (NPM) and (SC); but fails to meet (E). Thus (E) is not redundant.

(iii) **Non redundancy of Supplementation Consistency**

Consider the power index H defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ as follows:

$$H_i(G) = \frac{|\mathcal{M}_i^*(G)|}{\sum_{j \in N} |\mathcal{M}_j^*(G)|}$$

where

$$\mathcal{M}^*(G) = \{S \in \mathcal{M}(G) : |S| \leq |T| \text{ for all } T \in \mathcal{M}(G)\}$$

is the set of minimal winning coalitions with the minimal size and for all $i \in N$,

$$\mathcal{M}_i^*(G) = \{S \in \mathcal{M}^*(G) : i \in S\}.$$

The power index H satisfies (E) and (NPM), but H fails to meet (SC). To prove this, consider the simple game $G = (N, \mathcal{W}(G))$ such that $N = \{1, 2, 3, 4\}$ and $\mathcal{M}(G) = \{\{1, 3\}, \{2\}, \{3, 4\}\}$. Set $G' = (N \cup \{5\}, \mathcal{W}(G')) \in \mathcal{G}$ with $\mathcal{M}(G') = \{\{1, 3\}, \{2, 5\}, \{3, 4, 5\}\}$. Then G' is a 5-supplementation of G. Moreover, $\mathcal{M}^*(G) = \{\{2\}\}$ and $\mathcal{M}^*(G') = \{\{1, 3\}, \{2, 5\}\}$. It follows that $H(G) = (0, 1, 0, 0)$ and $H(G') = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right)$. It appears that the power index H does not satisfy (SC) since H satisfies (E) and $H_3(G') = \frac{1}{4} \neq 0 = (1 - H_5(G'))\, H_3(G)$. Thus (SC) is not redundant.

■

2.2.3 Further axiomatizations

We use here two other axioms introduced early in Chapter 1, namely: (IEM) and (IIM), to provide two new axiomatizations of the Public Good Index.

Theorem 2.2.2 (Safokem et al. (2021)).

A power index φ on $\mathcal{G}$ satisfies (E), (SC) and (IEM) if and only if $\varphi = \text{HP}$.

Proof.

Necessity. By Proposition 2.1.3 and Proposition 2.1.4, HP satisfies (E) and (SC); it remains to prove that HP satisfies (IEM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two independent players with $|T| \geq 2$. Items (c) and (d) in Proposition 1.2.2 imply that $\text{HP}_i(G^T) = \text{HP}_i(G)$ for all $i \in N \backslash T$. Therefore HP satisfies (IEM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (SC) and (IEM). Since φ satisfies (E) and (IEM), it follows by Proposition 1.2.7 that φ satisfies (NPM). Thus, Theorem 2.2.1 implies that $\varphi = \text{HP}$. ■

Theorem 2.2.3 (Safokem et al. (2021)).

A power index φ on $\mathcal{G}$ satisfies (E), (SC) and (IIM) if and only if $\varphi = \text{HP}$.

Proof.

Necessity. By Proposition 2.1.3 and Proposition 2.1.4, HP satisfies (E) and (SC); it remains to prove that HP satisfies (IIM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition T of independent players with $|T| \geq 2$ and a player $t \in T$. Given $i_T \in \mathcal{P} \backslash N$, note that $G^{T \to t}$ is obtained from G^T when i_T is replaced by t. Parts (c) and (d) in Proposition 1.2.2 imply that $\text{HP}_i(G^{T \to t}) = \text{HP}_i(G)$ for all $i \in N \backslash T$. Therefore HP satisfies (IIM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (SC) and (IIM). Since φ satisfies (IIM), it follows by Proposition 1.2.8 that φ also satisfies (IEM). Thus, Theorem 2.2.2 implies that $\varphi = \text{HP}$. ■

Remark 2.2.2. *For the independence of axioms used in Theorems* 2.2.2 *and* 2.2.3*, power indices defined in Proposition* 2.2.1 *still apply.*

2.3 Alternative axiomatizations

This section is devoted to other axiomatizations of the Public Good Index using Axiom (MEP) on equivalent games and a weak version of Axiom (NPM). We also provide characterization results of the coalitional versions of the Public Good Index on the domain of simple games with a priori unions and a variable electorate.

2.3.1 Axiomatization using Axiom (MEP)

In the next proposition, it is shown that the Public Good Index satisfies (MEP).

Proposition 2.3.1. *The Public Good Index HP satisfies (MEP).*

Proof. Consider two simple games $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ and a player $i \in N \cap N'$ such that $G\Delta_i G'$. It follows that there exists $(S,T) \in \mathcal{M}(G) \times \mathcal{M}(G')$ such that $\mathcal{M}(G') = (\mathcal{M}(G) \backslash \{S\}) \cup \{T\}$ with $|S| = |T|$ and ($i \in S \cap T$ or $i \notin S \cup T$). In the one hand, by Remark 1.2.3, we have:

- If $i \in S \cap T$, then $\mathcal{M}_i(G') = (\mathcal{M}_i(G) \backslash \{S\}) \cup \{T\}$, it follows that $|\mathcal{M}_i(G')| = |\mathcal{M}_i(G)|$;
- If $i \notin S \cup T$, then $\mathcal{M}_i(G') = \mathcal{M}_i(G)$, it follows that $|\mathcal{M}_i(G')| = |\mathcal{M}_i(G)|$.

In the other hand,

$$\begin{aligned}
\sum_{j \in N'} |\mathcal{M}_j(G')| &= \sum_{L \in \mathcal{M}(G')} |L| \\
&= |T| + \sum_{L \in \mathcal{M}(G') \backslash \{T\}} |L| \\
&= |T| + \sum_{L \in \mathcal{M}(G) \backslash \{S\}} |L| \text{ since } \mathcal{M}(G') \backslash \{T\} = \mathcal{M}(G) \backslash \{S\} \\
&= |S| + \sum_{L \in \mathcal{M}(G) \backslash \{S\}} |L| \text{ since } |S| = |T| \\
&= \sum_{L \in \mathcal{M}(G)} |L| \\
&= \sum_{j \in N} |\mathcal{M}_j(G)|.
\end{aligned}$$

Thus,

$$\begin{aligned}
HP_i(G') &= \frac{|\mathcal{M}_i(G')|}{\sum_{j \in N'} |\mathcal{M}_j(G')|} \\
&= \frac{|\mathcal{M}_i(G)|}{\sum_{j \in N} |\mathcal{M}_j(G)|} \\
&= \mathrm{HP}_i(G).
\end{aligned}$$

That is HP satisfies (MEP). ∎

We now prove that a power index that satisfies (E), (MEP) and (SC) coincides with the Public Good Index on simple games where each player belongs to at most one minimal winning coalition.

Lemma 2.3.1. *If φ is a power index on $\mathcal{G}$ that satisfies (E), (MEP) and (SC), then $\varphi(G) = \text{HP}(G)$ for a given simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.*

Proof. Suppose that a power index φ on $\mathcal{G}$ satisfies (E), (MEP) and (SC). We denote by $r(G) = max\{|S| : S \in \mathcal{M}(G)\}$ for all simple game $G \in \mathcal{G}$ and $\mathcal{G}_{\overline{r}} = \{G \in \mathcal{G} : r(G) = r\}$ for all positive integer r. We prove by induction on positive integer $r \geq 1$ that $\varphi(G) = \text{HP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r}}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Initialization: For $r = 1$, let $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{1}}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Then for all $i \in N \backslash N^0(G)$, $\mathcal{M}_i(G) = \{\{i\}\}$. Set $S = N \backslash N^0(G)$. We have $G \Delta_i G_S$ for all $i \in S$, where G_S is the singleton game associated to S. It follows that for all $i \in S$,

$$\begin{aligned}
\varphi_i(G) &= \varphi_i(G_S) \text{ since } \varphi \text{ satisfies (MEP) and } G\Delta_i G_S \\
&= \frac{1}{|S|} \text{ by Proposition 1.2.11 since } \varphi \text{ satisfies (E) and (MEP)} \\
&= \text{HP}_i(G_S) \text{ by Proposition 1.2.11 since HP satisfies (E) and (MEP)} \\
&= \text{HP}_i(G) \text{ since HP satisfies Axiom (MEP) and } G\Delta_i G_S.
\end{aligned}$$

Note that for all $i \in N^0(G)$, $\varphi_i(G) = 0 = \text{HP}_i(G)$ By Proposition 1.2.9. Thus, $\varphi(G) = \text{HP}(G)$.

Induction step: Suppose that for some positive integer $r \geq 1$, $\varphi(G) = \text{HP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r}}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$, we set $F(G) = \{S \in \mathcal{M}(G) : |S| = r + 1\}$. We prove by induction on $f = |F(G)| \geq 1$ that $\varphi(G) = \text{HP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $f = |F(G)|$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

For $f = 1$, consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|F(G)| = 1$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Set $F(G) = \{S\}$, then $|S| = r+1 \geq 2$. Fix two players $i, j \in S$ and define the simple games $G' = (N', \mathcal{W}(G')), G'' = (N'', \mathcal{W}(G'')) \in \mathcal{G}$ with $N' = N \backslash \{i\}$, $N'' = N \backslash \{j\}$, $\mathcal{M}(G') = (\mathcal{M}(G) \backslash \{S\}) \cup \{S'\}$ and $\mathcal{M}(G'') = (\mathcal{M}(G) \backslash \{S\}) \cup \{S''\}$ where $S' = S \backslash \{i\}$ and $S'' = S \backslash \{j\}$.

Then G is an i-supplementation of G', and it follows by Proposition 2.1.1 that $\varphi_k(G) = (1 - \varphi_i(G))\varphi_k(G')$ for all $k \in N \backslash \{i\}$. Note that $G' \in \mathcal{G}_{\overline{r}}$ and $|\mathcal{M}_k(G')| \leq 1$ for all $k \in N \backslash \{i\}$, it follows by induction assumption that $\varphi(G') = \text{HP}(G')$. But $\text{HP}_k(G') = \frac{1}{n - n_0 - 1}$ for all $k \in N \backslash (N^0(G) \cup \{i\})$ where $n_0 = |N^0(G)|$. Thus, $\varphi_j(G) = \varphi_k(G) = \frac{1 - \varphi_i(G)}{n - n_0 - 1}$ for all $k \in N \backslash (N^0(G) \cup \{i, j\})$.

Similarly, $\varphi_i(G) = \varphi_k(G) = \frac{1-\varphi_j(G)}{n-n_0-1}$ for all $k \in N \backslash (N^0(G) \cup \{i,j\})$ since G is an j-supplementation of G''.

It follows that, $\varphi_i(G) = \varphi_j(G) = \varphi_k(G)$ for all $k \in N \backslash (N^0(G) \cup \{i,j\})$.

We have by Proposition 1.2.9 that $\varphi_k(G) = 0 = \mathrm{HP}_k(G)$ for all $k \in N^0(G)$. It follows by Axiom (E) that $\varphi_k(G) = \frac{1}{n-n_0} = \mathrm{HP}_k(G)$ for all $k \in N \backslash N^0(G)$. Therefore, $\varphi(G) = \mathrm{HP}(G)$.

Now suppose that for some positive integer $f \geq 1$, $\varphi(G) = \mathrm{HP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|F(G)| = f$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|F(G)| = f + 1$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Set $S \in F(G)$, then $|S| = r + 1 \geq 2$. Fix two players $i, j \in S$ and define the simple games $G' = (N', \mathcal{W}(G')), G'' = (N'', \mathcal{W}(G'')) \in \mathcal{G}$ with $N' = N \backslash \{i\}$, $N'' = N \backslash \{j\}$, $\mathcal{M}(G') = (\mathcal{M}(G) \backslash \{S\}) \cup \{S'\}$ and $\mathcal{M}(G'') = (\mathcal{M}(G) \backslash \{S\}) \cup \{S''\}$ where $S' = S \backslash \{i\}$ and $S'' = S \backslash \{j\}$.

Then G is an i-supplementation of G', and it follows by Proposition 2.1.1 that $\varphi_k(G) = (1 - \varphi_i(G))\varphi_k(G')$ for all $k \in N \backslash \{i\}$. Note that $G' \in \mathcal{G}_{\overline{r+1}}$, $|F(G')| = f$ and $|\mathcal{M}_k(G')| \leq 1$ for all $k \in N \backslash \{i\}$, it follows by induction assumption on f that $\varphi(G') = \mathrm{HP}(G')$. But $\mathrm{HP}_k(G') = \frac{1}{n-n_0-1}$ for all $k \in N \backslash (N^0(G) \cup \{i\})$ where $n_0 = |N^0(G)|$. Thus, $\varphi_j(G) = \varphi_k(G) = \frac{1-\varphi_i(G)}{n-n_0-1}$ for all $k \in N \backslash (N^0(G) \cup \{i,j\})$.

Similarly, $\varphi_i(G) = \varphi_k(G) = \frac{1-\varphi_j(G)}{n-n_0-1}$ for all $k \in N \backslash (N^0(G) \cup \{i,j\})$ since G is an j-supplementation of G''.

It follows that, $\varphi_i(G) = \varphi_j(G) = \varphi_k(G)$ for all $k \in N \backslash (N^0(G) \cup \{i,j\})$.

We have by Proposition 1.2.9 that $\varphi_k(G) = 0 = \mathrm{HP}_k(G)$ for all $k \in N^0(G)$. It follows by Axiom (E) that $\varphi_k(G) = \frac{1}{n-n_0} = \mathrm{HP}_k(G)$ for all $k \in N \backslash N^0(G)$. Therefore, $\varphi(G) = \mathrm{HP}(G)$.

■

Results that precede now lead us to the following:

Theorem 2.3.1.

A power index φ on $\mathcal{G}$ satisfies (E), (SC) and (MEP) if and only if $\varphi = \mathrm{HP}$. ★

Proof.

Necessity: Proposition 2.1.3, Proposition 2.1.4 and Proposition 2.3.1, show that the Public Good Index HP satisfies (E), (SC) and (MEP).

Sufficiency: Now consider a power index φ on $\mathcal{G}$ that satisfies (E), (MEP) and (SC). For all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, we set $E(G) = \{i \in N : |\mathcal{M}_i(G)| \geq 2\}$. We denote $\mathcal{G}_{(m)} = \{G \in \mathcal{G} : |E(G)| = m\}$ and we prove by induction on positive integer m that $\varphi(G) = \mathrm{HP}(G)$ for all $G \in \mathcal{G}_{(m)}$.

Initialization: For $m = 0$, consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(0)}$. We have $E(G) = \emptyset$, that is $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. It follows by Lemma 2.3.1 that $\varphi(G) = \mathrm{HP}(G)$.

Induction step: Suppose that for some positive integer m, $\varphi(G) = \mathrm{HP}(G)$ for all $G \in \mathcal{G}_{(m)}$. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(m+1)}$ and a player $i \in E(G)$. Then $|\mathcal{M}_i(G)| \geq 2$, set $p = |\mathcal{M}_i(G)|, q = |\mathcal{M}(G)|, \mathcal{M}_i(G) = \{S_1, ..., S_p\}$ and $\mathcal{M}(G) = \{S_1, ..., S_p, S_{p+1}, ..., S_q\}$. Consider p players $i_1, ..., i_p \in \mathcal{P} \backslash N$ and set $T_l = (S_l \backslash \{i\}) \cup \{i_l\}$ for all $l \in \{1, ..., p\}$. Define the following simple games $(G_l)_{l \in \{0,...,p\}}$ by $G_0 = G$ and for all $l \in \{1, ..., p\}, G_l = (N_l, \mathcal{W}(G_l))$ with $N_l = N \cup \{i_1, ..., i_l\}$ and $\mathcal{M}(G_l) = \{T_1, ..., T_l, S_{l+1}, ..., S_q\}$.

Then $E(G_p) = E(G) \backslash \{i\}$, one obtains $|E(G_p)| = m$. It follows by the induction assumption that $\varphi(G_p) = \mathrm{HP}(G_p)$. Note that $G_p \Delta_j G_{p-1}$ for all $j \in N_{p-1} \backslash \{i\}$, therefore

$$\begin{aligned} \varphi_j(G_{p-1}) &= \varphi_j(G_p) \text{ since } \varphi \text{ satisfies Axiom (MEP)} \\ &= \mathrm{HP}_j(G_p) \text{ since } \varphi(G_p) = \mathrm{HP}(G_p) \\ &= \mathrm{HP}_j(G_{p-1}) \text{ since HP satisfies Axiom (MEP).} \end{aligned}$$

Moreover,

$$\begin{aligned} \varphi_i(G_{p-1}) &= 1 - \sum_{j \in N_{p-1} \backslash \{i\}} \varphi_j(G_{p-1}) \text{ since } \varphi \text{ satisfies Axiom (E)} \\ &= 1 - \sum_{j \in N_{p-1} \backslash \{i\}} \mathrm{HP}_j(G_{p-1}) \text{ since } \varphi_j(G_{p-1}) = \mathrm{HP}_j(G_{p-1}) \text{ for all } {\scriptstyle j \in N_{p-1} \backslash \{i\}} \\ &= \mathrm{HP}_i(G_{p-1}) \text{ since HP satisfies Axiom (E).} \end{aligned}$$

That is $\varphi(G_{p-1}) = \mathrm{HP}(G_{p-1})$.

Let $l \in \{1, ..., p\}$ and assume that $\varphi(G_l) = \mathrm{HP}(G_l)$. Note that for all $j \in N_{l-1} \backslash \{i\}, G_l \Delta_j G_{l-1}$; therefore

$$\begin{aligned} \varphi_j(G_{l-1}) &= \varphi_j(G_l) \text{ since } \varphi \text{ satisfies Axiom (MEP)} \\ &= \mathrm{HP}_j(G_l) \text{ since } \varphi(G_l) = \mathrm{HP}(G_l) \text{ by assumption} \\ &= \mathrm{HP}_j(G_{l-1}) \text{ since HP satisfies Axiom (MEP).} \end{aligned}$$

Moreover,

$$\begin{aligned} \varphi_i(G_{l-1}) &= 1 - \sum_{j \in N_{l-1} \backslash \{i\}} \varphi_j(G_{l-1}) \text{ since } \varphi \text{ satisfies Axiom (E)} \\ &= 1 - \sum_{j \in N_{l-1} \backslash \{i\}} \mathrm{HP}_j(G_{l-1}) \text{ since } \varphi_j(G_{l-1}) = \mathrm{HP}_j(G_{l-1}) \text{ for all } {\scriptstyle j \in N_{l-1} \backslash \{i\}} \\ &= \mathrm{HP}_i(G_{l-1}) \text{ since HP satisfies Axiom (E).} \end{aligned}$$

Thus, $\varphi(G_{l-1}) = \mathrm{HP}(G_{l-1})$.

For $l = 1$, one obtains $\varphi(G_0) = \mathrm{HP}(G_0)$. That is $\varphi(G) = \mathrm{HP}(G)$.

■

Next, we show in that none of the three axioms in Theorem 2.3.1 can not be dropped.

PROPOSITION 2.3.2. *Axioms (E), (SC) and (MEP) are logically independent.*

Proof.

(*i*) **Non redundancy of Membership Equivalence Property**

Consider the power index J defined by:

for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $J(G) = \begin{cases} \mathrm{HP}(G) \text{ if } |\mathcal{M}(G)| \geq 2 \\ \mathbb{1}_{\{i_0\}} \text{ if } \mathcal{M}(G) = \{S\} \text{ for some } S \subseteq N \end{cases}$

where $i_0 = min\{i : i \in S\}$.

It is easy to check that J satisfies Axiom (E).

To establish that J satisfies Axiom (SC), consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a k-supplementation $G' = (N', \mathcal{W}(G'))$ of G. Note that $|\mathcal{M}(G')| = |\mathcal{M}(G)|$ as in Proposition 1.2.4.

If $|\mathcal{M}(G)| \geq 2$, then $|\mathcal{M}(G')| \geq 2$; it follows by Corollary 2.1.4 that for all $i \in N, \mathrm{HP}_i(G') = (1 - \mathrm{HP}_k(G'))\,\mathrm{HP}_i(G)$. That is $J_i(G') = (1 - J_k(G'))J_i(G)$ for all $i \in N$.

If $|\mathcal{M}(G)| = 1$, set $\mathcal{M}(G) = \{S\}$ and $i_0 = min\{i : i \in S\}$. Then $\mathcal{M}(G') = \{S\}$ or $\mathcal{M}(G') = \{S \cup \{k\}\}$.

In the case $\mathcal{M}(G') = \{S\}$, we have $J_k(G') = 0$ and $J_i(G') = J_i(G) = (1 - J_k(G'))J_i(G)$ for all $i \in N$.

In the case $\mathcal{M}(G') = \{S \cup \{k\}\}$, $J_k(G') = \begin{cases} 1 \text{ if } k < i_0 \\ 0 \text{ if } k > i_0. \end{cases}$

If $k < i_0$, then $J_i(G') = 0 = (1 - J_k(G'))J_i(G)$ for all $i \in N$.

If $k > i_0$, then $J_i(G') = J_i(G) = (1 - J_k(G'))J_i(G)$ for all $i \in N$.

Thus, J satisfies Axiom (SC).

The power index J does not satisfy Axiom (MEP): In fact, consider the simple games $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ define by $N = \{2, 3, 4, 5\}, N' = \{1, 2, 3, 4, 5\}, \mathcal{M}(G) = \{\{2, 3, 4\}\}$ and $\mathcal{M}(G') = \{\{1, 2, 3\}\}$. It follows that $J(G) = (1; 0; 0; 0)$ meanwhile $J(G') = (1; 0; 0; 0; 0)$. Note that $G'\Delta_2 G$ meanwhile $J_2(G') \neq J_2(G)$. That is J does not satisfy Axiom (MEP).

(*ii*) **Non redundancy of Efficiency**

The power index $2\,\mathrm{HP}$ satisfies (MEP) and (SC); but fails to meet Axiom (E).

(*iii*) **Non redundancy of Supplementation Consistency**

It will be shown later in Chapter 3 that the Deegan-Packel index DP satisfies (E) and (MEP) (see Propositions 3.1.4 and 3.3.1), but fails to meet Axiom (SC). To see that DP does not meet Axiom (SC), consider what follows. Define the following simple games $G = (N, \mathcal{W}(G)), G' = (N \cup \{6\}, \mathcal{W}(G')) \in \mathcal{G}$ by $N =$

$\{1,2,3,4,5\}$, $N' = N \cup \{6\}$, $\mathcal{M}(G) = \{\{1,3,4\},\{2,4,5\},\{1,2\}\}$ and $\mathcal{M}(G') = \{\{1,3,4\},\{2,4,5\},\{1,2,6\}\}$. Note that G' is a 6-supplementation of G. We have $\mathrm{DP}(G) = \left(\frac{5}{18},\frac{5}{18},\frac{2}{18},\frac{4}{18},\frac{2}{18}\right)$ and $\mathrm{DP}(G') = \left(\frac{4}{18},\frac{4}{18},\frac{2}{18},\frac{4}{18},\frac{2}{18},\frac{2}{18}\right)$. It follows that $\frac{\mathrm{DP}_1(G')}{\mathrm{DP}_1(G)} \neq \frac{\mathrm{DP}_3(G')}{\mathrm{DP}_3(G)}$. Therefore, the Deegan-Packel index DP does not satisfy Axiom (SC).

■

Remark 2.3.1. *Note that another proof of Theorem 2.3.1 can be established using on the one hand, Proposition 2.1.3, Proposition 2.1.4 and Proposition 2.3.1 for the necessity part; and an the other hand, using Proposition 1.2.10 and Theorem 2.2.2 for the sufficiency part.*

2.3.2 Weakening Axiom (NPM)

Proposition 1.2.6 states that (NPM) implies (NP). It may be of interest to see how to weaken the (NPM) condition so that (NP) is explicitly needed; by so doing we aim at highlighting how far do these two axioms overlap. On this point, we show that when (NPM) is replaced in Theorem 2.2.1 with its weaker version obtained by restricting the merging operation only to independent players who are decisive in at least one coalition, incorporating (NP) is a necessary condition. Furthermore, to characterize the Public Good Index, we also need the classical axiom of symmetry.

Axiom 11. ***Symmetry (S)***: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all pairs $\{i,j\}$ of symmetric players in G, $\varphi_i(G) = \varphi_j(G)$.*

According to Axiom (S), two symmetric players in a simple game should enjoy the same voting power. To continue, we introduce the following definition.

Definition 2.3.1. *Given a simple game G and a coalition T, the merging operation from G to G^T is effective if T contains no null player in the game G.*

The following axiom is a weak version of Axiom (NPM) when only effective merging of independent players are considered.

Axiom 12. ***Non Profitable Effective Merging (NPEM)* of independent players**: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players,*

$$\varphi_{i_T}(G^T) = \sum_{i \in T} \varphi_i(G)$$

whenever T contains no null player and only independent players.

Proposition 2.3.3 (Safokem et al. (2021))**.** *If a power index φ on $\mathcal{G}$ satisfies (E), (SC), (NPEM), (NP) and (S), then φ satisfies (NPM).*

Proof.

Consider a power index φ on $\mathcal{G}$. Suppose that φ satisfies (E), (SC), (NPEM), (NP) and (S). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition $T \subseteq N$ containing only independent players with $|T| \geq 2$. Let S be the set of all members of T who are null players in G; and $G' = (N \backslash S, \mathcal{W}(G'))$ be the simple game such that $\mathcal{M}(G') = \mathcal{M}(G)$. The game G' is obtained from G when the members of S leave G without altering the status of any minimal winning coalition.

Step 1. First suppose that S is empty. Then by (NPEM), condition (1.3) is satisfied.

Step 2. Now suppose that S is not empty and $T\backslash S = \{i\}$, for some $i \in N$. Note that i is no longer a player in the game $G^T = (N^T, \mathcal{W}(G^T))$. Consider the game $G'' = (N^T \cup \{i\}, \mathcal{W}(G''))$ obtained from G^T when player i gets in the game by joining only all minimal winning coalitions that contain i_T; that is, for all $K \subseteq N^T \cup \{i\}$, $K \in \mathcal{M}(G'')$ if and only if $(i_T \notin K, i \notin K$ and $K \in \mathcal{M}(G^T))$, or $(i_T \in K$, $i \in K$ and $K\backslash\{i\} \in \mathcal{M}(G^T))$. On the one hand, G'' is an i-supplementation of the game G^T. Furthermore, i and i_T are symmetric players in G''. Therefore by (S), (E) and (SC), it follows that

$$\varphi_i(G'') = \varphi_{i_T}(G'') = (1 - \varphi_i(G''))\varphi_{i_T}(G^T). \tag{2.7}$$

On the other hand, G'' is an i_T-supplementation of G'. Thus, it follows that

$$\varphi_{i_T}(G'') = \varphi_i(G'') = (1 - \varphi_{i_T}(G''))\varphi_i(G'). \tag{2.8}$$

It follows from (2.7) and (2.8) that $\varphi_{i_T}(G^T) = \varphi_i(G')$. By noting that when a null player k leaves a simple game while the set of minimal winning coalitions remains unchanged, the shares of players other than k remain unchanged by applying both (E), (SC) and (NP). Therefore, $\varphi_i(G') = \varphi_i(G)$ since G' is obtained from G when the members of S leaves G and those members are null players in G. Therefore $\varphi_{i_T}(G^T) = \varphi_i(G)$. Hence condition (1.3) is satisfied.

Step 3. Finally, suppose that S is not empty and $|T\backslash S| \geq 2$. Set $T' = T\backslash S$. Then no null player belongs to T' and T' contains only independent players. By (NPEM),

$$\varphi_{i_{T'}}(G^{T'}) = \sum_{j \in T'} \varphi_j(G) = \sum_{j \in T} \varphi_j(G). \tag{2.9}$$

The second equality in (2.9) holds by (NP) since players in $S = T\backslash T'$ are null players in the game G. In the game $G^{T'}$, note that coalition $T'' = \{i_{T'}\} \cup S$ contains only independent players and all members of T'' are null players in $G^{T'}$ except player $i_{T'}$. As it is just shown in Step 2., merging in the game $G^{T'}$ the members of T'' into i_T implies that

$$\varphi_{i_T}\left(\left(G^{T'}\right)^{T''}\right) = \varphi_{i_{T'}}(G^{T'}). \tag{2.10}$$

Moreover, $\left(G^{T'}\right)^{T''} = G^{T}$. Therefore, condition (1.3) holds by (2.9) and (2.10).

In each of the three possible cases, condition (1.3) holds. That is φ satisfies (NPM). ■

The next result is a characterization of the Public Good Index using (NP).

Theorem 2.3.2 (Safokem et al. (2021)). A power index φ on $\mathcal{G}$ satisfies (E), (SC), (NPEM), (NP) and (S) if and only if $\varphi = \mathrm{HP}$. ★

Proof. The proof follows from Proposition 2.3.3 and Theorem 2.2.1. ■

Proposition 2.3.4. *Axioms in Theorem 2.3.2 are independent.*

Proof. Each of the three power indices presented in Section 2.2.2 satisfies both (NP) and (S). Those power indices can then be used to prove that in Theorem 2.3.2, none of the (E), (SC) and (NPEM) can be dropped. For (NP) and (S), we consider the two power indices below:

- Define the power index I on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by

$$I_i(G) = \begin{cases} \mathrm{HP}_i(G) & \text{if } |\mathcal{M}(G)| \geq 2 \\ \frac{1}{|N|} & \text{otherwise} \end{cases}$$

  The power index I satisfies (E), (SC), (NPEM) and (S); but I obviously fails to meet (NP) over the class of unanimity simple games or simple games with only a unique minimal coalition. This proves that (NP) is not redundant in Theorem 2.3.2.

- Consider the power index J defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by

$$J_i(G) = \begin{cases} \mathrm{HP}_i(G) & \text{if } {\scriptstyle |\mathcal{M}(G)| \geq 2} \\ 1 & \text{if } {\scriptstyle \mathcal{M}(G) = \{S\}} \text{ for some coalition } S \text{ such that } i \in S \text{ and } i \leq j \text{ for all } {\scriptstyle j \in S} \\ 0 & \text{otherwise.} \end{cases}$$

  The power index J satisfies (E), (SC), (NPEM) and (NP); but J obviously fails to meet (S) by considering simple games with a unique minimal winning coalition. This proves that (S) is not redundant in Theorem 2.3.2. ■

Using weak versions of (IEM) and (IIM) where the merging operation is restricted only to independent players who are decisive in at least one coalition, one can obtain two other characterizations of the Public Good Index. Since such results become obvious from what is already done here, they are simply omitted.

2.3.3 Axiomatization of coalitional versions of the Public Good Index

A simple game with a priori unions is a simple game where players are grouped into a coalition structure (that is a partition of the set of players). Alonso-Meijide et al. (2010b) has extended the Public Good Index to simple games with a priori unions as well as an axiomatization of the whole class of coalitional versions of the Public Good Index; these are power indices on simple games with a priori unions that yield, on simple games with a priori unions containing only singletons, the same distribution of voting power as with the Public Good Index of the main simple game with no a priori unions; see also Holler and Nohn (2009) for the introduction and axiomatization of four alternative versions of the Public Good Index for simple games with a priori unions. In a similar way, we provide below another characterization of coalitional versions of the Public Good Index. Following Alonso-Meijide et al. (2010b), we recall the following definitions.

Definition 2.3.2. *Given a finite nonempty set N of players:*

1. *A simple game with a priori unions is a 3-tuple (N, v, P), where (N, v) is a simple game and P, a partition of N called the set of a priori unions.*

2. *A coalitional power index is any mapping that associates each simple game with a priori unions (N, v, P) with an $n-$tuple of real numbers; hat is, $f(N, v, P) = (f_i(N, v, P))_{i\in N}$.*

3. *A coalitional power index f is said to be a coalitional version of a given power index g on simple games if $f(N, v, P^N) = g(N, v)$ for all $(N, v) \in \mathcal{G}$, where $P^N = \{\{i\} : i \in N\}$.*

In particular, a coalitional power index f is a coalitional version of the Public Good Index if:

$$f(N, v, P^N) = \mathrm{HP}(N, v), \forall (N, v) \in \mathcal{G}. \tag{2.11}$$

For power indices on simple games with a priori unions, the following axiom is due to Alonso-Meijide et al. (2010b).

Axiom 13. Singleton Efficiency (SE): *For all $G = (N, v) \in \mathcal{G}$, $\sum_{i\in N} f_i(N, v, P^N) = 1$.*

In a similar way, we now introduce the following coalitional versions of the axioms we early presented.

Axiom 14. Supplementation Consistency with a priori Unions (SCU): *For all $G = (N, v) \in \mathcal{G}$, for all $k \in \mathcal{P}\backslash N$, and for all k-supplementations $G' = (N', v')$ of G, there exists a real constant $\lambda_{G'}$ such that*

$$f_i\left(N', v', P^{N'}\right) = \lambda_{G'} f_i\left(N, v, P^N\right) \text{ for all } i \in N.$$

Axiom 15. No Profitable Merging with a priori Unions (NPMU): *For all $G = (N, v) \in \mathcal{G}$, for all $T \subseteq N$ with $|T| \geq 2$, for all $j_T \in \mathcal{P}\backslash N$, $f_{j_T}(N^T, v^T, P^{N^T}) = \sum_{i \in T} f_i(N, v, P^N)$ whenever T consists of independent players in G.*

Proposition 2.3.5 (**Safokem et al. (2021)**). *A coalitional power index f is a coalitional version of the Public Good Index if and only if f satisfies (SE), (SCU) and (NPMU).*

Proof.

Straightforward from Theorem 2.2.1 and equation (2.11). ∎

Proposition 2.3.6. *Axioms (SE), (SCU) and (NPMU) are independent.*

Proof.

(*i*) **Axioms (SE), (SCU) do not imply (NPMU)**

Define the power index $\widetilde{\varphi}$ for all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ as follows,

$$\widetilde{\varphi}_i(G) = \begin{cases} \dfrac{1}{|N\backslash N^0(N, \mathcal{W}(G))|} & \text{if } P = P^N \text{ and } i \in N\backslash N^0((N, \mathcal{W}(G)) \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\widetilde{\varphi}$, the restriction of $\widetilde{\varphi}$ on the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P^N)$ coincides with the power index φ defined in *Part (i)* of the proof of Proposition 2.2.1 (that is $\widetilde{\varphi}(N, \mathcal{W}(G), P^N) = \varphi(N, \mathcal{W}(G)))$. Since we have shown in the proof of Proposition 2.2.1 that φ satisfies (SE) and (SCU), but not (NPMU), then by (2.11), $\widetilde{\varphi}$ satisfies (SE) and (SCU), but fails to meet (NPMU).

(*ii*) **Axioms (NPMU) and (SCU) do not imply (SE)**

Define the power index $\widetilde{\mathrm{HP}}$ defined for all simple game with a priori unions $G = (N, \mathcal{W}(G), P)$ and for all $i \in N$ by,

$$\widetilde{\mathrm{HP}}_i(G) = \begin{cases} 2\,\mathrm{HP}_i(N, \mathcal{W}(G)) \text{ if } P = P^N \\ 0 \text{ otherwise.} \end{cases}$$

By definition, the restriction of $\widetilde{\mathrm{HP}}$ on the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ with $P = P^N$ coincides with the power index defined in *Part (ii)* of the proof of Proposition 2.2.1. By considering (2.11), the power index $\widetilde{\mathrm{HP}}$ satisfies (NPMU) and (SCU), but not (SE).

(*iii*) **Axioms (NPMU) and (SE) do not imply (SCU)**

Consider the power index $\widetilde{\mathrm{H}}$ defined for all simple game with a priori unions $G = (N, \mathcal{W}(G), P)$ and for all $i \in N$ by,

$$\widetilde{H}_i(G) = \begin{cases} \dfrac{|\mathcal{M}_i^*(N, \mathcal{W}(G))|}{\sum_{j\in N} |\mathcal{M}_j^*(N, \mathcal{W}(G))|} & \text{if } P = P^N \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\mathcal{M}^*(N, \mathcal{W}(G)) = \{S \in \mathcal{M}(N, \mathcal{W}(G)) : |S| \leq |T| \text{ for all } T \in \mathcal{M}(N, \mathcal{W}(G))\},$$

is the set of minimal winning coalitions with the minimal size and for all $i \in N$,

$$\mathcal{M}_i^*(N, \mathcal{W}(G)) = \{S \in \mathcal{M}^*(N, \mathcal{W}(G)) : i \in S\}.$$

On the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ with $P = P^N$, the restriction of $\widetilde{\mathrm{H}}$ coincides with the power index H defined in *Part (iii)* of the proof of Proposition 2.2.1. Therefore, it follows by (2.11) that the power index $\widetilde{\mathrm{H}}$ satisfies (NPMU) and (SE), but not (SCU).

■

Each of the theorems presented in this chapter on simple games can be extended to simple games with a priori unions using very similar arguments. We simply omit here all those results since it was not our primarily objective. What is important is that each of the characterization we present here enriches the literature on the Public Good Index on simple games as well as simple games with a priori unions (with very little effort). Furthermore, the tools developed in this chapter enable us to widen the scope of our analysis to the Deegan-Packel index. This is presented in the next chapter.

Chapter Three

Axiomatizations of the Deegan-Packel index on simple games with a variable electorate

The Deegan-Packel index was introduced in Deegan and Packel (1978). Two known axiomatizations of this power index are due, on the one hand, to Deegan and Packel (1978) who proved an (AN)+(E)+(NP)+(DPM) characterization; and on the other hand, to Lorenzo-Freire et al. (2007) who has presented another characterization by replacing Axiom (DPM) by a monotonicity condition. All these results are related to simple games with a fixed electorate. Our objective is to provide new axiomatizations of the Deegan-Packel index for simple games with a variable electorate on which no existing characterization has not yet been reported to the best of our knowledge.

For an overview of this chapter, we first provide in Section 3.1, some preliminaries on the Deegan-Packel index. A new axiom on the supplementation operation is introduced: the Supplementation Invariance (SI) and it is shown that the Deegan-Packel index satisfies (E), (SI) and (NPM). Secondly, we state and prove in Section 3.2 new axiomatizations of the Deegan-Packel index for simple games with a variable electorate together with the logical independence of the axioms we used in each result. We conclude this chapter in Section 3.3 with another axiomatization of the Deegan-Packel index using an axiom based on equivalent games; we also obtain axiomatization results of coalitional versions of the Deegan-Packel index.

3.1 Preliminaries

The present section is devoted to preliminaries on the Deegan-Packel index. This includes an overview of two related characterization results; the introduction of our axiom of Supplementation Invariance (SI) and the proof that the Deegan-Packel index satisfies (SI), (NPM) and (IEM).

3.1.1 On two known axiomatizations of the Deegan-Packel index

The following characterization of the Deegan-Packel index due to Deegan and Packel (1978) is in the same spirit with the characterization of the Public Good Index by Holler and Packel (1983); see Theorem 2.1.1. The two results differ only on their respective mergeability conditions.

Theorem 3.1.1 (Deegan and Packel (1978))**.**

Given a nonempty and finite subset N of $\mathcal{P}$, the Deegan-Packel index is the unique power index that simultaneously satisfies (NP), (E), (AN) and (DPM) on $\mathcal{G}_N$. $\star$

Alternatively, Lorenzo-Freire et al. (2007) characterized the Deegan-Packel index by replacing the Deegan-Packel Mergeability with the following axiom of Minimal Monotonicity.

Axiom 16. ***Minimal Monotonicity***: *A power index φ satisfies Minimal Monotonicity (MM) if for any two simple games $G = (N, \mathcal{W}(G))$ and $G' = (N, \mathcal{W}(G')) \in \mathcal{G}_N$, it holds that,*

$$\varphi_i(G)|\mathcal{M}(G)| \leq \varphi_i(G')|\mathcal{M}(G')| \text{ for all } i \in N \text{ such that } \mathcal{M}_i(G) \subseteq \mathcal{M}_i(G').$$

That is, if in a simple game $G = (N, \mathcal{W}(G))$, the set of minimal winning coalitions containing a voter $i \in N$ is a subset of minimal winning coalitions containing voter i in another simple game $G' = (N, \mathcal{W}(G'))$, then the power of voter i in game G' is not less than the power of voter i in G once the power is normalized by the number of minimal winning coalitions in G' and G, respectively. A similar axiom of monotonicity was later introduced by Alonso-Meijide et al. (2008) to built up what we refer to as a dual characterization of the Public Good Index as compared to the one stated by Lorenzo-Freire et al. (2007) on the Deegan-Packel index. This sort of duality between the Holler-Packel index and the Deegan-Packel index is certainly due to their respective formulations with an apparent closeness to minimal winning coalitions.

3.1.2 Axiom of Supplementation Invariance

As with the Supplementation Consistency (SC) condition we use to characterize the Public Good Index, we now introduce the following axiom of *Supplementation Invariance* which is another conceivable way changes due to a supplementation may be handled by a power index.

Axiom 17. ***Supplementation Invariance* (SI)**: *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \setminus N$, for all k-supplementation G' of G, for all $i \in N$, if $k \notin S$ for all $S \in \mathcal{M}_i(G')$, then $\varphi_i(G') = \varphi_i(G)$.*

When a simple game G' is a k-supplementation of a simple game G, Axiom (SI) says that, each time a player i in the initial game G is such that k belongs to no minimal winning coalition containing i in the new game G', then the i's voting power should not be affected.

A useful relationship between (SI), (E) and (NP) is provided in the next result.

Proposition 3.1.1. *Let φ be a power index on $\mathcal{G}$ that satisfies (E) and (SI).*

Then φ satisfies Axiom (NP).

Proof. Let φ be a power index on $\mathcal{G}$ that satisfies (E) and (SI). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a null player i in G. Pose $G_i = (N_i, \mathcal{W}(G_i))$ with $N_i = N \backslash \{i\}$ and $\mathcal{M}(G_i) = \mathcal{M}(G)$. Note that G is an i-supplementation of G_i such that for all $j \in N_i$, for all $S \in \mathcal{M}_j(G), i \notin S$. Since φ satisfies Axiom (SI), one gets $\varphi_j(G) = \varphi_j(G_i)$ for all $j \in N_i$. It follows that

$$\sum_{j \in N_i} \varphi_j(G) = \sum_{j \in N_i} \varphi_j(G_i). \tag{3.1}$$

Moreover,

$$\begin{aligned} \varphi_i(G) &= 1 - \sum_{j \in N_i} \varphi_j(G) \text{ by efficiency of } \varphi \text{ applied to the simple game } G \\ &= 1 - \sum_{j \in N_i} \varphi_j(G_i) \text{ by equation (3.1)} \\ &= 0 \text{ by efficiency of } \varphi \text{ applied to the simple game } G_i. \end{aligned}$$

■

The following propositions provide other properties of all power indices that satisfy (E) and (SI).

Proposition 3.1.2. *Let φ be a power index on $\mathcal{G}$ that satisfies (E) and (SI).*

If $G_0 = (N_0, \mathcal{W}(G_0))$ and $G = (N, \mathcal{W}(G))$ are two simple games such that $G = G_0[i_1, i_2, ..., i_l]$ for some voters $i_1, i_2, ..., i_l \in \mathcal{P} \setminus N_0$ with $l \in \mathbb{N}^*$; then $\varphi_{i_k}(G) = 0$ for all $k \in \{1, ..., l\}$ and $\varphi_i(G) = \varphi_i(G_0)$ for all $i \in N_0$.

Proof. Let φ be a power index on $\mathcal{G}$ that satisfies (E) and (SI). Consider two simple games $G_0 = (N_0, \mathcal{W}(G_0))$ and $G = (N, \mathcal{W}(G))$ such that $G = G_0[i_1, i_2, ..., i_l]$ for some $l \in \mathbb{N}^*$ and l players $i_1, i_2, ..., i_l \in \mathcal{P} \setminus N_0$. One have $N = N_0 \cup \{i_1, i_2, ..., i_l\}$. Since for all $k \in \{1, ..., l\}, i_k$ is a null player in the simple game G, it follows by Proposition 3.1.1 that $\varphi_{i_k}(G) = 0$. For all $i \in N_0, \varphi_i(G_0[i_1]) = \varphi_i(G_0)$ by Axiom (SI), since $G_0[i_1]$ is a i_1-supplementation of G_0 and for all $S \in \mathcal{M}_i(G_0[i_1]), i_1 \notin S$. Similarly, $\varphi_i(G_0[i_1, i_2, ..., i_k]) =$

$\varphi_i(G_0[i_1,...,i_{k-1}])$ for all $k \in \{2,...,l\}$ and $i \in N_0$, since $G_0[i_1,i_2,...,i_k]$ is a i_k-supplementation of $G_0[i_1,i_2,...,i_{k-1}]$ and for all $S \in \mathcal{M}_i(G_0[i_1,...,i_k]), i_k \notin S$. For $k=l$, one gets for all $i \in N_0, \varphi_i(G_0[i_1,...,i_l]) = \varphi_i(G_0[i_1,...,i_{l-1}]) = ... = \varphi_i(G_0[i_1]) = \varphi_i(G_0)$. That is $\varphi_i(G) = \varphi_i(G_0)$.

■

Proposition 3.1.2 enables us to proves the following result.

Proposition 3.1.3. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SI).*

Then for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $i \in N$ and for all $j \in \mathcal{P}\backslash N$, $\varphi_j\left(G^{i\leftrightarrow j}\right) = \varphi_i\left(G\right)$.

Proof.

Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SI). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $i \in N$ and $j \in \mathcal{P}\backslash N$. Let $k \in \mathcal{P}\backslash (N \cup \{j\})$. By Proposition 3.1.2, $\varphi_k(G[k]) = 0$ and $\varphi_i(G[k]) = \varphi_i(G)$. Moreover, $G^{i\leftrightarrow j}$ is obtained from $G[k]$ by merging i and k into j. Since φ satisfies Axiom (NPM), it follows that $\varphi_j\left(G^{i\leftrightarrow j}\right) = \varphi_k(G[k]) + \varphi_i(G[k]) = \varphi_i\left(G\right)$.

■

Proposition 3.1.3 tells us that, given any power index that satisfies (E), (NPM) and (SI), if from one simple game to another, only one player in the initial simple game is replaced by another player, then the new player simply inherits the replaced player's share. However, nothing is said about the shares of other players in the new simple game.

3.1.3 Preliminary results

As preliminary results to characterization results in subsequent sections, it is shown here that the Deegan-Packel index satisfies (E), (SI), (NPM) and (IEM). The following is straightforward and is also mentioned by Deegan and Packel (1978).

Proposition 3.1.4. *The Deegan-Packel index DP satisfies (E), (NP) and (AN).*

The next results are related to simple games with a variable electorate.

Corollary 3.1.1. *The Deegan-Packel index satisfies Axiom (SI).*

Proof.

Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$ and a k-supplementation $G' = (N', \mathcal{W}(G'))$ of G. Let $i \in N$ be a player such that for all $S \in \mathcal{M}_i(G'), k \notin S$, it follows as in the Proposition 1.2.4 that $|\mathcal{M}(G')| = |\mathcal{M}(G)|, \mathcal{M}_i(G') = \{S \cup \{k\} : S \in E \cap \mathcal{M}_i(G)\} \cup \{S : S \in$

$\mathcal{M}_i(G) \setminus E\}$ and $|\mathcal{M}_i(G')| = |\mathcal{M}_i(G)|$. One obtains $\mathcal{M}_i(G') = \mathcal{M}_i(G)$ and $|\mathcal{M}(G')| = |\mathcal{M}(G)|$. Thus,

$$\begin{aligned} \mathrm{DP}_i(G') &= \frac{1}{|\mathcal{M}(G')|} \sum_{S \in \mathcal{M}_i(G')} \frac{1}{|S|} \\ &= \frac{1}{|\mathcal{M}(G)|} \sum_{S \in \mathcal{M}_i(G)} \frac{1}{|S|} \\ &= \mathrm{DP}_i(G). \end{aligned}$$

Therefore DP satisfies Axiom (SI). ■

Corollary 3.1.2. *The Deegan-Packel index satisfies Axiom (NPM).*

Proof. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two players such that the merging operation from G to G^T consists of merging members of T, where T contains only pairs of independent players. Let $i \in T$. Note that for all $S \in \mathcal{M}_i(G)$, $S \cap T = \{i\}$ and $|S \cap T| = 1$ since T only contains pairs of independent players, then $S^T = (S \setminus T) \cup \{i_T\}$ and $|S^T| = |S| - |S \cap T| + 1 = |S|$. By Parts (b) and (d) of Proposition 1.2.2, one gets $|\mathcal{M}(G^T)| = |\mathcal{M}(G)|$ and $\mathcal{M}_{i_T}(G^T) = \cup_{i \in T}\{S^T : S \in \mathcal{M}_i(G)\}$ with $\mathcal{M}_i(G) \cap \mathcal{M}_j(G) = \emptyset$ for all pair of distinct players i, j in T. Since $S \mapsto S^T$ is injective, we get

$$\begin{aligned} \mathrm{DP}_{i_T}(G^T) &= \frac{1}{|\mathcal{M}(G^T)|} \sum_{S \in \mathcal{M}_{i_T}(G^T)} \frac{1}{|S|} \\ &= \frac{1}{|\mathcal{M}(G^T)|} \sum_{i \in T} \sum_{S \in \mathcal{M}_i(G)} \frac{1}{|S^T|} \\ &= \frac{1}{|\mathcal{M}(G)|} \sum_{i \in T} \sum_{S \in \mathcal{M}_i(G)} \frac{1}{|S|} \\ &= \sum_{i \in T} \left(\frac{1}{|\mathcal{M}(G)|} \sum_{S \in \mathcal{M}_i(G)} \frac{1}{|S|}\right) \\ &= \sum_{i \in T} \mathrm{DP}_i(G). \end{aligned}$$

Therefore, DP satisfies Axiom (NPM). ■

Using Parts (b) and (c) in Proposition 1.2.2, we show in the next corollary that DP even satisfies Axiom (IEM).

Corollary 3.1.3. *The Deegan-Packel index DP satisfies Axiom (IEM).*

Proof.
Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two players such that the game G^T is obtained from G by merging members of T, where T contains only pairs of independent players. Let $i \in N\backslash T$. Note that for all $S \in \mathcal{M}_i(G)$, if $S \cap T = \emptyset$, then $S^T = S$ and $|S^T| = |S|$; else $|S \cap T| = 1$ since T only contains pairs of independent players, then $S^T = (S \setminus T) \cup \{i_T\}$ and $|S^T| = |S| - |S \cap T| + 1 = |S|$. By Parts (b) and (c) of Proposition 1.2.2, one gets $|\mathcal{M}(G^T)| = |\mathcal{M}(G)|$ and $\mathcal{M}_i(G^T) = \{S^T : S \in \mathcal{M}_i(G)\}$. Since $S \mapsto S^T$ is injective, we get

$$\mathrm{DP}_i(G^T) = \frac{1}{|\mathcal{M}(G^T)|} \sum_{S \in \mathcal{M}_i(G^T)} \frac{1}{|S|} = \frac{1}{|\mathcal{M}(G)|} \sum_{S \in \mathcal{M}_i(G)} \frac{1}{|S|} = \mathrm{DP}_i(G).$$

Therefore, DP satisfies Axiom (IEM).

■

3.2 Axiomatizations of the Deegan-Packel index

We state and prove some axiomatizations of the Deegan-Packel index for simple games with a variable electorate using Axiom (SI) and some axioms related to merging operations. The logical independence among our axioms is also provided.

3.2.1 Main result

It is just shown above that the Deegan-Packel index satisfies (NP), (E), (SI), (NPM) and (IEM). We now prove that appropriate combinations of these axioms uniquely identify the Deegan-Packel index. But before stating such result, we need the following lemmas to ease its proof. We start by considering the case of singleton games.

Lemma 3.2.1. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SI). Then for all coalitions $S \subseteq \mathcal{P}$, $\varphi(G_S) = \mathrm{DP}(G_S)$.*

Proof.
Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SI). Consider a coalition $S \subseteq \mathcal{P}$ and the singleton game $G_S = (S, \mathcal{W}(G_S))$. Since φ satisfies (E), (NPM) and (SI), it follows from Proposition 3.1.3 that for all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $i \in N$ and for all $j \in \mathcal{P}\backslash N$, $\varphi_j(G^{i\leftrightarrow j}) = \varphi_i(G)$. Then using Remark 2.2.1 together with Lemma 2.2.1, we obtain $\varphi_s(G_S) = \dfrac{1}{|S|}$ for all $s \in S$. Considering the fact that $\mathrm{DP}_s(G_S) = \dfrac{1}{|S|}$ for all $s \in S$, we then conclude that for all $s \in S$, $\varphi_s(G_S) = \mathrm{DP}_s(G_S)$. Thus, $\varphi(G_S) = \mathrm{DP}(G_S)$.

■

Next, we consider the case of all simple games in which each voter is decisive in at most one minimal winning coalition.

Lemma 3.2.2. *Let φ be a power index on $\mathcal{G}$ that satisfies* (E), (NPM) *and* (SI).

Then $\varphi(G) = \mathrm{DP}(G)$ for all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Proof. Suppose that φ is a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SI). We denote by $N'(G) = \{i \in N : |\mathcal{M}_i(G)| = 1\}$ and $n' = |N'(G)|$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$. We prove by induction on integer $n' \geq 1$ the assertion $\mathcal{B}(n')$ that for all simple games $G \in \mathcal{G}$ with $|N'(G)| = n'$, $\varphi(G) = \mathrm{DP}(G)$. Note that $N = N'(G) \cup N^0(G)$ and $N'(G) \cap N^0(G) = \emptyset$.

Initialization step. For the initialization step, suppose that $n' = 1$ and let $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|N'(G)| = 1$. Then $N'(G) = \{i\}$ and $\mathcal{M}(G) = \{\{i\}\}$ for some $i \in \mathcal{P}$. By Proposition 3.1.1, we have $\varphi_j(G) = 0 = \mathrm{DP}_j(G)$ for all $j \in N^0(G)$ and $\varphi_i(G) = 1 = \mathrm{DP}_i(G)$ by Axiom (E). That is $\varphi(G) = \mathrm{DP}(G)$. Then $\mathcal{B}(1)$ holds.

Induction step. For the induction step, suppose that $\mathcal{B}(n')$ holds for some integer $n' \geq 1$. We prove that $\mathcal{B}(n'+1)$ necessarily holds. Consider a game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|N'(G)| = n' + 1$. Pose $\mathcal{M}(G) = \{S_1, S_2, ..., S_m\}$ with $S_p \cap S_q = \emptyset$ for all $p, q \in \{1, ..., m\}$ such that $p \neq q$. Set $L(G) = \{S \in \mathcal{M}(G) : |S| \geq 2\}$.

If $|L(G)| = 0$, then for all $T \in \mathcal{M}(G), |T| = 1$. That is $G = G_S[i_1, \ldots, i_m]$ where $S = N'(G)$ and $N^0(G) = \{i_1, \ldots, i_m\}$. It follows by Proposition 3.1.2 and Lemma 3.2.1 that $\varphi(G) = \mathrm{DP}(G)$.

Else, $|L(G)| \geq 1$ and there exists some $k \in \{1, ..., m\}$ such that $|S_k| \geq 2$. Pose $S_k = \{i, i_1, ..., i_s\}$ and $T_k = S_k \backslash \{i\} = \{i_1, ..., i_s\}$ with $s = |S_k| - 1 \geq 1$. Consider the simple game $G_0 = (N_0, \mathcal{W}(G_0))$ define by $N_0 = N \setminus \{i\}$ and $\mathcal{M}(G_0) = \{S_1, S_2, ..., S_{k-1}, T_k, S_{k+1}, ..., S_m\}$. Note that $|N'(G_0)| = n'$ and $|\mathcal{M}_j(G_0)| \leq 1$ for all $j \in N_0$. It follows by the induction assumption that $\varphi(G_0) = \mathrm{DP}(G_0)$. Moreover, G is an i-supplementation of G_0 such that for all $j \in N \setminus S_k$ and for all $S \in \mathcal{M}_j(G), i \notin S$ (since minimal winning coalitions of G are disjoint). Then by Axiom (SI), $\varphi_j(G) = \varphi_j(G_0)$ and $\mathrm{DP}_j(G) = \mathrm{DP}_j(G_0)$ for all $j \in N \setminus S_k$. Thus in the one hand, $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in N \setminus S_k$. In the other hand, to prove that $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in S_k$, we consider the following cases.

Case $|L(G)| \geq 2$: there exists some $S_{k'} \in \mathcal{M}(G) \setminus \{S_k\}$ such that $|S_{k'}| \geq 2$. It follows in a similar ways that $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in N \setminus S_{k'}$. Note that $S_k \subseteq N \setminus S_{k'}$ since $S_k \cap S_{k'} = \emptyset$. Then $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in S_k$ in this case.

<u>*Case* $|L(G)| = 1$</u>: If $\mathcal{M}(G)\backslash L(G) \neq \emptyset$, then there exists some $S_{k'} \in \mathcal{M}(G)$ with $|S_{k'}| = 1$. Consider a player $l \in \mathcal{P}\backslash N$, let $T_{k'} = S_{k'} \cup \{l\}$ and define the simple game $G_1 = (N_1, \mathcal{W}(G_1))$ by $N_1 = N \cup \{l\}$ and $\mathcal{M}(G_1) = (\mathcal{M}(G)\backslash\{S_{k'}\}) \cup \{T_{k'}\}$. We have $L(G_1) = \{S_k, T_{k'}\}$, it then follows from the case $|L(G_1)| \geq 2$ that $\varphi(G_1) = \mathrm{DP}(G_1)$. Note that G_1 is a l-supplementation of G such that for all $j \in N_1\backslash T_{k'}$ and for all $S \in \mathcal{M}_j(G_1), l \notin S$. It then follows from Axiom (SI) that $\varphi_j(G) = \varphi_j(G_1)$ and $\mathrm{DP}_j(G) = \mathrm{DP}_j(G_1)$ for all $j \in N_1\backslash T_{k'}$. Therefore, $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in N_1\backslash T_{k'}$. Note that $S_k \subseteq N_1\backslash T_{k'}$. Thus, $\varphi_j(G) = \mathrm{DP}_j(G)$ for all $j \in S_k$ in this case.

If $\mathcal{M}(G)\backslash L(G) = \emptyset$, then $\mathcal{M}(G) = L(G) = \{S_k\}$ with $|S_k| \geq 2$. Proposition 3.1.2 implies $\varphi_s(G) = \varphi_s(G_{S_k})$ and $\mathrm{DP}_s(G) = \mathrm{DP}_s(G_{S_k})$ for all $s \in S_k$ since $G = G_{S_k}[i_1, \ldots, i_m]$ where $N\backslash S_k = \{i_1, \ldots, i_m\}$. Noting by Lemma 3.2.1 that $\varphi_s(G_{S_k}) = \mathrm{DP}_s(G_{S_k})$ for all $s \in S_k$, it follows that $\varphi_s(G) = \mathrm{DP}_s(G)$ for all $s \in S_k$. Finally, $\varphi(G) = \mathrm{DP}(G)$.

■

We are now able to state and prove the following characterization of the Deegan-Packel index for simple games with a variable electorate.

Theorem 3.2.1.

A power index φ on $\mathcal{G}$ satisfies (E), (NPM) and (SI) if and only if $\varphi = \mathrm{DP}$. ★

Proof.

Necessity. By Proposition 3.1.4 together with Corollaries 3.1.1 and 3.1.2, DP satisfies (E), (SI) and (NPM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (NPM) and (SI). Given a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, we denote by $E(G)$ the set of all players i in G such that $|\mathcal{M}_i(G)| \geq 2$ and by $\mathcal{G}_{(m)}$ the set of all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ in which $E(G)$ contains exactly m players. We prove by induction on integer $m \geq 0$ the assertion $\mathcal{A}'(m)$ that for all games $G \in \mathcal{G}_{(m)}$, $\varphi(G) = \mathrm{DP}(G)$.

<u>Initialization step.</u> For the initialization step, we suppose that $m = 0$ and we consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(0)}$, then $E(G) = \emptyset$. That is $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Since φ satisfies (E), (NPM) and (SI), it follows by Lemma 3.2.2 that $\varphi(G) = \mathrm{DP}(G)$. This prove that $\mathcal{A}'(0)$ holds.

<u>Induction step.</u> For the induction step, we suppose that $\mathcal{A}'(m)$ holds for some integer $m \geq 0$. We prove that $\mathcal{A}'(m+1)$ necessarily holds. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(m+1)}$. Then $|E(G)| = m + 1 \geq 1$. Thus, there exists some player $i \in E(G)$. Let $q = |\mathcal{M}(G)|$ and $p = |\mathcal{M}_i(G)|$. We pose $\mathcal{M}_i(G) = \{S_1, S_2, ..., S_p\}$ and

$\mathcal{M}(G) = \{S_1, S_2, ..., S_p, S_{p+1}, ..., S_q\}$. Consider p distincts players $i_1, i_2, ..., i_p \in \mathcal{P}\backslash N$. We define the following simple game:

$G_1 = (N_1, \mathcal{W}(G_1))$ with $N_1 = (N\backslash \{i\}) \cup \{i_1, i_2, ..., i_p\}$ and

$$\mathcal{M}(G_1) = \{T_1 \cup \{i_1\}, ..., T_p \cup \{i_p\}, S_{p+1}, ..., S_q\}$$

where $T_t = S_t\backslash \{i\}$ for all $t \in \{1, 2, ..., p\}$.

Note that G_1 is obtained from G by replacing player i in each S_t by i_t for $t \in \{1, ..., p\}$. Since each new player belongs to exactly one minimal winning coalition in G_1, then $E(G_1) = E(G)\backslash\{i\}$. It follows that $|E(G_1)| = m$. Therefore by the induction assumption, $\varphi(G_1) = \mathrm{DP}(G_1)$. Also note that moving from G_1 to G consists of merging independent players $i_1, i_2, ..., i_p$ into i. We then deduce that

$$\begin{aligned}
\varphi_i(G) &= \sum_{t=1}^{p} \varphi_{i_t}(G_1) \text{ since } \varphi \text{ satisfies Axiom (NPM)} \\
&= \sum_{t=1}^{p} \mathrm{DP}_{i_t}(G_1) \text{ since } \varphi(G_1) = \mathrm{DP}(G_1) \\
&= \mathrm{DP}_i(G) \text{ since DP satisfies Axiom (NPM)}.
\end{aligned}$$

Moreover, for all $j \in N \setminus E(G)$, $|\mathcal{M}_j(G)| \leq 1$.

If $|\mathcal{M}_j(G)| = 0$, then $\mathcal{M}_j(G) = \emptyset$ and $j \in N^0(G)$. It follows by Proposition 1.2.6 that $\varphi_j(G) = 0 = \mathrm{DP}_j(G)$.

If $|\mathcal{M}_j(G)| = 1$, then $\mathcal{M}_j(G) = \{T\}$ for some $T \in 2^N$. Note that there exists $i \in E(G)$ such that $T \notin \mathcal{M}_i(G)$. Set $\mathcal{M}_i(G) = \{S_1, \ldots, S_p\}$, $\mathcal{M}(G) = \{S_1, \ldots, S_p, S_{p+1}, \ldots, S_q\}$ and define the simple game $G_2 = (N_2, \mathcal{W}(G_2))$ such that $N_2 = N\backslash\{i\}$ and $\mathcal{M}(G_2) = \{T_1, \ldots, T_p, S_{p+1}, \ldots, S_q\}$ where $T_l = S_l\backslash\{i\}$ for all $l \in \{1, \ldots, p\}$. It follows that G is an i-supplementation of G_2 such that for all $S \in \mathcal{M}_j(G)$, $i \notin S$. Axiom (SI) implies $\varphi_j(G) = \varphi_j(G_2)$ and $\mathrm{DP}_j(G) = \mathrm{DP}_j(G_2)$. Note that $E(G_2) = E(G)\backslash\{i\}$ and $|E(G_2)| = m$. Thus by the induction assumption, $\varphi_j(G_2) = \mathrm{DP}_j(G_2)$. Therefore $\varphi_j(G) = \mathrm{DP}_j(G)$.

Finally, we get $\varphi(G) = \mathrm{DP}(G)$. This proves that $\mathcal{A}'(m+1)$ holds. In conclusion, $\mathcal{A}'(m)$ holds for all integers $m \geq 0$.

■

Theorem 2.2.1 and Theorem 3.2.1 are both three-axiom characterizations of the Public Good Index and the Deegan-Packel index respectively. From one of the two axiomatizations to another, only one axiom changes on the way supplementation operations impact individual shares for each of the two power indices. This was already the case with Holler and Packel (1983) and Deegan and Packel (1978) whose respective results brought out the fact that among power indices that meet (E), (AN) and (NP), the Public Good Index and the

Deegan-Packel index differ only on the way mergeable simple games are treated. Similar dual results have been also reported by Lorenzo-Freire et al. (2007) and Alonso-Meijide et al. (2008) with the main difference being two distinct concepts of monotonicity.

3.2.2 Logical independence of axioms

It is shown here that none of the three axioms used in Theorem 3.2.1 can not be dropped.

Proposition 3.2.1. *Axioms (E), (SI) and (NPM) are logically independent.*

Proof.

(*i*) **Non redundancy of Non Profitable Merging of independent players**

Define the power index F on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ as follows:

– If $|\mathcal{M}(G)| \neq 1$, then we pose

$$F(G) = \mathrm{DP}(G).$$

– If $\mathcal{M}(G) = \{S\}$ for some $S \in 2^N$, then we pose

$$F(G) = \begin{cases} \mathbb{1}_{\{1\}} \text{ if } 1 \in S \\ \frac{1}{|S|}\mathbb{1}_S \text{ otherwise.} \end{cases}$$

Firstly, it is clear by the definition of F that F satisfies Axiom (E).

Secondly, to see that F satisfies Axiom (SI), consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P}\backslash N$ and $G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ such that G' is a k-supplementation of G. Also consider $i \in N$ such that $k \notin R$ for all $R \in \mathcal{M}_i(G')$. When i is a null player in G, it follows from the definition of F that $F_i(G) = F_i(G') = 0$. To continue, suppose that $\mathcal{M}_i(G) \neq \emptyset$. First suppose that $|\mathcal{M}(G)| \neq 1$, then $|\mathcal{M}(G')| \neq 1$ since $|\mathcal{M}(G')| = |\mathcal{M}(G)|$ by Proposition 1.2.4. Note that on the one hand, $F_i(G) = \mathrm{DP}_i(G)$ and $F_i(G') = \mathrm{DP}_i(G')$ by the definition of F; and on the other hand, $\mathrm{DP}_i(G') = \mathrm{DP}_i(G)$ since the Deegan-Packel index DP satisfies Axiom (SI) by Proposition 2.1.4. That is $F_i(G') = F_i(G)$. Now, suppose that $\mathcal{M}(G) = \{S\}$ for some $S \in 2^N$. Since G' is k-supplementation of G and $|\mathcal{M}(G')| = |\mathcal{M}(G)|$, either $\mathcal{M}(G') = \{S\}$; or $\mathcal{M}(G') = \{S \cup \{k\}\}$. Moreover, noting that $\mathcal{M}_i(G') = \mathcal{M}_i(G) \neq \emptyset$, it follows that $\mathcal{M}_i(G') = \mathcal{M}_i(G) = \{S\}$. Clearly,

– If $1 \in S$, then $1 \in S \cup \{k\}$ and by the definition of F,

$$F_i(G') = F_i(G) = \mathbb{1}_{\{1\}}(i).$$

– If $1 \notin S$, then $1 \notin S \cup \{k\}$ and by the definition of F,

$$F_i(G') = F_i(G) = \frac{1}{|S|}\mathbb{1}_S(i).$$

In both cases, $F_i(G') = F_i(G)$.

Finally, to see that F does not satisfy Axiom (NPM), consider the simple game $G = (N, \mathcal{W}(G))$ defined by $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 3, 5\}\}$ and pose $T = \{1, 2\}$. In G, voters 1 and 2 are independent players. Noting that $G^T = (N^T, \mathcal{W}(G^T))$ is the simple game where 1 and 2 are merged into some voter $i_T \in \mathcal{P}\backslash N$. That is $N^T = \{3, 4, 5, i_T\}$ and $\mathcal{M}(G^T) = \{\{i_T, 3, 5\}\}$. It follows that $F(G) = (1, 0, 0, 0, 0)$ and

$$F(G^T) = \left(F_3(G^T), F_4(G^T), F_5(G^T), F_{i_T}(G^T)\right) = \left(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\right).$$

Therefore $F_{i_T}(G^T) = \dfrac{1}{3} \neq F_1(G) + F_2(G) = 1$. Thus, F does not satisfy Axiom (NPM). We conclude that (NPM) can not be deduced from (E) and (SI).

(ii) **Non redundancy of Efficiency**

Clearly, the power index 2 DP satisfies (NPM) and (SI); but fails to meet Axiom (E). Thus Axiom (E) can not be deduced from (NPM) and (SI).

(iii) **Non redundancy of Supplementation Invariance**

Consider the Public Good Index HP; see Definition 1.1.13 at page 14. It was shown in Chapter 2 that HP satisfies (E) and (NPM). To see that HP fails to meet Axiom (SI), consider the simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and $G' = (N \cup \{6\}, \mathcal{W}(G'))$ such that $N = \{1, 2, 3, 4, 5\}$, $\mathcal{M}(G) = \{\{1, 3, 4\}, \{2, 4, 5\}, \{1, 2\}\}$. and $\mathcal{M}(G') = \{\{1, 3, 4\}, \{2, 4, 5\}, \{1, 2, 6\}\}$. Then G' is a 6-supplementation of G and $\mathcal{M}_3(G') = \{\{1, 3, 4\}\} = \mathcal{M}_3(G)$. Moreover,

$$\mathrm{HP}(G) = \left(\frac{2}{8}, \frac{2}{8}, \frac{1}{8}, \frac{2}{8}, \frac{1}{8}\right) \text{ and } \mathrm{HP}(G') = \left(\frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}\right).$$

It clearly appears that $6 \notin S$ for all $S \in \mathcal{M}_3(G')$; but $\mathrm{HP}_3(G') = \frac{1}{9} \neq \mathrm{HP}_3(G) = \frac{1}{8}$. Therefore, HP does not satisfy (SI). Thus, Axiom (SI) can not be deduced from (NPM) and (E).

■

3.2.3 Further axiomatizations

We provide two other axiomatizations of the Deegan-Packel index by successively replacing Axiom (NPM) in Theorem 3.2.1 by (IEM) and (IIM). The next theorem arises using

Proposition 1.2.7 and Theorem 3.2.1.

Theorem 3.2.2.

A power index φ on $\mathcal{G}$ satisfies (E), (SI) and (IEM) if and only if $\varphi = \mathrm{DP}$. ★

Proof.

Necessity. By Proposition 3.1.4 together with Corollaries 3.1.1 and 3.1.3, DP satisfies (E), (SI) and (IEM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (IEM) and (SI). Since φ satisfies (E) and (IEM), it also satisfies Axiom (NPM) (see Proposition 1.2.7). It then follows from Theorem 3.2.1 that $\varphi = \mathrm{DP}$. ■

Remark 3.2.1. *To check that (E), (SI) and (IEM) in Theorem 3.2.2 are independent, one can easily use the power indices defined in Proposition 3.2.1.*

Proposition 1.2.8 and Theorem 3.2.2 allow us to state and prove the following result.

Theorem 3.2.3.

A power index φ on $\mathcal{G}$ satisfies (E), (SI) and (IIM) if and only if $\varphi = \mathrm{DP}$. ★

Proof.

Necessity. By Proposition 3.1.4 together with Corollary 3.1.1, DP satisfies (E) and (SI). By mimicking the proof of Corollary 3.1.3, it is easy to establish that DP satisfies Axiom (IIM).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (IIM) and (SI). Since φ satisfies Axiom (IIM), it also satisfies Axiom (IEM) (see Proposition 1.2.8). It then follows from Theorem 3.2.2 that $\varphi = \mathrm{DP}$. ■

Remark 3.2.2. *Once again, to check that (E), (SI) and (IIM) in Theorem 3.2.3 are independent, one can easily use the power indices defined in Proposition 3.2.1.*

3.3 Alternative axiomatizations

We present here another characterization of the Deegan-Packel index using an axiom on equivalent simple games. Characterizations of some of its coalitional versions are also provided.

3.3.1 Axiomatization based on equivalent simple games

We start by stating some preliminary properties of the Deegan-Packel index.

Corollary 3.3.1. *The Deegan-Packel index satisfies Axiom (MEP).*

Proof. Consider two simple games $G = (N, \mathcal{W}(G))$ and $G' = (N', \mathcal{W}(G'))$. Let $i \in N \cap N'$ be a voter such that $G\Delta_i G'$. Since $G\Delta_i G'$, there exists $(S,T) \in \mathcal{M}(G) \times \mathcal{M}(G')$ such that $\mathcal{M}(G') = (\mathcal{M}(G)\backslash\{S\}) \cup \{T\}$ with $|S| = |T|$ and ($i \in S \cap T$ or $i \notin S \cup T$). It follows that $|\mathcal{M}(G')| = |\mathcal{M}(G)|$ and $|S| = |T|$.

If $i \notin S \cup T$, then it follows by Remark 1.2.3 $\mathcal{M}_i(G') = \mathcal{M}_i(G)$ and

$$\begin{aligned}
\mathrm{DP}_i(G') &= \frac{1}{|\mathcal{M}(G')|} \sum_{R \in \mathcal{M}_i(G')} \frac{1}{|R|} \\
&= \frac{1}{|\mathcal{M}(G)|} \sum_{R \in \mathcal{M}_i(G)} \frac{1}{|R|} \text{ since } \mathcal{M}_i(G') = \mathcal{M}_i(G) \text{ and } |\mathcal{M}(G')| = |\mathcal{M}(G)| \\
&= \mathrm{DP}_i(G).
\end{aligned}$$

If $i \in S \cap T$, then $\mathcal{M}_i(G') = (\mathcal{M}_i(G)\backslash\{S\}) \cup \{T\}$ and

$$\begin{aligned}
\mathrm{DP}_i(G') &= \frac{1}{|\mathcal{M}(G')|} \sum_{R \in \mathcal{M}_i(G')} \frac{1}{|R|} \\
&= \frac{1}{|\mathcal{M}(G)|} \sum_{R \in (\mathcal{M}_i(G)\backslash\{S\}) \cup \{T\}} \frac{1}{|R|} \text{ since } |\mathcal{M}(G')| = |\mathcal{M}(G)| \\
&= \frac{1}{|\mathcal{M}(G)|} \sum_{R \in \mathcal{M}_i(G)} \frac{1}{|R|} \text{ since } |S| = |T| \\
&= \mathrm{DP}_i(G).
\end{aligned}$$

Therefore DP satisfies Axiom (MEP).

■

Next, it is shown in the following lemma that any power index that satisfies (E), (MEP) and (SI) necessarily coincides with the Deegan-Packel index on any simple games where each player belongs to at most one minimal winning coalition. Formally,

Lemma 3.3.1. *If a power index φ on $\mathcal{G}$ satisfies (E), (MEP) and (SI), then $\varphi(G) = \mathrm{DP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.*

Proof. Suppose that a power index φ on $\mathcal{G}$ satisfies (E), (MEP) and (SI). We denote by $r(G) = max\{|S| : S \in \mathcal{M}(G)\}$ for all simple game $G \in \mathcal{G}$ and $\mathcal{G}_{\overline{r}} = \{G \in \mathcal{G} : r(G) = r\}$ for all positive integer r. We prove by induction on positive integer $r \geq 1$ that $\varphi(G) = \mathrm{DP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r}}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

-*Initialization*: For $r=1$, consider a simple game $G=(N,\mathcal{W}(G))\in\mathcal{G}_{\overline{1}}$ such that $|\mathcal{M}_i(G)|\leq 1$ for all $i\in N$. Then, $\mathcal{M}_i(G)=\{\{i\}\}$ for all $i\in N\backslash N^0(G)$. Set $S=N\backslash N^0(G)$. We have $G\Delta_i G_S$ for all $i\in S$, where G_S is the singleton game associated to S. It follows that for all $i\in S$,

$$\begin{aligned}\varphi_i(G) &= \varphi_i(G_S) \;\text{ since } \varphi \text{ satisfies Axiom (MEP) and } G\Delta_i G_S\\ &= \frac{1}{|S|} \;\text{ by Proposition 1.2.11 since } \varphi \text{ satisfies (E) and (MEP)}\\ &= \mathrm{DP}_i(G_S) \;\text{ by Proposition 1.2.11 since DP satisfies (E) and (MEP)}\\ &= \mathrm{DP}_i(G) \;\text{ since DP satisfies Axiom (MEP) and } G\Delta_i G_S.\end{aligned}$$

Moreover, for all $i\in N^0(G), \varphi_i(G)=0=\mathrm{DP}_i(G)$ by Proposition 3.1.1. Thus $\varphi(G)=\mathrm{DP}(G)$.

-*Induction step*: Suppose for some positive integer $r\geq 1$ that $\varphi(G)=\mathrm{DP}(G)$ for all simple game $G=(N,\mathcal{W}(G))\in\mathcal{G}_{\overline{r}}$ such that $|\mathcal{M}_i(G)|\leq 1$ for all $i\in N$.

Consider a simple game $G=(N,\mathcal{W}(G))\in\mathcal{G}_{\overline{r+1}}$ such that $|\mathcal{M}_i(G)|\leq 1$ for all $i\in N$, we set $F(G)=\{S\in\mathcal{M}(G):|S|=r+1\}$. We prove by induction on $f=|F(G)|\geq 1$ that $\varphi(G)=\mathrm{DP}(G)$ for all simple game $G=(N,\mathcal{W}(G))\in\mathcal{G}_{\overline{r+1}}$ such that $f=|F(G)|$ and $|\mathcal{M}_i(G)|\leq 1$ for all $i\in N$.

For $f=1$, consider a simple game $G=(N,\mathcal{W}(G))\in\mathcal{G}_{\overline{r+1}}$ such that $|F(G)|=1$ and $|\mathcal{M}_i(G)|\leq 1$ for all $i\in N$. Set $F(G)=\{S\}$, then $|S|=r+1\geq 2$. Fix a player $i\in S$ and define the simple game $G_1=(N_1,\mathcal{W}(G_1))\in\mathcal{G}$ with $N_1=N\backslash\{i\}$, and $\mathcal{M}(G_1)=(\mathcal{M}(G)\backslash\{S\})\cup\{S_1\}$ where $S_1=S\backslash\{i\}$. We have $G_1\in\mathcal{G}_{\overline{r}}$.

Then G is an i-supplementation of G_1 such that for all $l\in N\backslash S$, for all $L\in\mathcal{M}_l(G), i\notin L$. It follows that for all $l\in N\backslash S$,

$$\begin{aligned}\varphi_l(G) &= \varphi_l(G_1) \;\text{ since } \varphi \text{ satisfies Axiom (SI) and for all } L\in\mathcal{M}_l(G), i\notin L\\ &= \mathrm{DP}_l(G_1) \text{ since } \varphi(G_1)=\mathrm{DP}(G_1) \text{ by the induction assumption on } r=r(G_1)\\ &= \mathrm{DP}_l(G) \;\text{ since DP satisfies Axiom (SI) and for all } L\in\mathcal{M}_l(G), i\notin L.\end{aligned}$$

Consider three players $i,j\in S, k\in\mathcal{P}\backslash N$ and define the two simple games $G'=(N',\mathcal{W}(G')), G''=(N'',\mathcal{W}(G''))\in\mathcal{G}$ with $N'=(N\backslash\{i\})\cup\{k\}$, $N''=(N\backslash\{j\})\cup\{k\}$, $\mathcal{M}(G')=(\mathcal{M}(G)\backslash\{S\})\cup\{S'\}$ and $\mathcal{M}(G'')=(\mathcal{M}(G)\backslash\{S\})\cup\{S''\}$ where $S'=(S\backslash\{i\})\cup\{k\}$ and $S''=(S\backslash\{j\})\cup\{k\}$. We have in the one hand, for all $l\in N\backslash\{i\}, G'\Delta_l G$, it follows by Axiom (MEP) that $\varphi_l(G)=\varphi_l(G')$. Then

$$\begin{aligned}\varphi_i(G) &= 1-\sum_{l\in N\backslash\{i\}}\varphi_l(G) \;\text{ by Axiom (E) applied to the simple game } G\\ &= 1-\sum_{l\in N\backslash\{i\}}\varphi_l(G') \;\text{ since }\; \varphi_l(G)=\varphi_l(G') \;\text{ for all }\; l\in N\backslash\{i\}\\ &= \varphi_k(G') \;\text{ by Axiom (E) applied to the simple game } G'.\end{aligned}$$

In the other hand, for all $l\in N\backslash\{j\}, G''\Delta_l G$, it follows by Axiom (MEP) that $\varphi_l(G)=\varphi_l(G'')$. Then

$$
\begin{aligned}
\varphi_j(G) &= 1 - \sum_{l \in N\setminus\{j\}} \varphi_l(G) \ \text{ by Axiom (E) applied to the simple game } G \\
&= 1 - \sum_{l \in N\setminus\{j\}} \varphi_l(G'') \ \text{ since } \ \varphi_l(G) = \varphi_l(G'') \ \text{ for all } \ l \in N\setminus\{j\} \\
&= \varphi_k(G'') \ \text{ by Axiom (E) applied to the simple game } G''.
\end{aligned}
$$

Note that $G'\Delta_k G''$, it follows by Axiom (MEP) that $\varphi_k(G') = \varphi_k(G'')$. That is $\varphi_i(G) = \varphi_j(G) = \varphi_t(G)$ for all $t \in S$ since i and j were arbitrary choose in S. Then,

$$
\begin{aligned}
|S|\varphi_i(G) &= \sum_{l \in S} \varphi_l(G) \ \text{ since } \varphi_i(G) = \varphi_t(G) \text{ for all } t \in S \\
&= 1 - \sum_{l \in N\setminus S} \varphi_l(G) \ \text{ by Axiom (E)} \\
&= 1 - \sum_{l \in N\setminus S} \mathrm{DP}_l(G) \ \text{ since } \ \varphi_l(G) = \mathrm{DP}_l(G) \ \text{ for all } l \in N\setminus S \\
&= \sum_{l \in S} \mathrm{DP}_l(G) \ \text{ by Axiom (E)} \\
&= |S|\,\mathrm{DP}_i(G) \ \text{ since } \mathrm{DP}_i(G) = \mathrm{DP}_t(G) \text{ for all } t \in S.
\end{aligned}
$$

Therefore, $\varphi_i(G) = \mathrm{DP}_i(G)$ since $|S| \geq 2$. That is $\varphi_i(G) = \mathrm{DP}_i(G)$ for all $i \in S$. Thus $\varphi(G) = \mathrm{DP}(G)$.

Now suppose that for some positive integer $f \geq 1$, $\varphi(G) = \mathrm{DP}(G)$ for all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|F(G)| = f$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{\overline{r+1}}$ such that $|F(G)| = f + 1 \geq 2$ and $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Set $S, T \in F(G)$, then $|S| = |T| = r + 1 \geq 2$. Fix two players $i \in S, j \in T$ and define the two simple games

$G_1 = (N_1, \mathcal{W}(G_1)), G_2 = (N_2, \mathcal{W}(G_2)) \in \mathcal{G}$ with $N_1 = N\setminus\{i\}$, $N_2 = N\setminus\{j\}$, $\mathcal{M}(G_1) = (\mathcal{M}(G)\setminus\{S\}) \cup \{S_1\}$ and $\mathcal{M}(G_2) = (\mathcal{M}(G)\setminus\{S\}) \cup \{S_2\}$ where $S_1 = S\setminus\{i\}$ and $S_2 = T\setminus\{j\}$. Note that $|F(G_1)| = |F(G_2)| = f$ and G_1 as much as G_2 are simple games since any player in G belongs to at most one minimal winning coalition of G.

In the one hand, G is an i-supplementation of G_1 such that for all $l \in N\setminus S$, for all $L \in \mathcal{M}_l(G), i \notin L$. It follows that for all $l \in N\setminus S$,

$$
\begin{aligned}
\varphi_l(G) &= \varphi_l(G_1) \ \text{ since } \varphi \text{ satisfies Axiom (SI) and for all } L \in \mathcal{M}_l(G), i \notin L \\
&= \mathrm{DP}_l(G_1) \text{ since } \varphi(G_1) = \mathrm{DP}(G_1) \text{ by the induction assumption on } f \\
&= \mathrm{DP}_l(G) \ \text{ since DP satisfies Axiom (SI) and for all } L \in \mathcal{M}_l(G), i \notin L.
\end{aligned}
$$

In the other hand, G is an j-supplementation of G_2 such that for all $l \in N\setminus T$, for all $L \in \mathcal{M}_l(G), j \notin L$. It follows that for all $l \in N\setminus T$,

$$
\begin{aligned}
\varphi_l(G) &= \varphi_l(G_2) \ \text{ since } \varphi \text{ satisfies Axiom (SI) and for all } L \in \mathcal{M}_l(G), j \notin L \\
&= \mathrm{DP}_l(G_2) \text{ since } \varphi(G_2) = \mathrm{DP}(G_2) \text{ by the induction assumption on } f \\
&= \mathrm{DP}_l(G) \ \text{ since DP satisfies Axiom (SI) and for all } L \in \mathcal{M}_l(G), j \notin L.
\end{aligned}
$$

Note that $S \cap T = \emptyset$ since $S, T \in \mathcal{M}(G)$ and $|\mathcal{M}_l(G)| \leq 1$ for all $l \in N$. It follows that $S \subseteq N\setminus T$. Then $\varphi_l(G) = \mathrm{DP}_l(G)$ for all $l \in N$. Finally, $\varphi(G) = \mathrm{DP}(G)$.

■

The main result of this section is the following.

Theorem 3.3.1.

A power index φ on $\mathcal{G}$ satisfies (E), (SI) and (MEP) if and only if $\varphi = \text{DP}$. ★

Proof. *Necessity*. It follows from Proposition 3.1.4, Corollary 3.1.1 and Corollary 3.3.1, that the Deegan-Packel index DP satisfies axioms (E), (SI) and (MEP).

Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (MEP) and (SI). For all simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, we set $E(G) = \{i \in N : |\mathcal{M}_i(G)| \geq 2\}$. We denote $\mathcal{G}_{(m)} = \{G \in \mathcal{G} : |E(G)| = m\}$ and we prove by induction on positive integer m that $\varphi(G) = \text{DP}(G)$ for all $G \in \mathcal{G}_{(m)}$.

Initialization: For $m = 0$, consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(0)}$. We have $E(G) = \emptyset$, that is $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. It follows by Lemma 3.3.1 that $\varphi(G) = \text{DP}(G)$.

Induction step: Suppose that for some positive integer m, $\varphi(G) = \text{DP}(G)$ for all $G \in \mathcal{G}_{(m)}$. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(m+1)}$ and a player $i \in E(G)$. Then $|\mathcal{M}_i(G)| \geq 2$, set $p = |\mathcal{M}_i(G)|, q = |\mathcal{M}(G)|, \mathcal{M}_i(G) = \{S_1, ..., S_p\}$ and $\mathcal{M}(G) = \{S_1, ..., S_p, S_{p+1}, ..., S_q\}$. Consider p players $i_1, ..., i_p \in \mathcal{P} \backslash N$ and set $T_l = (S_l \backslash \{i\}) \cup \{i_l\}$ for all $l \in \{1, ..., p\}$. Define the following simple games $(G_l)_{l \in \{0,...,p\}}$ by $G_0 = G$ and for all $l \in \{1, ..., p\}, G_l = (N_l, \mathcal{W}(G_l))$ with $N_l = N \cup \{i_1, ..., i_l\}$ and $\mathcal{M}(G_l) = \{T_1, ..., T_l, S_{l+1}, ..., S_q\}$.

Then $E(G_p) = E(G) \backslash \{i\}$, one obtains $|E(G_p)| = m$. It follows by the induction assumption that $\varphi(G_p) = \text{DP}(G_p)$. Note that $G_p \Delta_j G_{p-1}$ for all $j \in N_{p-1} \backslash \{i\}$, therefore

$$\begin{aligned}
\varphi_j(G_{p-1}) &= \varphi_j(G_p) \text{ since } \varphi \text{ satisfies Axiom (MEP)} \\
&= \text{DP}_j(G_p) \text{ since } \varphi(G_p) = \text{DP}(G_p) \\
&= \text{DP}_j(G_{p-1}) \text{ since DP satisfies Axiom (MEP).}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\varphi_i(G_{p-1}) &= 1 - \sum_{j \in N_{p-1} \backslash \{i\}} \varphi_j(G_{p-1}) \text{ since } \varphi \text{ satisfies Axiom (E)} \\
&= 1 - \sum_{j \in N_{p-1} \backslash \{i\}} \text{DP}_j(G_{p-1}) \text{ since } \varphi_j(G_{p-1}) = \text{DP}_j(G_{p-1}) \text{ for all } {\scriptstyle j \in N_{p-1} \backslash \{i\}} \\
&= \text{DP}_i(G_{p-1}) \text{ since DP satisfies Axiom (E).}
\end{aligned}$$

That is $\varphi(G_{p-1}) = \text{DP}(G_{p-1})$.

Let $l \in \{1, ..., p\}$ and assume that $\varphi(G_l) = \text{DP}(G_l)$. Note that for all $j \in N_{l-1} \backslash \{i\}, G_l \Delta_j G_{l-1}$; therefore

$$\begin{aligned}
\varphi_j(G_{l-1}) &= \varphi_j(G_l) \text{ since } \varphi \text{ satisfies Axiom (MEP)} \\
&= \text{DP}_j(G_l) \text{ since } \varphi(G_l) = \text{DP}(G_l) \text{ by assumption} \\
&= \text{DP}_j(G_{l-1}) \text{ since DP satisfies Axiom (MEP).}
\end{aligned}$$

Moreover,

$$\begin{aligned}\varphi_i(G_{l-1}) &= 1 - \sum_{j \in N_{l-1}\setminus\{i\}} \varphi_j(G_{l-1}) \text{ since } \varphi \text{ satisfies Axiom (E)}\\ &= 1 - \sum_{j \in N_{l-1}\setminus\{i\}} \mathrm{DP}_j(G_{l-1}) \text{ since } \varphi_j(G_{l-1}) = \mathrm{DP}_j(G_{l-1}) \text{ for all } _{j\in N_{l-1}\setminus\{i\}}\\ &= \mathrm{DP}_i(G_{l-1}) \text{ since DP satisfies Axiom (E).}\end{aligned}$$

That is $\varphi(G_{l-1}) = \mathrm{DP}(G_{l-1})$.

For $l = 1$, one obtains $\varphi(G_0) = \mathrm{DP}(G_0)$. That is $\varphi(G) = \mathrm{DP}(G)$.

■

Remark 3.3.1. *To check that (E), (SI) and (MEP) in Theorem 3.3.1 are independent, one can easily use the power indices defined in Proposition 3.2.1.*

Remark 3.3.2. *Note that another proof of Theorem 3.3.1 can be established using on the one hand, Proposition 3.1.4, Corollary 3.1.1 and Corollary 3.3.1 for the necessity part. And on the other hand, using Proposition 1.2.10 and Theorem 3.2.2 for the sufficiency part.*

3.3.2 Axiomatizing coalitional versions of the Deegan-Packel index

Since Owen (1977) who provided an extension of the Shapley value to TU-games with a priori unions, the so-called Owen value, many contributions have emerged in this area; see Andjiga and Courtin (2015) for definition and characterization of a class of share functions that contains the Shapley value and the Banzhaf value for games with coalition configurations; or Alonso-Meijide et al. (2010a) and Alonso-Meijide et al. (2010b) for some extensions and axiomatizations of the Public Good Index to the class of simple games with a priori unions. In this line of inquiry, Alonso-Meijide et al. (2011a) has extended the Deegan-Packel index to simple games with a priori unions and provided an axiomatization of the whole class of coalitional versions of the Deegan-Packel index; these are power indices on simple games with a priori unions that coincide with the Deegan-Packel index when each union is formed by only one voter. We provide below other characterizations of coalitional versions of the Deegan-Packel index for simple games with a priori unions and a variable electorate.

Following Alonso-Meijide et al. (2011a), we first recall the definition of a coalitional version of the Deegan-Packel index. Since the definition of a simple game with a priori union has already been introduced in section 2.3.3 as well as the definition of a coalitional version of a power index; we will simply omit those concepts here.

Definition 3.3.1. *A coalitional power index f is a coalitional version of the Deegan-Packel index if:*

$$f(N, v, P^N) = \mathrm{DP}(N, v), \forall (N, v) \in \mathcal{G}_N, \tag{3.2}$$

where $P^N = \{\{i\} : i \in N\}$.

The following axioms are obtained by shifting some appropriate axioms we early presented from simple games to simple games with a priori unions.

Axiom 18. Supplementation Invariance with a priori Unions (SIU): *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \backslash N$, for all k-supplementations $G' = (N', \mathcal{W}(G'))$ of G, for all $i \in N$, if $k \notin S$ for all $S \in \mathcal{M}_i(G')$, then*

$$f_i\left(N', \mathcal{W}(G'), P^{N'}\right) = f_i\left(N, \mathcal{W}(G), P^N\right).$$

Axiom 19. Independence of External Merging independent players with a priori Unions (IEMU): *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $T \subseteq N$ with $|T| \geq 2$, for all $i_T \in \mathcal{P} \backslash N$, $f_i(N^T, \mathcal{W}(G^T), P^{N^T}) = f_i(N, v, P^N)$ for all $i \in N \setminus T$ whenever T consists of independent players in G.*

Proposition 3.3.1. *A coalitional power index f is a coalitional version of the Deegan-Packel index if and only if f satisfies* (SE), (SIU) *and* (NPMU).

Proof.

Straightforward from Theorem 3.2.1 and equation (3.2). ■

Proposition 3.3.2. *Axioms* (SE), (SIU) *and* (NPMU) *are independent.*

Proof.

(i) **Axioms (SE), (SIU) do not imply (NPMU)**

Define the power index $\widetilde{F}$ for all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ as follows,

– If $P = P^N$ and $|\mathcal{M}(G)| \neq 1$, then

$$\widetilde{F}(G) = \mathrm{DP}(N, \mathcal{W}(G)).$$

– If $P = P^N$ and $\mathcal{M}(G) = \{S\}$ for some $S \in 2^N$, then

$$\widetilde{F}(G) = \begin{cases} \mathbb{1}_{\{1\}} \text{ if } 1 \in S \\ \frac{1}{|S|}\mathbb{1}_S \text{ if } 1 \notin S. \end{cases}$$

– If $P \neq P^N$, then

$$\widetilde{F}_i(G) = 0 \text{ for all } i \in N.$$

By noting that the restriction of $\widetilde{F}$ on the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P^N)$ coincides with the power index defined in *Part (i)* of the proof of Proposition 3.2.1 and thanks to (3.2), it follows that the power index $\widetilde{F}$ satisfies (SE) and (SIU), but $\widetilde{F}$ fails to meet (NPMU).

(*ii*) **Axioms (NPMU) and (SIU) do not imply (SE)**

Define the power index $\widetilde{\mathrm{DP}}$ defined for all simple game with a priori unions $G = (N, \mathcal{W}(G), P)$ and for all $i \in N$ by,

$$\widetilde{\mathrm{DP}}_i(G) = \begin{cases} 2\,\mathrm{DP}_i(N, \mathcal{W}(G)) \text{ if } P = P^N \\ 0 \text{ otherwise.} \end{cases}$$

By definition, the restriction of $\widetilde{\mathrm{DP}}$ on the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ with $P = P^N$ coincides with the power index defined in *Part (ii)* of the proof of Proposition 3.2.1. By considering (3.2), the power index $\widetilde{\mathrm{DP}}$ satisfies (NPMU) and (SIU), but not (SE).

(*iii*) **Axioms (NPMU) and (SE) do not imply (SIU)**

Consider the power index $\widetilde{\mathrm{HP}}$ defined for all simple game with a priori unions $G = (N, \mathcal{W}(G), P)$ and for all $i \in N$ by,

$$\widetilde{\mathrm{HP}}_i(G) = \begin{cases} \mathrm{HP}_i(N, \mathcal{W}(G)) \text{ if } P = P^N \\ 0 \text{ otherwise.} \end{cases}$$

On the set of all simple games with a priori unions $G = (N, \mathcal{W}(G), P)$ with $P = P^N$, the restriction of $\widetilde{\mathrm{HP}}$ coincides with the power index defined in *Part (iii)* of the proof of Proposition 3.2.1. Therefore, it follows by (3.2) that the power index $\widetilde{\mathrm{DP}}$ satisfies (NPMU) and (SE), but not (SIU).

■

The next result is similar to the preceding and is based on Axiom (IEMU).

Proposition 3.3.3. *A coalitional power index f is a coalitional version of the Deegan-Packel index if and only if f satisfies (SE), (SIU) and (IEMU).*

Proof. Straightforward from Theorem 3.2.2 and equation (3.2). ■

Proposition 3.3.4. *Axioms (SE), (SIU) and (IEMU) are independent.*

Proof. The proof is straightforward by using the power indices defined in Proposition 3.3.2. ■

For power indices with a priori unions, we provide below further axioms on how individual shares are affected by an internal merging of independent players or in presence of an occurrence of a membership equivalence.

Axiom 20. Independence of Internal Merging independent players with a priori Unions (IIMU): *For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $T \subseteq N$ of at least two players, for all $t \in T$, $f_i(N^{T\to t}, \mathcal{W}(G^{T\to t}), P^{N^{T\to t}}) = f_i(N, \mathcal{W}(G), P^N)$ for all $i \in N \setminus T$ whenever T is a coalition of independent players.*

Proposition 3.3.5. *A coalitional power index f is a coalitional version of the Deegan-Packel index if and only if f satisfies (SE), (SIU) and (IIMU).*

Proof.
Straightforward from Theorem 3.2.3 and equation (3.2). ■

Remark 3.3.3. *To check that (SE), (SIU) and (IIMU) are independent, one can easily used the power indices defined in Proposition 3.3.2.*

Axiom 21. ***Membership Equivalence Property with a priori Unions*** **(MEPU)**: *For all $G = (N, \mathcal{W}(G)), G' = (N', \mathcal{W}(G')) \in \mathcal{G}$, for all player $i \in N \cap N'$, $\varphi_i(N, \mathcal{W}(G), P^N) = \varphi_i(N', \mathcal{W}(G'), P^{N'})$ whenever $G\Delta_i G'$.*

Proposition 3.3.6. *A coalitional power index f is a coalitional version of the Deegan-Packel index if and only if f satisfies (SE), (SIU) and (MEPU).*

Proof.
Straightforward from Theorem 3.3.1 and equation (3.2). ■

Remark 3.3.4. *To check that (SE), (SIU) and (MEPU) are independent, one can easily used the power indices defined in Proposition 3.3.2.*

In this chapter, we have presented several axiomatizations of the Deegan-Packel index. Following Holler and Packel (1983) and Deegan and Packel (1978), or Lorenzo-Freire et al. (2007) and Alonso-Meijide et al. (2008), each characterization result stated in this chapter mirrors a similar theorem from Chapter 2. By so doing, we have shed new light on the duality between the Holler-Packel index and the Deegan-Packel index. Results that we have already presented so far are all contributions within the supra-domain approach. In the next chapters, our concerns are rather intra-domain preoccupations.

Part II

An intra-domain study of the Shapley value

⋆ ⋆ Chapter Four ⋆ ⋆

On Shapley valid domains

Simple games are particular instances of a more general notion: the one of cooperative games with transferable utilities (TU-games). Keeping axiomatization issues in mind, we focus our attention in this chapter on a key solution of TU-games that has been recently considered a crown jewel of game theory by Thomson (2019): the so celebrated Shapley value. Our motivation is a sparse concern that very often emerges from characterization results: that is, does a given list of axioms that uniquely identifies a solution concept over a set of TU-games (called the initial domain) also prevails, when the attention is now focused only on some specific subsets of the initial domain? We refer to this concern as the intra-domain approach of analyzing axiomatization results. Our contribution in this chapter is an intra-domain analysis of Shapley's characterization of the Shapley value; see Shapley (1953).

There are two sections in the present chapter. Section 4.1 is devoted to the presentation of TU-games. Main issues in TU-games and solutions are described. The Shapley value is also presented. In Section 4.2, we introduce the notion of a valid domain to a characterization result. In the specific case of the characterization of the Shapley value by himself, we present existing results on Shapley valid domains. We then introduce the notion of conically consistent domains and prove that any conically consistent domain is a Shapley valid domain. To achieve this, we introduce an extension of the notion of cone and some other related concepts. Moreover, it is shown that previous Shapley valid domains we report here are specific instances of conically consistent domains. In the case of exactly two players, a characterization of all linear subspaces of the vector space of all TU-games on a nonempty and finite set of players is also provided.

4.1 TU-games

Cooperative games with transferable utilities (TU-games) are interactions where players can form coalitions through binding agreements to achieve a collective worth to be redistributed in a conceivable way. Basic concepts of TU-games such as the formal definition of a TU-game, the notion of Harsanyi dividends or the family of unanimity games. Single-valued solutions are presented with the Shapley value being our primary target.

4.1.1 Preliminaries on TU-games

Recall that, given any nonempty set X, we denote by $\mathcal{P}(X)$ the powerset of X.

Definition 4.1.1. A *transferable utility game* (or simply a *TU-game*) is any pair (N, v) such that :

(i) $N = \{1, 2, \cdots, n\}$ is a nonempty and finite set with $n \geq 2$;

(ii) $v : \mathcal{P}(N) \longrightarrow \mathbb{R}$ is a mapping satisfying $v(\emptyset) = 0$.

In this case, N is called the set of players and v the *characteristic function* of the game. When the set N of players is known, (N, v) will simply be identified to its characteristic function v.

Given a TU-game (N, v) and a coalition S, $v(S)$ is the collective payoff (or value) that the members of S can obtain regardless of the decisions of the players in $N \backslash S$.

Notation 4.1.1. We denote by Γ_N the set of all TU-games having N as the set of players. For the sake of simplicity and when there is no ambiguity, we will sometimes write $v(i)$ instead of $v(\{i\})$; or $v(S + i)$ instead of $v(S \cup \{i\})$ given $v \in \Gamma_N$, $i \in N$ and $S \subseteq N \backslash \{i\}$.

Before we continue, below is an illustrative example of TU-game inspired by Caulier (2009).

Example 4.1.1. Three neighbors in a town, say A, B and C, have to build connection facilities to the local sewer system. The connection cost (in hundreds of thousands CFA francs) is 2 for A alone , 6 for B and 7 for C. The cost for a joint connection is 7 for A and B, 8 for A and C, 10 for B and C. If the three neighbors are collectively served, the joint cost is 11.

Identifying A to 1, B to 2 and C to 3, this situation can be described by the TU-game (N, v) where $N = \{1, 2, 3\}$ and the characteristic function is given by: :

Coalitions S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Costs $v(S)$	0	-2	-6	-7	-7	-8	-10	-11

Will these three neighbors act together or solely? How will they share the collective cost in case of cooperation? How these two questions are addressed will be commented in the next section. We continue to introduce basics of TU-games.

Below is a definition of two usual operations on TU-games.

Definition 4.1.2. Let $u, v \in \Gamma_N$ and $\lambda \in \mathbb{R}$.

(i) The sum of u and v is the TU-game $u+v \in \Gamma_N$ defined for all $S \subseteq N$ by

$$(u+v)(S) = u(S) + v(S).$$

(ii) The scalar product of u by λ is the TU-game $\lambda u \in \Gamma_N$ defined for all $S \subseteq N$ by

$$(\lambda u)(S) = \lambda \times u(S).$$

Remark 4.1.1. The structure $(\Gamma_N, +, .)$ is a vector space. The null game is the TU-game $\widetilde{0} \in \Gamma_N$ defined for all $S \subseteq N$ by $\widetilde{0}(S) = 0$.

We now define two families of TU-games introduced by Shapley (1953) and which are very often used to analyze the structure of the set of all TU-games.

Definition 4.1.3. Given a coalition S,

- the unanimity game associated with S is the TU-game $\gamma_S \in \Gamma_N$ defined for all $T \subseteq N$ by

$$\gamma_S(T) = \begin{cases} 1 \text{ if } S \subseteq T \\ 0 \text{ else} \end{cases} \tag{4.1}$$

- the Dirac game associated with S is the TU-game $\delta_S \in \Gamma_N$ defined all $T \subseteq N$ by

$$\delta_S(T) = \begin{cases} 1 \text{ if } S = T \\ 0 \text{ else} \end{cases} \tag{4.2}$$

Remark 4.1.2. Shapley (1953) has shown that the space vector $(\Gamma_N, +, .)$ is of dimension $2^n - 1$, one of its basis is the collection $(\gamma_S)_{S \in 2^N}$ of unanimity games and for all $u \in \Gamma_N$,

$$u = \sum_{S \in 2^N} [u]_S \gamma_S \tag{4.3}$$

where for all $S \in 2^N$, $[u]_S$ is the Harsanyi dividend in the game u associated with S and is defined as follows:

$$[u]_S = \sum_{T \subseteq S} (-1)^{|S|-T|} u(T). \tag{4.4}$$

For recent and useful basis of $(\Gamma_N, +, .)$, we refer to Yokote et al. (2016).

4.1.2 TU-game solutions

As seen in the preceding example, any TU-game raises two problems:

- The problem of coalition formation: which coalition structure will emerge?
- The problem of payoff redistribution: how will the players share the collective payoff in case of cooperation?

A TU-game solution (or just solution for short) is any conceivable way to address the two issues mentioned above by matching each TU-game over a given set of players with a set of payoff vectors. A solution thus specifies how players are organized and how shares are derived. However, very few solutions have been set up combining the two problems. It is generally assumed that a coalition structure has already been reached and that the remaining question is how to share each coalition payoff. There are two main classes of solutions.

On the one hand, set-valued solutions attach to a TU-game a set of sharing vectors on which players will possibly agree. Very often, a payoff vector selected by a set-valued solution is an equilibrium with respect to some admissible types of objection or counter-objection. This is, for example, the case of the core by Gillies (1953); the notion of stable set by Von Neumann and Morgenstern (1944); the $\mathcal{B}$-core by Aumann and Dreze (1974); the rich panel of bargaining sets by Aumann and Maschler (1961), Zhou (1994), Mas-Colell (1989) or Dutta et al. (1989). To a TU-game, a set-valued solution may assign an infinite set of payoff vectors. In such cases, selecting a payoff raises new issues. A set-valued solution may also assign to a TU-game an empty set of payoff. Any such situation is clearly an impasse.

On the other hand, single-value solutions assign to a TU-game a single payoff that is supposed to best account for individual contributions in the game. It is therefore important to identify what are norms or standards behing each single-valued solution of TU-games. The more all the properties advocated under a single-valued solution are intuitively convincing, the better the desirability of the sharing vectors it generates. This is the basic principle of the normative approach: the so-called axiomatic approach, see Thomson (2001) for a nice guide of an axiomatic study. TU-games analyses have been enriched so far with a rich variety of single-valued solutions simply called values. A non exhaustive list of values includes for example the Shapley value by Shapley (1953), the Banzhaf value by Banzhaf (1965), the nucleolus by Schmeidler (1969), the Owen value by Owen (1977), the solidarity value by Nowak and Radzik (1994), the egalitarian nonpairwise-averaged contribution value by Driessen and Funaki (1997b), the consensus value by Ju et al. (2007) and the solidarity value with cooptation by Diffo Lambo and Wambo (2015). As we earlier mention, this chapter is devoted to shedding light on the Shapley value.

4.1.3 The Shapley value

The following definition is a formal statement of a value we use and is clearly related to a fixed set of players.

Definition 4.1.4. *A value (or a single-valued solution) on Γ_N is any mapping $\Psi : \Gamma_N \longrightarrow \mathbb{R}^N$.*

Given a value Ψ on Γ_N and $v \in \Gamma_N$, $\Psi(v)$ is the n-tuple $\Psi(v) = (\Psi_i(v))_{i\in N}$ where for each player $i \in N$, $\Psi_i(v)$ is the individual share of player i in the game v. Here, it is assumed that $\Psi_i(v)$ may be any real number. By so doing, it is implicitly assumed that coalitional payoff are infinitely divisible. Furthermore, no constraint about the feasibility of $\Psi(v)$ is required and is, however, $\Psi(v)$ is supposed to be derived from a reasonable procedure.

Definition 4.1.5. *The Shapley value is the value on Γ_N denoted by Shap and defined for all $v \in \Gamma_N$ and for all $i \in N$ by*

$$\mathrm{Shap}_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}[v(S+i) - v(S)]. \tag{4.5}$$

Equation (4.5) is also called the coalitional formula of the Shapley value and $v(S+i) - v(S)$ is the marginal contribution of player i to S.

Provided that the marginal contribution of player i to any coalition S is weighted by $|S|!\,(n-|S|-1)!/n!$, player i's share in a TU-game v by the Shapley value can be taken as his average marginal contribution across all the coalitions he can join. As it will be seen later, the sharing vector by the Shapley value is feasible since individual shares sum to the payoff of the grand coalition for all TU-games; that is, the Shapley value is essentially a solution to the problem of payoff redistribution. When a given coalition structure is embedded to any TU-game in Γ_N, we refer to the Owen value by Owen (1977) as a possible way of extending the Shapley value to those contexts.

Remark 4.1.3. *Alternative formulae of the Shapley value exist. Two such formulae are the following:*

$$\forall v \in \Gamma_N, \forall i \in N, \mathrm{Shap}_i(v) = \sum_{\pi\in\Pi_N} \frac{1}{n!}[v(P_\pi^i + i) - v(P_\pi^i)] \tag{4.6}$$

where Π_N is the set of all permutations of N and $P_\pi^i = \{j \in N : \pi(j) < \pi(i)\}$ is the set of all predecessors of player i with respect to a permutation $\pi \in \Pi_N$.

$$\forall v \in \Gamma_N, \forall i \in N, \mathrm{Shap}_i(v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{[v]_S}{s}. \tag{4.7}$$

where for all $S \in 2^N$, $[v]_S$ is the Harsanyi dividend of coalition S with respect to v defined in (4.4).

EXAMPLE 4.1.2. *The Shapley value of the TU-game of Example 4.1.1 is given by*

$$\text{Shap}(v) = \left(-\frac{4}{3}, -\frac{13}{3}, -\frac{16}{3}\right).$$

That is if A, B and C decide to cooperate and agree for a common linking facility, individual cost shares using the Shapley value are such that A will pay $\frac{4}{3}$, meanwhile B and C will pay $\frac{13}{3}$ and $\frac{16}{3}$ respectively.

To account on the merit of the Shapley value, Shapley (1953) provided a four-axiom characterization. Before a formal statement of that result, further concepts are needed. For example, in the following definition, some specific players in a TU-game are presented depending on their marginal contributions in the game.

DEFINITION 4.1.6. *Given a TU-game $v \in \Gamma_N$,*

- *A player i is said to be* **dummy player** *in v or simply* **v-dummy** *if*

$$\forall S \subseteq N\backslash\{i\}, v(S+i) - v(S) = v(i).$$

- *A* **null player** *in v is any player $i \in N$ such that*

$$\forall S \subseteq N\backslash\{i\}, v(S+i) - v(S) = 0.$$

- *Two players i and j are* **symmetric** *or simply* **v-symmetric** *if*

$$\forall S \subseteq N\backslash\{i,j\}, v(S+i) = v(S+j).$$

Consider a TU-game. A null player is any player whose marginal contribution to all coalitions he joins is null. A dummy player is any player whose marginal contribution to any coalition he joins is exactly what he obtains by standing alone. Thus, a null player is simply a dummy player who gains nothing by standing alone. Two symmetric players are any two players such that the replacement of one by the other does not change anything in any coalition; their respective marginal contributions coincide for any coalition containing neither of the two. For example, any two null players are symmetric.

DEFINITION 4.1.7. *Let Ψ be a value on Γ_N.*

(i) *Ψ satisfies* **efficiency** ***(E)*** *if for all $v \in \Gamma_N$,*

$$\sum_{i \in N} \Psi_i(v) = v(N).$$

(ii) *Ψ satisfies the* **null player property** ***(NP)*** *if for all $v \in \Gamma_N$ and for all $i \in N$ such that i is a null player in v, $\Psi_i(v) = 0$.*

(*iii*) Ψ satisfies the **dummy property** ***(D)*** if for all $v \in \Gamma_N$ and for all $i \in N$ such that i is v-dummy, $\Psi_i(v) = v(i)$.

(*iv*) Ψ is **additivite** ***(AD)*** if for all games $u, v \in \Gamma_N$ and for all player $i \in N$,

$$\Psi_i(u+v) = \Psi_i(u) + \Psi_i(v). \tag{4.8}$$

(*v*) Ψ satisfies **symmetry** ***(S)*** if for all game $v \in \Gamma_N$, and for all players $i, j \in N$ such that i and j are v-symmetric, $\Psi_i(v) = \Psi_j(v)$.

The Efficiency axiom can be seen as a feasibility condition when players form the grand coalition. It simply requires that individual shares should sum to the payoff of the grand coalition. The null player property states that any null player should receive a zero share (since he contributes noting to any coalition). A value that satisfies the dummy axiom assigns to each dummy player his stand alone worth in the game. Additivity is the requirement that summing any two TU-games should always results in summing the payoffs associated with the two games. Symmetry simply states that if two players in a TU-game are substitutes, then they should be rewarded the same amount.

Combining some axioms from Definition 4.1.7 yields the following result which is a characterization of the Shapley value by means of four axioms.

Theorem 4.1.1 (Shapley (1953)).

The Shapley value is the unique value on Γ_N that satisfies (E), (NP), (S) and (AD). ★

It can be easily checked that any value on Γ_N that satisfies Axiom (D) also satisfies Axiom (NP); and the converse also holds in presence of Axiom (AD). That is why some authors sometimes substitute the weak Axiom (NP) for the Axiom (D) in the preceding Shapley theorem to obtain the following.

Theorem 4.1.2 (Shapley (1953)).

The Shapley value is the unique value on Γ_N that satisfies (E), (D), (S) and (AD). ★

Whether the above characterization holds on some subsets of Γ_N is an important issue we address in the next sections.

4.2 An intra-domain analysis of the Shapley value

In this section, we introduce the notion of valid domain associated with a list of axioms on Γ_N the combination of which yields a given result. The case of Theorem 4.1.2 constitutes our main focus.

4.2.1 Shapley valid domains, subspaces and cones

We start with an abstract definition of the concept of a valid domain.

Definition 4.2.1. *A characterization result on Γ_N is any pair $(\mathcal{L}, \varphi_0)$ where $\mathcal{L}$ is a finite and nonempty list of axioms for solutions on Γ_N and φ is a value on Γ_N such that any value φ satisfies all the axioms in $\mathcal{L}$ if and only if $\varphi = \varphi_0$.*

Of course, our definition may be easily weaken to capture statements where axioms in $\mathcal{L}$ are just sufficient or necessary conditions; or statements where instead of Γ_N, the value φ_0 is defined on a proper subset of Γ_N.

Definition 4.2.2. *Given a characterization result $(\mathcal{L}, \varphi_0)$ on Γ_N, an $(\mathcal{L}, \varphi_0)$ valid domain is any nonempty subset Γ' of Γ_N such that any value satisfies all the axioms in $\mathcal{L}$ for games in Γ' if and only if $\varphi(v) = \varphi_0(v)$ holds for all $v \in \Gamma'$.*

Moving from Γ_N to one of its nonempty subset Γ', we consider only TU-games in Γ' and suppose that a solution applies only for games in Γ'. For example, the additivity axiom on Γ' is now the requirement that (4.8) holds only for all $u, v \in \Gamma'$ such that $u + v \in \Gamma'$. In the case of Theorem 4.1.2, we have the following definition.

Definition 4.2.3. *A* **Shapley valid domain** *is any $(\mathcal{L}, \varphi_0)$ valid domain $\mathcal{D}$ where $\mathcal{L} = \{(E), (D), (S), (AD)\}$ and $\varphi_0 = Shap$; that is, the Shapley value is the unique value on $\mathcal{D}$ that satisfies (E), (D), (S) and (AD).*

A Shapley valid domain can be of any kind since in Definition 4.2.3 no information is required on the structure of a Shapley valid domain. To see this, consider the following example.

Example 4.2.1. *By Theorem 4.1.2, Γ_N is obviously a Shapley valid domain. Moreover, for any TU-game v such that all players are v-dummy, the singleton $\mathcal{D} = \{v\}$ is a Shapley valid domain since any solution Ψ on $\mathcal{D}$ that satisfies (E), (D), (S) and (AD) necessarily satisfies $\Psi_i(v) = \mathrm{Shap}_i(v) = v(i)$ by (D). In this later case, (E), (S) and (AD) are clearly redundant over $\mathcal{D}$.*

Let Ψ be defined on the set $\mathcal{G}_N$ of all simple games on N for all $v \in \mathcal{G}_N$ by

$$\Psi_i(v) = \frac{1}{|N(v)|}\mathbb{1}_{N(v)}$$

where $N(v)$ is the set of non null players in v. Since $v(N) = 1$ for all $v \in \mathcal{G}_N$, the set $N(v)$ is always non empty. This proves that Ψ is well-defined on $\mathcal{G}_N$. Moreover, Ψ as well as any value on $\mathcal{G}_N$ satisfies (AD) since the sum of two simple games is never a simple game. Moreover, it can be easily checked that Ψ satisfies (E), (S), (D) and (AD). Therefore, $\mathcal{G}_N$ is not a Shapley valid domain.

Here below are some specific subsets of Γ_N that contain only TU-games derived from a fixed set of TU-games by linear combinations.

Definition 4.2.4. *Given a nonempty subset $\mathcal{F}$ of Γ_N, any TU-game*

$$u = \sum_{l=1}^{p} \alpha_l u_l$$

for some $p \in \mathbb{N}^*$, $(\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{R}$ and $(u_l)_{1 \leq l \leq p} \subseteq \mathcal{F}$ is called a linear combination of TU-games in $\mathcal{F}$. The set of linear combinations of TU-games in $\mathcal{F}$, denoted $\mathrm{Span}(\mathcal{F})$ is called the linear span of $\mathcal{F}$.

Any TU-game in $\mathcal{F}$ is called a generator of $\mathrm{Span}(\mathcal{F})$.

The linear span $\mathrm{Span}(\mathcal{F})$ of a nonempty subset $\mathcal{F}$ of Γ_N, involves linear combinations with arbitrary real coefficients. In the next definition, only linear combinations with integer coefficients are considered.

Definition 4.2.5. *Given a nonempty subset $\mathcal{F}$ of Γ_N, any TU-game*

$$u = \sum_{l=1}^{p} \alpha_l u_l$$

for some $p \in \mathbb{N}^*$, $(\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{Z}$ and $(u_l)_{1 \leq l \leq p} \subseteq \mathcal{F}$ is called an integer linear combination of TU-games in $\mathcal{F}$. The set of all integer linear combinations of TU-games in $\mathcal{F}$, denoted $\mathrm{Span}_{\mathbb{Z}}(\mathcal{F})$ is called the integer (linear) span of $\mathcal{F}$; formally,

$$\mathrm{Span}_{\mathbb{Z}}(\mathcal{F}) = \left\{ \sum_{l=1}^{p} \alpha_l u_l : \ p \in \mathbb{N}^*, \ (\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{Z} \text{ and } (u_l)_{1 \leq l \leq p} \subseteq \mathcal{F} \right\}.$$

Any TU-game in $\mathcal{F}$ is called a generator of $\mathrm{Span}_{\mathbb{Z}}(\mathcal{F})$.

For any nonempty subset $\mathcal{F}$ of Γ_N, it can be easily checked that any linear combination of two TU-games in $\mathrm{Span}(\mathcal{F})$ remains a TU-game in $\mathrm{Span}(\mathcal{F})$. This is a property of some well-known algebraic concepts in a vector space we recall below.

Definition 4.2.6. *Let $\mathcal{E}$ be a nonempty subset of Γ_N.*

- *The set $\mathcal{E}$ is called a subspace of Γ_N if $\alpha u + \beta v \in \mathcal{E}$ for all $u, v \in \mathcal{E}$ and for all $\alpha, \beta \in \mathbb{R}$.*
- *The set $\mathcal{E}$ is called an additive subgroup of Γ_N if $u - v \in \mathcal{E}$ for all $u, v \in \mathcal{E}$.*

For illustration, Γ_N is trivially both a subspace and an additive subgroup of Γ_N; and any subspace is an additive subgroup. More interestingly, we have the following remark.

Remark 4.2.1. *For any nonempty subset $\mathcal{F}$ of Γ_N, $\mathrm{Span}(\mathcal{F})$ is both a subspace and an additive subgroup of Γ_N; and $\mathrm{Span}_{\mathbb{Z}}(\mathcal{F})$ is an additive subgroup of Γ_N.*

Another subset of TU-games that is sometimes cited is the notion of convex cone presented in the next definition.

Definition 4.2.7. *Let $\mathcal{E}$ be a nonempty subset of Γ_N.*

- *The set $\mathcal{E}$ is called a (linear) cone in Γ_N if $\alpha u \in \mathcal{E}$ for each TU-game $u \in \mathcal{E}$ and for each positive $\alpha \in \mathbb{R}$.*
- *The set $\mathcal{E}$ is called a convex cone in Γ_N if $\alpha u + \beta v \in \mathcal{E}$ for all $u, v \in \mathcal{E}$ and for all positive $\alpha, \beta \in \mathbb{R}_{>0}$.*

It is immediate that any subspace of Γ_N is a cone of Γ_N. A rich panel of concepts on cones is provided by Bourbaki (1987) and Bernstein (2018).

Notation 4.2.1. *Given $\mathcal{F} \subseteq \Gamma_N$, we pose :*

$$Cone(\mathcal{F}) = \{v \in \Gamma_N : v = \alpha u \text{ for some } u \in \mathcal{F} \text{ and } \alpha > 0\}.$$

and

$$ConvCone(\mathcal{F}) = \left\{ \sum_{l=1}^{p} \alpha_l u_l : \; p \in \mathbb{N}^*, \; (\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{R}_{>0} \text{ and } (u_l)_{1 \leq l \leq p} \subseteq \mathcal{F} \right\}.$$

Clearly, $Cone(\mathcal{F})$ is a cone and $ConvCone(\mathcal{F})$ is a convex cone of Γ_N known, respectively, as the cone of $\mathcal{F}$ in Γ_N and the convex cone of $\mathcal{F}$ in Γ_N.

By definition, the null game $\widetilde{0}$ does not belong to some convex cones; but necessarily belongs to any convex cone of a nonempty subset $\mathcal{E}$ of Γ_N. We extend the notion of cone and related sets in the next definition.

Definition 4.2.8. *Let $\mathbb{E}$ be a nonempty subset of $\mathbb{R}$ and $\mathcal{F}$ a nonempty subset of Γ_N.*

The $\mathbb{E}$-span of $\mathcal{F}$ is the subset of Γ_N denoted by $\mathrm{Span}_{\mathbb{E}}(\mathcal{F})$ and defined as follows:

$$\mathrm{Span}_{\mathbb{E}}(\mathcal{F}) = \left\{ \sum_{l=1}^{p} \alpha_l u_l : \; p \in \mathbb{N}^*, \; (\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{E} \text{ and } (u_l)_{1 \leq l \leq p} \subseteq \mathcal{F} \right\}.$$

Any TU-game in $\mathcal{F}$ is called a generator of $\mathrm{Span}_{\mathbb{E}}(\mathcal{F})$.

The only change from $\mathrm{Span}(\mathcal{F})$ to $\mathrm{Span}_{\mathbb{E}}(\mathcal{F})$ is that only linear combinations of TU-games from $\mathcal{F}$ with coefficients in $\mathbb{E}$ are considered instead of arbitrary real coefficients. In the same way, we introduce the following definition:

Definition 4.2.9. *Let $\mathbb{E}$ be a nonempty subset of $\mathbb{R}$ and $\mathcal{F}$ a nonempty subset of Γ_N.*

- *The $\mathbb{E}$-cone of $\mathcal{F}$ is the subset of Γ_N denoted by $\mathrm{Cone}_{\mathbb{E}}(\mathcal{F})$ and defined as follows:*

$$\mathrm{Cone}_{\mathbb{E}}(\mathcal{F}) = \{\alpha u : \; \alpha \in \mathbb{E}_{>0} \text{ and } u \in \mathcal{F}\}, \text{ where } \mathbb{E}_{>0} = \{\alpha \in \mathbb{E} : \alpha > 0\}.$$

- *The convex $\mathbb{E}$-cone of $\mathcal{F}$ is the subset of Γ_N denoted by* $\text{ConvCone}_{\mathbb{E}}(\mathcal{F})$ *and defined as follows:*

$$\text{ConvCone}_{\mathbb{E}}(\mathcal{F}) = \left\{ \sum_{l=1}^{p} \alpha_l u_l : \; p \in \mathbb{N}^*, \; (\alpha_l)_{1 \leq l \leq p} \subseteq \mathbb{E}_{>0} \text{ and } (u_l)_{1 \leq l \leq p} \subseteq \mathcal{F} \right\}.$$

Any TU-game in $\mathcal{F}$ is called a generator of $\text{Cone}_{\mathbb{E}}(\mathcal{F})$ *and* $\text{ConvCone}_{\mathbb{E}}(\mathcal{F})$.

Clearly, $\text{ConvCone}_{\mathbb{E}}(\mathcal{F})$ may not be convex at all. But any convex combination of TU-games from $\mathcal{F}$ with coefficients from $\mathbb{E}$ belongs to $\text{ConvCone}_{\mathbb{E}}(\mathcal{F})$. What is important is that $\text{ConvCone}_{\mathbb{E}}(\mathcal{F})$ mimics the structure of $\text{ConvCone}(\mathcal{F})$ when the only admissible coefficients are elements of $\mathbb{E}$.

4.2.2 Existing Shapley valid domains

We present here some Shapley valid domains that have already been reported so far. One of the two related contributions we have identified is due to Neyman (1989). The author was addressing a remark due to Hart and Mas-Colell (1989) on the fact that some "standard axiomatizations require the application of the axioms to a large class of games (e.g., all games; or, all simple games; etc.) in order to uniquely determine it (the value) for any single game".

Definition 4.2.10. *Given a TU-game $v \in \Gamma_N$ and a coalition S, the subgame of v associated with S is the TU-game denoted by v_S and defined for all $T \subseteq N$ by $v_S(T) = v(S \cap T)$.*

The set of all subgames of v is denoted by $\Gamma(v)$.

Hart and Mas-Colell (1989) used a distinct notion of subgame to provide an alternative characterization of the Shapley value. They simple referred to the subgame associated with a coalition S as the restriction of the initial game to only subsets of S; this is rather a TU-game in Γ_S; but not necessary a TU-game in Γ_N. Definition 4.2.10, due to Neyman (1989), can be seen as a way to deal with games in Γ_N although only players in S matter.

To fit our vocabulary on Shapley valid domains, we introduce the following definition.

Definition 4.2.11. *A nonempty subset $\mathcal{D}$ of Γ_N is a Neyman domain if there exists a TU-game $v \in \Gamma_N$ such that* $\mathcal{D} = \text{Span}_{\mathbb{Z}}(\Gamma(v))$.

To rephrase the preceding definition, a Neyman domain is any additive subgroup of Γ_N generated by a TU-game and all its subgames.

Theorem 4.2.1 (Neyman (1989)).

On the additive subgroup generated by any TU-game and all its subgames, the Shapley value is the unique value that satisfies (E), (NP), (AD) and (S). ⋆

Since (D) implies (NP), the following result is an immediate consequence of Theorem 4.2.1.

Corollary 4.2.1. *All Neyman domains are Shapley valid domains.*

Another family of Shapley valid domain is due to Peleg and Sudhölter (2007). The authors also provided some specific instances of their main condition.

Definition 4.2.12. *A subset $\mathcal{D}$ of Γ_N is a Peleg-Sudhölter domain if $\mathcal{D}$ is a convex cone which contains the null game and all the unanimity games.*

It is clear that Γ_N is a Peleg-Sudhölter domain of TU-games.

Theorem 4.2.2 (Peleg and Sudhölter (2007))**.**

On any Peleg-Sudhölter domain, the Shapley value is the unique value that satisfies (E), (NP), (AD) and (S).

As we recall above, (D) implies (NP). Thus, the following result is an immediate consequence of Theorem 4.2.2.

Corollary 4.2.2. *All Peleg-Sudhölter domains are Shapley valid domains.*

We present below some particular Peleg-Sudhölter domains.

Definition 4.2.13. *A TU-game $u \in \Gamma_N$ is*

- **superadditive** *if for all $S, T \in 2^N$,*

$$u(S) \geq u(S \cap T) + u(S \backslash T).$$

- **monotonic** *if for all $S, T \in 2^N$,*

$$S \subseteq T \Longrightarrow u(S) \leq u(T).$$

- **convex** *if for all $S, T \in 2^N$,*

$$u(S) + u(T) \leq u(S \cup T) + u(S \cap T).$$

- **additive** *if all players in N are dummy players in u.*

Following Peleg and Sudhölter (2007), we denote by $\mathcal{V}^{(s)}$ the set of all superadditive TU-games in Γ_N, by $\mathcal{V}^{(m)}$ the set of all monotonic TU-games in Γ_N, by $\mathcal{V}^{(c)}$ the set of all convex TU-games in Γ_N and by $\mathcal{V}^{(a)}$ the set of all convex TU-games in Γ_N.

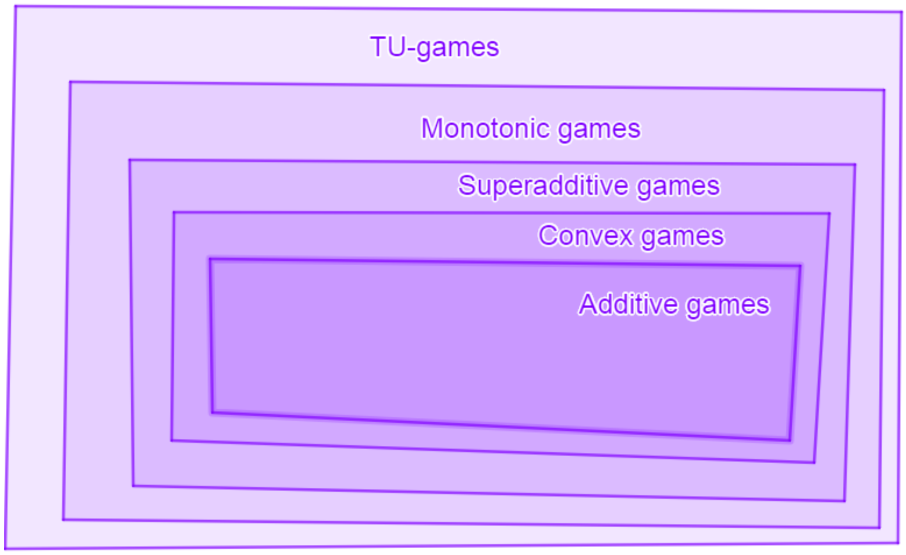

Figure 4.1: Representation of some classic families of TU-games.

As depicted in Figure 4.1, the four classic families of TU-games presented in Definition 4.2.13 are closely related and are each a convex cone that contains all the unanimity games. Moreover, $\mathcal{V}^{(a)} \subset \mathcal{V}^{(c)} \subset \mathcal{V}^{(s)} \subset \mathcal{V}^{(m)} \subset \Gamma_N$. That is, any additive game is a convex game; any convex game is a superadditive game; any superadditive game is a monotonic game. This leads to the following result reported by Peleg and Sudhölter (2007).

Corollary 4.2.3 (**Peleg and Sudhölter (2007)**). *For $\mathcal{E} \in \{\mathcal{V}^{(s)}, \mathcal{V}^{(m)}, \mathcal{V}^{(c)}, \mathcal{V}^{(a)}\}$, the Shapley value is the unique value that satisfies* (E), (NP), (AD) *and* (S) *on $\mathcal{E}$.*

The next result holds as an immediate consequence of Corollary 4.2.3.

Corollary 4.2.4. *Each of the four subsets $\mathcal{V}^{(s)}$, $\mathcal{V}^{(m)}$, $\mathcal{V}^{(c)}$ and $\mathcal{V}^{(a)}$ is a Shapley valid domain.*

In the next section, we provide new Shapley valid domains.

4.2.3 New Shapley valid domains : conically consistency

In what follows, for each TU-game $u \in \Gamma_N$, we denote the set of all dummy players in u by $Dum(u)$ and the partition of all non dummy players in u by $Sym(u)$; that is

$$Dum(u) = \{i \in N : i \text{ is dummy in } u\}$$

and for all $S \in Sym(u)$, for all $i, j \in N \backslash Dum(u)$,

- $i \in T$ for some $T \in Sym(u)$;
- $(i \in S \text{ and } j \in S) \implies i$ and j are symmetric players in u;
- $(i \in S \text{ and } j \notin S) \implies i$ and j are non symmetric players in u.

Definition 4.2.14. *Given a TU-game $u \in \Gamma_N$, any coalition in $Sym(u)$ is called a symmetry class of non dummy players in u.*

Remark 4.2.2. *Given $u \in \Gamma_N$, consider the binary relation $\sim_u$ defined for all $i, j \in N \backslash Dum(u)$ by $i \sim_u j$ if i and j are symmetric players in u. Then $\sim_u$ is an equivalence relation on $N \backslash Dum(u)$. Moreover, $Sym(u)$ is the set of all equivalence classes of $\sim_u$ on $N \backslash Dum(u)$.*

For illustration, we provide the following example.

Example 4.2.2. *Consider the TU-game v defined in Example 4.1.1. We have $Dum(v) = \emptyset$ and $Sym(v) = \{\{1\}, \{2\}, \{3\}\}$ since any two players in v are not symmetric. Moreover, given any coalition S, $Dum(\gamma_S) = N \backslash S$ and $Sym(\gamma_S) = \{S\}$; and for any additive game u, $Dum(u) = N$ and $Sym(u) = \emptyset$.*

It can be easily checked that a necessary and sufficient condition for a TU-game u to be additive is that

$$u(S) = \sum_{l \in S} u(l) \text{ for all } S \in 2^N.$$

Furthermore, the following result holds for all TU-games.

Proposition 4.2.1. *For all $u \in \Gamma_N$ and all player $i \in N$, we have $D_u \neq N \backslash \{i\}$.*

Proof.
Consider a TU-game $u \in \Gamma_N$ and a player $i \in N$. Suppose that $D_u = N \backslash \{i\}$. Let $S = \{i_1, ..., i_s\} \in 2^N$. First suppose that $i \notin S$. Then $u(S) = \sum_{k=1}^{s} u(i_k) = \sum_{j \in S} u(j)$ since all members of $N \backslash \{i\}$ are dummy players. Now suppose that $i \in S$. We show by induction on $s \in \{1, ..., n\}$ the assertion $\mathcal{A}(s)$ that $u(S) = \sum_{l \in S} u(l)$ for all $S \in 2^N$ such that $i \in S$.

Initialization: For $s = 1$, $S = \{i\}$ and $\mathcal{A}(1)$ obviously holds.

Induction step: Let $k \in \{1, 2, ..., n-1\}$ and assume that $u(S) = \sum_{l \in S} u(l)$ for all $S \in 2^N$ such that $i \in S$ and $|S| \leq k$. Consider $S \in 2^N$ such that $i \in S$ and $s = k + 1$. Then,

$$\begin{aligned} u(S) &= u((S \backslash \{j\}) + j) \text{ where } j \in S \backslash \{i\} \\ &= u(S \backslash \{j\}) + u(j) \text{ since } j \in D_u \\ &= u(j) + \sum_{l \in S \backslash \{j\}} u(l) \text{ by induction assumption} \\ &= \sum_{l \in S} u(l). \end{aligned}$$

Conclusion: $u(S) = \sum_{l \in S} u(l)$ for all $S \in 2^N$ such that $i \in S$.

This proves that u is additive and that i is a dummy player in u. A contradiction arises. Therefore $D_u \neq N \backslash \{i\}$. ■

By Proposition 4.2.1, if at least $n-1$ players are dummy in a TU-game, then all the n players are dummy in that TU-game. We now move to a necessary and sufficient condition for the linear span Span($\{u\}$) of a TU-game to be a Shapley valid domain.

Proposition 4.2.2. *Let $u \in \Gamma_N$ and $\mathcal{D}$ be any nonempty subset of* Span($\{u\}$).

The Shapley value is the unique value on $\mathcal{D}$ that satisfies (E), (S), (AD) *and* (D) *if and only if u admits at most one symmetry class of non dummy players.*

Proof. Consider $u \in \Gamma_N$.

$\Longleftarrow$) Assume that $|Sym(u)| \leq 1$ and consider a value on $\mathcal{D}$ that satisfies (E), (S), (AD) and (D). First suppose that $|Sym(u)| = 0$. Then $Dum(u) = N$. In this case, for all $v \in$ Span($\{u\}$), $v = \alpha u$ for some $\alpha \in \mathbb{R}$ and therefore $Dum(v) = N$. By (D), it follows that $\varphi_i(v) = v(i) = Shap_i(v)$ for all $v \in \mathcal{D}$ and for all $i \in N$. Therefore, $\varphi(v) = Shap(v)$ for all $v \in \mathcal{D}$.

Now suppose that $|Sym(u)| = 1$. That is $Sym(u) = \{S\}$ and $Dum(u) = T$ for some two-coalition partition $\{S, T\}$ of N. Let $v \in$ Span($\{u\}$) . Then $v = \alpha u$ for some $\alpha \in \mathbb{R}$. Suppose that $\alpha = 0$. Then $v = \widetilde{0}$ and it follows by (D) that $\varphi_i(v) = Shap_i(v) = 0$ for all $i \in N$; that is $\varphi(v) = Shap(v)$. Suppose that $\alpha \neq 0$. Then $Sym(v) = \{S\}$ and $Dum(v) = T$. Therefore by (D), $\varphi_i(v) = v(i) = Shap_i(v)$ for all $i \in T$ and by (S), $\varphi_i(v) = \varphi_j(v)$ for all $i, j \in S$. By (E), $\varphi_i(v) = Shap_i(v) = \frac{1}{|S|}\left(v(N) - \sum_{j \in T} v(j)\right)$ for all $i \in S$. Therefore, $\varphi(v) = Shap(v)$ for all $v \in \mathcal{D}$.

$\Longrightarrow$) Suppose that $|Sym(u)| \geq 2$ and consider $S, S' \in Sym(u)$. Moreover, consider $K \in 2^N$ such that $u(K) \neq 0$. Of course, such a coalition K exists; otherwise $u = \widetilde{0}$ and $Sym(u) = \emptyset$, a contradiction. For each $\lambda \in \mathbb{R}$, define the value φ_λ for all $v \in \mathcal{D}$ by $\varphi_\lambda(v) = (\varphi_{\lambda,i}(v))_{i \in N}$ such that for all $i \in N$,

$$\varphi_{\lambda,i}(v) = \begin{cases} v(i) \text{ if } i \in Dum(u) \\ \dfrac{\lambda}{|S|} v(K) \text{ if } i \in S \\ \dfrac{1}{|N \setminus (S \cup Dum(u))|}\left[v(N) - \lambda v(K) - \displaystyle\sum_{j \in Dum(u)} v(j).\right] \text{ otherwise} \end{cases}$$

Each φ_λ satisfies (E), (S), (AD) and (D) on $\mathcal{D}$. To see this, consider $\lambda \in \mathbb{R}$. The value φ_λ is well defined since $|S| \neq 0$ and $|N \setminus (S \cup Dum(u))| \geq |S'| \geq 1$ by noting the fact that $S' \subseteq N \setminus (S \cup Dum(u))$ and $S, S' \in Sym(u)$. By definition, $\varphi_\lambda(v)$ linearly depends on v and therefore φ_λ satisfies (AD). To continue, consider $v \in \mathcal{D}$. Then $v = \alpha u$ for some $\alpha \in \mathbb{R}$.

If $\alpha = 0$, then $v(L) = 0$ for all $L \subseteq N$, $Dum(v) = N$ and by definition, $\varphi_\lambda(\alpha u) = 0 = v(i)$ for all $i \in N$. Otherwise, $\alpha \neq 0$, $Dum(v) = Dum(u)$ and it follows by the definition of φ_λ that $\varphi_{\lambda,i}(\alpha u) = v(i)$ for all $i \in Dum(u) = Dum(v)$. This proves that φ_λ meets (D).

By adding individual shares for $\varphi_\lambda(v) = (\varphi_{\lambda,i}(v))_{i\in N}$, the following holds by definition:

$$\sum_{l\in N}\varphi_{\lambda,l}(v) = (\sum_{l\in Dum(u)} v(l)) + \lambda v(K) + v(N) - \lambda v(K) - \left(\sum_{l\in Dum(u)} v(l)\right) = v(N).$$

Thus, φ_λ satisfies (E).

Consider two symmetric players $i, j \in N$ in v. We prove that $\varphi_{\lambda,i}(v) = \varphi_{\lambda,j}(v)$. First suppose that $\alpha = 0$, then v is null and $\varphi_{\lambda,i}(v) = \varphi_{\lambda,j}(v) = 0$ by the definition of φ_λ. Now suppose that $\alpha \neq 0$, then i and j are symmetric players in u and thus, $u(i) = u(j)$. There are three possible cases: *(i)* $i, j \in Dum(u)$; *(ii)* $i, j \in S$; or *(iii)* $i, j \in N\backslash(Dum(u) \cup S)$. In each of the three possible cases, $\varphi_{\lambda,i}(v) = \varphi_{\lambda,j}(v)$ by the definition of φ_λ. Thus, φ_λ satisfies (S).

To conclude, the family $(\varphi_\lambda)_{\lambda\in\mathbb{R}}$ is a family of distinct values on $\mathcal{D}$ that satisfies (E), (D), (S) and (AD). Therefore the Shapley value is not the only value on $\mathcal{D}$ that satisfies (E), (D), (S) and (AD). Thus $\mathcal{D}$ is not a Shapley valid domain. ∎

It is worth mentioning that for a TU-game $u \in \Gamma_N$ such that $|Sym(u)| \geq 2$, it is still possible to define a value that satisfies (E), (S) and (D) but not (AD) on Span($\{u\}$). This may be achieved by simply substituting $(v(K))^2$ to $v(K)$ in the definition of φ_λ in the proof of Proposition 4.2.2. The next result is straightforward from Proposition 4.2.2.

Corollary 4.2.5. *For any TU-game $u \in \Gamma_N$, any nonempty subset $\mathcal{D}$ of* Span($\{u\}$) *is a Shapley valid domain if and only if u admits at most one symmetry class of non dummy players.*

In the next result, we identify the set of all TU-games $u \in \Gamma_N$ such that $|Sym(u)| \leq 1$.

Proposition 4.2.3. *A TU-game $u \in \Gamma_N$ admits at most one symmetry class of non dummy players if and only if*

$$u = \sum_{i\in D}\alpha_i\gamma_{\{i\}} + \sum_{l=1}^{|S|}\beta_l \sum_{T\subseteq S, |T|=l}\gamma_T \tag{4.9}$$

for some $D \subseteq N, S \subseteq N, (\alpha_i)_{i\in D} \subset \mathbb{R}$ and $(\beta_l)_{l\in\{1,\ldots,|S|\}} \subset \mathbb{R}$ such that $S \cap D = \emptyset$ and $S \cup D = N$.

Proof.

$\Longrightarrow$) It is obvious that any TU-game $u \in \Gamma_N$ that satisfies (4.9) also satisfies $|Sym(u)| \leq 1$.

$\Longleftarrow$) Suppose a that a TU-game $u \in \Gamma_N$ is such that $|Sym(u)| \leq 1$. First suppose that $|Sym(u)| = 0$. Then $Dum(u) = N$ and therefore $u = \sum_{i\in N} u(i)\gamma_i$. Thus u satisfies (4.9) for $D = N$, $S = \emptyset$ and $\alpha_i = u(i)$ for all $i \in N$. Now suppose that $|Sym(u)| = 1$.

Let $Sym(u) = S$ and $Dum(u) = D$. It follows that $S \cap D = \emptyset$ and $S \cup D = N$. Then for all $T \in 2^N$,

$$\begin{aligned}
u(T) &= u((T \cap D) \cup (T \cap S)) \\
&= \sum_{i \in D \cap T} u(i) + u(T \cap S) \text{ since all players in } D \text{ are dummy players in } u \\
&= \sum_{i \in D \cap T} u(i) + \sum_{L \in 2^N} [u]_L \gamma_L(T \cap S) \text{ by (4.3)} \\
&= \sum_{i \in D \cap T} u(i) + \sum_{L \in 2^S / L \subseteq T} [u]_L \\
&= \sum_{i \in D} u(i)\gamma_i(T) + \sum_{L \in 2^S} [u]_L \gamma_L(T)
\end{aligned}$$

Let $s = |S|$ and for each $k \in \{1, 2, \ldots, s\}$, consider a subset S_k of S such that $|S_k| = k$. Given that $L \in 2^S$ and $l = |L|$, the following holds

$$\begin{aligned}
[u]_L &= \sum_{K \subseteq L} (-1)^{l-|K|} u(K) \\
&= \sum_{k=1}^{s} (-1)^{l-k} \binom{l}{k} u(S_k) \text{ since for all } K \in 2^S,\ |K| = k \text{ implies } u(K) = u(S_k) \\
&= [u]_{S_l} \text{ since only the size of } L \text{ matters}
\end{aligned}$$

We deduce that for all $T \in 2^N$,

$$\begin{aligned}
u(T) &= \sum_{i \in D} u(i)\gamma_i(T) + \sum_{L \in 2^S} [u]_L \gamma_L(T) \\
&= \sum_{i \in D} u(i)\gamma_i(T) + \sum_{k=1}^{s} [u]_{S_l} \sum_{L \in 2^S / |L| = l} \gamma_L(T)
\end{aligned}$$

Therefore u fits (4.9) with $\alpha_i = u(i)$ for all $i \in D$ and $\beta_l = [u]_{S_l}$ for all $l \in \{1, 2, \ldots, s\}$. ■

We are now ready to state and prove a sufficient condition on a nonempty subset of Γ_N to be a Shapley valid domain. But, before we need the following notation.

Notation 4.2.2. *For any nonempty subset $\mathbb{E}$ of $\mathbb{R}$, we denote by $\mathbb{E}^{\rightarrow}$ and $\mathbb{E}^{\leftrightarrow}$ the subset of $\mathbb{R}$ defined by*

$$\mathbb{E}^{\rightarrow} = \{|\alpha| / \alpha \in \mathbb{E}\} \text{ and } \mathbb{E}^{\leftrightarrow} = \{|\alpha| / \alpha \in \mathbb{E}\} \cup \{-|\alpha| / \alpha \in \mathbb{E}\}.$$

That is

$$\mathbb{E}^{\rightarrow} = \mathbb{E}_{\geq 0} \cup \{-\alpha / \alpha \in \mathbb{E}_{\leq 0}\} \text{ and } \mathbb{E}^{\leftrightarrow} = \mathbb{E}^{\rightarrow} \cup \{-\alpha / \alpha \in \mathbb{E}^{\rightarrow}\}.$$

In particular, $\mathbb{Z}^{\rightarrow} = \mathbb{Z}_{\geq 0}$ and $\mathbb{R}^{\rightarrow} = \mathbb{R}_{\geq 0}$. Similarly, $\mathbb{Z}^{\leftrightarrow} = \mathbb{Z}$ and $\mathbb{R}^{\leftrightarrow} = \mathbb{R}$.

Theorem 4.2.3.

Let $\mathbb{E}$ be a nonempty subset of $\mathbb{R}$, $\mathcal{F}$ a nonempty subset of Γ_N such that $|Sym(u)| \leq 1$ for all $u \in \mathcal{F}$.
Any nonempty subset $\mathcal{D}$ of Γ_N that satisfies

$$\mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F}) \subseteq \mathcal{D} \subseteq \mathrm{Span}_{\mathbb{E}^{\leftrightarrow}}(\mathcal{F}) \tag{4.10}$$

is a Shapley valid domain.

Proof. Consider a nonempty subset $\mathbb{E}$ of $\mathbb{R}$ and $\mathcal{F}$ a nonempty subset of Γ_N such that $|Sym(u)| \leq 1$ for all $u \in \mathcal{F}$. Suppose $\mathcal{D}$ is a nonempty subset of Γ_N that satisfies (4.10).

It is well-known that the Shapley value *Shap* satisfies (E), (D), (AD) and (S) on Γ_N, and thus on $\mathcal{D}$. Now, let φ be a value on $\mathcal{D}$ that satisfies (E), (D), (AD) and (S). Consider $v \in \mathcal{D}$. Clearly if $v = \widetilde{0}$, then $\varphi(v) = Shap(v)$ by (E) and (S).

To continue, suppose that $v \neq \widetilde{0}$. Since $\mathcal{D} \subseteq \mathrm{Span}_{\mathbb{E}^{\leftrightarrow}}(\mathcal{F})$, there exist $p \in \mathbb{N}^*, (\alpha_l)_{l\in\{1,\ldots,p\}} \subseteq \mathbb{E}^{\leftrightarrow}$, and $(v_l)_{l\in\{1,\ldots,p\}} \subset \mathcal{F}$ such that $v = \sum\limits_{l=1}^{p} \alpha_l v_l$. Set $\mathcal{I}^+ = \{l \in \{1,\ldots,p\} : \alpha_l > 0\}$ and $\mathcal{I}^- = \{l \in \{1,\ldots,p\} : \alpha_l < 0\}$. The game v can be rewritten as follows:

$$v = \sum_{l\in\mathcal{I}^+} \alpha_l v_l + \sum_{l\in\mathcal{I}^-} \alpha_l v_l.$$

Note that we suppose that both $\mathcal{I}^+$ and $\mathcal{I}^-$ are nonempty since v can be rewritten as $v = v + \alpha_0 v_0 - \alpha_0 v_0$ for some $\alpha_0 \in \mathbb{E}^{\rightarrow}$ and $v_0 \in \mathcal{F}$. Equivalently,

$$v + \sum_{l\in\mathcal{I}^-} (-\alpha_l) v_l = \sum_{l\in\mathcal{I}^+} \alpha_l v_l.$$

Note that for all $p \in \mathcal{I}^+$ and for all $q \in \mathcal{I}^-$,

$$\alpha_p v_p, (-\alpha_q) v_q, \sum_{l\in\mathcal{I}^+} \alpha_l v_l, \sum_{l\in\mathcal{I}^-} (-\alpha_l) v_l \in \mathcal{D}$$

since by assumption $\mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F}) \subseteq \mathcal{D}$. It follows that

$$\varphi\left(v + \sum_{l\in\mathcal{I}^-} (-\alpha_l) v_l\right) = \varphi\left(\sum_{l\in\mathcal{I}^+} \alpha_l v_l\right).$$

Furthermore, by (AD), we deduce that

$$\varphi(v) + \sum_{l\in\mathcal{I}^-} \varphi((-\alpha_l) v_l) = \sum_{l\in\mathcal{I}^+} \varphi(\alpha_l v_l).$$

Therefore

$$\varphi(v) = \sum_{l \in \mathcal{I}^+} \varphi(\alpha_l v_l) - \sum_{l \in \mathcal{I}^-} \varphi((-\alpha_l) v_l).$$

Noting that for all $l \in \mathcal{I}^+ \cup \mathcal{I}^-$, $\mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\{v_l\}) \subseteq \mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F}) \subseteq \mathcal{D}$ and that by assumption on $\mathcal{F}$, $|Sym(v_l)| \leq 1$, Corollary 4.2.5 implies that $\mathrm{Span}(\{v_l\})$ is a Shapley valid domain. Hence, $\varphi(\alpha_l v_l) = Shap(\alpha_l v_l)$ for all $l \in \mathcal{I}^+$ and $\varphi((-\alpha_l)v_l) = Shap((-\alpha_l)v_l)$ for all $l \in \mathcal{I}^-$. Finally, we deduce that

$$\varphi(v) = \sum_{l \in \mathcal{I}^+} Shap(\alpha_l v_l) - \sum_{l \in \mathcal{I}^-} Shap((-\alpha_l)v_l) = Shap(v).$$

This proves that any value φ on $\mathcal{D}$ that satisfies (E), (D), (AD) and (S) on $\mathcal{D}$ necessarily coincides on $\mathcal{D}$ with $Shap$. Thus $\mathcal{D}$ is a Shapley valid domain. ■

In condition (4.10), no algebraic property is required on the structure of $\mathbb{E}$. This underlines the diversity of Shapley valid domain, having some cone-wise structure. To describe all such families of TU-games, we introduce the following definition.

Definition 4.2.15. *Any non empty subset $\mathcal{D}$ of Γ_N is said to be conically consistent if $\mathcal{D}$ satisfies*

$$\mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F}) \subseteq \mathcal{D} \subseteq \mathrm{Span}_{\mathbb{E}^{\leftrightarrow}}(\mathcal{F})$$

for some nonempty subset $\mathbb{E}$ of $\mathbb{R}$ and for some nonempty subset $\mathcal{F}$ of Γ_N such that $|Sym(u)| \leq 1$ for all $u \in \mathcal{F}$.

In this case, we say that $\mathcal{D}$ is conically consistent with respect to $\mathbb{E}$ and $\mathcal{F}$.

From now on, we consider the following notation.

Notation 4.2.3. *We denote by $\mathcal{U}_N$ the set of all unanimity games γ_S in Γ_N and by $\mathcal{F}_N$ the set of all TU-games that have at most one symmetry class of non dummy players. That is,*

$$\mathcal{U}_N = \{v \in \Gamma_N / v = \gamma_S \text{ for some } S \in 2^N\}$$

and

$$\mathcal{F}_N = \{v \in \Gamma_N / |Sym(v)| \leq 1\}.$$

It is immediate that $\emptyset \neq \mathcal{U}_N \subseteq \mathcal{F}_N$.

The next result provides some trivial cases of conically consistent domains.

Proposition 4.2.4. *For any nonempty subset $\mathbb{E}$ of $\mathbb{R}$ and for any nonempty subset $\mathcal{F}$ of $\mathcal{F}_N$, $\mathrm{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F})$ and $\mathrm{Span}_{\mathbb{E}}(\mathcal{F})$ are conically consistent in Γ_N with respect to $\mathbb{E}$ and $\mathcal{F}$.*

Proof.

The proof immediately follows from the definition of $\mathcal{F}_N$ and the fact that for any nonempty subset $\mathbb{E}$ of $\mathbb{R}$ and for any nonempty subset $\mathcal{F}$ of $\mathcal{F}_N$, $\text{ConvCone}_{\mathbb{E}^{\rightarrow}}(\mathcal{F}) \subseteq \text{Span}_{\mathbb{E}}(\mathcal{F}) \subseteq \text{Span}_{\mathbb{E}^{\leftrightarrow}}(\mathcal{F})$. ■

Proposition 4.2.5. *Γ_N is conically consistent with respect to $\mathbb{R}$ and $\mathcal{U}_N$.*

Proof.
Noting that $\mathcal{U}_N \subseteq \mathcal{F}_N$, the proof immediately follows from Proposition 4.2.4 and the fact that $\text{Span}_{\mathbb{R}}(\mathcal{U}_N) = \Gamma_N$ since $\mathcal{U}_N$ is a basis of the linear space Γ_N. ■

In the next section, we prove that previous results we earlier mentioned are particular instances of conically consistent domains of Γ_N.

4.2.4 Comparison with existing results and further directions

We prove here that Neyman domains as well as Peleg-Sudhölter domains are particular instances of conically consistent domains. New directions are also presented.

The case of Neyman domains

For an arbitrary TU-game $v \in \Gamma_N$, a subgame v_S of v is defined with respect to any coalition, say, S. However, players in S are not necessarily symmetric players in v_S. The Neyman domain associated with v is the additive subgroup $\text{Span}_{\mathbb{Z}}(\Gamma(v))$ generated by the collection of its subgames. We prove below that any Neyman domain is conically consistent with respect to $\mathbb{Z}$ and a subset of Γ that contains only TU-games with at most one symmetry class of non dummy players. For this purpose, we introduce the following definition.

Definition 4.2.16. *Given a TU-game $v \in \Gamma_N$, the Harsanyi component game of v associated to a coalition S is the TU-game $[v]_S\gamma_S$ where $[v]_S$ is the Harsanyi dividend of the members of S in v.*

We denote by $\mathcal{H}(v)$ the set all Harsanyi component games of v.

Obviously, the Harsanyi component game $[v]_S\gamma_S$ of v is null as soon as $[v]_S = 0$. Moreover $[v]_S\gamma_S$ and $[v]_T\gamma_T$ are distinct games whenever $S \neq T$. To show that the Neyman domain associated to v coincides with the additive subgroup generated by the collection of Harsanyi component games of v, we first prove the following:

Proposition 4.2.6. *Given $v \in \Gamma_N$ and $S \in 2^N$, $v_S = \sum_{T \in 2^S} [v]_T\gamma_T$.*

Proof.

Let $S, L \in 2^N$. On the one hand, $v_S(L) = v(S \cap L)$ by definition of v_S. Therefore, it follows from (4.3) that

$$\begin{aligned} v_S(L) &= \sum_{T \in 2^N} [v]_T \gamma_T(S \cap L) \\ &= \sum_{T \subseteq S \cap L} [v]_T \text{ since } \gamma_T(S \cap L) = 0 \text{ for } T \nsubseteq S \cap L \\ &= \sum_{T \in 2^S / T \subseteq L} [v]_T \\ &= \sum_{T \in 2^S} [v]_T \gamma_T(L) \text{ since } \gamma_T(L) = 1 \text{ for } T \subseteq L \end{aligned}$$

This proves that $v_S = \sum_{T \in 2^S} \lambda_T(v) \gamma_T$. ■

In the next result, it is shown that the additive group of Γ_N generated by the collection of all subgames of a given TU-game coincides with the additive subgroup of Γ_N generated by the collection of all Harsanyi component games of the same TU-game.

Proposition 4.2.7. *For all $v \in \Gamma_N$, we have $\text{Span}_{\mathbb{Z}}(\Gamma(v)) = \text{Span}_{\mathbb{Z}}(\mathcal{H}(v))$.*

Proof. To prove that $\text{Span}_{\mathbb{Z}}(\Gamma(v)) \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{H}(v))$, it is sufficient to prove that $\Gamma(v) \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{H}(v))$. Consider $u \in \Gamma(v)$. Then by definition of $\Gamma(v)$, $u = v_S$ for some coalition S. Therefore by Proposition 4.2.6, $v_S = \sum_{T \in 2^S} [v]_T \gamma_T$ and thus, $v_S \in \text{Span}_{\mathbb{Z}}(\mathcal{H}(v))$. We conclude that $\Gamma(v) \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{H}(v))$.

Similarly, to show that $\text{Span}_{\mathbb{Z}}(\mathcal{H}(v)) \subseteq \text{Span}_{\mathbb{Z}}(\Gamma(v))$, we simply prove that $\mathcal{H}(v) \subseteq \text{Span}_{\mathbb{Z}}(\Gamma(v))$. To do this, we prove by induction on $t \in \{1, 2, \ldots, n\}$ the assertion $\mathcal{A}(t)$ that $[v]_T \gamma_T \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ for all $T \in 2^N$ such that $|T| \leq t$.

Initialization: Consider $T \in 2^N$ such that $|T| = 1$. Then $T = \{i\}$ for some $i \in N$ and by Proposition 4.2.6, $v_T = [v]_T \gamma_T$ since $T = \{i\}$ is the only coalition contained in 2^T in this case. Therefore $[v]_T \gamma_T = v_T \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$.

Induction step: Consider $t \in \{1, \ldots, n-1\}$ and suppose that $[v]_L \gamma_L \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ for all $L \in 2^N$ such that $|L| \leq t$. Consider $T \in 2^N$ such that $|T| = t + 1$. By Proposition 4.2.6, the following holds:

$$v_T = \sum_{L \in 2^T} [v]_L \gamma_L = [v]_T \gamma_T + \sum_{2^T \backslash \{T\}} [v]_L \gamma_L.$$

We deduce that

$$[v]_T \gamma_T = v_T - \sum_{L \in 2^T \backslash \{T\}} [v]_L \gamma_L.$$

Therefore $[v]_T \gamma_T \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ since $v_T \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ by definition of $\Gamma(v)$ and for all $L \in 2^T \backslash \{T\}$, $[v]_L \gamma_L \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ by the induction assumption.

Conclusion: $[v]_T\gamma_T \in \text{Span}_{\mathbb{Z}}(\Gamma(v))$ for all $L \in 2^N$. ■

By Proposition 4.2.6, the subgame of a TU-game $u \in \Gamma_N$ associated to a coalition S is non null as soon as the corresponding Harsanyi component game is non null. But the Harsanyi component game associated to a coalition can be null (its Harsanyi dividend is null) while the corresponding subgame is non null (there exists a proper subset with a non zero Harsanyi dividend). It is then immediate that the number of Harsanyi component games of a TU-game is at most equal to the total number of its subgames. Proposition 4.2.7 is therefore a description of the additive subgroup generated by the collection of all subgames of a TU-game with possibly a smaller number of generators. Furthermore, we have the following result.

Corollary 4.2.6. *All Neyman domains are conically consistent.*

Proof. Suppose that a nonempty subset $\mathcal{D}$ of Γ_N is a Neyman domain. Then there exists a TU-game $u \in \Gamma_N$ such that $\mathcal{D} = \text{Span}_{\mathbb{Z}}(\Gamma(u))$. Then by Proposition 4.2.7, it follows that $\mathcal{D} = \text{Span}_{\mathbb{Z}}(\mathcal{H}(u))$. It is obvious that each Harsanyi component game in $\mathcal{H}(u)$ admits at most one symmetry class of non dummy players. Moreover, $\text{ConvCone}_{\mathbb{Z}}(\mathcal{H}(u)) \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{H}(u)) = \mathcal{D} \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{H}(u))$. Therefore, $\mathcal{D}$ is conically consistent with respect to $\mathbb{E} = \mathbb{Z}$ and $\mathcal{F} = \mathcal{H}(u)$. ■

By Corollary 4.2.6, the scope of Theorem 4.2.3 is larger than the one of Theorem 4.2.1.

The case of Peleg-Sudhölter domains

As already mentioned in Remark 4.1.2, Shapley (1953) has shown that the collection $\mathcal{U}_N$ of all unanimity games is a basis of the vector space Γ_N. The following result is thus straightforward.

Proposition 4.2.8. *The linear space Γ_N is such that*

$$\text{Span}_{\mathbb{R}}(\mathcal{U}_N) = \Gamma_N.$$

The next result provides the relationship between Peleg-Sudhölter domains and conically consistent domains.

Proposition 4.2.9. *All Peleg-Sudhölter domains are conically consistent.*

Proof. Suppose that a nonempty subset $\mathcal{D}$ of Γ_N is a Peleg-Sudhölter domain. Then $\mathcal{D}$ is a convex cone of Γ_N that contains the set $\mathcal{U}_N$ of all unanimity games in Γ_N. Therefore $\text{ConvCone}_{\mathbb{R}}(\mathcal{U}_N) \subseteq \mathcal{D}$. Each unanimity game in Γ_N admits exactly one symmetry class of non dummy players. The result then follows from the fact that $\text{ConvCone}_{\mathbb{R}}(\mathcal{U}_N) \subseteq \mathcal{D} \subseteq \text{Span}_{\mathbb{R}}(\mathcal{U}_N) = \Gamma_N$. ■

Further directions

The notion of conically consistent domain is closely related to the structure of a cone in Γ_N and on whether or not a domain contains some specific combinations of some games. An alternative approach is to focus on some specific domains in Γ_N with a well-known algebraic structure as it is, for example, the case with linear subspaces of Γ_N.

Notation 4.2.4. *For any nonempty subset $\mathcal{D}$ of Γ_N. We pose $\mathcal{D}^* = \mathcal{D} \cap \mathcal{F}_N$; that is*

$$\mathcal{D}^* = \{v \in \mathcal{D} : |Symv| \leq 1\}.$$

To the question of what subspace of Γ_N are Shapley valid domains, the next result provides a necessary and sufficient condition in the case of exactly two players.

Theorem 4.2.4.

With exactly two players, any subspace $\mathcal{E}$ of Γ_N is a Shapley valid domain if and only if $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$.

Proof. Assume that $N = \{1, 2\}$ and consider an arbitrary subspace $\mathcal{E}$ of Γ_N.

First suppose that $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$. Then by Proposition 4.2.4, $\mathcal{E}$ is a Shapley valid domain since $\mathcal{E}^* \subseteq \mathcal{F}_N$ by definition of $\mathcal{E}^*$.

Now suppose that $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) \neq \mathcal{E}$. To prove that $\mathcal{E}$ is not a Shapley valid domain, suppose on the contrary, that $\mathcal{E}$ is a Shapley valid domain. By Proposition 4.2.8, $\Gamma_N = \text{Span}_{\mathbb{R}}(\{\gamma_{\{1\}}, \gamma_{\{2\}}, \gamma_{\{1,2\}}\})$. It follows that $\mathcal{E}$ is of dimension at most 3. We then distinguish three distinct cases.

Firstly, suppose that $dim(\mathcal{E}) \leq 1$. Then by Corollary 4.2.5, $\mathcal{E} = \text{Span}_{\mathbb{R}}(\{u\})$ for some $u \in \mathcal{F}_N$. Therefore $\mathcal{E}^* = \text{Span}_{\mathbb{R}}(\{u\}) = \mathcal{E}$ and thus, $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$. This is a contradiction holds to the assumption that $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) \neq \mathcal{E}$.

Secondly, suppose that $dim(\mathcal{E}) = 3$. Then $\mathcal{E} = \Gamma_N$. In this case $\{\gamma_{\{1\}}, \gamma_{\{2\}}, \gamma_{\{1,2\}}\} \subseteq \mathcal{E}^*$ and thus, $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$ by Proposition 4.2.8; a contradiction since $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) \neq \mathcal{E}$.

Thirdly and finally, suppose that $dim(\mathcal{E}) = 2$. Since $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) \neq \mathcal{E}$, then $dim\,\text{Span}_{\mathbb{R}}(\mathcal{E}^*) \leq 1$ and there exists $u \in \mathcal{E}$ such that $u \notin \text{Span}_{\mathbb{R}}(\mathcal{E}^*)$. Moreover, there exists $v \in \mathcal{E}$ such that $\mathcal{E} = \text{Span}_{\mathbb{R}}(\{u, v\})$. To continue, we identify each game $w \in \Gamma_N$ with the triplet (x, y, z) such that $x = w(\{1\})$, $y = w(\{2\})$ and $z = w(\{1, 2\})$. In this sense, we set $u = (a, b, c)$ and $v = (a', b', c')$. Noting that $u \notin \text{Span}_{\mathbb{R}}(\mathcal{E}^*)$, it follows that $c \neq a + b$ and $a \neq b$. We pose $\lambda = \dfrac{c' - a' - b'}{c - a - b}$ and $w = v - \lambda u$. The game w is such that:

$$w(\{1\}) = \frac{a'(c - b) - a(c' - b')}{c - a - b} \text{ and } w(\{2\}) = \frac{b'(c - a) - b(c' - a')}{c - a - b}$$

and

$$w(\{1\}) + w(\{2\}) = w(\{1,2\}) = \frac{ca' + cb' - ac' - bc'}{c - b - a}.$$

Therefore, $Dum(w) = N$, $Sym(w) = \emptyset$ and thus, $w \in \mathcal{E}^* \subseteq \text{Span}_{\mathbb{R}}(\mathcal{E}^*)$. Recalling that u and v are linearly independent vectors, $w \neq \widetilde{0}$ and thus, $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \text{Span}_{\mathbb{R}}(\{w\})$. Furthermore, u and w are linearly independent otherwise $u \in \text{Span}_{\mathbb{R}}(\{w\}) \subseteq \text{Span}_{\mathbb{R}}(\mathcal{E}^*)$ which stands in contradiction to the fact that $u \notin \text{Span}_{\mathbb{R}}(\mathcal{E}^*)$. Therefore $\mathcal{E}$ can also be written by $\mathcal{E} = \text{Span}_{\mathbb{R}}(\{u, w\})$.

Suppose that $\alpha \neq \beta$. Then for $t = \dfrac{b' - a'}{\beta - \alpha}$, the TU-game $w' = u - t * w$ is such that $w'(\{1\}) = w'(\{2\})$. Therefore $w' \in \text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \text{Span}_{\mathbb{R}}(\{w\})$. It follows that there exists $k \in \mathbb{R}$ such that $u - t * w = k * w$. We get $u = (k + t) * w$ and it appears that u and w are linearly dependent; a contradiction arises since $\mathcal{E} = \text{Span}_{\mathbb{R}}(\{u, w\})$ and $dim(\mathcal{E}) = 2$.

We deduce that $\alpha = \beta \neq 0$ and we pose $w_0 = \dfrac{1}{\alpha} w = (1, 1, 2)$. We note that $\mathcal{E} = \text{Span}_{\mathbb{R}}(\{u, w_0\})$ and define the value Φ on $\mathcal{E}$ for all $g \in \mathcal{E}$ by:

$\Phi(g) = (\Phi_1(g), \Phi_2(g)) = (x + y \times u(N), x)$ provided that $g = x \times w_0 + y \times u$ with $x, y \in \mathbb{R}$.

Below, we prove that Φ satisfies (E), (D), (S) and (AD). For this purpose, let $N = \{i, j\}$.

<u>(E)</u>: Consider $g = x \times w_0 + y \times u \in \mathcal{E}$. We have

$$\Phi_1(g) + \Phi_2(g) = 2x + y \times u(N) = x \times w_0(N) + y \times u(N) = g(N).$$

Thus Φ satisfies (E).

<u>(D)</u>: Consider $g = x \times w_0 + y \times u \in \mathcal{E}$ and a player $i \in N$. If player i is dummy in g, then by Proposition 4.2.1, player j is also dummy. It follows that $g(N) = g(\{1\}) + g(\{2\})$. Equivalently, $2x + y \times u(N) = x + y \times u(\{1\}) + x + y \times u(\{2\})$. That is $yc = y(a+b)$ and since $c \neq a + b$, we obtain $y = 0$. Then $g = x \times w_0$ and $\Phi_1(g) = x = \Phi_2(g) = g(\{1\}) = g(\{2\})$. Thus Φ satisfies (D).

<u>(S)</u>: Consider $g = x \times w_0 + y \times u \in \mathcal{E}$. If players 1 and 2 are symmetric players in g, then $g(1) = g(2)$. That can be rewritten as $x + y \times u(\{1\}) = x + y \times u(\{2\})$. That is $ya = yb$ and since $a \neq b$, we obtain $y = 0$. Then $g = x \times w_0$ and $\Phi_1(g) = x = \Phi_2(g)$. Thus Φ satisfies (S).

<u>(AD)</u>: Consider $g, g' \in \mathcal{E}$. Then $g = x \times w_0 + y \times u, g' = x' \times w_0 + y' \times u$ and $g + g' = (x + x') \times w_0 + (y + y') \times u$ for some $x, y, x', y' \in \mathbb{R}$. It follows that $\Phi(g + g') = (x + x' + (y + y') \times u(N), x + x') = (x + y \times u(N); x) + (x' + y' \times u(N); x') = \Phi(g) + \Phi(g')$. Thus Φ satisfies (AD).

To conclude, we recall that $\mathcal{E}$ is a Shapley valid domain by assumption. This implies that $\Phi(g) = Shap(g)$. Note that $u = 0 \times w_0 + 1 \times u$. Therefore $(c, 0) = \left(a + \dfrac{c - a - b}{2}, b + \dfrac{c - a - b}{2}\right)$. Hence $c = a + \dfrac{c - a - b}{2}$ and $0 = b + \dfrac{c - a - b}{2}$. Therefore $a = b + c$ and $c = a + b$. A contradiction arises since $c \neq a + b$. In each of the possible cases, $\mathcal{E}$ is a Shapley valid domain only if $\text{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$. ■

The proof of Theorem 4.2.4 is heavily related to some specificities due to the assumption $N = \{1, 2\}$. For example, a TU-game $u \in \Gamma_N$ admits at most one symmetry class of non dummy players if and only if $u(\{1\}) = u(\{2\})$. This is no longer valid with at least three players. A characterization of all subspaces of Γ_N that are Shapley valid is thus remains an open issue. Of course, the condition $\mathrm{Span}_{\mathbb{R}}(\mathcal{E}^*) = \mathcal{E}$ is, in general, a sufficient condition; and also a necessary condition in the case of exactly two players.

In the next chapter, we revisit the Van den Brink's axiomatization of the Shapley value using a fairness condition. We essentially present a valid domain to this later characterization to solve an issue that was left open.

Chapter Five

On the Van den Brink's type of characterization results

An axiomatization of the Shapley value (see Shapley (1953)) for TU-games was provided by Van den Brink (2001) using an axiom that combines the axioms of additivity and symmetry: the so-called *fairness condition*. We refer to this characterization of the Shapley value as the VDB characterization of the Shapley value. The author also proved that the set of all $\{0, 1\}$-valued TU-games (also known as voting games) is a VDB valid domain. In a follow up paper, Casajus (2011) proves that any Peleg-Sudhölter domain presented in the previous chapter is also a VDB valid domain. Whether the set of all simple games is a VDB valid domain was abandoned due to the formulation of the fairness axiom. In this chapter, we revisit the VDB characterization to reach new characterization results mainly on the set of simple games where the initial condition of fairness no longer works.

To present the results so far obtained, this chapter comprises three sections. In Section 5.1, we present the axiom of fairness due to Van den Brink (2001). We then state the VDB characterization result together with a new proof built only on the fact that each TU-game is a linear combination of unanimity games. Still in the same section, we provide a new version of the VDB fairness condition together with some useful comparison results. Due to some technical considerations on how minimal winning coalitions of a simple game overlap, a decisiveness graph as well as some key definitions on simple games are presented. In Section 5.2, we state and prove a three-axiom characterization of the Shapley value on the set of all simple games using the new condition of fairness. This is achieved thanks to some insights on some specific symmetry relationships we provide between players in a simple game. Section 5.3 is devoted to two further fairness analyses. We first combine the axioms of null player property and symmetry into a new single axiom called the null player fairness. This leads us to another characterization of the Shapley value on Γ_N. We also consider a strengthened condition of fairness that emerges to another three-axiom charaterization of the Shapley-Shubik index the proof of which is simple and short.

5.1 On the VDB axiomatization of the Shapley value

Concepts and axioms presented in previous chapters are simply used here without any restatement. We mainly present two known characterization results stated using an axiom that combines the additivity condition and another classical axiom to obtain a three-axiom characterization of the Shapley value. In this section, we present a fairness condition due to Van den Brink (2001).

5.1.1 Van den Brink's theorem on Γ_N

Consider two TU-games and suppose from one of these two games to another, one simply adds a TU-game in which two given players are symmetric. Then the VDB fairness requirement is that the two players' payoffs should increase by the same (algebraic) amount. More formally,

Axiom 22. *Van den Brink's Fairness* (VDB-F):
For all $u, v \in \Gamma_N$ and for all $i, j \in N$,

$$\varphi_i(u+v) - \varphi_i(u) = \varphi_j(u+v) - \varphi_j(u)$$

whenever i and j are symmetric players in v.

Suppose that i and j are symmetric players in v. Then noting that $v = (u+v) - u$, it appears that i and j play symmetric roles in the changes that occur from u to $u+v$. The condition $\varphi_i(u+v) - \varphi_i(u) = \varphi_j(u+v) - \varphi_j(u)$ simply means that the shares of the two players increase or decrease by the same amount. Clearly, (VDB-F) is a fairness condition on how individual shares vary when changes occur from one game to another. The following remark has been pointed out by Van den Brink (2001).

Remark 5.1.1. *If a given value φ on a domain that contains the null game $\widetilde{0}$ satisfies (VDB-F) and (NP), then φ also satisfies Axiom (S).*

The next result is a characterization of the Shapley value where two axioms, namely Additivity (AD) and Symmetry (S), are replaced by the fairness condition we just present above.

Theorem 5.1.1 (Van den Brink (2001)).

A value φ on Γ_N satisfies (E), (NP) and (VDB-F) if and only if $\varphi = \text{Shap}$. ⋆

The original proof of Theorem 5.1.1 by Van den Brink (2001) is based on the collection of unanimity games in Γ_N and some graph concepts. Casajus (2011) proved the same result using TU-games that have positive Harsanyi dividends on Γ_N as well as on any Peleg-Sudhölter domain. We propose here a new proof of that theorem based only on unanimity games using an induction reasoning. For this purpose, we first state and prove a lemma that would help us to ease the proof.

Lemma 5.1.1. *Consider a value φ on Γ_N that satisfies (E), (NP) and (VDB-F).*

Given any TU-game $u_0 \in \Gamma_N$, if $\varphi(u_0+\alpha_S\gamma_S) = \mathrm{Shap}(u_0+\alpha_S\gamma_S)$ holds for all $S \in 2^N$ and for all $\alpha_S \in \mathbb{R}$, then for all $S, T \in 2^N$ and $\alpha_S, \alpha_T \in \mathbb{R}$, $\varphi(u_0 + \alpha_S\gamma_S + \alpha_T\gamma_T) = \mathrm{Shap}(u_0 + \alpha_S\gamma_S + \alpha_T\gamma_T)$.

Proof.

Let φ be a value on Γ_N that satisfies (E), (NP) and (VDB-F). Suppose that $u_0 \in \Gamma_N$ is a TU-game such that for all $S \in 2^N$ and for all $\alpha_S \in \mathbb{R}$, $\varphi(u_0+\alpha_S\gamma_S) = \mathrm{Shap}(u_0+\alpha_S\gamma_S)$.

Consider $S, T \in 2^N$; and $\alpha_S, \alpha_T \in \mathbb{R}$. We pose $\overline{S} = N\backslash S$, $\overline{T} = N\backslash T$,

$$u = u_0 + \alpha_S\gamma_S + \alpha_T\gamma_T,$$

$$u_{-T} = u - \alpha_T\gamma_T = u_0 + \alpha_S\gamma_S$$

and

$$u_{-S} = u - \alpha_S\gamma_S = u_0 + \alpha_T\gamma_T.$$

Since all members of T are $\alpha_T\gamma_T$-symmetric and φ satisfies Axiom (VDB-F), then there exists $x \in \mathbb{R}$ such that for all $k, l \in T$

$$x = \varphi_k(u) - \varphi_k(u_{-T}) = \varphi_l(u) - \varphi_l(u_{-T}).$$

Similarly, since members of $\overline{T} = N\backslash T$ are null players in the game $\alpha_T\gamma_T$, they are also $\alpha_T\gamma_T$-symmetric. Then there exists $y \in \mathbb{R}$ such that for all $k \in \overline{T}$,

$$y = \varphi_k(u) - \varphi_k(u_{-T}).$$

Noting that $\varphi(u_{-T}) = \mathrm{Shap}(u_{-T})$, we obtain

$$\varphi(u) = \varphi(u_{-T}) + x\mathbb{1}_T + y\mathbb{1}_{\overline{T}} = \mathrm{Shap}(u_{-T}) + x\mathbb{1}_T + y\mathbb{1}_{\overline{T}}. \tag{5.1}$$

Similarly, there exist $x', y' \in \mathbb{R}$ such that

$$\varphi(u) = \mathrm{Shap}(u_{-S}) + x'\mathbb{1}_S + y'\mathbb{1}_{\overline{S}}. \tag{5.2}$$

By observing that φ meets (E), it follows by (5.1) that $u_{-T}(N) + tx + (n-t)y = u(N)$. That is

$$tx + (n-t)y = \alpha_T. \tag{5.3}$$

Similarly, by (5.2), one obtains

$$sx' + (n-s)y' = \alpha_S. \tag{5.4}$$

Note that in the case $S\cap T\neq\emptyset$, one obtains, by combining (5.1) and (5.2), that for any player $i\in S\cap T$, $\varphi_i(u)=\text{Shap}_i(u_{-T})+x=\text{Shap}_i(u_{-S})+x'$. Since $\text{Shap}_i(u_{-T})-\text{Shap}_i(u_{-S})=\frac{\alpha_S}{s}-\frac{\alpha_T}{t}$, it follows that

$$x'-x=\frac{\alpha_S}{s}-\frac{\alpha_T}{t}. \tag{5.5}$$

In the case $S\backslash T\neq\emptyset$, one obtains, by combining (5.1) and (5.2), that for any player $j\in S\backslash T$, $\varphi_j(u)=\text{Shap}_j(u_{-T})+y=\text{Shap}_j(u_{-S})+x'$. Since $\text{Shap}_j(u_{-T})-\text{Shap}_j(u_{-S})=\frac{\alpha_S}{s}$, it follows that

$$x'-y=\frac{\alpha_S}{s}. \tag{5.6}$$

In the case $T\backslash S\neq\emptyset$, one obtains, by combining (5.1) and (5.2), that for any player $k\in T\backslash S$, $\varphi_k(u)=\text{Shap}_k(u_{-T})+x=\text{Shap}_k(u_{-S})+y'$. Since $\text{Shap}_k(u_{-T})-\text{Shap}_k(u_{-S})=-\frac{\alpha_T}{t}$, it follows that

$$y'-x=-\frac{\alpha_T}{t}. \tag{5.7}$$

We distinguish five possible cases as follows:

<u>Case 1:</u> $S\cap T\neq\emptyset$, $S\backslash T\neq\emptyset$ and $T\backslash S\neq\emptyset$. Consider $i\in S\cap T, j\in S\backslash T$ and $k\in T\backslash S$. Considering (5.3), (5.4), (5.5), (5.6) and (5.7), the following linear system arises:

$$(S_1):\begin{cases} tx+(n-t)y=\alpha_T\\ sx'+(n-s)y'=\alpha_S\\ x'-x=\frac{\alpha_S}{s}-\frac{\alpha_T}{t}\\ x'-y=\frac{\alpha_S}{s}\\ y'-x=-\frac{\alpha_T}{t}.\end{cases}$$

Summing (5.6) and (5.7), one gets $x'-y+y'-x=\frac{\alpha_S}{s}-\frac{\alpha_T}{t}$ and using equation (5.5), we have $x'-y+y'-x=x'-x$ and thus, $y'=y$. Therefore (5.7) leads us to $y=y'=x-\frac{\alpha_T}{t}$. Replacing in (5.3), it follows that $x=\frac{\alpha_T}{t}$ and from (5.7), $y=0$. Then replacing in (5.1), one finally obtains

$$\begin{aligned}\varphi(u)&=\text{Shap}(u_{-T})+\tfrac{\alpha_T}{t}\mathbb{1}_T+0*\mathbb{1}_{\overline{T}}\\ &=\text{Shap}(u_{-T})+\text{Shap}(\alpha_T\gamma_T)\\ &=\text{Shap}(u_{-T}+\alpha_T\gamma_T)\ \text{since Shap satisfies Axiom (AD)}\\ &=\text{Shap}(u).\end{aligned}$$

<u>Case 2:</u> $S\cap T\neq\emptyset$, $S\backslash T=\emptyset$ and $T\backslash S=\emptyset$. Then $S=T$ and $u=u_0+(\alpha_S+\alpha_T)\gamma_S$. It follows by assumption on u_0 that $\varphi(u)=\text{Shap}(u)$.

<u>Case 3:</u> $S\cap T\neq\emptyset$, $S\backslash T\neq\emptyset$ and $T\backslash S=\emptyset$. Then (5.3), (5.4), (5.5) and (5.6) hold and lead us to the following linear system:

$$(S_3):\begin{cases} tx+(n-t)y=\alpha_T\\ sx'+(n-s)y'=\alpha_S\\ x'-x=\frac{\alpha_S}{s}-\frac{\alpha_T}{t}\\ x'-y=\frac{\alpha_S}{s}.\end{cases}$$

Substracting (5.5) from (5.6) leads to $y = x - \frac{\alpha_T}{t}$. By replacing y in equation (5.3), one obtains $x = \frac{\alpha_T}{t}$. Finally, $x = \frac{\alpha_T}{t}$ and $y = 0$. Thus $\varphi(u) = \mathrm{Shap}(u)$ as in Case 1.

Case 4: $S \cap T \neq \emptyset$, $S \backslash T = \emptyset$ and $T \backslash S \neq \emptyset$. Then (5.3), (5.4), (5.5) and (5.7) hold and lead us to the following linear system:

$$(S_4)\begin{cases} tx + (n-t)y = \alpha_T \\ sx' + (n-s)y' = \alpha_S \\ x' - x = \frac{\alpha_S}{s} - \frac{\alpha_T}{t} \\ y' - x = -\frac{\alpha_T}{t}. \end{cases}$$

Substracting equation (5.5) from equation (5.7) leads to $y' = x' - \frac{\alpha_S}{s}$. By replacing y' in equation (5.4), one obtains $x' = \frac{\alpha_S}{s}$. Finally, $x' = \frac{\alpha_S}{s}$ and $y' = 0$. Thus $\varphi(u) = \mathrm{Shap}(u)$.

Case 5 : $S \cap T = \emptyset$. Then $S \subseteq \overline{T}$ and $T \subseteq \overline{S}$. Therefore $S \backslash T = S \neq \emptyset$ and $T \backslash S = T \neq \emptyset$. Considering $j \in S \backslash T$ and $k \in T \backslash S$, (5.3), (5.4), (5.6) and (5.7) hold and lead to the following linear system:

$$(S_2)\begin{cases} tx + (n-t)y = \alpha_T \\ sx' + (n-s)y' = \alpha_S \\ x' - y = \frac{\alpha_S}{s} \\ y' - x = -\frac{\alpha_T}{t}. \end{cases}$$

By (5.6) and (5.7), one gets $x' = y + \frac{\alpha_S}{s}$ and $y' = x - \frac{\alpha_T}{t}$. Replacing (5.4), one obtains

$$(n-s)x + sy = \frac{n-s}{t}\alpha_T. \tag{5.8}$$

By (5.3), $x = \frac{\alpha_T - (n-t)y}{t}$ and replacing in (5.8), leads to $n(s+t-n)y = 0$. If $n \neq s+t$, then $y = 0$ and $x = \frac{\alpha_T}{t}$. It follows that

$$\begin{aligned} \varphi(u) &= \mathrm{Shap}(u_{-T}) + \tfrac{\alpha_T}{t}\mathbb{1}_T + 0 * \mathbb{1}_{\overline{T}} \\ &= \mathrm{Shap}(u_{-T}) + \mathrm{Shap}(\alpha_T\gamma_T) \\ &= \mathrm{Shap}(u_{-T} + \alpha_T\gamma_T) \text{ since Shap satisfies Axiom (AD)} \\ &= \mathrm{Shap}(u). \end{aligned}$$

Now, suppose that $n = s+t$. Then $S = \overline{T}$ and $T = \overline{S}$. We pose $L = S\backslash\{j\}$ and $w = u_0 + \alpha_T\gamma_T - \alpha_S\gamma_{L+k}$ where $j \in S$ and $k \in T$. Then the TU-game u satisfies:

$$u = \alpha_S(\gamma_{L+j} + \gamma_{L+k}) + w.$$

Note that $\varphi(u_0 + \alpha_T\gamma_T) = \mathrm{Shap}(u_0 + \alpha_T\gamma_T)$ and $\varphi(u_0 - \alpha_S\gamma_{L+k}) = \mathrm{Shap}(u_0 - \alpha_S\gamma_{L+k})$ by assumption on u_0. Since $T \cap (L+k) = \{k\} \neq \emptyset$, it follows from cases 1, 2, 3 and 4 that $\varphi(w) = \mathrm{Shap}(w)$. Since j and k are $\alpha_S(\gamma_{L+j} + \gamma_{L+k})$-symmetric and φ satisfies (VDB-F),

$$\varphi_j(u) - \varphi_j(w) = \varphi_k(u) - \varphi_k(w).$$

That is

$$y + \mathrm{Shap}_j(u_{-T}) - \varphi_j(w) = x + \mathrm{Shap}_k(u_{-T}) - \varphi_k(w). \tag{5.9}$$

Moreover $\mathrm{Shap}_j(u_{-T}) = \mathrm{Shap}_j(u_0) + \frac{\alpha_S}{s}$, $\mathrm{Shap}_j(w) = \mathrm{Shap}_j(u_0)$, $\mathrm{Shap}_k(u_{-T}) = \mathrm{Shap}_k(u_0)$ and $\mathrm{Shap}_k(w) = \mathrm{Shap}_k(u_0) + \frac{\alpha_T}{t} - \frac{\alpha_S}{s}$. Replacing in (5.9), one obtains

$$y = x - \frac{\alpha_T}{t}. \tag{5.10}$$

Replacing y in (5.3), it follows that $x = \frac{\alpha_T}{t}$ and replacing x in (5.10), it follows that $y = 0$. Therefore

$$\begin{aligned} \varphi(u) &= \mathrm{Shap}(u_{-T}) + \tfrac{\alpha_T}{t}\mathbb{1}_T + 0 * \mathbb{1}_{\overline{T}} \\ &= \mathrm{Shap}(u_{-T}) + \mathrm{Shap}(\alpha_T\gamma_T) \\ &= \mathrm{Shap}(u_{-T} + \alpha_T\gamma_T) \text{ since Shap satisfies Axiom (AD)} \\ &= \mathrm{Shap}(u). \end{aligned}$$

In each of the five possible cases, $\varphi(u) = \mathrm{Shap}(u)$. ■

We now prove Theorem 5.1.1.

Proof.

$\Longleftarrow$) It is well-known that the Shapley value satisfies (E) and (NP). Now consider two games $u, v \in \Gamma_N$ and two players $i, j \in N$ such that i and j are v-symmetric. Then

$$\begin{aligned} \mathrm{Shap}_i(v+u) - Shap_i(u) &= \mathrm{Shap}_i(v) + \mathrm{Shap}_i(u) - Shap_i(u) \text{ by (AD)} \\ &= \mathrm{Shap}_i(v) \\ &= \mathrm{Shap}_j(v) \text{ by (S)} \\ &= \mathrm{Shap}_j(v) + \mathrm{Shap}_j(u) - \mathrm{Shap}_j(u) \\ &= \mathrm{Shap}_j(v+u) - \mathrm{Shap}_j(u) \text{ by AD)}. \end{aligned}$$

Thus, the Shapley value satisfies Axiom (VDB-F).

$\Longrightarrow$) Suppose that φ is a value on Γ_N that satisfies (E), (NP) and (VDB-F). We show that $\varphi = \mathrm{Shap}$.

Suppose that $n = 2$. Pose $N = \{1, 2\}$, consider $u \in \Gamma_N$ and define the TU-games u_1 and u_2 as shown in the following table:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$u(S)$	0	a	b	c
$u_1(S)$	0	a	a	$a+c-b$
$u_2(S)$	0	0	$b-a$	$b-a$

Since player 1 is a null player in u_2 and φ satisfies (NP), one obtains $\varphi_1(u_2) = 0$. The value φ also satisfies (E). Then $\varphi_2(u_2) = b - a - \varphi_1(u_2) = b - a$. Noting that $u = u_1 + u_2$ and that players 1 and 2 are u_1-symmetric. Then by (VDB-F), it holds that $\varphi_1(u_1 + u_2) - \varphi_1(u_2) = \varphi_2(u_1 + u_2) - \varphi_2(u_2)$. That is $\varphi_1(u) - \varphi_1(u_2) = \varphi_2(u) - \varphi_2(u_2)$. Moreover, φ satisfies (E), one obtains $\varphi_1(u) + \varphi_2(u) = u(\{1, 2\}) = c$. Then the following linear system arises:

$$(S_5) : \begin{cases} \varphi_1(u) - \varphi_2(u) = a - b \\ \varphi_1(u) + \varphi_2(u) = c. \end{cases}$$

Solving (S_5) yields $\varphi(u) = (\varphi_1(u), \varphi_2(u)) = (\frac{c+a-b}{2}, \frac{c+b-a}{2}) = \mathrm{Shap}(u)$.

Now suppose that $n \geq 3$. Given an integer $m \geq 1$, we pose

$$E_m = \left\{ u \in \Gamma_N / u = \sum_{T \in E} \alpha_T \gamma_T \text{ for some } E \subseteq 2^N \text{ and } (\alpha_T)_{T \in E} \subseteq \mathbb{R} \text{ such that } |E| \leq m. \right\}$$

We prove by induction on $m \in \{1, 2, \ldots, 2^n - 1\}$ that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_m$.

Initialization: Consider $u \in E_1$. Then $u = \alpha\gamma_T$ for some $T \in 2^N$ and $\alpha \in \mathbb{R}$. Any player $i \in N \backslash T$ is a null player in u and thus, $\varphi_i(u) = 0 = \mathrm{Shap}_i(u)$ since φ satisfies (NP). Moreover, players in T are symmetric players in u. Since φ satisfies (VDB-F) and (NP), it follows by Remark 5.1.1 that φ also satisfies (S) since $\widetilde{0} \in \Gamma_N$. It then follows that $\varphi_i(u) = \frac{\alpha}{|T|} = \mathrm{Shap}_i(u)$. Thus $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_1$.

Induction step: Consider an integer $m \in \{1, 2, \ldots, 2^n - 2\}$ and assume that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_m$. We show that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_{m+1}$. By induction assumption, it is sufficient to prove that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_{m+1}$ such that there exists $T_1, T_2, \cdots, T_{m+1} \in 2^N$ and $\alpha_1, \alpha_2, \cdots, \alpha_{m+1} \in \mathbb{R}$ such that:

$$u = \sum_{l=1}^{m+1} \alpha_l \gamma_{T_l} = v + \alpha_{m+1}\gamma_{T_{m+1}} \text{ for } v = \sum_{l=1}^{m} \alpha_l \gamma_{T_l}.$$

Let $u \in E_{m+1}$ be one such game.

If $m = 1$, then $v = \alpha_1\gamma_{T_1} + \widetilde{0}$ and since for all $T \in 2^N$ and $\alpha_T \in \mathbb{R}$, $\widetilde{0} + \alpha_T\gamma_T \in E_m$, it follows by the induction assumption that $\varphi(\widetilde{0} + \alpha_S\gamma_S) = \mathrm{Shap}(\widetilde{0} + \alpha_S\gamma_S)$. Finally, from Lemma 5.1.1, $\varphi(\widetilde{0} + \alpha_1\gamma_{T_1} + \alpha_2\gamma_{T_2}) = \mathrm{Shap}(\widetilde{0} + \alpha_1\gamma_{T_1} + \alpha_2\gamma_{T_2})$. That is $\varphi(u) = \mathrm{Shap}(u)$.

Now, suppose that $m \geq 2$. Then $v = u_0 + \alpha_m\gamma_{T_m}$ with $u_0 = \sum_{l=1}^{m-1} \alpha_l\gamma_{T_l}$. Since for all $T \in 2^N$ and $\alpha_T \in \mathbb{R}$, $u_0 + \alpha_T\gamma_T \in E_m$, it follows by the induction assumption that $\varphi(u_0 + \alpha_S\gamma_S) = \mathrm{Shap}(u_0 + \alpha_S\gamma_S)$. From Lemma 5.1.1, it follows that $\varphi(u_0 + \alpha_m\gamma_{T_m} + \alpha_{m+1}\gamma_{T_{m+1}}) = \mathrm{Shap}(u_0 + \alpha_m\gamma_{T_m} + \alpha_{m+1}\gamma_{T_{m+1}})$.

Conclusion: $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_{2^n-1} = \Gamma_N$. ∎

Van den Brink (2001) proved that Theorem 5.1.1 still holds on the set of all voting games. In other words, the set of all voting games is a valid domain for the characterization result $(\mathcal{L}_f, \mathrm{Shap})$ for $\mathcal{L}_f = \{(E), (NP), (VDB-F)\}$. Casajus (2011) extended the scope of valid domains to the same result to cover all convex cones that contain all the unanimity games (known as Peleg-Sudhölter domain). Conically consistent domains we introduce in the previous chapter are potential $(\mathcal{L}_f, \mathrm{Shap})$ valid domains. But we omit investigation in that direction to focus on the validity of $(\mathcal{L}_f, \mathrm{Shap})$ on the set of simple games. Of course, the importance of the Shapley value for simple games (called Shapley-Shubik index) is unquestionable after Shapley and Shubik (1954) made it a suitable measure of voting power. It is therefore of great interest to give the Shapley-Shubik index a characterization in terms of fairness. This is the subject of the next section.

5.1.2 Revisiting Van den Brink's fairness

We recall that $\mathcal{G}_N$ is the set of all simple games with the same set $N = \{1, ..., n\}$ of n players. Hereafter, the restriction of the Shapley value on $\mathcal{G}_N$ is called the *Shapley-Shubik index* and is denoted SS. Our objective is to give the Shapley-Shubik index a characterization that is somewhat a version of Theorem 5.1.1 when only simple games are admissible games. The main difficulty is that adding two simple games is no more a feasible operation since the sum of any two simple games is never a simple game. A way out from this is due to Dubey (1975) who proposed the transfer property as a substitute of the additivity property to characterize the Shapley-Shubik index. Our solution here consists in an equivalent version of (VDB-F) that better fits possible restrictions to any nonempty subset $\mathcal{K}$ of Γ_N; that is the set of admissible TU-games.

Axiom 23. *Fairness* on $\mathcal{K}$ (F): *For all $u, v \in \mathcal{K}$, for all players $i, j \in N$ such that i and j are symmetric in $v - u$,*

$$\varphi_j(v) - \varphi_j(u) = \varphi_i(v) - \varphi_i(u).$$

Note that in the above axiom, $v - u$ is seen as an element of Γ_N rather than an element of $\mathcal{K}$ since $\mathcal{K}$ may not be closed under the difference of two games (as it is, for instance, the case for $\mathcal{K} = \mathcal{G}_N$). The TU-game $v - u$ is seen as a measure of changes that occurs from u to v. In other words, the fairness Axiom (F) is the disposition that whenever two players play symmetric roles for changes that occur from one game to another, the shares of these two players should increase or decrease by the same amount. This is exactly the main intuition in (VDB-F). Of course, the small and inessential difference is that it is necessary in (VDB-F) that the difference which measures the changes in the two games is also an admissible game. More precisely, the restriction of (VDB-F) on a nonempty subset of $\mathcal{K}$ of Γ_N suggested by Van den Brink (2001) himself is as follows:

Axiom 24. *(VDB-F)* on $\mathcal{K}$: *For all $u, v \in \mathcal{K}$, for all players $i, j \in N$ such that i and j are symmetric in v,*

$$\varphi_j(u + v) - \varphi_j(u) = \varphi_i(u + v) - \varphi_i(u)$$

whenever $u + v \in \mathcal{K}$.

Definition 5.1.1. *A nonempty subset $\mathcal{K}$ of Γ_N is closed under difference if $v - u \in \mathcal{K}$ for all $u, v \in \mathcal{K}$.*

This formulation contains an inessential requirement. That is the fact that $v \in \mathcal{K}$ since the game v is simply a measure of how players are involved in the changes that occur from u to $u+v$. Below, we present some existing relationships between (VDB-F) and (F).

Proposition 5.1.1. *Consider a nonempty subset $\mathcal{K}$ of Γ_N.*

(i) Any value that satisfies (F) on $\mathcal{K}$ also satisfies (VDB-F) on $\mathcal{K}$.

(ii) If $\mathcal{K}$ is closed under the difference, then any value that satisfies (VDB-F) on $\mathcal{K}$ also satisfies (F) on $\mathcal{K}$.

Proof.

Let $\mathcal{K}$ be a nonempty subset of Γ_N.

First suppose that a value φ satisfies (F) on $\mathcal{K}$. Consider two TU-games $u, v \in \mathcal{K}$ such that $u+v \in \mathcal{K}$ and two players $i, j \in N$ such that i and j are symmetric players in v. Noting $v = (u+v) - u$, it follows by (F) that $\varphi_j(u+v) - \varphi_j(u) = \varphi_i(u+v) - \varphi_i(u)$. Therefore, φ satisfies (VDB-F) on $\mathcal{K}$.

Now, suppose that $\mathcal{K}$ is closed under the difference. Let φ be a value that satisfies (VDB-F) on $\mathcal{K}$. Consider two TU-games $u, v \in \mathcal{K}$ and two players $i, j \in N$ such that i and j are symmetric players in $v-u$. Noting $v-u \in \mathcal{K}$ and $v = u + (v-u)$, it follows by (VDB-F) that $\varphi_j(v) - \varphi_j(u) = \varphi_i(v) - \varphi_i(u)$. Therefore, φ satisfies (F) on $\mathcal{K}$. ■

As underlined in Proposition 5.1.1, (F) implies (VDB-F) on any nonempty subset of Γ_N. It then appears that (F) is a strengthening version of (VDB-F). Furthermore, (F) and (VDB-F) are equivalent on any nonempty subset of Γ_N which is closed under difference. In particular, (F) and (VDB-F) are equivalent on Γ_N and on any subspace of Γ_N. The following result is thus straightforward.

Theorem 5.1.2 (Van den Brink (2001) reformulated)**.**

A value φ on Γ_N satisfies (E), (NP) and (F) if and only if $\varphi = \text{Shap}$. ★

Van den Brink (2001) (see Proposition 2.4., p. 311) proved that (NP) and (VDB-F) imply (S) on Γ_N. Since (VDB-F) and (F) are equivalent on Γ_N, it also holds that (NP) and (F) imply (S) on Γ_N. When we replace Γ_N by $\mathcal{G}_N$, this later result still holds for $n \geq 3$; but no longer for $n = 2$. This is developed in the next proposition and remark.

Proposition 5.1.2. *For $n \geq 3$, (F) and (NP) implies (S) on $\mathcal{G}_N$.*

Proof.

Assume $n \geq 3$. Let φ be a value on $\mathcal{G}_N$ that meets (F) and (NP). Consider $i, j \in N$ and $v \in \mathcal{G}_N$ such that i and j are v-symmetric. Note that for $k \in N \backslash \{i, j\}$, i and j are symmetric players in $v - \gamma_{\{k\}}$. Therefore, (F) implies $\varphi_i(v) - \varphi_i(\gamma_{\{k\}}) = \varphi_j(v) - \varphi_j(\gamma_{\{k\}})$. Since both i and j are null players in $\gamma_{\{k\}}$, then $\varphi_i(\gamma_{\{k\}}) = \varphi_j(\gamma_{\{k\}}) = 0$ by (NP). Hence $\varphi_i(v) = \varphi_j(v)$. Thus φ satisfies (S). ■

Remark 5.1.2. For $n = 2$, $N = \{1, 2\}$ and $\mathcal{G}_N = \{v_1, v_2, v_3, v_4\}$ where $v_1 = \gamma_{\{1\}}$, $v_2 = \gamma_{\{2\}}$, $v_3 = \gamma_{\{1\}} \vee \gamma_{\{2\}}$ and $v_4 = \gamma_{\{1,2\}}$; see the following tabular:

$v_l \backslash S$	$\{1\}$	$\{2\}$	$\{1,2\}$
v_1	1	0	1
v_2	0	1	1
v_3	1	1	1
v_4	0	0	1

Consider a value φ on $\mathcal{G}_N$ that satisfies (E), (NP) and (F). It follows by (E) and (NP) that $\varphi(v_1) = (1, 0) = \mathrm{SS}(v_1)$ and $\varphi(v_2) = (0, 1) = \mathrm{SS}(v_2)$. Set $\varphi(v_3) = (x, y)$ and $\varphi(v_4) = (x', y')$. Note that players 1 and 2 are symmetric in $v_3 - v_4$, it follows by (F) that $x - x' = y - y'$. Since $x + y = x' + y' = 1$ by Axiom (E), we obtain $x = x'$ and $y = y'$. Players 1 and 2 are neither symmetric in $v_2 - v_1$, nor in $v_3 - v_1$, nor in $v_4 - v_1$, nor in $v_3 - v_2$, nor in $v_4 - v_2$. Thus all value φ on $\mathcal{G}_N$ satisfies (E), (NP) and (F) if and only if $\varphi(v_1) = (1, 0) = \mathrm{SS}(v_1)$, $\varphi(v_2) = (0, 1) = \mathrm{SS}(v_2)$ and $\varphi(v_3) = (x, 1 - x) = \varphi(v_4)$ for some $x \in \mathbb{R}$. For $x = \frac{1}{3}$, one obtains $\varphi(v_3) = (\frac{1}{3}, \frac{2}{3}) \neq \mathrm{SS}(v_3)$,$\varphi_1(v_3) \neq \varphi_2(v_3)$ meanwhile 1 and 2 are symmetric players in v_3. That is φ satisfies (F) and (NP); but not (S).

To continue, we need some further definitions and notations.

Definition 5.1.2. *Given $v, w \in \mathcal{G}_N$, the games $v \wedge w$ and $v \vee w$ are defined as follows:*

$$\forall S \subseteq N, (v \wedge w)(S) = \min\{v(S), w(S)\} \text{ and } (v \vee w)(S) = \max\{v(S), w(S)\}. \tag{5.11}$$

A given coalition is winning in $v \wedge w$ if it is winning in both v and w, meanwhile for a coalition to be winning in $v \vee w$, it should be winning at least in v or in w. The set $\mathcal{G}_N$ of simple games is closed under the operators $\vee$ and $\wedge$, that is $v \wedge w, v \vee w \in \mathcal{G}_N$ for all $v, w \in \mathcal{G}_N$.

It can be easily checked that

$$u \vee v + u \wedge v = u + v \text{ and } u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w) \tag{5.12}$$

and that for all coalitions S and T,

$$\gamma_S \wedge \gamma_T = \gamma_{S \cup T}. \tag{5.13}$$

Furthermore, for any $v \in \mathcal{G}_N$,

$$v = \bigvee_{T \in \mathcal{M}(v)} \gamma_T. \tag{5.14}$$

The above mentioned properties of $\vee$ and $\wedge$ were of crucial importance for Dubey (1975) who provided a characterization of the Shapley-Shubik index. They are also useful in the sequel.

Notation 5.1.1. *Assuming that v is a simple game, $S \in \mathcal{M}(v)$ and $|\mathcal{M}(v)| \geq 2$, the games v^{-S} and v^{+S} are, respectively, obtained by*

$$v^{-S} = \bigvee_{T \in \mathcal{M}(v) \setminus \{S\}} \gamma_T \text{ and } v^{+S} = \gamma_S \wedge v^{-S} = \bigvee_{T \in \mathcal{M}(v) \setminus \{S\}} \gamma_{T \cup S}.$$

From v, the game v^{-S} is obtained by cancelling S among minimal winning coalitions. To be winning in v^{+S}, a coalition should simultaneously comprise all members of S and all members of at least one other minimal winning coalition in v. The last equation holds thanks to the distributivity of $\wedge$ and $\vee$ in the lattice $\mathcal{G}_N$.

To apply fairness disposals, one needs some specific relationships between simple games that exhibit symmetric roles between players. On this issue, the following decomposition of a simple game is very often used to link a given simple game with other simple games with fewer minimal winning coalitions.

Proposition 5.1.3. *For any simple game v and for any $S \in \mathcal{M}(v)$ such that $|\mathcal{M}(v)| \geq 2$,*

$$v = \gamma_S + v^{-S} - v^{+S}. \tag{5.15}$$

Moreover, players in S are symmetric players in $v - v^{-S}$.

Proof. Equation (5.15) is straightforward from (5.12) and (5.13) by noting that $v = \gamma_S \vee v^{-S}$. Now, players in S are symmetric players in γ_S as well as in $\gamma_{T \cup S}$ for all $T \in \mathcal{M}(v) \setminus \{S\}$. Therefore, players in S are symmetric players in $\gamma_S - v^{+S}$, and thus in $v - v^{-S}$ since $v - v^{-S} = \gamma_S - v^{+S}$ by (5.15). ■

The preceding decomposition of a simple game appears in the proof of several results; see for example Dubey and Shapley (1979) (proof of Theorem 1), Weber (1988) (proof of Theorem 7) or Peleg and Sudhölter (2007) (proof of Theorem 8.6.4), among others. Moreover, the explicit expression of v^{+S} in terms of a conjunction of unanimity games is also reported by Peleg and Sudhölter (2007).

5.1.3 A decisiveness graph of a simple game

In a simple game v, a player i is decisive in a coalition S if S is winning while $S\backslash\{i\}$ is losing. One way to describe the decisiveness structure in a simple game consists in associating to this game a graph whose edges coincide with pairs of winning coalitions having common decisive players. Such a graph may be built using only minimal winning coalitions since a player i is decisive in a coalition S if and only if i is a member of a minimal winning coalition contained in S, see Safokem et al. (2021). We opt for the following definition of a *decisiveness graph* [1] associated to a simple game.

Definition 5.1.3. *Given $v \in \mathcal{G}_N$, the decisiveness graph of v is the ordered pair $G_v = (\mathcal{M}(v), E_v)$ where E_v is defined by*

$$E_v = \{\{S,T\} \subseteq \mathcal{M}(v) : S \cap T \neq \emptyset\} \cup \{\{S\} \subseteq \mathcal{M}(v) : S \cap T = \emptyset \text{ for all } T \in \mathcal{M}(v)\backslash\{S\}\}.$$

In G_v, $\mathcal{M}(v)$ is the set of vertices and E_v contains all edges of G_v; and any loop corresponds to a minimal winning coalition which overlaps with no other.

Definition 5.1.4. *Given $v \in \mathcal{G}_N$ and two minimal winning coalitions S and T, a path from S to T is any sequence $p = (S_1, \ldots, S_k)$ of minimal winning coalitions such that $S_1 = S$, $S_k = T$ and $\{S_t, S_{t+1}\} \in E_v$ for all $t \in \{1, 2, \ldots, k-1\}$. In this case, the length of p is $k-1$ and we write $l(p) = k-1$.*

A path is simple if it contains only distinct coalitions.

Definition 5.1.5. *A connected set in the decisiveness graph G_v of a game $v \in \mathcal{G}_N$ is any set $\mathcal{C} \subseteq \mathcal{M}(v)$ such that $|\mathcal{C}| = 1$; or $|\mathcal{C}| \geq 2$ and for all pairs $\{S,T\} \subseteq \mathcal{C}$, there is a path in G_v from S to T.*

In particular, a component of G_v is any maximal connected set $\mathcal{C}$ in G_v; $S \in \mathcal{C}$ and $T \in \mathcal{M}(v)\backslash\mathcal{C}$. The total number of components of G_v is denoted by $c(v)$ and the set of all components by $\mathcal{C}(v)$.

Example 5.1.1. Let $N = \{1,2,3,4,5,6,7,8,9,10\}$ and $v \in \mathcal{G}_N$ be a simple game such that $\mathcal{M}(v) = \{\{10\},\{1,2\},\{3,5\},\{7,8\},\{8,9\},\{2,3,4\},\{3,4,6\}\}$. The graph of the game, see Figure 5.1, has three components. That is $\mathcal{C}(v) = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ with $\mathcal{C}_1 = \{\{1,2\},\{3,5\},\{2,3,4\}\{3,4,6\}\}$, $\mathcal{C}_2 = \{\{7,8\},\{8,9\}\}$ and $\mathcal{C}_3 = \{\{10\}\}$.

Definition 5.1.6. *A peripheral coalition in a component $\mathcal{C}$ of the graph G_v of a game $v \in \mathcal{G}_N$ is any coalition $S \in \mathcal{C}$ such that $\mathcal{C} = \{S\}$, or $\mathcal{C} \neq \{S\}$ and removing S from $\mathcal{C}$ again gives a connected subset of coalitions in G_v.*

[1] A similar but distinct graph is introduced and used by Van den Brink who defines edges between players from coalitions with non null Harsanyi dividends instead of edges between minimal winning coalitions presented here.

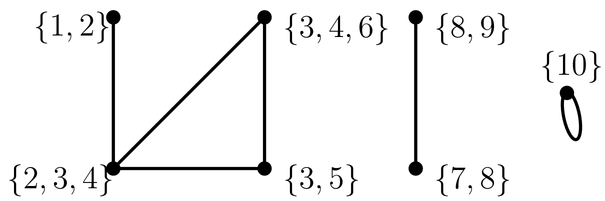

Figure 5.1: Decisiveness graph with $\mathcal{M}(v) = \{\{10\}, \{1,2\}, \{3,5\}, \{7,8\}, \{8,9\}, \{2,3,4\}, \{3,4,6\}\}$

Note that removing a peripheral coalition reduces by one the total number of components or simply leaves this number unchanged.

Example 5.1.2. *In Example 5.1.1, coalitions in $\mathcal{C}_1$ are all peripheral except $\{2,3,4\}$.*

In the next proposition, it is established that the set of peripheral coalitions in a given consistent component of a simple game is always nonempty.

Proposition 5.1.4. *Given $v \in \mathcal{G}_N$, any component of v admits at least a peripheral coalition.*

Proof. Consider $v \in \mathcal{G}_N$ and let $\mathcal{C}$ be a component of G_v. First note that if $\mathcal{C} = \{S\}$ for some $S \in \mathcal{M}(v)$, then S is, by definition, a peripheral coalition in $\mathcal{C}$. Next, suppose that $\mathcal{C} = \{S, T\}$ for some $S, T \in \mathcal{M}(v)$, then both S and T are peripheral coalitions in $\mathcal{C}$ by definition. From now on, $|\mathcal{C}| \geq 3$. Let $\mathcal{P}(\mathcal{C})$ be the set of all simple paths $p = (T_1, T_2, \ldots, T_k)$ such that $T_h \in \mathcal{C}$ for all $h \in \{1, 2, \ldots, k\}$ and denote by $\mathcal{P}^*(\mathcal{C})$ the set of all paths in $\mathcal{P}(\mathcal{C})$ of maximal length. The set $\mathcal{P}^*(\mathcal{C})$ is nonempty since $\mathcal{M}(v)$ is a finite set. Consider $p = (T_1, T_2, \ldots, T_k) \in \mathcal{P}^*(\mathcal{C})$.

To prove that $S = T_1$ is peripheral in $\mathcal{C}$, suppose on the contrary that it is not. Then there exists $\{K, L\} \subseteq \mathcal{C} \backslash \{S\}$ with no path in $G_{v^{-S}}$ from K to L. Since $\mathcal{C}$ is a component in G_v, then there exists a simple path $(S_1, S_2, \ldots, S_g, S, S_{g+1}, \ldots, S_h)$ in v from K to L (which necessarily includes S). Let $\mathcal{K} = \cup_{1 \leq t \leq g} S_t$ and $\mathcal{L} = \cup_{g+1 \leq t \leq h} S_t$. If $S_g \notin \{T_2, T_3, \ldots, T_k\}$, then $p' = (S_g, S, T_2, \ldots, T_k) \in \mathcal{P}(\mathcal{C})$ and $l(p') = l(p) + 1$. A contradiction arises since $p \in \mathcal{P}^*(\mathcal{C})$. Thus, there exists $x \in \{2, 3, \ldots, k\}$ such that $S_g = T_x$. Similarly, there exists $y \in \{2, 3, \ldots, k\}$ such that $S_{g+1} = T_y$. There are two possible cases. If $x < y$, then $(S_1, S_2, \ldots, S_g, T_{x+1}, \ldots, T_{y-1}, S_{g+1}, \ldots, S_k)$ is a path in $G_{v^{-S}}$ from K to L. Similarly, if $x > y$, then $(S_h, S_{h-1}, \ldots, S_{g+2}, T_y, T_{y-1}, \ldots, T_{x+1}, S_g, S_{g-1}, \ldots, S_1)$ is a path in $G_{v^{-S}}$ from L to K. In both cases, a contradiction arises since, by assumption, there is no path in v^{-S} from K to L.

■

We have the following properties of the components of the decisiveness graph of a simple game.

Proposition 5.1.5. *Consider $v \in \mathcal{G}_N$ and suppose that G_v admits a component $\mathcal{C}$ such that $|\mathcal{M}(v)\backslash\mathcal{C}| \geq 2$. Then there exist $\{R, S\} \subseteq \mathcal{M}(v)\backslash\mathcal{C}$ and $i \in S\backslash R$ such that $R' \cup \{i\}$ is losing for all proper subset R' of R or $R' \cup S\backslash\{i\}$ is losing for all proper subset R' of R.*

Proof.
Consider a game $v \in \mathcal{G}_N$ and suppose that the graph G_v admits a component $\mathcal{C}$ such that $|\mathcal{M}(v)\backslash\mathcal{C}| \geq 2$. Consider $\{R, L\} \subseteq \mathcal{M}(v)\backslash\mathcal{C}$ and $i \in L\backslash R$. First suppose $R' \cup \{i\}$ is losing in v for all proper subset R' of R. The result follows by taking $S = L$. To complete the proof, suppose that $R' \cup \{i\}$ is winning in v for some proper subset R' of R. It follows that $R'' \cup \{i\} \in \mathcal{M}(v)$ for some proper subset R'' of R. Let $S = R'' \cup \{i\}$. Then $R \cup S\backslash\{i\} = R$. Since $R \in \mathcal{M}(v)$, it follows that $R' \cup S\backslash\{i\}$ is losing for all proper subset R' of R.

■

5.2 Van den Brink's theorem on $\mathcal{G}_N$

In the sequel, we make use of the following notation:

Notation 5.2.1. *Given $v \in \mathcal{G}_N$ and $\mathcal{F} \subseteq 2^N$, we pose*

$$m(v) = |\mathcal{M}(v)|, c(v) = |\mathcal{C}(v)| \text{ and } s(v) = min\{|S| : S \in \mathcal{M}(v)\}.$$

In words, $m(v)$ is the total number of minimal winning coalitions in v, $c(v)$ is the total number of components in G_v and $s(v)$ is the smallest size of a minimal winning coalition.

Furthermore, given $\mathcal{F} \subseteq 2^N$ and an integer $p \geq 1$, we set

$$\cup\mathcal{F} = \cup_{S\in\mathcal{F}} S \text{ and } \mathcal{G}_{N,p} = \{v \in \mathcal{G}_N : m(v) = p\}.$$

Clearly, $\cup\mathcal{F}$ is the union of all coalitions in the collection $\mathcal{F}$. In particular, $\cup\mathcal{M}(v) = \cup_{S\in\mathcal{M}(v)} S$ is the set of all players who belong each to at least one minimal winning coalition of the game $v \in \mathcal{G}_N$. The set $\mathcal{G}_{N,p}$ consists of all simple games on N with exactly p minimal winning coalitions.

5.2.1 Preliminary results

Before a formal statement of the Van den Brink's theorem on $\mathcal{G}_N$, we first state and prove some preliminary results we later use to ease the presentation of the main proof.

Proposition 5.2.1. *Consider two efficient values φ and ψ on $\mathcal{G}_N$, $v \in \mathcal{G}_N$, a coalition S and $k \in N$.*

If $\varphi_k(v) - \varphi_l(v) = \psi_k(v) - \psi_l(v)$ for all $l \in S$ and $\varphi_l(v) = \psi_l(v)$ for all $l \in N \backslash S$, then $\varphi_l(v) = \psi_l(v)$ for all $l \in N$.

Proof. Consider two efficient values φ and ψ on $\mathcal{G}_N$, $v \in \mathcal{G}_N$, a coalition S and $k \in N$ such that $\varphi_k(v) - \varphi_l(v) = \psi_k(v) - \psi_l(v)$ for all $l \in S$ and $\varphi_l(v) = \psi_l(v)$ for all $l \in N \backslash S$. Then, summing all these equalities over $l \in S$ yields, by efficiency,

$$|S|\varphi_k(v) - \left(1 - \sum_{l \in N \backslash S} \varphi_l(v)\right) = |S|\psi_k(v) - \left(1 - \sum_{l \in N \backslash S} \psi_l(v)\right).$$

Therefore, $\varphi_k(v) = \psi_k(v)$ since $\varphi_l(v) = \psi_l(v)$ for all $l \in N \backslash S$. The result follows by considering this latter equality in each of the $|S|$ previous others associated to players in S.

■

Proposition 5.2.1 still holds on any nonempty subset $\mathcal{K} \subseteq \Gamma_N$. We simply state it here on $\mathcal{G}_N$ to share the same scope with other subsequent results.

Remark 5.2.1. *Let φ and ψ be two efficient values on $\mathcal{G}_N$, $v \in \mathcal{G}_N$ and $k \in \cup\mathcal{M}(v)$. Further assume that both φ and ψ satisfy (NP). It is an immediate consequence of Proposition 5.2.1 that $\varphi_l(v) = \psi_l(v)$ for $l \in N$ whenever $\varphi_k(v) - \varphi_l(v) = \psi_k(v) - \psi_l(v)$ for all $l \in \cup\mathcal{M}(v)$.*

The next result highlights a symmetry relationship that links a dictator and other players in a game. This is of course useful to scrutinize how a power index that satisfies (F) varies from some games to other.

Proposition 5.2.2. *Assume that a simple game v admits a dictator i and a minimal winning coalition S such that $|S| \geq 2$. Then for any member j of S, i and j are symmetric players in $v \vee \gamma_{S\backslash\{j\}} - v$. Moreover, $\mathcal{M}(v \vee \gamma_{S\backslash\{j\}}) \subseteq \{S\backslash\{j\}\} \cup (\mathcal{M}(v)\backslash\{S\})$.*

Proof. Let v be a simple game that admits a dictator i and a minimal winning coalition S such that $|S| \geq 2$. Consider $j \in S$ and $K \subseteq N\backslash\{i, j\}$. Since i is a dictator in v, $\left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K \cup \{i\}) = 0$. To evaluate $\left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K \cup \{j\})$, two cases arise. First suppose that $\gamma_{S\backslash\{j\}}(K \cup \{j\}) = 1$. Then $S\backslash\{j\} \subseteq K$. This implies that

$v(K\cup\{j\}) = 1$ and $\left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K\cup\{j\}) = 0$. Now suppose that $\gamma_{S\backslash\{j\}}(K\cup\{j\}) = 0$. Then $v \vee \gamma_{S\backslash\{j\}}(K\cup\{j\}) = v(K\cup\{j\})$ and thus, $\left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K\cup\{j\}) = 0$. In both cases, $\left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K\cup\{i\}) = \left(v \vee \gamma_{S\backslash\{j\}} - v\right)(K\cup\{j\}) = 0$.

Finally, since $\gamma_{S\backslash\{j\}} \vee \gamma_S = \gamma_{S\backslash\{j\}}$, it follows that

$$v \vee \gamma_{S\backslash\{j\}} = \gamma_{S\backslash\{j\}} \vee \left(\bigvee_{T\in\mathcal{M}(v)\backslash\{S\}} \gamma_T\right). \tag{5.16}$$

Therefore $\mathcal{M}(v \vee \gamma_{S\backslash\{j\}}) \subseteq \{S\backslash\{j\}\} \cup (\mathcal{M}(v)\backslash\{S\})$.

■

In Proposition 5.2.2, the main change from v to $v \vee \gamma_{S\backslash\{j\}}$ is that player j is no longer decisive in the winning coalition S. In the new game, $S\backslash\{j\}$ is a minimal winning coalition. However, it is important to note that change can also turn down the status of some other minimal winning coalition. It then appears that there are at most as many minimal winning coalitions in $v \vee \gamma_{S\backslash\{j\}}$ as in v.

The next result is another symmetry relationship between some players from distinct components of asimple game.

Proposition 5.2.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct components of a simple game v such that $m(v) \geq 3$, $s(v) \geq 2$ and $|\mathcal{C}_1| \leq |\mathcal{C}_2|$. Then for all $T \in \mathcal{C}_1$ and $j \in T$, there exists $\{R, S\} \subseteq \mathcal{M}(v)\backslash\mathcal{C}_1$ and $i \in S\backslash R$ such that $R' \cup \{i\}$ is losing in v for all proper subset R' of R or $R' \cup S\backslash\{i\}$ is losing in v for all proper subset R' of R.*

Moreover, for

$$S' = S\backslash\{i\},\ T' = T\backslash\{j\} \text{ and } w = v^{-R} \vee \gamma_{S'\cup R\cup\{j\}} \vee \gamma_{T'\cup R\cup\{i\}},$$

(i) i and j are symmetric players in $v - w$;

(ii) $\mathcal{M}(v)\backslash\{R\} \subseteq \mathcal{M}(w) \subseteq (\mathcal{M}(v)\backslash\{R\}) \cup \{S' \cup R \cup \{j\}, T' \cup R \cup \{i\}\}$;

(iii) $S' \cup R \cup \{j\} \in \mathcal{M}(w)$ or $T' \cup R \cup \{i\} \in \mathcal{M}(w)$.

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct components of a simple game v such that $m(v) \geq 3$, $s(v) \geq 2$ and $|\mathcal{C}_1| \leq |\mathcal{C}_2|$. Consider $T \in \mathcal{C}_1$ and $j \in T$. Since $m(v) \geq 3$ and $|\mathcal{C}_1| \leq |\mathcal{C}_2|$, it follows that $|\mathcal{M}(v)\backslash\mathcal{C}_1| \geq 2$. Then by Proposition 5.1.5, there exist $\{R, S\} \subseteq \mathcal{M}(v)\backslash\mathcal{C}_1$ and $i \in S\backslash R$ such that $R' \cup \{i\}$ is losing for all proper subset R' of R or $R' \cup S\backslash\{i\}$ is losing for all proper subset R' of R. Let $S' = S\backslash\{i\}$, $T' = T\backslash\{j\}$ and $w = v^{-R} \vee \gamma_{S'\cup R\cup\{j\}} \vee \gamma_{T'\cup R\cup\{i\}}$.

(i) To prove that i and j are symmetric players in $v - w$, consider $K \subseteq N\backslash\{i, j\}$. First suppose that $S' \cup R \subseteq K$. Then $R \subseteq K$ and $S = S' \cup \{i\} \subseteq K \cup \{i\}$. Therefore $w(K \cup \{j\}) = v(K \cup \{j\}) = v(K \cup \{i\}) = 1$ and $w(K \cup \{i\}) = v^{-R}(S' \cup \{i\}) = 1$. It

follows that $(v-w)(K\cup\{j\})=(v-w)(K\cup\{i\})=0$. The same reasoning applies when $T'\cup R\subseteq K$. Now, suppose that neither $S'\cup R\subseteq K$; nor $T'\cup R\subseteq K$. In this case, both $K\cup\{j\}$ and $K\cup\{i\}$ do not contain R. Therefore $v(K\cup\{l\})=v^{-R}(K\cup\{l\})=w(K\cup\{l\})$ for $l\in\{i,j\}$. Thus $(v-w)(K\cup\{j\})=(v-w)(K\cup\{i\})=0$.

(ii) By the definition of w, it is immediate that $\mathcal{M}(v)\backslash\{R\}\subseteq\mathcal{M}(w)\subseteq(\mathcal{M}(v)\backslash\{R\})\cup\{S'\cup R\cup\{j\},T'\cup R\cup\{i\}\}$.

(iii) Suppose that $T'\cup R\cup\{i\}\notin\mathcal{M}(w)$. By the definition of w, $w(T'\cup R\cup\{i\})=1$ and $w(R)=0$. It follows that there exists $L\in\mathcal{M}(w)\backslash\{R\}$ such that $L\subsetneq T'\cup R\cup\{i\}$. By assumption, $T\in\mathcal{C}_1$ and $\{R,S\}\subseteq\mathcal{M}(v)\backslash\mathcal{C}_1$. Therefore, L overlaps with either T' or with $R\cup\{i\}$. But T and L are both minimal winning coalitions in v. Thus, $L\subsetneq R\cup\{i\}$ and $R'\cup\{i\}$ is winning in v for some proper subset R' of R. By assumption, $R'\cup S'$ is losing in v for all proper subset R' of R. To see that $S'\cup R\cup\{j\}\in\mathcal{M}(w)$, note that $w(S'\cup R\cup\{j\})=1$ and suppose that some proper subset, say L', of $S'\cup R\cup\{j\}$ is winning in w. Then $w(L')=v^{-R}(L')=v(L')=1$. It follows that $L'\neq\{j\}$ since $s(v)\geq 2$. Clearly, $L\subsetneq R$ or $L\subseteq S'\cup R'$ for some proper subset R' of R. In both cases, $w(L)=v_{-R}(L)=v(L)=0$. A contradiction arises. ■

5.2.2 Van den Brink's theorem for simple games

We proceed by introducing four lemmas, the first of which concerns simple games containing a dictator.

Lemma 5.2.1. *Given $n\geq 3$ and a value φ on $\mathcal{G}_N$ that meets* (E), (NP) *and* (F), *then $\varphi(v)=\mathrm{SS}(v)$ for all simple games $v\in\mathcal{G}_N$ such that $s(v)=1$.*

Proof. Consider a value φ that satisfies (E), (NP) and (F) on $\mathcal{G}_N$ with $n\geq 3$. We prove by induction on $\lambda(v)=\sum_{S\in\mathcal{M}(v)}|S|$ the assertion $\mathcal{A}(p)$ that $\varphi(v)=\mathrm{SS}(v)$ for all $v\in\mathcal{G}_N$ such that $s(v)=1$ and $1\leq\lambda(v)\leq p$. It is worth noticing that, by Proposition 5.1.2, φ satisfies (S).

Initialization : Let $v\in\mathcal{G}_N$ be such that $s(v)=1$ and $\lambda(v)=1$. Then $\mathcal{M}(v)=\{\{i\}\}$ for some $i\in N$. Therefore by (E) and (NP), $\varphi_j(v)=0=\mathrm{SS}_j(v)$ for all $j\in N\backslash\{i\}$ and $\varphi_i(v)=1=\mathrm{SS}_i(v)$.

Induction step : Assume that $\mathcal{A}(p)$ holds for some integer $p\geq 1$. Consider $v\in\mathcal{G}_N$ such that $s(v)=1$ and $\lambda(v)=p+1$. Let $k\in\cup\mathcal{M}(v)$ such that $\{k\}\in\mathcal{M}(v)$ and consider $i\in\cup\mathcal{M}(v)$. First note that, if $\{i\}\in\mathcal{M}(v)$, then k and i are symmetric players in v. Therefore $\varphi_k(v)-\varphi_i(v)=\mathrm{SS}_k(v)-\mathrm{SS}_i(v)=0$. Now suppose that $i\in S$ for some $S\in\mathcal{M}(v)$ such that $|S|\geq 2$. It follows, by Proposition 5.2.2, that k and i

are symmetric players in $v \vee \gamma_{S\setminus\{i\}} - v$. Since φ and SS satisfy Axiom (F), it follows that $\varphi_k(v) - \varphi_i(v) = \varphi_k(v \vee \gamma_{S\setminus\{i\}}) - \varphi_i(v \vee \gamma_{S\setminus\{i\}})$ and $\mathrm{SS}_k(v) - \mathrm{SS}_i(v) = \mathrm{SS}_k(v \vee \gamma_{S\setminus\{i\}}) - \mathrm{SS}_i(v \vee \gamma_{S\setminus\{i\}})$. Furthermore, $\{k\} \in \mathcal{M}(v \vee \gamma_{S\setminus\{i\}})$ and by Proposition 5.2.2, $\mathcal{M}(v \vee \gamma_{S\setminus\{i\}}) \subseteq \{S\setminus\{i\}\} \cup (\mathcal{M}(v)\setminus\{S\})$. Therefore, $\lambda(v \vee \gamma_{S\setminus\{i\}}) \leq p$ and $s(v \vee \gamma_{S\setminus\{i\}}) = 1$. Thus, by induction assumption, $\varphi(v \vee \gamma_{S\setminus\{i\}}) = \mathrm{SS}(v \vee \gamma_{S\setminus\{i\}})$. It follows that $\varphi_k(v) - \varphi_i(v) = \mathrm{SS}_k(v \vee \gamma_{S\setminus\{i\}}) - \mathrm{SS}_i(v \vee \gamma_{S\setminus\{i\}}) = \mathrm{SS}_k(v) - \mathrm{SS}_i(v)$. Thus, by Proposition 5.2.1, $\varphi(v) = \mathrm{SS}(v)$. This proves that $\mathcal{A}(p+1)$ holds.

∎

In the following lemma, we show that when a power index agrees with the Shapley-Shubik index on all simple games with at most p minimal winning coalitions, players' shares for the two indices vary by the same amount for any pair of players belonging to the same component of a simple game with exactly $p+1$ minimal winning coalitions.

Lemma 5.2.2. *Given $n \geq 3$, assume that φ is a power index on $\mathcal{G}_N$ that meets (E), (NP) and (F). If $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$, then $\varphi_j(v) - \varphi_i(v) = \mathrm{SS}_j(v) - \mathrm{SS}_i(v)$ for all $v \in \mathcal{G}_{N,p+1}$ and for all connected players $i, j \in N$.*

Proof. Suppose that $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$. Let $v \in \mathcal{G}_{N,p+1}$ and $i, j \in N$ be such that i and j are connected. Then, there exists $\{S_1, S_2, \ldots, S_q\} \subseteq \mathcal{M}(v)$ and $I = \{i_1, i_2, \ldots, i_q\}$ such that $i = i_1 \in S_1$, $j = i_q \in S_q$ and for all $l \in \{2, \ldots, q\}$, $i_l \in S_l \cap S_{l-1}$. We prove by induction on $l \in \{1, 2, \ldots, q\}$ the assertion $\mathcal{A}(l)$ that for all $k \in S_l$, $\varphi_k(v) - \varphi_i(v) = \mathrm{SS}_k(v) - \mathrm{SS}_i(v)$.

Initialization: Consider $k \in S_1\setminus\{i\}$. It follows by Proposition 5.1.3 that i and k are symmetric players in $v - v^{-S_1}$, then

$$\begin{aligned}
\varphi_k(v) - \varphi_i(v) &= \varphi_k(v^{-S_1}) - \varphi_i(v^{-S_1}) \text{ since } \varphi \text{ satisfies } (F) \\
&= \mathrm{SS}_k(v^{-S_1}) - \mathrm{SS}_i(v^{-S_1}) \text{ since } m(v^{-S_1}) < m(v) \text{ and } m(v) \leq p \text{ by assumption} \\
&= \mathrm{SS}_k(v) - \mathrm{SS}_i(v) \text{ since } SS \text{ also satisfies } (F)
\end{aligned}$$

Therefore $\mathcal{A}(1)$ holds.

Induction step: Assume that $\mathcal{A}(l-1)$ holds for some $l \in \{2, \ldots, q\}$ and consider $k \in S_l\setminus\{i_l\}$. As in the previous step, it holds by Proposition 5.1.3 and Axiom (F) that $\varphi_k(v) - \varphi_{i_l}(v) = \mathrm{SS}_k(v) - \mathrm{SS}_{i_l}(v)$. Moreover, $i_l \in S_{l-1}$. Thus, by induction assumption, $\varphi_{i_l}(v) - \varphi_i(v) = \mathrm{SS}_{i_l}(v) - \mathrm{SS}_i(v)$. Summing these two equations gives $\varphi_k(v) - \varphi_i(v) = \mathrm{SS}_k(v) - \mathrm{SS}_i(v)$. This proves $\mathcal{A}(l)$.

Conclusion: For all $l \in \{1, 2, \ldots, q\}$, $\mathcal{A}(l)$ holds.

∎

In the next lemma, we deal with simple games with at most two minimal winning coalitions. It is shown that the Shapley-Shubik index is the only power index that simultaneously meets (E), (NP) and (F) on $\mathcal{G}_{N,1} \cup \mathcal{G}_{N,2}$.

Lemma 5.2.3. *Given $n \geq 3$ and a value φ on $\mathcal{G}_N$ that meets (E), (NP) and (F), then $\varphi(v) = \mathrm{SS}(v)$ for all simple games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq 2$.*

Proof.

Assume that $n \geq 3$ and consider a value φ that satisfies (E), (NP) and (F) on $\mathcal{G}_N$. By Proposition 5.1.2, φ satisfies (S). We partition the set $E = \{v \in \mathcal{G}_N : 1 \leq m(v) \leq 2\}$ into three disjoint subsets E_1, E_2 and E_3 defined by $E_1 = \{v \in \mathcal{G}_N : m(v) = 1\}$; $E_2 = \{v \in \mathcal{G}_N : \mathcal{M}(v) = \{S, T\}$ with $S \cap T \neq \emptyset\}$; and $E_3 = \{v \in \mathcal{G}_N : \mathcal{M}(v) = \{S, T\}$ with $S \cap T = \emptyset\}$. Firstly, consider $v \in E_1$. Then $\mathcal{M}(v) = \{S\}$ for some coalition S. Therefore by (NP), $\varphi_i(v) = \mathrm{SS}_i(v) = 0$ for all $i \in N \backslash S$. Moreover, by (NP), (S) and (E), $\varphi_i(v) = \mathrm{SS}_i(v) = 1/|S|$ for all $i \in S$. Hence $\varphi(v) = \mathrm{SS}(v)$. Secondly, consider $v \in E_2$. Then $\mathcal{M}(v) = \{S, T\}$ for some coalitions S and T such that $S \cap T \neq \emptyset$. Then players in $\cup\mathcal{M}(v) = S \cup T$ are connected to each other; and by Lemma 5.2.2 and Remark 5.2.1, $\varphi(v) = \mathrm{SS}(v)$. In the rest of the proof, $S \cap T = \emptyset$. To complete the proof, we prove by induction on $s(v) = \min(|S|, |T|)$ the assertion $\mathcal{A}(k)$ that $\varphi(v) = \mathrm{SS}(v)$ for all $v \in E_3$ such that $1 \leq s(v) \leq k$.

Initialization : Let $v \in E_3$ be such that $s(v) = 1$. Then $\mathcal{M}(v) = \{S, T\}$ for some coalitions S and T such that $S \cap T = \emptyset$ and $s(v) = 1$. Then, $|S| = 1$ or $|T| = 1$. It follows, by dictator lemma (see Lemma 5.2.1), that $\varphi(v) = \mathrm{SS}(v)$. Thus $\mathcal{A}(1)$ holds.

Induction step : Assume that $\mathcal{A}(k)$ holds for some integer $k \geq 1$. Consider $v \in E_3$ such that $\mathcal{M}(v) = \{S, T\}$ with $S \cap T = \emptyset$ and $s(v) = k + 1$. Without loss of generality, assume that $|S| = k + 1 \leq |T|$. Consider $\{i, j\} \subseteq S$ and $\{k, l\} \subseteq T$. Note by Proposition 5.1.3 that i and j are symmetric players in $v - \gamma_T$. Then it follows, that

$$\begin{aligned} \varphi_j(v) - \varphi_i(v) &= \varphi_j(\gamma_T) - \varphi_i(\gamma_T) \text{ since } \varphi \text{ satisfies } (F) \\ &= \mathrm{SS}_j(\gamma_T) - \mathrm{SS}_i(\gamma_T) \text{ since } \varphi(\gamma_T) = \mathrm{SS}(\gamma_T) \\ &= \mathrm{SS}_j(v) - \mathrm{SS}_i(v) \text{ since } SS \text{ also satisfies } (E),\ (NP) \text{ and } (F). \end{aligned}$$

Similarly, $\varphi_j(v) - \varphi_i(v) = \mathrm{SS}_j(v) - \mathrm{SS}_i(v)$ for all $j \in S$; and $\varphi_k(v) - \varphi_l(v) = \mathrm{SS}_k(v) - \mathrm{SS}_l(v)$ for all $l \in T$.

Also note by Proposition 5.1.3 that i and k are symmetric players in $v \vee \gamma_{\{i,k\}} - v$. Then it follows, that

$$\begin{aligned} \varphi_k(v) - \varphi_i(v) &= \varphi_k(v \vee \gamma_{\{i,k\}}) - \varphi_i(v \vee \gamma_{\{i,k\}}) \text{ since } \varphi \text{ satisfies } (F) \\ &= \mathrm{SS}_k(v \vee \gamma_{\{i,k\}}) - \mathrm{SS}_i(v \vee \gamma_{\{i,k\}}) \text{ since } \varphi(v \vee \gamma_{\{i,k\}}) = \mathrm{SS}(v \vee \gamma_{\{i,k\}}) \\ &= \mathrm{SS}_k(v) - \mathrm{SS}_i(v) \text{ since } SS \text{ also satisfies } (E),\ (NP) \text{ and } (F). \end{aligned}$$

It follows that $\varphi_l(v) - \varphi_i(v) = \mathrm{SS}_l(v) - \mathrm{SS}_i(v)$ for all $l \in \cup\mathcal{M}(v) = S \cup T$. Thus, from Proposition 5.2.1 and Remark 5.2.1, we conclude that $\varphi(v) = \mathrm{SS}(v)$.

Conclusion : $\varphi(v) = \mathrm{SS}(v)$ for all $v \in E_3$.

■

In the next and final lemma, we show that whenever a power index agrees with the Shapley-Shubik index on the set of all simple games with at most p minimal winning coalitions, the two power indices necessarily agree on the set of all simple games with exactly $p+1$ minimal winning coalitions.

Lemma 5.2.4. *Given $n \geq 3$ and $p \geq 2$, assume that φ is a power index on $\mathcal{G}_N$ that meets (E), (NP) and (F). If $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$, then $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_{N,p+1}$.*

Proof.

Assume that $n \geq 3$, $p \geq 2$ and consider a value φ that satisfies (E), (NP) and (F) on $\mathcal{G}_N$. Also assume that $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$. By Lemma 5.2.2, $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_{N,p+1}$ such that $c(v) = 1$. Furthermore, by the dictator lemma (see Lemma 5.2.1), $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_{N,p+1}$ such that $s(v) = 1$. Now, consider $v \in \mathcal{G}_{N,p+1}$ such that $c(v) \geq 2$ and $s(v) \geq 2$. Consider two components $\mathcal{C}_1$ and $\mathcal{C}_2$ of v such that $|\mathcal{C}_1| \leq |\mathcal{C}_2|$, $j_0 \in \cup\mathcal{C}_1$ and $i_0 \in \cup\mathcal{C}_2$. First suppose that $\mathcal{C}_1 = \mathcal{C}_2$. Then i_0 and j_0 are connected and by Lemma 5.2.2, $\varphi_{j_0}(v) - \varphi_{i_0}(v) = \mathrm{SS}_{j_0}(v) - \mathrm{SS}_{i_0}(v)$.

Now, suppose that $\mathcal{C}_1 \neq \mathcal{C}_2$. Since $m(v) = p+1 \geq 3$ and by Proposition 5.1.5, there exist $T \in \mathcal{C}_1$, $\{R, S\} \subseteq \mathcal{M}(v)\backslash\mathcal{C}_1$, $j \in T$ and $i \in S\backslash R$ such that $R' \cup \{i\}$ is losing for all proper subset R' of R or $R' \cup S\backslash\{i\}$ is losing for all proper subset R' of R. Then $j_0, j \in \cup\mathcal{C}_1$ and $i_0, i \in \cup\mathcal{C}_2$. By Lemma 5.2.2, $\varphi_{j_0}(v) - \varphi_j(v) = \mathrm{SS}_{j_0}(v) - \mathrm{SS}_j(v)$ and $\varphi_{i_0}(v) - \varphi_i(v) = \mathrm{SS}_{i_0}(v) - \mathrm{SS}_i(v)$. Thus, to prove that $\varphi_{i_0}(v) - \varphi_{j_0}(v) = \mathrm{SS}_{i_0}(v) - \mathrm{SS}_{j_0}(v)$, we only have to prove that $\varphi_i(v) - \varphi_j(v) = \mathrm{SS}_i(v) - \mathrm{SS}_j(v)$. For this purpose, let $I = S\backslash\{i\}$, $J = T\backslash\{j\}$ and $w = v^{-R} \vee \gamma_{I\cup R\cup\{j\}} \vee \gamma_{J\cup R\cup\{i\}}$. By Proposition 5.2.3, i and j are symmetric players in $v - w$. It follows that $\varphi_i(v) - \varphi_j(v) = \varphi_i(w) - \varphi_j(w)$. To complete the proof, we show that $\varphi_i(w) - \varphi_j(w) = \mathrm{SS}_i(w) - \mathrm{SS}_j(w)$. Note that by Proposition 5.2.3, $\mathcal{M}(w) \subseteq (\mathcal{M}(v) \setminus \{R\}) \cup \{I \cup R \cup \{j\}, J \cup R \cup \{i\}\}$ and $m(w) \leq p+2$.

First suppose that $I \cup R \cup \{j\} \notin \mathcal{M}(w)$. Then by Proposition 5.2.3, $J \cup R \cup \{i\} \in \mathcal{M}(w)$ and $m(w) = p+1$. Moreover, i and j are connected in w since $S \in \mathcal{M}(w)$, $T \in \mathcal{M}(w)$, $i \in S \cap (J \cup R \cup \{i\})$ and $\emptyset \neq J \subseteq T \cap (J \cup R \cup \{i\})$. Thus, by Lemma 5.2.2, $\varphi_i(w) - \varphi_j(w) = \mathrm{SS}_i(w) - \mathrm{SS}_j(w)$. Similarly, suppose that $J \cup R \cup \{i\} \notin \mathcal{M}(w)$. Then by Proposition 5.2.3, $I \cup R \cup \{j\} \in \mathcal{M}(w)$ and $m(w) = p+1$. This implies that i and j are connected in w since $S \in \mathcal{M}(w)$, $T \in \mathcal{M}(w)$, $j \in T \cap (I \cup R \cup \{j\})$ and $\emptyset \neq I \subseteq S \cap (I \cup R \cup \{j\})$. By Lemma 5.2.2, $\varphi_i(w) - \varphi_j(w) = \mathrm{SS}_i(w) - \mathrm{SS}_j(w)$.

Now suppose that $I \cup R \cup \{j\} \in \mathcal{M}(w)$ and $J \cup R \cup \{i\} \in \mathcal{M}(w)$ and consider $l \in R$. Then $\mathcal{M}(w) = (\mathcal{M}(v) \setminus \{R\}) \cup \{I \cup R \cup \{j\}, J \cup R \cup \{i\}\}$. By Proposition 5.1.3, l and j are symmetric players in $w - w^{-I\cup R\cup\{j\}}$. Therefore, $\varphi_l(w) - \varphi_j(w) = \varphi_l(w^{-I\cup R\cup\{j\}}) - \varphi_j(w^{-I\cup R\cup\{j\}})$ since φ satisfies (F). Moreover, i and j are still connected in $w^{-I\cup R\cup\{j\}}$ since $J\cup R\cup\{i\}, S, T \in \mathcal{M}(w^{-I\cup R\cup\{j\}})$. Noting that $m(w^{-I\cup R\cup\{j\}}) = p+1$, it follows by Lemma 5.2.2 that $\varphi_l(w) - \varphi_j(w) = \mathrm{SS}_l(w^{-I\cup R\cup\{j\}}) - \mathrm{SS}_j(w^{-I\cup R\cup\{j\}})$. Recalling that SS also satisfies satisfies (E), (NP) and (F), we deduce that $SS_l(w) - SS_j(w) = \mathrm{SS}_l(w^{-I\cup R\cup\{j\}}) - \mathrm{SS}_j(w^{-I\cup R\cup\{j\}}) = \varphi_l(w) - \varphi_j(w)$. In the same way, $\varphi_l(w) - \varphi_i(w) = \mathrm{SS}_l(w) - \mathrm{SS}_i(w)$. Hence $\varphi_i(w) - \varphi_j(w) = \mathrm{SS}_i(w) - \mathrm{SS}_j(w)$.

In summary, $\varphi_{i_0}(v) - \varphi_{j_0}(v) = \mathrm{SS}_{i_0}(v) - \mathrm{SS}_{j_0}(v)$ for all $i_0, j_0 \in \cup\mathcal{M}(v)$. We conclude by Proposition 5.2.1 and Remark 5.2.1 that $\varphi(v) = \mathrm{SS}(v)$.

■

We are now ready to state and prove the main result of this section.

Theorem 5.2.1.

For $n \geq 3$, if a value φ on $\mathcal{G}_N$ meets (E), (NP) and (F), then $\varphi(v) = \mathrm{SS}(v)$ for all simple games $v \in \mathcal{G}_N$. ★

Proof. Consider a power index φ that satisfies (E), (NP) and (F) on $\mathcal{G}_N$. By Proposition 5.1.2, φ satisfies (S). We prove by induction on $p \geq 1$ the assertion $\mathcal{A}(p)$ that $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$.

Initialization : Let $v \in \mathcal{G}_N$ be such that $1 \leq m(v) \leq 2$. Then by Lemma 5.2.3, $\varphi(v) = \mathrm{SS}(v)$.

Induction step : Given an integer $p \geq 2$, assume that $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $1 \leq m(v) \leq p$. Let $v \in \mathcal{G}_N$ be a simple game such that $m(v) = p + 1$. It follows by Lemma 5.2.4 that $\varphi(v) = \mathrm{SS}(v)$.

Conclusion : It holds that $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_N$.

■

In view of this result, some remarks and comments deserve our attention.

REMARK 5.2.2. *Note that Theorem 5.2.1 does not hold on $\mathcal{G}_N$ for $n = 2$ since in Remark 5.1.2, we provide a value on $\mathcal{G}_N$ for $n = 2$ that satisfies (E), (NP) and (F) without being the Shapley-Shubik index.*

REMARK 5.2.3. *By Lemma 5.2.1, the set of all simple games that each admit a dictator is a valid domain of the result stated in Theorem 5.2.1; that is a valid domain for the characterization result $(\mathcal{L}, \mathrm{Shap})$ for*

$$\mathcal{L} = \{(E), (NP), (F)\}.$$

Furthermore, by Lemma 5.2.4 and Lemma 5.2.3, it follows that for all integers $p \geq 1$, the set $\mathcal{G}_{N,\leq p}$ with

$$\mathcal{G}_{N,\leq p} = \bigcup_{k=1}^{p} \mathcal{G}_{N,k}.$$

is also a valid domain for the characterization result $(\mathcal{L}, \mathrm{Shap})$.

For further highlights, we have the following result that simply shows that the set of all proper simple games v on N (that is $c(v) = |\mathcal{C}(v)| = 1$) is a valid domain for the characterization result $(\mathcal{L}, \mathrm{Shap})$.

PROPOSITION 5.2.4. *For $n \geq 3$, if a value φ on $\mathcal{G}_N$ meets (E), (NP) and (F), then $\varphi(v) = \mathrm{SS}(v)$ for all proper simple games v.*

Proof.

Assume that $n \geq 3$ and consider a value φ that satisfies (E), (NP) and (F) on $\mathcal{G}_N$. Denote by $\mathcal{G}_N^*$ the set of all proper simple games $v \in \mathcal{G}_N$. Note that for $v \in \mathcal{G}_N^*$, $\mathcal{C}(v) = \{\mathcal{M}(v)\}$ and $\mathcal{M}(v)$ is the unique component of the decisiveness graph associated to v.

To prove that $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_N^*$, we proceed by induction on the cardinality of the unique consistent component $\mathcal{M}(v)$ of v.

Initialization : Suppose that $|\mathcal{M}(v)| = 1$. Then $\mathcal{M}(v) = \{S\}$ for some coalition S; that is $v = \gamma_S$. Then all players in $N \backslash S$ are null players in v while any two players in S are v-symmetric. By (NP), $\varphi_i(v) = 0 = \mathrm{SS}_i(v)$ for all $i \in N \backslash S$; and by Proposition 5.1.2, $\varphi_i(v) = \varphi_j(v)$ for all $i, j \in S$. Therefore, (E) implies $\varphi_i(v) = \dfrac{1}{|S|} = \mathrm{SS}_i(v)$ for all $i \in S$. Hence $\varphi(v) = \mathrm{SS}(v)$.

Induction step : Given a positive integer m, assume that $\varphi(v) = \mathrm{SS}(v)$ for all simple games $v \in \mathcal{G}_N^*$ such that $|\mathcal{M}(v)| \leq m$. Then let $v \in \mathcal{G}_N^*$ be a simple game such $|\mathcal{M}(v)| = m + 1$. First note that any player $i \in N$ such that $\mathcal{M}_i(v)$ is empty is a null player in v and therefore, $\varphi_i(v) = 0 = \mathrm{SS}_i(v)$ by (NP). Since v is a proper simple game, $\mathcal{M}(v) = \{S_1, S_2, \ldots, S_{m+1}\}$ and $S_t \cap S_{t+1} \neq \emptyset$ for all $t \in \{1, 2, \ldots, m\}$. Thus, consider $i_t \in S_t \cap S_{t+1}$ for all $t \in \{1, 2, \ldots, m\}$ and set $i_{m+1} = i_m$.

Given $t \in \{1, 2, \ldots, m+1\}$, note that by Proposition 5.1.3, $v - v^{-S_t} = \gamma_{S_t} - v^{+S_t}$ and players in S_t are symmetric players in $v - v^{-S_t}$. Therefore it follows that for all $i \in S_t$,

$$\mathrm{SS}_i(v) \overset{(F)}{=} \mathrm{SS}_{i_t}(v) + \mathrm{SS}_i(v^{-S_t}) - \mathrm{SS}_{i_t}(v^{-S_t})$$

and

$$\varphi_i(v) \overset{(F)}{=} \varphi_{i_t}(v) + \varphi_i(v^{-S_t}) - \varphi_{i_t}(v^{-S_t}) = \varphi_{i_t}(v) + \mathrm{SS}_i(v^{-S_t}) - \mathrm{SS}_{i_t}(v^{-S_t})$$

where the second equality holds by induction assumption on v^{-S_t}. A substraction using the previous identities yields,

$$\varphi_i(v) - \mathrm{SS}_i(v) = \varphi_{i_t}(v) - \mathrm{SS}_{i_t}(v) \text{ for all } i \in S_t.$$

By construction, $i_t \in S_{t+1}$ for all $t \in \{1, 2, \ldots, m\}$. Therefore,

$$\varphi_{i_t}(v) - \mathrm{SS}_{i_t}(v) = \varphi_{i_{t+1}}(v) - \mathrm{SS}_{i_{t+1}}(v).$$

This proves that the sequence $(\varphi_{i_t}(v) - \mathrm{SS}_{i_t}(v))_{1\leq t\leq m+1}$ is constant. Hence

$$\varphi_{i_t}(v) - \mathrm{SS}_{i_t}(v) = \varphi_{i_1}(v) - \mathrm{SS}_{i_1}(v),\ t \in \{1, 2, \ldots, m+1\}.$$

and for all $t \in \{1, 2, \ldots, m+1\}$,

$$\varphi_i(v) - \mathrm{SS}_i(v) = \varphi_{i_1}(v) - \mathrm{SS}_{i_1}(v) \text{ for all } i \in S_t.$$

By summing over all players $i \in \cup\mathcal{M}(v)$,

$$\sum_{i\in\cup\mathcal{M}(v)} (\varphi_i(v) - \mathrm{SS}_i(v)) = |\cup\mathcal{M}(v)|\,(\varphi_{i_1}(v) - \mathrm{SS}_{i_1}(v)) \overset{(E)}{=} 0.$$

Finally, $\varphi_{i_1}(v) - \mathrm{SS}_{i_1}(v) = 0$ and therefore, $\varphi_i(v) - \mathrm{SS}_i(v) = 0$ for all $i \in S_t$ and for all $t \in \{1, 2, \ldots, m+1\}$; that is $\varphi(v) = \mathrm{SS}(v)$.

Conclusion : It holds that $\varphi(v) = \mathrm{SS}(v)$ for all simple games $v \in \mathcal{G}_N^*$.

■

Proposition 5.2.5. *Axioms (E), (NP) and (F) are logically independent on $\mathcal{G}_N$.*

Proof. We prove that no axiom in Theorem 5.2.1 can not be dropped.

(a) Defined ξ on $\mathcal{G}_N$ by $\xi_i(v) = 1/n$ for all $v \in \mathcal{G}_N$ and for all $i \in N$. Then, the power index ξ satisfies (E) and (F), but fails to meet (NP).

(b) The power index $2SS$ satisfies (F) and (NP), but fails to meet (E).

(c) The Public Good Index HP, see Definition 1.1.13, satisfies (E) and (NP), but not (F). To see that HP does not satisfy (F), consider $\{i, j, k\} \subseteq N$ and pose $N' = N\backslash\{i, j, k\}$. Let u and v be the simple games defined on N by $u = \gamma_{\{i\}} \vee \gamma_{\{j,k\}}$ and $v = \gamma_{\{i\}} \vee \gamma_{\{j\}}$. It follows that

$$v - u = \bigvee_{S\subset N'} \gamma_{S\cup\{j\}}.$$

Therefore players i and k play symmetric roles in $v - u$. Moreover,

$$\mathrm{HP}_i(v) - \mathrm{HP}_i(u) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ and } \mathrm{HP}_k(v) - \mathrm{HP}_k(u) = \frac{1}{3} - 0 = \frac{1}{3}.$$

It then appears that $\mathrm{HP}_i(v) - \mathrm{HP}_i(u) \neq \mathrm{HP}_k(v) - \mathrm{HP}_k(u)$ although i and k play symmetric roles in changes that occur from u to v. Thus, HP does not satisfies (F).

■

In the next section, we present further insights on some aspects of fairness considerations.

5.3 A new axiomatization of the Shapley value

In this section, we provide two other three-axiom axiomatization of the Shapley value on Γ_N using two fairness conditions: the Null Player Fairness and the Strong Fairness. We also provide a new axiomatization of the Shapley value on simple games that use (E), (NP) and a strong version of the fairness Van den Brink (2001) used. This last result also holds on Γ_N.

5.3.1 The null player fairness condition

Recall that a fairness condition captures how some changes impact on individual shares from one game to another. For example, suppose that v is a TU-game in which a given player, say i, is a null player. Then if v describes changes that occur from a TU-game to another, player i contributes nothing in all changes that are observed and this may be a reasonable condition to require player i's share not to be affected. These considerations leads us to the following axiom.

Axiom 25. ***Null Player Fairness* (NPF)**: *For all $v \in \Gamma_N$, for all players $i \in N$ such that i is a null player in v,*

$$\varphi_i(u+v) = \varphi_i(v) \text{ for all } u \in \Gamma_N.$$

The (NPF) condition requires a null increment of the share of a player from a TU-game to another whenever the move is obtained by only adding a TU-game in which that player is a null player.

Proposition 5.3.1. *Consider a value $\varphi : \Gamma_N \to \mathbb{R}^N$.*

(a) If φ satisfies (NP) and (AD), then it also satisfies (NPF).

(b) If φ satisfies (NPF) and (AD), then φ also satisfies (NP).

(c) If φ satisfies (E), (S) and (NPF), then φ also satisfies (NP).

Proof.

Consider a value φ on Γ_N.

(a) Suppose that φ satisfies (NP) and (AD). For all $u, v \in \Gamma_N$, for all $i \in N$ such that i is a null player in v, one have:

$$\begin{aligned} \varphi_i(u+v) &= \varphi_i(u) + \varphi_i(v) \text{ since } \varphi \text{ satisfies Axiom (AD)} \\ &= \varphi_i(u) \text{ since } \varphi \text{ satisfies Axiom (NP) and } i \text{ is a null player in } v. \end{aligned}$$

(b) Suppose that φ satisfies (NPF) and (AD). Let $v \in \Gamma_N$ and $i \in N$ such that i is a null player in v. By applying the additivity of φ to the zero-game $\widetilde{0}$, we obtain $\varphi_i(\widetilde{0}) = \varphi_i(\widetilde{0}+\widetilde{0}) = \varphi_i(\widetilde{0}) + \varphi_i(\widetilde{0})$. That is $\varphi_i(\widetilde{0}) = 0$. Moreover,

$$\begin{aligned}\varphi_i(v) &= \varphi_i(\widetilde{0}+v)\\ &= \varphi_i(\widetilde{0}) \text{ since } \varphi \text{ satisfies Axiom (NPF) and } i \text{ is a null player in } v\\ &= 0 \text{ since } \varphi_i(\widetilde{0})=0.\end{aligned}$$

Thus φ satisfies (NP).

(*c*) Suppose that φ satisfies (E), (S) and (NPF). Let $v \in \Gamma_N$ and $i \in N$ such that i is a null player in v. Note that all players are symmetric in the zero-game $\widetilde{0}$ and since φ satisfies (S) and (E), one obtains $\varphi_i(\widetilde{0}) = \frac{\widetilde{0}(N)}{n} = 0$. Moreover,

$$\begin{aligned}\varphi_i(v) &= \varphi_i(\widetilde{0}+v)\\ &= \varphi_i(\widetilde{0}) \text{ since } \varphi \text{ satisfies Axiom (NPF) and } i \text{ is a null player in } v\\ &= 0 \text{ since } \varphi_i(\widetilde{0})=0.\end{aligned}$$

Thus φ satisfies (NP).

■

Note from Parts (*a*) and (*b*) of Proposition 5.3.1 that in presence of Axiom (AD), axioms (NPF) and (NP) are logically equivalent on Γ_N. The next result is a characterization of the Shapley value where condition (NPF) is substituted to the classical conditions (AD) and (NP).

Theorem 5.3.1.

A value φ on Γ_N satisfies (E), (S) and (NPF) if and only if $\varphi = \text{Shap}$. ★

Proof.

$\Longleftarrow$) It is well-known that the Shapley value satisfies (E) and (S). Since the Shapley value also satisfies (NP) and (AD), we deduce from part (*a*) of Proposition 5.3.1 that the Shapley value satisfies Axiom (NPF).

$\Longrightarrow$) Suppose that φ is a value on Γ_N that satisfies (E), (S) and (NPF). We show that $\varphi = \text{Shap}$. To do this, we consider, for each positive integer $m \in \mathbb{N}^*$, the set E_m of all TU-games u which are each a linear combination of at most m unanimity games; that is there exist some $T_1, T_2, \cdots, T_m \in 2^N$ and some $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{R}$ such that

$$u = \sum_{l=1}^{m} \alpha_l \gamma_{T_l}.$$

We prove by induction on $m \in \mathbb{N}^*$ the assertion that $\varphi(u) = \text{Shap}(u)$ for all $u \in E_m$.

Initialization: For $m = 1$, consider $u \in E_1$. Then there exists $T_1 \in 2^N$ and $\alpha_1 \in \mathbb{R}$ such that $u = \alpha_1 \gamma_{T_1}$. Let $i \in N$.

First suppose that $i \in N \backslash T_1$. Then i is a null player in u. By Part (c) of Proposition 5.3.1, φ satisfies (NP). Therefore, $\varphi_i(u) = 0 = \text{Shap}_i(u)$.

Now, suppose that $i \in T_1$. Since all members of T_1 are u-symmetric, we have $|T_1|\varphi_i(u) = \alpha_1$ since φ satisfies (E) and (S). Therefore, $\varphi_i(u) = \frac{\alpha_1}{|T_1|} = \mathrm{Shap}_i(u)$.

This proves that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_1$.

Induction step: Consider $m \in \mathbb{N}^*$ and suppose that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_m$. We show that $\varphi(u) = \mathrm{Shap}(u)$ for all $u \in E_{m+1}$. Let $u \in E_{m+1}$. By definition of E_m, there exist $T_1, T_2, \cdots, T_{m+1} \in 2^N$ and $\alpha_1, \alpha_2, \cdots, \alpha_{m+1} \in \mathbb{R}$ such that

$$u = \sum_{l=1}^{m+1} \alpha_l \gamma_{T_l}.$$

Consider $i \in N$. To show that $\varphi_i(u) = \mathrm{Shap}_i(u)$, we pose $T = \cap_{l=1}^{m+1} T_l$. There are two possible cases;

First suppose that $i \notin T$. Then there exist $l_0 \in \{1, \cdots, m+1\}$ such that $i \notin T_{l_0}$. This implies that i is a null player in the game $\alpha_{l_0}\gamma_{T_{l_0}}$. We then deduce the following:

$$\begin{aligned}
\varphi_i(u) &= \varphi_i(v + \alpha_{l_0}\gamma_{T_{l_0}}) \text{ where } v = \sum_{l=1, l\neq l_0}^{m+1} \alpha_l \gamma_{T_l} \\
&= \varphi_i(v) \text{ since } \varphi \text{ satisfies Axiom (NPF)} \\
&= \mathrm{Shap}_i(v) \;\text{ by induction assumption on } v \in E_m \\
&= \mathrm{Shap}_i(v + \alpha_{l_0}\gamma_{T_{l_0}}) \;\text{ since Shap satisfies Axiom (NPF)} \\
&= \mathrm{Shap}_i(u) \text{ since } u = v + \alpha_{l_0}\gamma_{T_{l_0}}.
\end{aligned}$$

Now, suppose that $i \in T$. Noting that all members of T are $\alpha_l\gamma_{T_l}$-symmetric for all $l \in \{1, \cdots, m+1\}$, then they are also u-symmetric. Moreover, since φ satisfies (E) and (S), it then follows that

$$\begin{aligned}
\varphi_i(u) &= \frac{1}{|T|}\left(u(N) - \sum_{j\notin T} \varphi_j(u)\right) \\
&= \frac{1}{|T|}\left(u(N) - \sum_{j\notin T} \mathrm{Shap}_j(u)\right) \;\text{ since } \varphi_j(u) = \mathrm{Shap}_j(u)\ \forall j \in N\backslash T \\
&= \mathrm{Shap}_i(u) \;\text{ since Shap satisfies (E) and (S).}
\end{aligned}$$

In each possible case, $\varphi_i(u) = \mathrm{Shap}_i(u)$. Therefore $\varphi(u) = \mathrm{Shap}(u)$.

Conclusion: For all $m \in \mathbb{N}^*$, for all $u \in E_m, \varphi(u) = \mathrm{Shap}(u)$.

Recalling the collection of all unanimity games is a basis of Γ_N, one gets $\Gamma_N = E_{2^n-1}$ and thus $\varphi = \mathrm{Shap}$. ■

The proof of Theorem 5.3.1 is built around the decomposition of a TU-game as a linear combination of unanimity games. Therefore, the following remark is straightforward.

Remark 5.3.1. *Theorem 5.3.1 still holds when one replaces Γ_N by any Peleg-Sudhölter domain.*

Proposition 5.3.2. *Axioms (E), (S) and (NPF) are logically independent on Γ_N.*

Proof.

(a) The *egalitarian rule* δ defined on Γ_N for all $v \in \Gamma_N$ and for all $i \in N$ by $\delta_i(v) = \frac{v(N)}{n}$, satisfies (E) and (S), but fails to meet (NPF).

(b) The *Banzhaf value* (see Banzhaf (1965) or Owen (1975)) β defined on Γ_N by:

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} [v(S+i) - v(S)] \ \forall v \in \Gamma_N, \forall i \in N,$$

satisfies (S) and (NPF), but fails to fulfill (E).

To see that β satisfies Axiom (NPF), we simply note that β obviously satisfies (NP) and (AD); and then apply Part (a) of Proposition 5.3.1 to conclude.

(c) The value φ defined on Γ_N given a player $i_o \in N$ by

$$\forall v \in \Gamma_N, \forall i \in N, \varphi_i(v) = \begin{cases} \text{Shap}_{i_o}(v) - 1 \text{ if } i = i_o \\ \text{Shap}_i(v) + \frac{1}{n-1} \text{ otherwise.} \end{cases}$$

with $n = |N|$. The value φ satisfies (E) and (NPF), but fails to meet Axiom (S).

To show that φ satisfies Axiom (NPF), consider two games $u, v \in \Gamma_N$ and a player $i \in N$ such that i is a null player in the game v. There are two possible cases.

For $i = i_0$, we have

$$\begin{aligned} \varphi_i(u+v) &= \text{Shap}_i(u+v) - 1 \\ &= \text{Shap}_i(u) + \text{Shap}_i(v) - 1 \text{ since Shap satisfies Axiom (AD)} \\ &= \text{Shap}_i(u) - 1 \text{ since Shap satisfies Axiom (NP) and } i \text{ is a null player in } v \\ &= \varphi_i(u). \end{aligned}$$

For $i \neq i_0$, we have

$$\begin{aligned} \varphi_i(u+v) &= \text{Shap}_i(u+v) + \tfrac{1}{n-1} \\ &= \text{Shap}_i(u) + \text{Shap}_i(v) + \tfrac{1}{n-1} \text{ since Shap satisfies Axiom (AD)} \\ &= \text{Shap}_i(u) + \tfrac{1}{n-1} \text{ since Shap satisfies Axiom (NP) and } i \text{ is a null player in } v \\ &= \varphi_i(u). \end{aligned}$$

In both cases, $\varphi_i(u+v) = \varphi_i(u)$. Thus φ satisfies Axiom (NPF).

■

5.3.2 The strong fairness condition

In the statement of condition (F), the attention is focused on a pair of TU-games and on the difference between those two TU-games. Given a value, it is an interesting issue whether the same set of changes produces the same impact on individual shares in all circumstances. In other words, how do individual shares vary when two pairs of TU-games yield the same difference? This may be handled in various ways. To present our formulation, we need some further notation.

Consider $v, w \in \Gamma_N$, $i, j \in N$ (with $i = j$ eventually) and a subset $S \subseteq N\backslash\{i\}$. We denote by $v_{i\leftarrow j}$ and $S_{i\leftarrow j}$ the game and the coalition obtained from v and S by interchanging the roles of i and j in v and S respectively. Formally,

$$v_{i\leftarrow j}(S) = v(S_{i\leftarrow j}) \text{ with } S_{i\leftarrow j} = \begin{cases} (S\backslash\{i\}) \cup \{j\} & \text{if } i \in S \text{ and } j \notin S \\ (S\backslash\{j\}) \cup \{i\} & \text{if } j \in S \text{ and } i \notin S \\ S & \text{otherwise.} \end{cases}$$

In particular, $v_{i\leftarrow i} = v$ and $S_{i\leftarrow i} = S$.

Thus, two players i and j are *symmetric* in v if $v_{i\leftarrow j} = v$. Given a subset $\mathcal{K}$ of Γ_N, formulation of a fairness condition when one considers the same set of changes from two pairs of TU-games is as follows:

Axiom 26. ***Strong Fairness* on $\mathcal{K}$ (SF)**: *For all $u, u', v, v' \in \mathcal{K}$, for all players $i, j \in N$ (possibly $i = j$) such that $(v' - v)_{i\leftarrow j} = u' - u$,*

$$\varphi_j(v') - \varphi_j(v) = \varphi_i(u') - \varphi_i(u).$$

Consider two sets of changes in two games that result in two new games. Then given any two players who play identical roles in the two moves, the payoff differential of each of them coincides with the payoff differential of his correspondent. It is worth noticing that $v_{i\leftarrow j}$ and u or $v'_{i\leftarrow j}$ and u' may be distinct games; what is important is that $(v' - v)_{i\leftarrow j} = u' - u$.

Remark 5.3.2. *If a value φ on $\mathcal{K}$ satisfies Axiom (SF), then for all $u, u', v, v' \in \mathcal{K}$,*

$$\varphi(v') - \varphi(v) = \varphi(u') - \varphi(u) \text{ whenever } v' - v = u' - u.$$

In fact, given $u, u', v, v' \in \mathcal{K}$ and $i \in N$ such that $v' - v = u' - u$,

$$\begin{aligned} \varphi_i(u') - \varphi_i(u) &= \varphi_i(v'_{i\leftarrow i}) - \varphi_i(v_{i\leftarrow i}) \quad \text{since} \quad u' - u = (v'_{i\leftarrow i} - v_{i\leftarrow i})_{i\leftarrow i} \\ &= \varphi_i(v') - \varphi_i(v) \quad \text{since} \quad v'_{i\leftarrow i} = v' \quad \text{and} \quad v_{i\leftarrow i} = v. \end{aligned}$$

The next proposition arises showing the relationship between (SF) and (F).

Proposition 5.3.3. *Consider a value φ on $\mathcal{K}$.*

If φ satisfies (SF) on $\mathcal{K}$, then φ satisfies (F) on $\mathcal{K}$.

Proof.

Consider a value $\varphi : \mathcal{K} \to \mathbb{R}^N$ and suppose that φ satisfies Axiom (SF) on $\mathcal{K}$. Consider $u, v \in \mathcal{K}$ (with $u + v \in \mathcal{K}$) and two players $i, j \in N$ such that i and j are *symmetric* in v. Note that

$$\begin{aligned}(v + u - u)_{i \leftarrow j} &= v_{i \leftarrow j} \\ &= v \text{ since } i \text{ and } j \text{ are symmetric in } v \\ &= v + u - u.\end{aligned}$$

Thus Axiom (SF) entails that $\varphi_j(v+u) - \varphi_j(u) = \varphi_i(v+u) - \varphi_i(u)$. That is φ satisfies Axiom (F).

■

Remark 5.3.3. *The converse of Proposition 5.3.3 does not hold. To see this, define the value φ on Γ_N by $\varphi_i(v) = (v(N))^2$ for all $v \in \Gamma_N$ and $i \in N$. Consider two games $u, v \in \Gamma_N$ and two players $i, j \in N$ such that i and j are symmetric in $v - u$. Then by definition of φ, $\varphi_j(v) - \varphi_j(u) = (v(N))^2 - (u(N))^2 = \varphi_i(v) - \varphi_i(u)$. Therefore φ satisfies (F). Now, pose $u = \gamma_{\{1\}}$, $v = \gamma_{\{1\}} + \gamma_N$, $u' = 2\gamma_{\{2\}}$ and $v' = 2\gamma_{\{2\}} + \gamma_N$. Then, we obtain $v - u = v' - u' = \gamma_N$ and therefore $(v' - u')_{1 \leftarrow 2} = v - u$. Moreover, $\varphi_1(v) - \varphi_1(u) = 2^2 - 1^2 = 3$ while $\varphi_2(v') - \varphi_2(u') = 3^2 - 2^2 = 5$. It then appears that $\varphi_1(v) - \varphi_1(u) \neq \varphi_2(v') - \varphi_2(u')$. This clearly proves that φ fails to satisfy (SF).*

The next result underlines a useful relationship between (NP), (SF) and (S).

Proposition 5.3.4. *(SF) and (NP) imply (S) on $\mathcal{G}_N$.*

Proof.

Consider a value φ that satisfies (SF) and (NP) on $\mathcal{G}_N$. Consider $i, j \in N$ and $v \in \mathcal{G}_N$ such that i and j are v-symmetric. Note that $\gamma_{\{i\}}, \gamma_{\{j\}} \in \mathcal{G}_N$ and $(v - \gamma_{\{i\}})_{i \leftarrow j} = v - \gamma_{\{j\}}$. Therefore $\varphi_i(v) - \varphi_i(\gamma_{\{j\}}) = \varphi_j(v) - \varphi_j(\gamma_{\{i\}})$ by (SF). Since $\varphi_i(\gamma_{\{j\}}) = \varphi_j(\gamma_{\{i\}}) = 0$ by (NP), it follows that $\varphi_i(v) = \varphi_j(v)$.

■

Remark 5.3.4. *For $n \geq 3$, Proposition 5.3.4 follows from Proposition 5.1.2 and Proposition 5.3.3. We have opted for the proof presented above since it is short and immediately involves the case of $n = 2$.*

Remark 5.3.5. *A value may satisfy (SF) without satisfying neither (S) nor (NP). To see this, consider the value φ defined in Part (c) in the proof of Proposition 5.3.2.*

With at least three players, the next result can be viewed as a weak version of Theorem 5.2.1 since (SF) is stronger than (F). The advantage of its presentation is that it remains valid with two players and admits a short proof.

Theorem 5.3.2.

A power index φ on $\mathcal{G}_N$ satisfies (E), (NP) and (SF) if and only if $\varphi = \mathrm{SS}$. ★

Proof.

Necessity: It is well-known that SS satisfies (E) and (NP) on $\mathcal{G}_N$. Now, to prove that SS also satisfies (SF) on $\mathcal{G}_N$, consider four simple games $u, v, u', v' \in \mathcal{G}_N$ and two players $i, j \in N$ such that $(v'-v)_{i\leftarrow j} = u' - u$. We have

$$\begin{aligned} \mathrm{SS}_j(v') - \mathrm{SS}_j(v) &= \mathrm{SS}_j(v'-v) \text{ since on } \mathcal{G}_N,\ \mathrm{SS} = \mathrm{Shap} \text{ and Shap is (AD) on } \Gamma_N \\ &= \mathrm{SS}_i\left((v'-v)_{i\leftarrow j}\right) \text{ since on } \mathcal{G}_N,\ \mathrm{SS} = \mathrm{Shap} \text{ and Shap is (S) on } \Gamma_N \\ &= \mathrm{SS}_i\left(u'-u\right) \text{ since by assumption, } (v'-v)_{i\leftarrow j} = u'-u \\ &= SS_i(u') - SS_i(u) \text{ since on } \mathcal{G}_N,\ \mathrm{SS} = \mathrm{Shap} \text{ and Shap is (AD)} \end{aligned}$$

This proves that SS satisfies Axiom (SF).

Sufficiency: Consider a power index φ that satisfies (SF), (E) and (NP) on $\mathcal{G}_N$, a game $v \in \mathcal{G}_N$ and the set $\mathcal{M}(v)$ of minimal winning coalitions of v. We prove by induction on the cardinality $m(v)$ of $\mathcal{M}(v)$ that $\varphi(v) = \mathrm{SS}(v)$.

Initialization : Assume that $m(v) = 1$. Then $\mathcal{M}(v) = \{S\}$ for some coalition S. Therefore by (NP), $\varphi_i(v) = 0 = \mathrm{SS}_i(v)$ for all $i \in N \backslash S$. Moreover, by (S) and (E), $\varphi_i(v) = 1/s = \mathrm{SS}_i(v)$ for all $i \in S$. Hence $\varphi(v) = \mathrm{SS}(v)$.

Induction step : Given a positive integer m, assume that $\varphi(v) = \mathrm{SS}(v)$ for all games $v \in \mathcal{G}_N$ such that $m(v) \leq m$. Then let $v \in \mathcal{G}_N$ be a game such that $m(v) = m+1$ and pose $\mathcal{M}(v) = \{S_1, S_2, \ldots, S_m, S_{m+1}\}$. Let $v', u', u \in \mathcal{G}_N$ be the games defined by

$$v' = \gamma_{S_1}, u' = v^{-S_1} \text{ and } u = v^{-S_1} \wedge \gamma_{S_1}.$$

Then note that v', u' and u are such that

$$\mathcal{M}(v') = \{S_1\}, \mathcal{M}(u') = \{S_2, \ldots, S_m, S_{m+1}\} \text{ and } \mathcal{M}(u) = \{S_2 \cup S_1, S_3 \cup S_1, \ldots, S_{m+1} \cup S_1\}.$$

It follows from Proposition 5.1.3 that $v - v' = u' - u$. Therefore, by (SF) and Remark 5.3.2, $\varphi(v) - \varphi(v') = \varphi(u') - \varphi(u)$. Moreover, $m(w) \leq m$ for $w \in \{v', u', u\}$. We deduce by the induction assumption that $\varphi(w) = \mathrm{SS}(w)$ for $w \in \{v', u', u\}$. We deduce that

$$\varphi(v) = \varphi(v') + \varphi(u') - \varphi(u) = \mathrm{SS}(v') + \mathrm{SS}(u') - \mathrm{SS}(u) = \mathrm{SS}(v).$$

Conclusion : It holds that $\varphi(v) = \mathrm{SS}(v)$ for all $v \in \mathcal{G}_N$ (since each simple game $v \in \mathcal{G}_N$ admits a finite number of minimal winning coalitions).

■

The relationship between (F) and (SF) calls for some remarks.

Remark 5.3.6. *Since (SF) implies (F) on any nonempty subset of* Γ_N, *it follows that (E), (NP) and (SF) uniquely identifies Shap on any nonempty subset* $\mathcal{K}$ *of* Γ_N

that is $(\{(E),(NP),(F)\},\mathrm{Shap})$ valid. This is for example the case of Peleg-Sudhölter domains.

Remark 5.3.7. *An immediate consequence of Theorem 5.2.1 and Theorem 5.3.2 is that (F) and (SF) are equivalent on $\mathcal{G}_N$ in the presence of at least three players.*

Remark 5.3.8. *Both (F) and (SF) share the same intuition with the initial condition (VDB-F) of fairness presented by Van den Brink (2001). These two reformulations of the initial condition permit us to obtain new characterization results of the Shapley value on domains such as the set of all simple games where (VDB-F) no longer works.*

It is shown in the next result that axioms in Theorem 5.3.2 are independent.

Proposition 5.3.5. *Axioms (E), (NP) and (SF) are logically independent on $\mathcal{G}_N$.*

Proof. We prove that no axiom in Theorem 5.3.2 can not be dropped.

(a) Defined ξ on $\mathcal{G}_N$ by $\xi_i(v) = 1/n$ for all $v \in \mathcal{G}_N$ and for all $i \in N$. Then, the power index ξ satisfies (E) and (SF), but fails to meet (NP).

(b) The power index $2SS$ satisfies (SF) and (NP), but fails to meet (E).

(c) The Public Good Index HP, see Definition 1.1.13, satisfies (E) and (NP), but not (F) as shown in Proposition 5.2.5. Therefore, HP also fails to meet (SF) by Proposition 5.3.3. ■

At the end of this chapter, we would like to recall our initial objective that was to look for new versions of the fairness condition provided by Van den Brink (2001). Our motivation was a will to reach new characterization results mainly on the set of simple games where the (VDB-F) condition no longer works. On this issue, two conditions have been introduced, namely, (F) and (SF). These two axioms effectively lead us to new characterization results of the Shapley-Shubik index; see Theorem 5.2.1 and Theorem 5.3.2. Further investigations relegated to future works include for example the search for valid domains of each of the newly established characterization result.

⋆ ⋆ Conclusion ⋆⋆

In this thesis, we have been concerned with axiomatization issues: the design, under a variable electorate, of new axioms that allow a full description of a power index and the search of subdomains of TU-games on which a given characterization result remains valid. The results we obtain are related to three well-known single-valued solutions: the Public Good Index, the Deegan-Packel index and the Shapley value.

For variable electorates, we have inaugured the characterization of the Public Good Index and the Deegan-Packel index with newly stated axioms. To achieve this, we have introduced some operations and binary relations on simple games: merging, supplementation and equivalence. The merging operation has emerged to the statement of some normative requirements such as the axiom of non profitable merging (NPM) of independent players, the axiom of independence of external merging (IEM) of independent players or the axiom of independence of internal merging (IIM) of independent players. Other axioms are related to the supplementation operation. This is for example the case for the axiom of supplementation consistency (SC) and the axiom of supplementation invariance (SI). Thanks to these newly introduced axioms, several characterizations results are stated and proved in Chapter 2 for the Public Good Index and in Chapter 3 for the Deegan-Packel index. The performances of these two power indices are summarized below where we indicate whether each axiom is satisfied (✓) by each of the two power indices or not (✗).

Axioms	NP	E	NPM	IEM	IIM	MEP	SC	SI
Public Good Index	✓	✓	✓	✓	✓	✓	✓	✗
Deegan-Packel index	✓	✓	✓	✓	✓	✓	✗	✓

Table 5.1: HP and DP performances on axioms for variable electorate

In Chapter 4 and Chapter 5, we have addressed some domain validity issues. With respect to Shapley's characterization result $(\{(E),(D),(S),(AD)\},\text{Shap})$, we have introduced the concept of conically consistent domain and Shapley valid domain. Further, we have shown in Chapter 4 that the condition of being a conically consistent domain generalizes previous known results on this topic such as Peleg-Sudhölter domains and Neyman domains. Still on the validity domain issue, we reconsidered in Chapter 5 a characterization of the Shapley value by Van den Brink (2001) that no longer holds on the set of simple games. We have mainly provided an equivalent reformulation of the author's condition of fairness and proved that the new characterization remains valid on the set of simple games. The same intuition leads us to some other results by considering axioms such as the null player fairness or the strong fairness conditions.

It is worth mentioning that, for each characterization result we presented here, it has been established that the axioms used are logically independent.

In terms of discussion, for our results on axiomatization of the Public Good Index and the Deegan-Packel index it is not surprising that our characterization results use three axioms instead of four axioms as those of Deegan and Packel (1978) who gave an (AN)+(E)+(NP)+(DPM) characterization of the Deegan-Packel index, Lorenzo-Freire et al. (2007) who provided an alternative characterization of the Deegan-Packel index by replacing a monotonicity axiom to the Deegan-Packel Mergeability; Holler and Packel (1983) who proved an (AN)+(E)+(NP)+(HPM) characterization of the Public Good Index, Alonso-Meijide et al. (2008) who provided an alternative characterization by substituting a monotonicity axiom to the Holler-Packel Mergeability since our results are on the domain of simple games with a variable electorate meanwhile their are on the domains of simple games with a fix electorate. Moreover, a combination of some of our axioms implies some of their axiom. For instance, we shown that (E) and (NPM) imply (NP) as well as (E) and (SI) imply (NP). We should notice that the only axiom that is common to their results and our is (E).

For the choice of the Shapley value in the supra-domain approach, we recognise the this is the main redistribution methods used in Game Theory and this fact played a main role in our choice despite many other values can be very interesting to study, for the solidarity value see Nowak and Radzik (1994) or the Owen value see Owen (1977).

We believe that the idea behind the geometrical properties of a cone could help us to definitely solve the problem of Shapley valid domains in terms of necessary and sufficient condition. Since just succeed to prove that our condition is sufficient in general.

There are still enough concerns for future work. Especially, the question whether a given subset of Γ_N is a Shapley valid domain or not remains open. The richness of conically consistent domains we presented tells us that Shapley valid domains can be very coarse and that the search for a general condition seems very challenging. There is a huge variety of characterization results in game theory. Each such result merits a validity domain analysis. Another direction for furure works consists in analyzing under variable electorates some other classical power indices (such as the Shapley-Shubik index, the Banzhaf index, the Johnston index, ...) together with their associated coalitional versions.

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⋆ ⋆ Appendix ⋆⋆

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Axiomatizing the Public Good Index via merging and new arrival properties

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ABSTRACT

Power indices are methods for numerical evaluation of players' voting power in simple (voting) games. We present alternative characterizations of the well-known Holler–Packel index, also known as the Public Good Index. To achieve this, we replace the Null Player Property, Anonymity and the Mergeability Condition with two new axioms. These axioms are based on the merging operation and the new arrival property.

1. Introduction

Since Shapley and Shubik [43] who have considered the Shapley value as an appropriate tool for measuring voting power, several indices have emerged, from game theory and related fields, offering many perspectives to measure the ability of a player to influence the social outcome in a decision-making process; see Andjiga et al. [7] for a rich selection of power indices. One of the main concerns for a given power index is to bring out characterization results that highlight some of its key features. For example, the Shapley–Shubik index has been scrutinized from various perspectives; see Dubey [15],Laruelle and Valenciano [37], or Einy and Haimanko [16]. For the Holler–Packel index introduced by Holler [22] and called Public Good Index, we provide new characterization results without the axiom of mergeability used by Holler and Packel [30].

Conceptually, the Public Good Index, in its normalized version, proportionally measures the relative share of each player when only minimal winning coalitions are formed and each player is endowed with the probability of belonging to the ruling coalition. In support of the intuition of forming only minimal winning coalitions, Holler and Nurmi [29] point out the fact that a larger player is less welcome in a losing coalition than a smaller one if some players of the losing coalition will not be critical (decisive) in the winning coalition after the arrival of the new player. For further motivation, see Holler [24,25],Holler and Napel [27] or Holler [23].

From a normative approach, Holler and Packel [30] prove that the Public Good Index is the unique power index that satisfies the null player property, efficiency, anonymity, and mergeability; see Napel [41] for the independence of these axioms. Alonso-Meijide et al. [5] provide an alternative characterization by substituting a monotonicity axiom into the axiom of mergeability. Related contributions include, for example, Holler and Li [26] for the non-normalized version (efficiency is dropped), Freixas and Kurz [18] for some type of monotonic power indices obtained as convex combinations

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of the Public Good Index and the Banzhaf index, Courtin and Tchantcho [12] for an extension to $(j, 2)$ simple games[1] and Alonso-Meijide et al. [1] for an extension to simple games with externalities.[2]

In this paper, we characterize the Public Good Index by means of efficiency and two new axioms. One of the two new axioms relates to what happens when two or more players are merged into a single new player: when players are independent in some sense, merging should not impact on the power of partners. Commenting their mergeability axiom, Holler and Packel [30] say: "While the axiom is mathematically appealing we offer no compelling story for its plausibility, finding it neither more nor less plausible then the corresponding final characterizing axioms for the other indices." Our attempt can thus be seen as an alternative mergeability property with a plausible story: the merging operation of some players which is a common practice in real-world situations. For example, in order to increase their influence, two or more parties (or shareholders) may merge into a new one. This well-established practice is analyzed by Aziz et al. [8] and Lasisi and Allan [38] in the case of weighted voting games in which several voters may unite into one by annexation or merging the weights of voters in the coordination; see also Felsenthal and Machover [17] who even propose a conceivable illustration with the Council of Ministers of the European Community.

The other axiom deals with the arrival of a new player: changes on players' shares, if any, due to the arrival of a new player who enters the game by joining some existing winning coalitions without disrupting any minimal winning coalition are proportional to the players' shares in the initial game. In other words, if a new player is added in a simple game without essentially changing the set of minimal winning coalitions, then the relative comparison of the powers of the old players should not change. A similar proportionality axiom is used by Barua et al. [9] in the specific case of merging exactly two voters in a unanimity (voting) game. We would also like to mention that this latter axiom has the same intuition as the consistency axiom of Hougaard and Moulin [31] in the context of cost allocation problems.

Merging players or allowing partnership are considered by many other authors; see Kalai and Samet [32], Carreras [11], Rodríguez-Veiga et al. [42], Koshevoy et al. [36] and Giménez et al. [19]. In the context of n-person simple bargaining games, Harsanyi [20] studies the impact on the total share of two or more players who opt for merging; Moulin [40] characterizes the proportional method using merging/splitting axioms; see also Lehrer [39] who uses bilateral mergers to obtain a characterization of the Banzhaf value; or Haviv [21], Derks and Tijs [14], Aziz et al. [8] or Besner [10] for some other models with TU games.

The rest of the paper is organized as follows: Section 2 provides preliminaries on simple games and power indices. The merging and the supplementation operations are also introduced as well as a formal statement of our primary axioms. The main result stating a new axiomatization of the Public Good Index is presented in Section 3 together with the independence of the axioms we use. Variants of the newly introduced axioms lead us to further characterization results. Section 4 concludes the paper with some perspectives on how to extend the results in this paper to other power indices on the one hand, or to coalitional versions of the Public Good Index on the other hand. All lengthy or technical proofs are relegated to the appendix section.

2. Simple games and power indices

2.1. Preliminaries

Let $\mathcal{P} = \{1, 2, \ldots\}$ be an infinite set of potential players indexed by positive integers. Each finite and nonempty subset S of $\mathcal{P}$ is called a coalition and $|S|$ denotes the cardinality of S.

Definition 2.1. A *simple game* is a pair $G = (N, v)$, where N is a coalition and v is a $\{0, 1\}$-valued map defined on the subsets of N such that: (i) $v(\emptyset) = 0$; (ii) $v(N) = 1$; and (iii) v is *monotonic*, that is, for all $S, T \subseteq N$, $S \subseteq T$ implies $v(S) \leq v(T)$.

The set of all simple games is denoted by $\mathcal{G}$. Given $G = (N, v) \in \mathcal{G}$, a coalition S is called *winning* if $v(S) = 1$, and *losing* otherwise. A winning coalition S is called *minimal* if $v(S \setminus \{i\}) = 0$ for all $i \in S$. The set of all minimal winning coalitions in the simple game G will be denoted by $\mathcal{M}(G)$ and the set of all minimal winning coalitions that contain a player $i \in N$ by $\mathcal{M}_i(G)$. The simple game G will also be denoted by $G = (N, \mathcal{W}(G))$, since G is completely determined by the player set N and the corresponding set $\mathcal{W}(G)$ of winning coalitions (or by the set $\mathcal{M}(G)$ of minimal winning coalitions).

Definition 2.2. Given a coalition S, a player $i \in S$ is *decisive* in S if $S \in \mathcal{W}(G)$, but $S \setminus \{i\} \notin \mathcal{W}(G)$. In this case, player i is called *complementary* to $S \setminus \{i\}$.

The set $\mathcal{D}_i(G)$ is the collection of all winning coalitions of the simple game G in which player i is decisive. A *null player* in G is any player i such that $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$; that is, $\mathcal{D}_i(G) = \emptyset$. The set of all null players in the simple game G is denoted by $N^0(G)$.

[1] A $(j, 2)$ simple game is a simple game with j ordered qualitative options in the input and 2 ordered quantitative options in the outcome.

[2] The worth of a coalition depends not only on the members of that coalition but on the whole coalition structure.

Remark 2.1. Given a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $S \subseteq N$.
If $S \in \mathcal{D}_i(G)$ then $K \in \mathcal{M}_i(G)$ for some $K \subseteq S$.

Before proceeding, here are some specific simple games that will be useful for what follows. With each coalition $S \subseteq \mathcal{P}$, we associate the simple game $G_S = (S, \mathcal{W}(G_S))$ with $\mathcal{M}(G_S) = \{\{i\} : i \in S\}$; that is, a coalition is winning in the game G_S if and only if it overlaps with S. Next, given a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $j \in \mathcal{P}\backslash N$, we denote by $G^{i\leftrightarrow j}$ the simple game obtained from G by only replacing i by j. Note that for all $T \subseteq \mathcal{P}\backslash\{i, j\}$, $G_{T\cup\{i\}} = \left(G_{T\cup\{j\}}\right)^{j\leftrightarrow i}$. Furthermore, given a positive integer p and $\left\{i_1, i_2, \ldots, i_p\right\} \subseteq \mathcal{P}\backslash N$, we denote by $G\left[i_1, i_2, \ldots, i_p\right]$ the simple game obtained from G by successively introducing players $i_1, i_2, \ldots, i_{p-1}$ and i_p as null players in such a way that the player set in $G\left[i_1, i_2, \ldots, i_p\right]$ is $N \cup \left\{i_1, i_2, \ldots, i_p\right\}$ and the set of minimal winning coalitions is still $\mathcal{M}(G)$. Note that player i_1 gets in first, player i_2 second and so on. Moreover, by definition, we have

$$G\left[i_1, i_2, \ldots, i_p\right] = G\left[j_1, j_2, \ldots, j_p\right] \text{ whenever } \left\{i_1, i_2, \ldots, i_p\right\} = \left\{j_1, j_2, \ldots, j_p\right\}.$$

2.2. Merging independent players

Players in a coalition T can enter the simple game $G = (N, \mathcal{W}(G))$ with a *prior agreement* to act as a single player $i_T \in \mathcal{P}\backslash N$. In this case, players in T are merged into i_T, their representative in the new simple game. The modified simple game that follows is denoted by $G^T = (N^T, v^T)$ and defined such that $N^T = (N \setminus T) \cup \{i_T\}$ and for all coalitions $S \subseteq N^T$, $v^T(S) = v(S \setminus \{i_T\}) \cup T$ if $i_T \in S$; and $v^T(S) = v(S)$ otherwise. We will sometimes replace G^T by $G^{T\to k}$ to specify that i_T is identified with a given player $k \in \mathcal{P}\backslash N$. Moving from G to G^T is called a *merging operation*. Hereafter, for each $S \subseteq N$, we denote by S^T the subset of N^T defined by $S^T = (S \setminus T) \cup \{i_T\}$ if $S \cap T \neq \emptyset$; and $S^T = S$ otherwise.

The merging operation is well-known in the literature although authors sometimes used distinct notation options, see Knudsen and Østerdal [33] or Slavov and Evans [44]. It is also known under the name of *amalgamation* as in [39] or [21].

Example 2.1. Consider the simple game $G = (N, \mathcal{W}(G))$ such that $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$. Let $T = \{1, 5\}$. Here, $N^T = \{i_T, 2, 3, 4\}$ and $\mathcal{M}(G^T) = \{\{i_T, 2\}, \{i_T, 3, 4\}\}$. If we set $i_T = 6$, then $N^{T\to 6} = \{6, 2, 3, 4\}$ and $\mathcal{M}(G^{T\to 6}) = \{\{6, 2\}, \{6, 3, 4\}\}$. Similarly, for $K = \{4, 5\}$, $N^K = \{1, 2, 3, i_K\}$ and $\mathcal{M}(G^K) = \{\{1, 2\}, \{2, 3, i_K\}, \{1, 3, i_K\}\}$. Still for illustration with $T = \{1, 5\}$, note that $\{1, 2\}^T = \{i_T, 2\}$, $\{1, 2, 5\}^T = \{i_T, 2\}$ and $\{2, 3, 4\}^T = \{2, 3, 4\}$.

Definition 2.3. Two players i and j are disconnected if $|S \cap \{i, j\}| \leq 1$ for all $S \in \mathcal{M}(G)$.

Two players are disconnected in a simple game if no minimal winning coalition contains both players at the same time.

Definition 2.4. Given a simple game $G = (N, \mathcal{W}(G))$, two players i and j are *independent* if i and j are disconnected and for all losing coalitions $S \subseteq N \setminus \{i, j\}$:

$$\left(\begin{array}{c} S \cup \{i\} \in \mathcal{D}_i(G) \\ \text{and} \\ S \cup \{j\} \in \mathcal{D}_j(G) \end{array}\right) \Rightarrow \text{for some } S', S'' \subseteq S, \left(\begin{array}{c} S' \cup \{i\} \in \mathcal{D}_i(G), S' \cup \{j\} \notin \mathcal{D}_j(G) \\ \\ S'' \cup \{j\} \in \mathcal{D}_j(G) \text{ and } S'' \cup \{i\} \notin \mathcal{D}_i(G) \end{array}\right).$$

Furthermore, we say that a coalition T consists of independent players in G if any two players in T are independent in G.

Intuitively, two players are independent in a simple game if each time they are both complementary to the same losing coalition S, each of the two players is complementary to some subset of S while the other is not. In Example 2.1, players 4 and 5 are disconnected in the simple game G and there is no losing coalition S such that $S \cup \{4\}$ and $S \cup \{5\}$ are winning. Thus, 4 and 5 are independent in G. Now, players 1 and 5 are disconnected in G; but for $S = \{2, 3\}$, S is losing, $S \cup \{1\}$ and $S \cup \{5\}$ are winning, but $S' \cup \{5\}$ is losing for all $S' \subsetneq S$. Thus, 1 and 5 are not independent in G.

It is worth mentioning that if i is a null player in a simple game G, then for any other player j, i and j are independent. In the general case, we have the following result.

Proposition 2.1. *Given a simple game $G = (N, \mathcal{W}(G))$, two disconnected players i and j are independent if and only if for all $S \in \mathcal{M}_i(G)$ and for all $S' \in \mathcal{M}_j(G)$, $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$.*

Proof. Consider two disconnected players i and j in a simple game $G = (N, \mathcal{W}(G))$. First suppose that players i and j are independent. Consider $S \in \mathcal{M}_i(G)$ and $S' \in \mathcal{M}_j(G)$. To prove that $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$, suppose the contrary. W.l.o.g., suppose that $S \setminus (S' \cup \{i\}) = \emptyset$; that is, $S \subseteq S' \cup \{i\}$. Since i and j are disconnected, $j \notin S$. Therefore $S \setminus \{i\} \subseteq S' \setminus \{j\}$. Let $K = S' \setminus \{j\}$. It follows that K is losing, $K \cup \{i\} \in \mathcal{D}_i(G)$ and $K \cup \{j\} \in \mathcal{D}_j(G)$. Since i and j are independent, there exists $S'' \subseteq K$ such that $S'' \cup \{j\} \in \mathcal{D}_j(G)$ and $S'' \cup \{i\} \notin \mathcal{D}_i(G)$. Note that $K \cup \{i\}$ is winning while $S'' \cup \{i\}$ is losing.

We deduce that S'' is a proper subset of K and that $S'' \cup \{j\}$ is a proper winning subset of S'; that is $S'' \cup \{j\} \subseteq S'$ and $S'' \cup \{j\} \neq S'$. A contradiction arises since $S' \in \mathcal{M}(G)$.

Now suppose that for all $S \in \mathcal{M}_i(G)$ and for all $S' \in \mathcal{M}_j(G)$, $S \setminus (S' \cup \{i\}) \neq \emptyset$ and $S' \setminus (S \cup \{j\}) \neq \emptyset$. Consider a losing coalition $S \subseteq N \setminus \{i, j\}$ such that both $S \cup \{i\} \in \mathcal{D}_i(G)$ and $S \cup \{j\} \in \mathcal{D}_j(G)$. Then, by Remark 2.1, there exist $S', S'' \subseteq S$ such that $S' \cup \{i\} \in \mathcal{M}_i(G)$ and $S'' \cup \{j\} \in \mathcal{M}_j(G)$. To prove that $S' \cup \{j\} \notin \mathcal{D}_j(G)$ and $S'' \cup \{i\} \notin \mathcal{D}_i(G)$, suppose the contrary. W.l.o.g., suppose that $S' \cup \{j\} \in \mathcal{D}_j(G)$. Then by Remark 2.1, there exists $L \subseteq S'$ such that $L \cup \{j\} \in \mathcal{M}_j(G)$. It holds that $S' \cup \{i\} \in \mathcal{M}_i(G)$, and $L \cup \{j\} \in \mathcal{M}_j(G)$ with $L \setminus S' = \emptyset$. A contradiction arises since by assumption we should have $L \setminus S' = (L \cup \{j\}) \setminus (S' \cup \{i, j\}) \neq \emptyset$. □

2.3. Supplementation of a simple game

The scenario we now consider, for a given simple game $G = (N, \mathcal{W}(G))$, is the arrival of a new player $k \notin N$.

Definition 2.5. A *k-supplementation* of a simple game $G = (N, \mathcal{W}(G))$ is any simple game $G' = (N', \mathcal{W}(G'))$ such that $k \in \mathcal{P} \setminus N$, $N' = N \cup \{k\}$ and for all coalitions $S \subseteq N$:

$$S \in \mathcal{M}(G) \Longleftrightarrow (S \in \mathcal{M}(G') \text{ or } S \cup \{k\} \in \mathcal{M}(G')).$$

From G to a k-supplementation G' of G, the arrival of player k is such that, for each minimal winning coalition S in G, either S or $S \cup \{k\}$ remains a minimal winning coalition in G'. For each winning coalition S in G, we say that k becomes *supplementary* to S from G to G' when S is losing in G', while $S \cup \{k\}$ is winning in G'. For cost allocation problems, a similar definition was considered by Hougaard and Moulin [31] in their definition of an item that is supplementary to an agent needs. From G to a k -supplementation of G, the arrival of player k changes the set of minimal winning coalition of G exactly as the introduction of a supplementary resource to the needs of an agent, say i, reshapes the collection of agent i's minimal serving sets.

Example 2.2. Let $G = (N, \mathcal{W}(G))$ be the simple game defined in Example 2.1 with $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 2\}, \{2, 3, 5\}, \{1, 3, 4\}\}$. Then G is a 3 -supplementation of $G_1 = (N_1, \mathcal{W}(G_1))$ where $N_1 = \{1, 2, 4, 5\}$ and $\mathcal{M}(G_1) = \{\{1, 2\}, \{2, 5\}, \{1, 4\}\}$. From G_1 to G, player 3 is supplementary not only to the minimal winning coalitions $\{2, 5\}$ and $\{1, 4\}$, but also to some non minimal winning coalitions such as $\{2, 4, 5\}$. Similarly, the simple game G is a 5-supplementation of $G_2 = (N_2, \mathcal{W}(G_2))$ with $N_2 = \{1, 2, 3, 4\}$ and $\mathcal{M}(G_2) = \{\{1, 2\}, \{2, 3\}, \{1, 3, 4\}\}$.

Proposition 2.2. *Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a simple game $G' = (N', \mathcal{W}(G'))$ such that $N' = N \cup \{k\}$. Then G' is a k-supplementation of G if and only if there exists a subset E of $\mathcal{M}(G)$ such that*

$$\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}.$$

Proof. See Appendix A. □

2.4. Axioms of power indices

Definition 2.6. A *power index* φ is a map defined on $\mathcal{G}$ such that $\varphi(G) \in \mathbb{R}^N$ for every $G \in \mathcal{G}$.

For many authors, the definition of a power index includes the disposition that players' shares are non negative numbers; see Kóczy [34] or Kong and Peters [35]. But in general, the positivity fulfillment is deduced from some reasonable properties of power indices. Among classical axioms for power indices, we have the following:

Axiom 1. ***Null Player* (NP)**: For all $G \in \mathcal{G}$, $\varphi_i(G) = 0$ whenever i is a null player in G.

Axiom 2. ***Efficiency* (E)**: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $\sum_{i \in N} \varphi_i(G) = 1$.

A power index that meets (E) is called *normalized*. Note that for any power index φ such that individual shares in all simple games always sum to non-zero values, one gets an efficient power index $\overline{\varphi}$ defined for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by:

$$\overline{\varphi}_i(G) = \frac{\varphi_i(G)}{\sum_{j \in N} \varphi_j(G)}.$$

The power index $\overline{\varphi}$ is called the normalized version of φ. The Public Good Index Φ, we focus on, is the normalized form of the power index that associates each simple game $G = (N, \mathcal{W}(G))$ with the collection $\left(\frac{|\mathcal{M}_i(G)|}{|\mathcal{M}(G)|}\right)_{i \in N}$. More formally, for all $i \in N$,

$$\Phi_i(G) = \frac{|\mathcal{M}_i(G)|}{\sum_{j \in N} |\mathcal{M}_j(G)|}.$$

In this paper, we also consider two new axioms:

Axiom 3. ***Supplementation Consistency* (SC)**: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \setminus N$, for all k-supplementations G' of G, for all $i \in N$, $\varphi_i(G') = \varphi_i(G)\lambda_{G'}$ for some constant $\lambda_{G'}$.

From a simple game G to a k-supplementation G' of G, the change, if any, on each player's share is proportional to his/her power in G. More specifically, when the power index is normalized, the constant $\lambda_{G'}$ becomes more explicit as shown below:

Proposition 2.3. *If a power index φ satisfies (E) and (SC), then for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all $k \in \mathcal{P} \setminus N$, for all k-supplementations G' of G, for all $i \in N$,*

$$\varphi_i(G') = (1 - \varphi_k(G'))\varphi_i(G). \tag{1}$$

Proof. Suppose that a power index φ satisfies (E) and (SC). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a k-supplementation G' of G. Then, there exists a constant $\lambda_{G'}$ such that $\varphi_i(G') = \varphi_i(G)\lambda_{G'}$ for all $i \in N$. Then by (E), $1 - \varphi_k(G') = \sum_{i\in N} \varphi_i(G') = \lambda_{G'} \left(\sum_{i\in N} \varphi_i(G)\right) = \lambda_{G'}$. □

Eq. (1) effectively matches the consistency requirement by Hougaard and Moulin [31] as an agent is removed (or introduced) in a cost allocation problem.

Axiom 4. ***Non Profitable Merging (NPM) of independent players***: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players,

$$\varphi_{i_T}(G^T) = \sum_{i\in T} \varphi_i(G) \tag{2}$$

whenever T contains only independent players.

Axiom (NPM)[3] is a weak condition of the lack of incentive for independent players to merge in a simple game; for similar requirements, see Knudsen and Østerdal [33]. When the merging equality (2) is considered only for coalitions of size 2 (without any restriction), one obtains the 2-efficiency condition of Lehrer [39].[4]

3. Axiomatizations of the Public Good Index

3.1. Preliminary results

In this section, it is shown that the Public Good Index meets (NP), (E), (NPM) and (SC). As observed by Holler and Packel [30], the following result is straightforward:

Proposition 3.1. *The Public Good Index Φ satisfies (NP) and (E).*

The next result tells us how merging independent players impacts the structure of a simple game. It allows us to prove that Φ satisfies (NPM).

Proposition 3.2. *Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two players . If T is a coalition of independent players, then*

(a) For all $S, R \in \mathcal{M}(G)$, $S \neq R$ implies $S^T \neq R^T$;
(b) $\mathcal{M}(G^T) = \{S^T : S \in \mathcal{M}(G)\}$;
(c) $|\mathcal{M}_j(G^T)| = |\mathcal{M}_j(G)|$ for all $j \in N \setminus T$;
(d) $|\mathcal{M}_{i_T}(G^T)| = \sum_{i\in T} |\mathcal{M}_i(G)|$.

Proof. See Appendix B. □

Corollary 3.1. *The Public Good Index Φ satisfies (NPM).*

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3 It can be checked that the power index that associates each simple game $G = (N, \mathcal{W}) \in \mathcal{G}$ with the n-tuple $(1/n; 1/n; \cdots; 1/n)$ satisfies the Holler–Packel mergeability axiom but not (NPM). And the Deegan–Packel index (see [13]) satisfies axiom (NPM) but fails to meet the Holler–Packel mergeability axiom. Furthermore, the Holler–Packel mergeability condition applies to a fix electorate while (NPM) is designed for variable electorates.

4 The author proves that the Banzhaf value for TU-games is uniquely determined by the 2-efficiency condition among all values that coincide with the Shapley value on all 2-player games. It follows that the Banzhaf power index (normalized or not) satisfies (2) for all coalitions T of size 2.

Proof. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of independent players such that $|T| \geq 2$. Parts (c) and (d) in Proposition 3.2 imply that

$$\sum_{j \in N^T} |\mathcal{M}_j(G^T)| = \sum_{j \in N} |\mathcal{M}_j(G)|$$

Furthermore,

$$\Phi_{i_T}(G^T) = \frac{|\mathcal{M}_{i_T}(G^T)|}{\sum_{j \in N^T} |\mathcal{M}_j(G^T)|} = \frac{\sum_{i \in T} |\mathcal{M}_i(G)|}{\sum_{j \in N} |\mathcal{M}_j(G)|} = \sum_{i \in T} \Phi_i(G).$$

Therefore Φ satisfies (NPM). □

Using Parts (c) and (d) in Proposition 3.2, it can be shown that Φ even satisfies a stronger version of (NPM) when one also considers that, for any coalition T of independent players, merging the members of T should affect no player in $N \setminus T$. We now prove that Φ satisfies (SC).

Proposition 3.3. *The Public Good Index Φ satisfies (SC).*

Proof. Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$, a k-supplementation $G' = (N', \mathcal{W}(G'))$ of G and $i \in N$. It follows from Proposition 2.2 that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$ for some $E \subseteq \mathcal{M}(G)$. First note that $\mathcal{M}_i(G') = \{S \in \mathcal{M}(G') : i \in S\} = \{S \cup \{k\} : i \in S \in E\} \cup \{S : i \in S \in \mathcal{M}(G) \setminus E\}$. That is $\mathcal{M}_i(G') = \{S \cup \{k\} : S \in E \cap \mathcal{M}_i(G)\} \cup \{S : S \in \mathcal{M}_i(G) \setminus E\}$. Since $E \cap \mathcal{M}_i(G)$ and $\mathcal{M}_i(G) \setminus E$ are disjoint sets, it follows that $|\mathcal{M}_i(G')| = |E \cap \mathcal{M}_i(G)| + |\mathcal{M}_i(G) \setminus E| = |\mathcal{M}_i(G)|$.

Now, we have

$$\begin{aligned}
\Phi_i(G') &= \frac{|\mathcal{M}_i(G')|}{\sum_{j \in N'} |\mathcal{M}_j(G')|} \\
&= \frac{|\mathcal{M}_i(G)|}{|\mathcal{M}_k(G')| + \sum_{j \in N} |\mathcal{M}_j(G)|} \\
&= \frac{\sum_{j \in N} |\mathcal{M}_j(G)|}{|\mathcal{M}_k(G')| + \sum_{j \in N} |\mathcal{M}_j(G)|} \times \frac{|\mathcal{M}_i(G)|}{\sum_{j \in N} |\mathcal{M}_j(G)|} = (1 - \Phi_k(G'))\Phi_i(G)
\end{aligned}$$

Therefore Φ satisfies (SC). □

For power indices that meet (SC) or (NPM), we now present some results we use in the next section.

Proposition 3.4. *Consider a simple game $G = (N, \mathcal{W}(G))$, a player $k \in \mathcal{P} \setminus N$, a k-supplementation G' of G and two power indices φ and ψ that both satisfy (E) and (SC) .*

Then ($\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1$) implies $\psi(G) = \varphi(G)$.

Proof. Suppose that two power indices φ and ψ satisfy (E) and (SC). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a player $k \in \mathcal{P} \setminus N$ and a k-supplementation G' of G such that $\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1$. Then, by Proposition 2.3, it follows that for all $i \in N$, $\psi_i(G') = (1 - \psi_k(G'))\psi_i(G)$ and $\varphi_i(G') = (1 - \varphi_k(G'))\varphi_i(G)$. Since $\psi(G') = \varphi(G')$ and $\varphi_k(G') \neq 1$, then $\psi_i(G) = \varphi_i(G)$ for all $i \in N$. That is $\psi(G) = \varphi(G)$. □

Proposition 3.5. *All power indices that satisfy (NPM) also satisfy (NP).*

Proof. See Appendix C. □

Remark 3.1. By Proposition 3.5, any power index φ that satisfies (NPM) also satisfies (NP). Moreover, if φ satisfies both (E), (SC) and (NPM), then $\varphi_i(G) = \varphi_i(G_0)$ for all $i \in N_0$ whenever $G = G_0[i_1, \ldots, i_p]$.

Proposition 3.6. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC).*

Then for all simple games $G = (N, \mathcal{W}(G))$, for all $i \in N$ and for all $j \in \mathcal{P} \backslash N$, $\varphi_j\left(G^{i \leftrightarrow j}\right) = \varphi_i(G)$.

Proof. Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Consider a simple game $G = (N, \mathcal{W}(G))$, $i \in N$ and $j \in \mathcal{P} \backslash N$. Let $k \in \mathcal{P} \backslash (N \cup \{j\})$. By Proposition 3.5, $\varphi_k(G[k]) = 0$ and therefore, $\varphi_i(G[k]) = \varphi_i(G)$ by (E) and (SC). Moreover, $G^{i \leftrightarrow j}$ is obtained from $G[k]$ by merging i and k into j. Therefore by (NPM), $\varphi_j\left(G^{i \leftrightarrow j}\right) = \varphi_k(G[k]) + \varphi_i(G[k]) = \varphi_i(G)$. □

Proposition 3.6 tells us that, if from one simple game to another, only one player in the initial simple game is replaced by another player, then the new player simply inherits the replaced player's share. However, nothing is said about the shares of other players in the new simple game.

Remark 3.2. Suppose φ is a power index that satisfies (E), (NPM) and (SC). We denote by $\varphi_{T,i}$ the share of player i in the simple game $G_{T\cup\{i\}}$ for all coalitions $T \subseteq \mathcal{P}$ and for all $i \in \mathcal{P}\backslash T$. Note that for any other player $j \in \mathcal{P}\backslash(T \cup \{i\})$, $G_{T\cup\{i\}} = \left(G_{T\cup\{j\}}\right)^{j\leftrightarrow i}$. Therefore, it follows from Proposition 3.6 that

$$\varphi_{T,i} = \varphi_i\left(G_{T\cup\{i\}}\right) = \varphi_j\left(G_{T\cup\{j\}}\right) = \varphi_{T,j} \ .$$

This shows that $\varphi_{T,i}$ only depends on T but not on the player i we choose from $\mathcal{P}\backslash T$. That is why, from now on, we simply denote by φ_T the share, with respect to φ, of an arbitrary player $i \in \mathcal{P}\backslash T$ in the simple game $G_{T\cup\{i\}}$. By efficiency, it follows that for all coalitions $S \subseteq \mathcal{P}$ of at least two players, the collection $(\varphi_T)_{T\subseteq S/|T|=|S|-1}$ satisfies the following equation (E_S) :

$$\sum_{T\subseteq S/|T|=|S|-1} \varphi_T = 1. \tag{3}$$

Recall that the set $\mathcal{P}$ of potential players is infinite. So, there is an infinite number of equations similar to (3). As we will show in the next section, this is sufficient to determine the collection $(\varphi_T)_{\emptyset\neq T\subseteq\mathcal{P}}$ for any power index that satisfies (E), (NPM) and (SC). For illustration, we show, in the example below, how to determine φ_T when T is a singleton.

Example 3.1. Suppose that $|S| = 2$ and set $S = \{i, j\}$. We have to determine $\varphi_{\{i\}} = \varphi_j\left(G_{\{i,j\}}\right)$ and $\varphi_{\{j\}} = \varphi_i\left(G_{\{i,j\}}\right)$ assuming that φ is a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). We first bring into consideration a new player, say $k \in \mathcal{P}\backslash\{i, j\}$. As stated in (3), we have

$$E_{\{i,j\}} : \varphi_{\{i\}} + \varphi_{\{j\}} = 1,\ E_{\{i,k\}} : \varphi_{\{i\}} + \varphi_{\{k\}} = 1,\ E_{\{j,k\}} : \varphi_{\{j\}} + \varphi_{\{k\}} = 1.$$

Equivalently

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_{\{i\}} \\ \varphi_{\{j\}} \\ \varphi_{\{k\}} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since the corresponding matrix is invertible, we get

$$\begin{pmatrix} \varphi_{\{i\}} \\ \varphi_{\{j\}} \\ \varphi_{\{k\}} \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Instead of inverting the matrix, it is sufficient to note that $\frac{1}{2}\left(E_{\{i,j\}}\right) + \frac{1}{2}\left(E_{\{i,k\}}\right) - \frac{1}{2}\left(E_{\{j,k\}}\right)$ would have immediately lead us to $\varphi_{\{i\}} = \frac{1}{2}$. In the general case, we provide in the next section appropriate combinations to obtain φ_T.

3.2. Main result

It is shown in the preceding section that the Public Good Index satisfies (NP), (E), (NPM) and (SC). We now prove that it is the unique power index on $\mathcal{G}$ that meets (E), (NPM) and (SC). Before the statement of the main result, we need the following lemmas to ease its proof.

Lemma 3.1. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC).*
Then for all coalitions $C \subseteq \mathcal{P}$, $\varphi(G_C) = \Phi(G_C)$.

Proof. See Appendix D. □

Lemma 3.2. *Let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC).*
Then $\varphi(G) = \Phi(G)$ for all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ such that $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$.

Proof. See Appendix E. □

Theorem 3.1. *A power index φ on $\mathcal{G}$ satisfies (E), (NPM) and (SC) if and only if $\varphi = \Phi$.*

Proof. By Proposition 3.1, Corollary 3.1 and Proposition 3.3, Φ necessary satisfies (E), (NPM) and (SC). For the sufficiency part of the proof, see Appendix F. □

3.3. Independence of axioms

Theorem 3.1 provides an axiomatic characterization of the Public Good Index. Moreover, we prove in this section that those axioms are independent (none of the three axioms is redundant).

3.3.1. Axiom of Non Profitable Merging of independent players

Consider the power index φ defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by:

$$\varphi_i(G) = \begin{cases} \dfrac{1}{|N \backslash N^0(G)|} & \text{if } i \in N\backslash N^0(G) \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that φ satisfies (E) and (SC). To see that φ fails to meet (NPM), let $G = (N, \mathcal{W}(G))$ with $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{M}(G) = \{\{1, 3, 5\}, \{2, 3, 4\}\}$. Set $T = \{4, 5\}$. Then T is a pair of independent players. Moreover $N^T = \{1, 2, 3, i_T\}$, $\varphi_4(G) + \varphi_5(G) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \neq \varphi_{i_T}(G^T) = \frac{1}{4}$. Hence φ does not satisfy (NPM). It follows that (NPM) is not redundant.

3.3.2. Axiom of Efficiency

The power index 2Φ (where Φ is the Public Good Index) satisfies (NPM) and (SC); but fails to meet (E). Thus (E) is not redundant.

3.3.3. Axiom of Supplementation Consistency

Consider the power index H defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ as follows:

$$H_i(G) = \frac{|\mathcal{M}_i^*(G)|}{\sum_{j\in N} |\mathcal{M}_j^*(G)|}$$

where $\mathcal{M}^*(G) = \{S \in \mathcal{M}(G) : |S| \leq |T| \text{ for all } T \in \mathcal{M}(G)\}$ and $\mathcal{M}_i^*(G) = \{S \in \mathcal{M}^*(G) : i \in S\}$ for all $i \in N$. The power index H satisfies (E) and (NPM), but H fails to meet (SC). To prove that (SC) is not satisfied, consider the simple game $G = (N, \mathcal{W}(G))$ such that $N = \{1, 2, 3, 4\}$ and $\mathcal{M}(G) = \{\{1, 3\}, \{2\}, \{3, 4\}\}$. Set $G' = \left(N \cup \{5\}, \mathcal{W}(G')\right) \in \mathcal{G}$ with $\mathcal{M}(G') = \{\{1, 3\}, \{2, 5\}, \{3, 4, 5\}\}$. Then G' is a 5-supplementation of G. Moreover, $\mathcal{M}^*(G) = \{\{2\}\}$ and $\mathcal{M}^*(G') = \{\{1, 3\}, \{2, 5\}\}$. It follows that $H(G) = (0, 1, 0, 0)$ and $H(G') = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\right)$. It appears that the index H does not satisfy (SC) since H satisfies (E) and $H_3\left(G'\right) = \frac{1}{4} \neq 0 = \left(1 - H_5\left(G'\right)\right) H_3\left(G\right)$. Thus (SC) is not redundant.

3.4. Further results

In this section, two new axioms are introduced by reformulating Axiom (NPM) when by merging independent players, our attention is focused on the impact of this operation on the shares of other players.

Axiom 5. ***Independence of External Merging (IEM) independent players*** : For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players, $\varphi_i(G^T) = \varphi_i(G)$ for all $i \in N \setminus T$ whenever T is a coalition of independent players.

While a merging operation involves independent players from one game to another, the shares of other players remain unchanged.

Proposition 3.7. *If a power index φ satisfies (E) and (IEM), then φ satisfies (NPM).*

Proof. Suppose that a power index φ satisfies (E) and (IEM). Consider $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition $T \subseteq N$ of independent players. We have:

$$\begin{aligned}\varphi_{i_T}(G^T) &= 1 - \sum_{i\in N\backslash T} \varphi_i(G^T) \text{ by (E)} \\ &= 1 - \sum_{i\in N\backslash T} \varphi_i(G) \text{ by (IEM)} \\ &= \sum_{i\in T} \varphi_i(G) \text{ by (E).}\end{aligned}$$

Therefore φ satisfies (NPM). □

Theorem 3.2. *A power index φ on $\mathcal{G}$ satisfies (E), (SC) and (IEM) if and only if $\varphi = \Phi$.*

Proof. Necessity. It remains to prove that Φ satisfies (IEM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a coalition T of at least two independent players with $|T| \geq 2$. Parts (c) and (d) in Proposition 3.2 imply that $\Phi_i(G^T) = \Phi_i(G)$ for all $i \in N\backslash T$. Therefore Φ satisfies (IEM).

Sufficiency. By Proposition 3.7 and Theorem 3.1. □

In the following axiom, the members of a coalition T of independent players in a simple game G have to choose their representative in T. This is the case in an annexation described by Aziz et al. [8] when a player in T takes the voting weight of other members of T. The notation $G^{T\to t}$ is extended below so that the possibility for t to be a player from T is now included.

Axiom 6. ***Independence of Internal Merging (IIM) independent players***: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players, for $t \in T$, $\varphi_i(G^{T\to t}) = \varphi_i(G)$ for all $i \in N \setminus T$ whenever T is a coalition of independent players.

When the members of a coalition T of independent players are offered the possibility to merge into a player in T, (IIM) requires that this should not impact the shares of players out of T.

Proposition 3.8. *All power indices on $\mathcal{G}$ that satisfy (IIM) also satisfy (IEM).*

Proof. Suppose that a power index φ on $\mathcal{G}$ satisfies (IIM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition T of independent players with $|T| \geq 2$ and a player $i_T \in \mathcal{P}\backslash N$. Set $G_1 = G[i_T]$. By noting that $|N| \geq |T| \geq 2$, consider for all $i \in N$, some $j \in N\backslash\{i\}$. Since $G = G_1^{\{j,i_T\}\to j}$, then by (IIM), it follows that $\varphi_i(G) = \varphi_i(G_1)$. Now, note that $G^T = (G_1)^{T\cup\{i_T\}\to i_T}$. Therefore, (IIM) implies that for all $i \in N\backslash T$, $\varphi_i(G^T) = \varphi_i(G_1)$ and therefore, $\varphi_i(G^T) = \varphi_i(G)$. This proves that φ satisfies (IEM). □

Theorem 3.3. *A power index φ on $\mathcal{G}$ satisfies (E), (SC) and (IIM) if and only if $\varphi = \Phi$.*

Proof. Necessity. It remains to prove that Φ satisfies (IIM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition T of independent players with $|T| \geq 2$ and a player $t \in T$. Given $i_T \in \mathcal{P}\backslash N$, note that $G^{T\to t}$ is obtained from G^T when i_T is replaced by t. Parts (c) and (d) in Proposition 3.2 imply that $\Phi_i(G^{T\to t}) = \Phi_i(G)$ for all $i \in N\backslash T$. Therefore Φ satisfies (IIM).

Sufficiency. See Proposition 3.8 and Theorem 3.2. □

It can be checked, using the power indices presented in Section 3.3, that axioms in Theorem 3.2 as well as in Theorem 3.3 are independent.

4. Some perspectives

We present in this section three plausible directions that deserve further attention.

4.1. A weak version of the merging condition

Proposition 3.5 states that (NPM) implies (NP). It may be of interest to see how to weaken the (NPM) condition so that (NP) is explicitly needed; thus highlighting how far do these two axioms overlap. On this line, we show that when (NPM) is replaced in Theorem 3.1 with its weaker version obtained by restricting the merging operation only to independent players who are decisive in at least one coalition, incorporating (NP) is a necessary condition. Furthermore, to characterize the Public Good Index, we also need the classical axiom of symmetry we now recall. Two payers i and j are called symmetric in a simple game $G = (N, \mathcal{W}(G))$ if for all $S \subseteq N\backslash\{i, j\}$, $S \cup \{i\}$ and $S \cup \{j\}$ are both winning or both losing.

Axiom 7. ***Symmetry (S)***: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all pairs $\{i, j\}$ of symmetric players in G, $\varphi_i(G) = \varphi_j(G)$.

We say that a merging operation from G to G^T is *effective* if T contains no null player in the game G. The following axiom is a weak version of Axiom (NPM) when only effective merging of independent players are considered.

Axiom 8. ***Non Profitable Effective Merging (NPEM)* of independent players**: For all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, for all coalitions $T \subseteq N$ of at least two players,

$$\varphi_{i_T}(G^T) = \sum_{i\in T} \varphi_i(G)$$

whenever T contains no null player and only independent players.

Proposition 4.1. *If a power index φ on $\mathcal{G}$ satisfies (E), (SC), (NPEM), (NP) and (S), then φ satisfies (NPM).*

Proof. See Appendix G. □

Theorem 4.1. *A power index φ on $\mathcal{G}$ satisfies (E), (SC), (NPEM), (NP) and (S) if and only if $\varphi = \Phi$.*

Proof. The proof follows from Proposition 4.1 and Theorem 3.1. □

Each of the three power indices presented in Section 3.3 satisfies both (NP) and (S). Those power indices can then be used to prove that in Theorem 4.1, none of the axioms (E), (SC) and (NPEM) can be dropped. For (NP) and (S), we consider the two power indices below:

- Define the power index I on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by

$$I_i(G) = \begin{cases} \Phi_i(G) & \text{if } |\mathcal{M}(G)| \geq 2 \\ \frac{1}{|N|} & \text{otherwise} \end{cases}$$

The power index I satisfies (E), (SC), (NPEM) and (S); but I obviously fails to meet (NP) over the class of unanimity simple games or simple games with only a unique minimal coalition. This proves that (NP) is not redundant in Theorem 4.1.

- Consider the power index J defined on $\mathcal{G}$ for all $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and for all $i \in N$ by

$$J_i(G) = \begin{cases} \Phi_i(G) & \text{if } |\mathcal{M}(G)| \geq 2 \\ 1 & \text{if } \mathcal{M}(G) = \{S\} \text{ for some coalition } S \text{ such that } i \in S \text{ and } i \leq j \text{ for all } j \in S \\ 0 & \text{otherwise} \end{cases}$$

The power index J satisfies (E), (SC), (NPEM) and (NP); but J obviously fails to meet (S) by considering simple games with a unique minimal winning coalition. This proves that (S) is not redundant in Theorem 4.1.

4.2. Relations with other power indices

Several other power indices on simple games depend only on the size of some subsets of the set of minimal winning coalitions. This is the case for the Deegan–Packel Index by Deegan and Packel [13] and the Shift Power Index by Alonso-Meijide and Freixas [6]. It can be checked that the Deegan–Packel Index satisfies both (E) and (NPM); but fails to meet (SC). One can also check that the Shift Power Index satisfies (E); but fails to meet neither (NPM) nor (SC). Therefore it is an interesting issue to find, for each of these two indices, which appropriate versions of the axioms presented in this paper are satisfied.

4.3. Relations with coalitional versions of the Public Good Index

Alonso-Meijide et al. [3] has extended the Public Good Index to simple games with a priori unions and provided an axiomatization of the whole class of coalitional versions of the Public Good Index; these are power indices on simple games with a priori unions that coincide with the Public Good Index on simple games which are identifiable with simple games with a priori unions containing only singletons; see also Holler and Nohn [28] for some alternative versions of the Public Good Index for simple games with a priori unions. This is done by reformulating the axioms used by Holler and Packel [30]. In a similar way, we provide below another characterization of coalitional versions of the Public Good Index. Following Alonso-Meijide et al. [3], we recall the following definitions.

Definition 4.1. Given a finite nonempty set N of players:

1. A simple game with a priori unions is a 3-tuple (N, v, P), where (N, v) is a simple game and P, a partition of N called the set of a priori unions.
2. A coalitional power index is any mapping that associates a simple game with a priori unions (N, v, P) with an $n-$tuple of real numbers. That is, $f(N, v, P) = (f_i(N, v, P))_{i\in N}$.
3. A coalitional power index f is said to be a coalitional version of a given power index g on simple games if $f(N, v, P^N) = g(N, v)$ for all $(N, v) \in \mathcal{G}_N$, where $P^N = \{\{i\} : i \in N\}$.

In particular, a coalitional power index f is a coalitional version of the Public Good Index if:

$$f(N, v, P^N) = \Phi(N, v), \forall (N, v) \in \mathcal{G}_N. \tag{4}$$

For power indices on simple games with a priori unions, the following axiom is due to Alonso-Meijide et al. [3].

Axiom 9. Singleton Efficiency (SE): For all $G = (N, v) \in \mathcal{G}$, $\sum_{i\in N} f_i(N, v, P^N) = 1$.

In a similar way, we now introduce the following versions of the axioms we early presented.

Axiom 10. Supplementation Consistency with a priori Unions (SCU): For all $G = (N, v) \in \mathcal{G}$, for all $k \in \mathcal{P}\backslash N$, and for all k-supplementations $G' = (N', v')$ of G, there exists a real constant $\lambda_{G'}$ such that

$$f_i\left(N', v', P^{N'}\right) = \lambda_{G'} f_i\left(N, v, P^N\right) \text{ for all } i \in N.$$

Axiom 11. No Profitable Merging with a priori Unions (NPMU): For all $G = (N, v) \in \mathcal{G}$, for all $T \subseteq N$ with $|T| \geq 2$, for all $i_T \in \mathcal{P}\backslash N$, $f_{i_T}(N^T, v^T, P^{N^T}) = \sum_{i\in T} f_i(N, v, P^N)$ whenever T consists of independent players in G.

Proposition 4.2. *A coalitional power index f is a coalitional version of the Public Good Index if and only if f satisfies (SE), (SCU) and (NPMU).*

Proof. Straightforward from Theorem 3.1 and (4). $\square$

Each of the theorems presented in this paper on simple games can be extended to simple games with a priori unions using very similar arguments. Furthermore, Alonso-Meijide et al. [2] characterized two particular coalitional versions of the Public Good Index (the Solidarity PGI and the Union PGI). These two power indices have been also extended and characterized by Alonso-Meijide et al. [4] in the context of simple games with coalition configurations (a given player might belong to more than one coalition of the configuration). It would be interesting to scrutinize and enrich all these results in the context of a variable electorate.

Appendix A. Proof of Proposition 2.2

Proof. Let $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, $k \in \mathcal{P} \setminus N$ and $G' = (N', \mathcal{W}(G')) \in \mathcal{G}$ such that $N' = N \cup \{k\}$.

Necessity. Suppose that G' is a k-supplementation of G. Let $E = \mathcal{M}(G) \setminus \mathcal{W}(G')$. We prove that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. We first show that $\mathcal{M}(G') \supseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. To see this, first consider a coalition $R \subseteq N'$ such that $R = S \cup \{k\}$ for some $S \in E$. By definition of E, $R \setminus \{k\} = S \notin \mathcal{W}(G')$, then $S \notin \mathcal{M}(G')$. Since $S \in E \subseteq \mathcal{M}(G)$, it follows by assumption on G' that $S \cup \{k\} \in \mathcal{M}(G')$. That is $R \in \mathcal{M}(G')$. Now, consider a coalition $R \subseteq N'$ such that $R \in \mathcal{M}(G) \setminus E = \mathcal{M}(G) \cap \mathcal{W}(G')$. Then $R \in \mathcal{M}(G)$ and $R \cup \{k\} \notin \mathcal{M}(G')$ since $R \in \mathcal{W}(G')$. It follows by assumption on G' that $R \in \mathcal{M}(G')$.

We now prove that $\mathcal{M}(G') \subseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$. For this purpose, consider $R \in \mathcal{M}(G')$. If $k \notin R$, then by definition of G', $R \in \mathcal{M}(G)$ and therefore $R \in \mathcal{M}(G) \cap \mathcal{M}(G') \subseteq \mathcal{M}(G) \cap \mathcal{W}(G') = \mathcal{M}(G) \setminus E$. Now suppose that $k \in R$, that is $R = S \cup \{k\}$ for some $S \subseteq N$. It follows that $S \in \mathcal{M}(G)$ since G' is a k-supplementation of G by assumption. Moreover, $R \setminus \{k\} = S \notin \mathcal{W}(G')$ Therefore $S \in \mathcal{M}(G) \setminus \mathcal{W}(G') = E$. We conclude that $\mathcal{M}(G') \subseteq \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$.

In summary, $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$.

Sufficiency. Assume that $\mathcal{M}(G') = \{S \cup \{k\} : S \in E\} \cup \{S : S \in \mathcal{M}(G) \setminus E\}$ for some subset E of $\mathcal{M}(G)$. We prove that G' is a k-supplementation of G. Consider $S \subseteq N$.

First suppose that $S \in \mathcal{M}(G)$. If $S \notin E$, then $S \in \mathcal{M}(G) \setminus E$ and this implies that $S \in \mathcal{M}(G')$. If $S \in E$, then it follows by assumption that $S \cup \{k\} \in \mathcal{M}(G')$. We have prove that $S \in \mathcal{M}(G)$ implies $S \in \mathcal{M}(G')$ or $S \cup \{k\} \in \mathcal{M}(G')$. Now suppose that $S \in \mathcal{M}(G')$ or $S \cup \{k\} \in \mathcal{M}(G')$. If $S \in \mathcal{M}(G')$ then $S \in \mathcal{M}(G) \setminus E$ since $k \notin S$. Thus $S \in \mathcal{M}(G)$. If $S \cup \{k\} \in \mathcal{M}(G')$ then $S \in E \subseteq \mathcal{M}(G)$. Therefore in both cases, $S \in \mathcal{M}(G)$. In summary, G' is a k-supplementation of G. $\square$

Appendix B. Proof of Proposition 3.2

Proof. Suppose that T is a coalition of independent players with $|T| \geq 2$.

Part (a). Consider $S, R \in \mathcal{M}(G)$ such that $S \neq R$. First suppose that $S \cap T = R \cap T = \emptyset$. By definition, $S^T = S \neq R = R^T$. Now suppose that $S \cap T \neq \emptyset$ and $R \cap T = \emptyset$. Then $i_T \in S^T$ and $i_T \notin R^T$. Therefore $S^T \neq R^T$. Similarly, $S^T \neq R^T$ if $S \cap T = \emptyset$ and $R \cap T \neq \emptyset$. Finally suppose that $S \cap T \neq \emptyset$ and $R \cap T \neq \emptyset$. Since T contains only pairs of independent (disconnected) players, then $S \cap T = \{i\}$ and $R \cap T = \{j\}$ for some $i, j \in T$. If $i = j$, then $S^T \setminus R^T = S \setminus R \neq \emptyset$ since $R \in \mathcal{M}(G)$. Otherwise $i \neq j$. Players i and j are independent, $S \in \mathcal{M}_i(G)$ and $R \in \mathcal{M}_j(G)$, thus by Proposition 2.1, $S \setminus (R \cup \{i\}) \neq \emptyset$. Since $S \setminus (R \cup \{i\}) \subseteq S^T \setminus R^T$, it follows that $S^T \neq R^T$. In each possible case, $S^T \neq R^T$.

Part (b). We first prove that $\mathcal{M}(G^T) \supseteq \{S^T : S \in \mathcal{M}(G)\}$. Let $S \in \mathcal{M}(G)$. If $S \cap T = \emptyset$, then $v^T(S^T) = v(S) = 1$ and for all $i \in S$, $v^T(S \setminus \{i\}) = v(S \setminus \{i\}) = 0$ since $S \in \mathcal{M}(G)$. Thus, $S^T \in \mathcal{W}(G^T)$ and $S \setminus \{i\}$ is losing in G^T for all $i \in S$. That is $S^T \in \mathcal{M}(G^T)$. Otherwise, $|S \cap T| = 1$ since $S \in \mathcal{M}(G)$ and T contains only pairs of disconnected players. Set $S \cap T = \{i\}$. First note that $v^T(S^T) = v(S \cup T) = 1$ and $v^T(S^T \setminus \{i_T\}) = v(S \setminus \{i\}) = 0$ and thus $S^T \setminus \{i_T\} \notin \mathcal{W}(G^T)$. Now consider $j \in S^T \setminus \{i_T\}$ and suppose that $S^T \setminus \{j\} \in \mathcal{W}(G^T)$. By definition, $v^T(S^T \setminus \{j\}) = v((S \setminus \{i, j\}) \cup T) = 1$. It follows that $(S \setminus \{i, j\}) \cup T \in \mathcal{W}(G)$ and therefore, there exists $A \subseteq (S \setminus \{i, j\}) \cup T$ such that $A \in \mathcal{M}(G)$. Set $A = A' \cup A''$ with $A' \subseteq S \setminus \{i, j\}$ and $A'' \subseteq T$. Note that $A'' \neq \emptyset$ since $A' \subseteq S \setminus \{i, j\} \notin \mathcal{W}(G)$. Moreover $A'' = \{k\}$ for some $k \in T$ since T contains only pairs of disconnected players. Recalling that $S \setminus \{j\} \notin \mathcal{W}(G)$, it follows that $k \neq i$. Note that $\{i, k\} \subseteq T$, $S = (S \setminus \{i\}) \cup \{i\} \in \mathcal{D}_i(G)$ and $A \subseteq (S \setminus \{i\}) \cup \{k\} \in \mathcal{D}_k(G)$. Since i and k are independent, therefore there exists $K \subseteq S \setminus \{i\}$ such that $K \cup \{i\} \in \mathcal{D}_i(G)$ and $K \cup \{k\} \notin \mathcal{D}_k(G)$. Hence $K \subsetneq S \setminus \{i\}$ and thus $K \cup \{i\} \subsetneq S$. A contradiction arises since $K \cup \{i\} \in \mathcal{W}(G)$ and $S \in \mathcal{M}(G)$. This proves that $S^T \setminus \{j\} \notin \mathcal{W}(G^T)$ for all $j \in S^T \setminus \{i_T\}$. Therefore $S^T \in \mathcal{M}(G^T)$. We conclude that $\mathcal{M}(G^T) \supseteq \{S^T : S \in \mathcal{M}(G)\}$.

Now, we prove that $\mathcal{M}(G^T) \subseteq \{S^T : S \in \mathcal{M}(G)\}$. Let $R \in \mathcal{M}(G^T)$. If $i_T \notin R$, then $v^T(R) = v(R) = 1$ and for all $i \in R$, $v^T(R \setminus \{i\}) = v(R \setminus \{i\}) = 0$. This implies that $R \in \mathcal{M}(G)$ and $R = R^T \in \{S^T : S \in \mathcal{M}(G)\}$. Otherwise, $i_T \in R$. Two possible cases arise. First suppose that $R = \{i_T\}$. Then $v^T(R) = v(T) = 1$. This implies that T contains in the simple game G some minimal winning coalition K. Therefore $R = K^T \in \{S^T : S \in \mathcal{M}(G)\}$. Now suppose that $R \neq \{i_T\}$. Since $R \in \mathcal{M}(G^T)$, we have $v^T(R) = v((R \setminus \{i_T\}) \cup T) = 1$ while $v^T(R \setminus \{i_T\}) = v(R \setminus \{i_T\}) = 0$. It follows that there exists some nonempty subset L of T such that $(R \setminus \{i_T\}) \cup L \in \mathcal{D}_i(G)$ for all $i \in L$. Such a coalition L can be obtained from $(R \setminus \{i_T\}) \cup T$ by removing, one by one, some members of T. Note that $[(R \setminus \{i_T\}) \cup L]^T = R$. To prove that $R \in \{S^T : S \in \mathcal{M}(G)\}$, we only have to prove that $(R \setminus \{i_T\}) \cup L$ is necessary a minimal winning coalition in the simple game G. Suppose on the contrary that this is not the case. Then, there exists $j \in R \setminus \{i_T\}$ such that $(R \setminus \{i_T, j\}) \cup L$ is still a winning coalition in the simple game G. By the definition of G^T, $[(R \setminus \{i_T, j\}) \cup L]^T = R \setminus \{j\}$ is a proper subset of R which is a winning coalition in G^T. A contradiction arises since $R \in \mathcal{M}(G^T)$. Therefore $(R \setminus \{i_T\}) \cup L$ is a minimal winning coalition in the simple game G. In each possible case, $R \in \{S^T : S \in \mathcal{M}(G)\}$. We conclude that $\mathcal{M}(G^T) \subseteq \{S^T : S \in \mathcal{M}(G)\}$.

Part (c). Given $j \in N \setminus T$, note that for all coalitions $S \subseteq N$, $j \in S^T$ if and only if $j \in S$. Therefore, Part (b) implies that $\mathcal{M}_j(G^T) = \{S^T : S \in \mathcal{M}(G) \text{ and } j \in S^T\} = \{S^T : S \in \mathcal{M}_j(G)\}$. By Part (a), the operator $S \longmapsto S^T$ is injective in $\mathcal{M}(G)$. Therefore $|\mathcal{M}_j(G^T)| = |\{S^T : S \in \mathcal{M}_j(G)\}| = |\mathcal{M}_j(G)|$.

Part (d). By Part (b), $\mathcal{M}_{i_T}(G^T) = \{S^T : S \in \mathcal{M}(G) \text{ and } T \cap S \neq \emptyset\}$. Recall that T contains only pairs of independent (disconnected) players, then, $|S \cap T| \leq 1$ for all $S \in \mathcal{M}(G)$. It follows that $\mathcal{M}_{i_T}(G^T) = \{S^T : S \in \mathcal{M}(G) \text{ and } i \in S \text{ for some } i \in T\} = \cup_{i \in T}\{S^T : S \in \mathcal{M}_i(G)\}$. Moreover $\mathcal{M}_i(G) \cap \mathcal{M}_j(G) = \emptyset$ for pairs $\{i, j\}$ of distinct players in T. Taking into account that the operator $S \longmapsto S^T$ is injective in $\mathcal{M}(G)$, one finally gets $|\mathcal{M}_{i_T}(G^T)| = \sum_{i \in T} |\mathcal{M}_i(G)|$. □

Appendix C. Proof of Proposition 3.5

Proof. Suppose that φ is a power index that satisfies (NPM). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ and a null player $k \in N$ in G. Denote by G_0 the simple game obtained when k leaves the game G while the set of minimal winning coalitions is unchanged; that is $G = G_0[k]$. To prove that $\varphi_k(G) = 0$, we consider $\{k_1, k_2, k_3, k_4, k_5, k_6\} \subset \mathcal{P} \backslash N$ and the following simple games :

$$G_1 = G_0[k_1, k_2, k_3, k_4],\ \ G_2 = G_0[k_1, k_2, k_5], G_3 = G_0[k_5, k_6] \text{ and } G_4 = G_0[k_1, k_2, k_3, k_6].$$

Note that any player from $\{k_1, k_2, k_3, k_4, k_5, k_6\}$ who is involved in a games G_j for $j = 1, 2, 3, 4$ is a null player. Moreover, in terms of the merging operation, the following holds :

$$G = G_1^{\{k_1,k_2,k_3,k_4\}\to k} = G_2^{\{k_1,k_2,k_5\}\to k}, G_2 = G_1^{\{k_3,k_4\}\to k_5} \text{ and } G_3 = G_2^{\{k_1,k_2\}\to k_6}. \tag{C.1}$$

Furthermore, applying (NPM) on each of the above mentioned merging operations leads to

$$\begin{aligned}
\varphi_{k_1}(G_1) + \varphi_{k_2}(G_1) &= \varphi_k(G) - \varphi_{k_3}(G_1) - \varphi_{k_4}(G_1) \text{ since by (C.1), } G = G_1^{\{k_1,k_2,k_3,k_4\}\to k}\\
&= \varphi_k(G) - \varphi_{k_5}(G_2) \text{ since by (C.1), } G_2 = G_1^{\{k_3,k_4\}\to k_5}\\
&= \varphi_{k_1}(G_2) + \varphi_{k_2}(G_2) \text{ since by (C.1), } G = G_2^{\{k_1,k_2,k_5\}\to k}\\
&= \varphi_{k_6}(G_3) \text{ since by (C.1), } G_3 = G_2^{\{k_1,k_2\}\to k_6}.
\end{aligned}$$

In a similar way, $\varphi_i(G_1) + \varphi_j(G_1) = \varphi_{k_6}(G_3)$ for all pairs $\{i, j\} \subseteq \{k_1, k_2, k_3, k_4\}$. This proves that $\varphi_i(G_1) + \varphi_j(G_1)$ does not depend on the pair $\{i, j\} \subseteq \{k_1, k_2, k_3, k_4\}$. Since $\varphi_{k_1}(G_1) + \varphi_{k_2}(G_1) + \varphi_{k_3}(G_1) + \varphi_{k_4}(G_1) = \varphi_k(G)$, it follows that $\varphi_i(G_1) = \frac{1}{4}\phi_k(G)$ for all $i \in \{k_1, k_2, k_3, k_4\}$ and $\varphi_{k_6}(G_3) = \frac{1}{2}\varphi_k(G)$. In the same way, $\varphi_i(G_4) = \frac{1}{4}\varphi_k(G)$ for all $i \in \{k_1, k_2, k_3, k_6\}$.

Noting that $G_3 = G_4^{\{k_1,k_2,k_3\}\to k_5}$, it follows by (NPM) that $\varphi_{k_5}(G_3) = \frac{3}{4}\varphi_k(G)$. Moreover $G = G_3^{\{k_5,k_6\}\to k}$. Thus $\varphi_{k_6}(G_3) = \varphi_k(G) - \varphi_{k_5}(G_3) = \frac{1}{4}\varphi_k(G)$. Hence $\frac{1}{2}\varphi_k(G) = \frac{1}{4}\varphi_k(G)$ and therefore $\varphi_k(G) = 0$. □

Appendix D. Proof of Lemma 3.1

Proof. To ease the proof, we introduce, for all integers $p \geq 2$, the sequence $(c_m)_{1\leq m\leq p}$ defined by

$$c_m = \frac{(-1)^{m-1}}{\binom{p-1}{m-1}}.$$

where $\binom{p-1}{m-1}$ is the binomial coefficient. Note that it can be easily checked that for $0 \leq m < p$,

$$(p - m)\, c_{m+1} + mc_m = 0. \tag{D.1}$$

Now, let φ be a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Consider a coalition $C \subseteq \mathcal{P}$ and set $|C| = p$. If $p = 1$, then $C = \{k\}$ and $\mathcal{M}\,(G_C) = \{\{k\}\}$ for some $k \in \mathcal{P}$. By efficiency, $\varphi_k\,(G_C) = \Phi_k\,(G_C) = 1$. Clearly, $\varphi\,(G_C) = \Phi\,(G_C)$. Now suppose that $p \geq 2$. As we announced earlier, we consider a coalition $C' \subseteq \mathcal{P}\backslash S$ of $p - 1$ players. Set $N = C \cup C'$ and denote by $\mathcal{H}$ the set of all simple games G_S such that $S \subseteq N$ and $|S| = p$. Note that there are exactly $\binom{2p-1}{p}$ simple games G_S in $\mathcal{H}$ that lead to $\binom{2p-1}{p}$ equation $(E_S)_{S\subseteq N,|S|=p}$, with exactly $\binom{2p-1}{p-1}$ variables $(X_T)_{T\subseteq N,|T|=p-1}$.

Since $\Phi_i\,(G_C) = \frac{1}{p}$ for all $i \in C$, we have to prove that $\varphi_i\,(G_C) = \frac{1}{p}$ for all $i \in C$. That is, $X_{C\backslash\{i\}} = \frac{1}{p}$ for all $i \in C$. Consider $i \in C$ and set $K = C\backslash\{i\}$. In (3) (see Remark 3.2), we multiply by $c_{p-|S\cap K|}$ the left-hand-side and the right-hand-side of each equation (E_S) such that $S \subseteq N$ and $|S| = p$. By summing over all left-hand-sides and over all right-hand-sides, we obtain

$$\sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \sum_{T\subseteq S,|T|=p-1} X_T = \sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \times 1 \tag{D.2}$$

On the one hand, the right-hand-side of (D.2), say $\sum_R$, is simplified as follows:

$$\sum_R = \sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \times 1 = \sum_{k=0}^{p-1} \sum_{S\subseteq N,|S|=p,|S\cap K|=k} c_{p-k}$$

Each coalition S such that $S \subseteq N$, $|S| = p$ and $|S \cap K| = k$ consists in k players from K and $p-k$ players from $N\backslash K$. Since $|K| = p-1$ and $|N\backslash K| = p$, there are exactly $\binom{p-1}{k}\binom{p}{p-k}$ such coalitions. Noting that $0 \le |S \cap K| \le p-1$, it follows that

$$\sum_R = \sum_{k=0}^{p-1} \binom{p}{p-k}\binom{p-1}{k} \frac{(-1)^{p-1-k}}{\binom{p-1}{p-1-k}}$$

Since $\binom{p-1}{p-1-k} = \binom{p-1}{k}$, it follows that

$$\sum_R = (-1)^{p-1} \sum_{k=0}^{p-1} (-1)^k \binom{p}{k} = (-1)^{p-1}\left(0-(-1)^p\right) = 1. \tag{D.3}$$

On the other hand, the left-hand-side of (D.2), say $\sum_L$, is simplified as follows:

$$\sum_L = \sum_{S\subseteq N,|S|=p} c_{p-|S\cap K|} \sum_{T\subseteq S,|T|=p-1} X_T = \sum_{T\subseteq N,|T|=p-1} X_T \sum_{S\subseteq N,|S|=p,T\subseteq S} c_{p-|S\cap K|}.$$

Given $T \subseteq N$ such that $|T| = p-1$, each coalition $S \subseteq N$ such that $|S| = p$ and $T \subseteq S$ can be rewritten as $S = T \cup \{l\}$ for some $l \in N\backslash T$. Furthermore, $N\backslash T = (K\backslash T) \cup (N\backslash(K \cup T))$. Thus,

$$\sum_L = \sum_{T\subseteq N,|T|=p-1} X_T \left(\sum_{l\in K\backslash T,S=T\cup\{l\}} c_{p-|S\cap K|} + \sum_{l\in N\backslash(T\cup K),S=T\cup\{l\}} c_{p-|S\cap K|} \right).$$

Consider a coalition $T \subseteq N$ such that $|T| = p-1$. First suppose that $T = K$. Then no coalition S exists such that $S = T \cup \{l\}$ for some $l \in K\backslash T$, since $K\backslash T = \emptyset$. And there are exactly p coalitions S such that $S = T\cup\{l\}$ for some $l \in N\backslash(T \cup K) = N\backslash K$. Now suppose that $T \neq K$. For all $l \in K\backslash T$ and $S = T \cup \{l\}$, $|S \cap K| = |T \cap K| + 1 \le p-1$. And for all $l \in N\backslash(T \cup K)$ and $S = T \cup \{l\}$, $|S \cap K| = |T \cap K|$. Therefore,

$$\sum_L = pX_K + \sum_{T\subseteq N,T\neq K,|T|=p-1} \left[|K\backslash T|\, c_{p-1-|T\cap K|} + |N\backslash(T\cup K)|\, c_{p-|T\cap K|} \right] X_T.$$

Since $|K| = p-1$ and $|N| = 2p-1$, it follows that for all $T \subseteq N$ such that $T \neq K$ and $|T| = p-1$, we have $|K\backslash T| = p-1-|T\cap K|$ and $|N\backslash(T\cup K)| = 2p-1-|T|-|K\backslash T| = p-|K\backslash T| = |T\cap K|+1$. Therefore,

$$\sum_L = pX_K + \sum_{T\subseteq N,T\neq K,|T|=p-1} \left[(p-1-|T\cap K|)\, c_{p-1-|T\cap K|} + (|T\cap K|+1)\, c_{p-|T\cap K|} \right] X_T.$$

For all $T \subseteq N$ such that $T \neq K$ and $|T| = p-1$, note that, the relation (D.1), taking $m = p-1-|T\cap K|$, implies that

$$(p-1-|T\cap K|)\, c_{p-1-|T\cap K|} + (|T\cap K|+1)\, c_{p-|T\cap K|} = 0 \text{ and } \sum_L = pX_K. \tag{D.4}$$

We conclude from (D.2), (D.3) and (D.4) that $X_K = \frac{1}{p}$ for all coalitions $K \subseteq \mathcal{P}$. Thus, $\varphi(G_C) = \Phi(G_C)$. □

Appendix E. Proof of Lemma 3.2

Proof. Suppose that φ is a power index on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). We denote by $\mathcal{G}_n$ the set of all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ with n players such that $|\mathcal{M}_i(G)| \le 1$ for all $i \in N$. We prove by induction on $n = |N| \ge 1$ the assertion $\mathcal{A}(n)$ that for all simple games $G \in \mathcal{G}_n$, $\varphi(G) = \Phi(G)$. In the initialization step, we consider $n \in \{1, 2\}$.

Suppose that $n = 1$ and let $G = (N, \mathcal{W}(G)) \in \mathcal{G}_1$. Then $N = \{i\}$ for some $i \in \mathcal{P}$. By efficiency, $\varphi(G) = \Phi(G)$.

Suppose that $n = 2$ and let $G = (N, \mathcal{W}(G)) \in \mathcal{G}_2$. Then $\mathcal{M}(G) = \{\{i\}\}$; or $\mathcal{M}(G) = \{\{j\}\}$; or $\mathcal{M}(G) = \{\{i\},\{j\}\}$; or $\mathcal{M}(G) = \{\{i,j\}\}$ with $N = \{i,j\}$ for some $i,j \in \mathcal{P}$. First suppose that $\mathcal{M}(G) = \{\{i\}\}$ with $N = \{i,j\}$. Then by efficiency and Proposition 3.5, $\varphi_i(G) = \Phi_i(G) = 1$ and $\varphi_j(G) = \Phi_j(G) = 0$ since j is a null player in G. That is $\varphi(G) = \Phi(G)$. Similarly, if

$\mathcal{M}(G) = \{\{j\}\}$, then $\varphi_j(G) = \Phi_j(G) = 1$ and $\varphi_i(G) = \Phi_i(G) = 0$. That is $\varphi(G) = \Phi(G)$. Now suppose that $\mathcal{M}(G) = \{\{i\}, \{j\}\}$ with $N = \{i, j\}$. Then $G = G_{\{i,j\}}$ and $\varphi(G) = \Phi(G)$ by Lemma 3.1. Finally, suppose that $\mathcal{M}(G) = \{\{i, j\}\}$ with $N = \{i, j\}$. Consider a player $k \in \mathcal{P}\backslash\{i, j\}$. Let $G_1 = (N_1, \mathcal{W}(G_1))$ and $G_2 = (N_2, \mathcal{W}(G_2))$ be the simple games defined by $N_1 = \{i, k\}$, $N_2 = \{j, k\}$, $\mathcal{M}(G_1) = \{\{i, k\}\}$ and $\mathcal{M}(G_2) = \{\{j, k\}\}$. Note that $G_1 = G^{j\leftrightarrow k}$, $G_2 = G^{i\leftrightarrow k}$ and $G_2 = G_1^{i\leftrightarrow j}$. By Proposition 3.6, it follows that

$$\varphi_i(G) = \varphi_k(G_2), \varphi_j(G) = \varphi_k(G_1) \text{ and } \varphi_i(G_1) = \varphi_j(G_2).$$

It then follows by efficiency with respect to G, G_1 and G_2,

$$\varphi_i(G) + \varphi_j(G) = 1, \varphi_i(G_1) + \varphi_j(G) = 1 \text{ and } \varphi_i(G_1) + \varphi_i(G) = 1.$$

Solving this three equations leads to $\varphi_i(G) = \varphi_j(G) = \varphi_i(G_1) = \frac{1}{2}$. Since $\Phi_i(G) = \Phi_j(G) = \frac{1}{2}$, it holds that $\varphi(G) = \Phi(G)$.

For the induction step, suppose that $\mathcal{A}(n)$ holds for some integer $n \geq 2$. We prove that $\mathcal{A}(n+1)$ necessarily holds. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{n+1}$. Set $C = \{i \in N : |\mathcal{M}_i(G)| = 1\}$ and $N^0(G) = \{i_1, i_2, \ldots, i_{n_0}\}$ with $n_0 = |N^0(G)|$ that is the number of null players in the simple game G. Note that $N = C \cup N^0(G)$. First suppose that $|S| = 1$ for all $S \in \mathcal{M}(G)$. Then $G = G_C[i_1, i_2, \ldots, i_{n_0}]$ and $\varphi_i(G) = 0 = \Phi_i(G)$ for all $i \in N^0(G)$ by Proposition 3.5. And for all $i \in C$, one have in the one hand $\varphi_i(G) = \varphi_i(G_C)$ and $\Phi_i(G) = \Phi_i(G_C)$ by Remark 3.1, and in the other hand $\varphi_i(G_C) = \Phi_i(G_C)$ by Lemma 3.1; therefore $\varphi_i(G) = \Phi_i(G)$. Then $\varphi(G) = \Phi(G)$. Now suppose that there exists some $S \in \mathcal{M}(G)$ such that $|S| \geq 2$. Consider three distinct players i, j and k in N such that $i, j \in S$ and $k \in C$. Let $S_i = S\backslash\{i\}$ and $S_j = S\backslash\{j\}$. Define the simple games $G_1 = (N_1, \mathcal{W}(G_1))$ and $G_2 = (N_2, \mathcal{W}(G_2))$ by $N_1 = N\backslash\{i\}$, $N_2 = N\backslash\{j\}$, $\mathcal{M}(G_1) = [\mathcal{M}(G)\backslash\{S\}] \cup \{S_i\}$ and $\mathcal{M}(G_2) = [\mathcal{M}(G)\backslash\{S\}] \cup \{S_j\}$. Since $G \in \mathcal{G}_{n+1}$, coalitions in $\mathcal{M}(G)$ are disjoints. Thus, no coalition in $\mathcal{M}(G)\backslash\{S\}$ contains S_i or S_j. This guarantees that the simple games G_1 and G_2 are well-defined. By the induction assumption, $\varphi_t(G_1) = \Phi_t(G_1) = \frac{1}{n_1}$ for all $t \in C \setminus \{i\}$ and $\varphi_t(G_2) = \Phi_t(G_2) = \frac{1}{n_2}$ for all $t \in C \setminus \{j\}$. Note that $n_1 = |N_1 \setminus N^0(G_1)| = |N \setminus (N^0(G) \cup \{i\})| = |N \setminus (N^0(G) \cup \{j\})| = |N_2 \setminus N^0(G_2)| = n_2$. Moreover, G is an i -supplementation of G_1 as well as a j-supplementation of G_2. Therefore, moving from G_1 to G, (SC) implies that

$$\varphi_j(G) = (1 - \varphi_i(G))\,\varphi_j(G_1) = \frac{1 - \varphi_i(G)}{n_1} \text{ and } \varphi_t(G) = (1 - \varphi_i(G))\,\varphi_t(G_1) = \frac{1 - \varphi_i(G)}{n_1}.$$

Similarly, from G_2 to G, (SC) implies that

$$\varphi_i(G) = \left(1 - \varphi_j(G)\right)\varphi_i(G_2) = \frac{1 - \varphi_j(G)}{n_1} \text{ and } \varphi_t(G) = \left(1 - \varphi_j(G)\right)\varphi_t(G_2) = \frac{1 - \varphi_j(G)}{n_1}.$$

We then deduce that

$$\varphi_t(G) = \frac{1 - \varphi_i(G)}{n_1} = \frac{1 - \varphi_j(G)}{n_1} \text{ with } \varphi_i(G) = \frac{1 - \varphi_j(G)}{n_1}.$$

Therefore

$$\varphi_i(G) = \varphi_j(G) = \frac{1}{n_1 + 1}.$$

Recalling that G is an i-supplementation of G_1, we deduce by (SC) that for all $t \in C\backslash\{i\}$,

$$\varphi_t(G) = (1 - \varphi_i(G))\,\varphi_t(G_1) = \frac{1 - \varphi_i(G)}{n_1} = \frac{1}{n_1 + 1}.$$

This proves that $\varphi_t(G) = \Phi_t(G) = \frac{1}{n_1+1}$ for all $t \in C$ where $n_1 + 1 = |\{t \in N : |\mathcal{M}_t(G)|= 1\}| = |N \setminus N^0(G)|$. Also, one have $\varphi_t(G) = 0 = \Phi_t(G)$ for all $t \in N^0(G)$ by Proposition 3.5. Finally, $\varphi(G) = \Phi(G)$. This proves that $\mathcal{A}(n+1)$ holds. Therefore, we conclude that $\mathcal{A}(n)$ holds for all integers $n \geq 2$. □

Appendix F. Proof of the sufficiency part of Theorem 3.1

Proof. Sufficiency. Consider a power index φ on $\mathcal{G}$ that satisfies (E), (NPM) and (SC). Given a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, we denote by $E(G)$ the set of all players i in G such that $|\mathcal{M}_i(G)| \geq 2$ and by $\mathcal{G}_{(m)}$ the set of all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}$ in which $E(G)$ contains exactly m players. We prove by induction on integer $m \geq 0$ the assertion $\mathcal{A}'(m)$ that for all simple games $G \in \mathcal{G}_{(m)}$, $\varphi(G) = \Phi(G)$.

For the initialization step, suppose that $m = 0$. For all simple games $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(0)}$, $E(G) = \emptyset$. That is $|\mathcal{M}_i(G)| \leq 1$ for all $i \in N$. Therefore, $\varphi(G) = \Phi(G)$ by Lemma 3.2. This prove that $\mathcal{A}'(0)$ holds.

For the induction step, suppose that $\mathcal{A}'(m)$ holds for some integer $m \geq 0$. We prove that $\mathcal{A}'(m+1)$ necessarily holds. Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}_{(m+1)}$. Then $|E(G)| = m + 1 \geq 1$. Thus, there exists some player $i \in E(G)$. Let $p = |\mathcal{M}(G)|$ and $q = |\mathcal{M}_i(G)|$. We write $\mathcal{M}_i(G) = \{S_1, S_2, \ldots, S_q\}$ and $\mathcal{M}(G) = \{S_1, S_2, \ldots, S_q, S_{q+1}, \ldots, S_p\}$. Consider $p + q$ distinct players $j_1, j_2, \ldots, j_p, i_1, i_2, \ldots, i_q \in \mathcal{P}\backslash N$. We define the simple games:

- $G_1 = (N_1, \mathcal{W}(G_1))$ with $N_1 = (N\backslash\{i\}) \cup \{j_1, j_2, \ldots, j_p\} \cup \{i_1, i_2, \ldots, i_q\}$ and
$$\mathcal{M}(G_1) = \left\{T_1 \cup \{i_1, j_1\}, \ldots, T_q \cup \{i_q, j_q\}, S_{q+1} \cup \{j_{q+1}\}, \ldots, S_p \cup \{j_p\}\right\}$$
where $T_t = S_t\backslash\{i\}$ for all $t \in \{1, 2, \ldots, q\}$.
- $G_2 = (N_2, \mathcal{W}(G_2))$ with $N_2 = N \cup \{j_1, j_2, \ldots, j_p\}$ and
$$\begin{aligned}\mathcal{M}(G_2) &= \left\{T_1 \cup \{i, j_1\}, \ldots, T_q \cup \{i, j_q\}, S_{q+1} \cup \{j_{q+1}\}, \ldots, S_p \cup \{j_p\}\right\} \\ &= \left\{S_1 \cup \{j_1\}, \ldots, S_q \cup \{j_q\}, S_{q+1} \cup \{j_{q+1}\}, \ldots, S_p \cup \{j_p\}\right\}.\end{aligned}$$
- $G_3 = (N_3, \mathcal{W}(G_3))$ with $N_3 = N_2\backslash\{i\}$ and
$$\mathcal{M}(G_3) = \left\{T_1 \cup \{j_1\}, \ldots, T_q \cup \{j_q\}, S_{q+1} \cup \{j_{q+1}\}, \ldots, S_p \cup \{j_p\}\right\}.$$

Note that G_1 is obtained from G by adding j_t to S_t for $t \in \{1, \ldots, p\}$ and replacing player i in each S_t by i_t for $t \in \{1, \ldots, q\}$. Since each new player belongs to exactly one minimal coalition in G_1, then $E(G_1) = E(G)\backslash\{i\}$. It follows that $|E(G_1)| = m$. Therefore by the induction assumption, $\varphi(G_1) = \Phi(G_1)$. Also note that moving from G_1 to G_2 consists in merging players $i_1, i_2, \ldots, i_q$ into i. Since $\{i_1, i_2, \ldots, i_q\}$ is a coalition of independent players in the simple game G_1, we then deduce that

$$\begin{aligned}\varphi_i(G_2) &= \sum_{t=1}^{q} \varphi_{i_t}(G_1) \text{ since } \varphi \text{ satisfies (NPM)} \\ &= \sum_{t=1}^{q} \Phi_{i_t}(G_1) \text{ since } \varphi(G_1) = \Phi(G_1) \\ &= \Phi_i(G_2) \text{ since } \Phi \text{ satisfies (NPM).}\end{aligned}$$

To continue, also note that $E(G_3) = E(G)\backslash\{i\}$, it follows that $|E(G_3)| = m$. Therefore by induction assumption, $\varphi(G_3) = \Phi(G_3)$. By observing that G_2 is an i-supplementation of G_3, it follows that for all $k \in N_2\backslash\{i\}$,

$$\begin{aligned}\varphi_k(G_2) &= (1 - \varphi_i(G_2))\,\varphi_k(G_3) \text{ since } \varphi \text{ satisfies (SC)} \\ &= (1 - \Phi_i(G_2))\,\Phi_k(G_3) \text{ since } \varphi_i(G_2) = \Phi_i(G_2) \text{ and } \varphi(G_3) = \Phi(G_3) \\ &= \Phi_k(G_2) \text{ since } \Phi \text{ satisfies (SC).}\end{aligned}$$

This proves that $\varphi(G_2) = \Phi(G_2)$. Finally, we define the simple games $\left(G'_t\right)_{0\le t\le p}$ by $G'_0 = G$ and for all $t \in \{1, 2, \ldots, p\}$, $G'_t = \left(N'_t, \mathcal{W}(G'_t)\right)$ with

$$N'_t = N \cup \{j_1, j_2, \ldots, j_t\} \text{ and } \mathcal{M}(G'_t) = \left\{S_1 \cup \{j_1\}, \ldots, S_t \cup \{j_t\}, S_{t+1}, \ldots, S_p\right\}.$$

Note that $G'_p = G_2$ and that G'_t is an j_t -supplementation of G'_{t-1} for all $t \in \{1, 2, \ldots, p\}$. Moreover $i \in N'_t$ and $\mathcal{M}_i(G'_t) \neq \emptyset$ for all $t \in \{1, 2, \ldots, p\}$. By the definition of Φ, it follows that $\Phi_i(G'_t) > 0$ for $t \in \{1, 2, \ldots, p\}$. This proves that, for all $k \in N'_t\backslash\{i\}$, $\Phi_k(G'_t) < 1$. Since $\varphi(G_2) = \Phi(G_2)$ and G'_p is an j_p-supplementation of G'_{p-1}, we deduce from Proposition 3.4 that $\varphi(G'_{p-1}) = \Phi(G'_{p-1})$ with $\Phi_{j_{p-1}}(G'_{p-1}) < 1$. By iterating this procedure for $t = p, p-1, \ldots, 1$, it holds that $\varphi(G'_0) = \Phi(G'_0)$. Since $G'_0 = G$, we get $\varphi(G) = \Phi(G)$. This proves that $\mathcal{A}'(m+1)$ holds. In conclusion, $\mathcal{A}'(m)$ holds for all integers $m \geq 0$. □

Appendix G. Proof of Proposition 4.1

Proof. Consider a power index φ on $\mathcal{G}$. Suppose that φ satisfies (E), (SC), (NPEM), (NP) and (S). Consider a simple game $G = (N, \mathcal{W}(G)) \in \mathcal{G}$, a coalition $T \subseteq N$ containing only independent players with $|T| \geq 2$. Let S be the set of all members of T who are null players in G; and $G' = (N\backslash S, \mathcal{W}(G'))$ be the simple game such that $\mathcal{M}(G') = \mathcal{M}(G)$. The game G' is obtained from G when the members of S leave G without altering the status of any minimal winning coalition.

Step 1. First suppose that S is empty. Then by (NPEM), condition (2) is satisfied.

Step 2. Now suppose that S is not empty and $T\backslash S = \{i\}$. Note that i is no longer a player in the game $G^T = (N^T, \mathcal{W}(G^T))$. Consider the game $G'' = (N^T \cup \{i\}, \mathcal{W}(G''))$ obtained from G^T when player i gets in the game by joining only all minimal winning coalitions that contain i_T; that is, for all $K \subseteq N^T \cup \{i\}$, $K \in \mathcal{M}(G'')$ if and only if ($i_T \notin K, i \notin K$ and $K \in \mathcal{M}(G^T)$), or ($i_T \in K$, $i \in K$ and $K\backslash\{i\} \in \mathcal{M}(G^T)$). On the one hand, G'' is an i-supplementation of the game G^T. Furthermore, i and i_T are symmetric players in G''. Therefore by (S), (E) and (SC), it follows that

$$\varphi_i(G'') = \varphi_{i_T}(G'') = (1 - \varphi_i(G''))\varphi_{i_T}(G^T). \tag{G.1}$$

On the other hand, G'' is an i_T-supplementation of G'. Thus, it follows that

$$\varphi_{i_T}(G'') = \varphi_i(G'') = (1 - \varphi_{i_T}(G''))\varphi_i(G'). \tag{G.2}$$

It follows from (G.1) and (G.2) that $\varphi_{i_T}(G^T) = \varphi_i(G')$. By noting that when a null player k leaves a simple game while the set of minimal winning coalitions remains unchanged, the shares of players other than k remain unchanged by applying both (E), (SC) and (NP). Therefore, $\varphi_i(G') = \varphi_i(G)$ since G' is obtained from G when the members of S leaves G. Therefore $\varphi_{i_T}(G^T) = \varphi_i(G)$. Hence condition (2) is satisfied.

Step 3. Finally, suppose that S is not empty and $|T\backslash S| \geq 2$. Set $T' = T\backslash S$. Then no null player belongs to T' and T' contains only independent players. By (NPEM),

$$\varphi_{i_{T'}}(G^{T'}) = \sum_{j\in T'} \varphi_j(G) = \sum_{j\in T} \varphi_j(G). \tag{G.3}$$

The second equality in (G.3) holds by (NP) since players in $S = T\backslash T'$ are null players in the game G. In the game $G^{T'}$, note that coalition $T'' = \{i_{T'}\} \cup S$ contains only independent players and all members of T'' are null players in $G^{T'}$ except player $i_{T'}$. As it is just shown in Step 2., merging in the game $G^{T'}$ the members of T'' into i_T implies that

$$\varphi_{i_T}\left(\left(G^{T'}\right)^{T''}\right) = \varphi_{i_{T'}}(G^{T'}). \tag{G.4}$$

Moreover, $\left(G^{T'}\right)^{T''} = G^T$. Therefore, condition (2) holds by (G.3) and (G.4).

In each of the three possible cases, condition (2) holds. That is φ satisfies (NPM). □

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