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Title :

Noncommutative Solvable Models in Quantum and Field Theories

Jury:

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Abstract

The dissertation work substantially bears our contribution to the development of noncommutative (NC) theories. It mainly provides a comparative study of ordinary complex scalar $\phi^4_{\star D}$ and complex Grosse-Wulkenhaar (GW) model. In this context, relevant physical quantities such as energy momentum tensors (EMTs) are explicitly computed and improved to satisfy known physics based properties in line with the Wilson and Jackiw techniques. In all these theories, the dilatation symmetry is broken and the breaking terms are discussed. As expected, all computed physical quantities for ordinary complex $\phi^4_{\star D}$ noncommutative field theory (NCFT) are easily recovered from the results obtained for the complex GW NCFT by setting $\Omega = 0$.

A generalization of the Hamiltonian formulation developed by Gomis *et al* is perfomed and analyzed for the renormalizable Grosse-Wulkenhaar $\phi^4_{\star D}$ model.

The dynamical noncommutativity introduced by Aschieri *et al* [3] is implemented and discussed in the case of a new class of renormalizable NC field theories (RNCFT) built on the GW $\phi^{\star 4}$ scalar field model defined in Euclidean space. Our investigations show that the twisted GW action is not invariant under global translation. Such an undesirable feature has been got round by imposing a constraint on the Lagrangian action, which is nothing but the equation of motion governing the GW harmonic term. Contrarily to pevious works, both ordinary GW and twisted GW models provide nonlocally conserved and nonsymmetric EMT, angular momentum tensor (AMT) and dilatation current (DC) due to the presence of the harmonic term Ω. Fortunately, all these physical quantities can be subjected to well known Jackiw and Wilson regularization procedures to acquire the local conservation property. We define the twisted connections in NC spaces and discuss NC gauge transformations. Then, the Yang-Mills (YM) action, twisted in the dynamical Moyal space, is proved to be invariant under $U_\star(1)$ local gauge transformation with the parameter $\alpha = \alpha_0 + \epsilon_\mu x^\mu$, where ϵ_μ is an infinitesimal parameter and α_0 a constant. The NC gauge invariant currents are explicitly computed. These currents are locally conserved.

Besides, the main properties of the harmonic oscillator in the framework of a dynamical noncommutativity realized through a twisted Moyal product are discussed. Working in the NC configuration space, explicit spectrums of harmonic oscillator with non-vanished momentum-momentum bracket are derived and the spectrums computed. It should be pointed out that, in order to maintain the Bose-Einstein statistics, the model parameters Θ and $\bar{\Theta}$ must satisfy the relation $\Theta^2 - \bar{\Theta}^2 = 0$. Therefore, the parameters Θ and $\bar{\Theta}$ reflect the intrinsic noncommutativity between positions and momenta, respectively, (as a Planck constant encodes the noncommutativity of position and momentum).

Résumé

Cette thèse porte sur notre contribution au développement des théories noncommutatives des champs. Elle prévoit notamment une étude comparative du modèle scalaire ordinaire et scalaire complexe de Grosse-Wulkenhaar (GW). Dans ce contexte, les quantités physiques pertinentes telles que le tenseur d'énergie impulsion (EMT) et les courants de Noether en général sont explicitement calculés et régularisés pour répondre aux propriétés physique de conservation locale, basée essentielement sur les techniques de Wilson et Jackiw. Dans toutes ces théories, la symétrie par dilatation est brisée et le terme de brisure est calculé. Comme prévu, toutes les quantités physiques en théorie des champs ordinaire sont facilement retrouvées en posant $\Omega = 0$. Une généralisation de la formulation hamiltonienne développée par Gomis et al est adapté pour le champ scalaire de GW. La noncommutativité dynamique introduite par Aschieri et al est mis en oeuvre et utilisée dans le cas d'une nouvelle classe de théories de champ renormalisables, celui de Grosse et Wulkenhaar définit dans l'espace euclidien. Notre analyse montrent que l'action de GW dynamique n'est pas invariante sous les translations d'espace temps. Une telle caractéristique indésirable a été contourné en imposant une contrainte sur le lagrangien de la théorie, qui n'est rien d'autre qu'une nouvelle équation du mouvement liée au terme harmonique de GW. Contrairement aux résultats antérieurs, le tenseur d'énergie impulsion et le tenseur moment angulaire du modèle ordinaire et dynamique de GW possèdent la pathologie de la non-conservation locale à cause de la présence du terme harmonique additionel. Les tenseurs sont de même non-symétriques. Les courants de dilatation sont aussi calculés dans cette étude. Heureusement, toutes ces quantités physiques peuvent être soumises à l'analyse de régularité de Jackiw et de Wilson pour acquérir la propriété de conservation locale. Nous définissons la connection de jauge dans un espace dynamique noncommutatif et discutons des transformations de jauge NC appropriées. Ensuite, Nous faisons une analyse sur la théorie de Yang-Mills pure dans un tel espace déformé et il s'avère que l'action correspondante est invariante sous les transformations de jauge locale $U_\star(1)$ définient par le parametre infinitésimal $\alpha = \alpha_0 + \epsilon_\nu x^\nu$, ϵ étant un vecteur infinitésimal. Les courants issus de l'invariance de jauge NC sont explicitement calculés. Ces courants sont localement conservés. Par ailleurs, les principales propriétés de l'oscillateur harmonique dans un tel espace dynamique réalisé grace au produit de Moyal déformé encore appelé produit de Moyal dynamique sont discutées. Le spectre, de l'oscillateur harmonique dans l'espace de phase NC avec la condition que les impulsions sont noncommutatives est analysé. Il convient de souligner que, afin de maintenir la statistique de Bose-Einstein, les paramètres du modèle de la théorie doivent satisfaire la relation $\Theta^2 - \bar{\Theta}^2 = 0$. Par conséquent, ces paramètres encodent la propriété de noncommutativité de l'espace temps (comme la constante de Planck encode la noncommutativité des positions et

des mouvements).

Dedication

TO MY FAMILY

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Contents

Publications

1. M.N. Hounkonnou and D. Ousmane Samary.: 2008, 'Noncommutative complex Grosse-Wulkenhaar model', *American Institute of Physics: Proc* **Vol. no. 1079**, pp. 82-88.

2. M.N. Hounkonnou and D. Ousmane Samary.: 2009, 'Energy momentum tensor in noncommutative complex scalar field theories', *Copromath Proc*, in press.

3. M.N. Hounkonnou and D. Ousmane Samary.: 2010 'Twisted Grosse-Wulkenhaar $\phi^{\star 4}$ model: dynamical noncommutativity and Noether currents', *J. Phys. A: Math. and Theor.* **43** 155202.

4. M.N. Hounkonnou and D. Ousmane Samary: 2010 'Harmonic oscillator in twisted Moyal plane: eigenvalue problem and relevant properties', *J. Math. Phys.***51** 102108 *[e-print math-ph/1008.1325]*.

5. M.N. Hounkonnou and D. Ousmane Samary: 2011 'Twisted Yang-Mills field theory: connections and Noether currents' *J. Phys. A: Math. and Theor.* **44** 315401.

6. M.N. Hounkonnou and D. Ousmane Samary.: 2011, 'Deformation quantization of harmonic oscillator in a general noncommutative phase space: Spectrum in relevant representations' *American Institut of Physics: Proc*, in press.

Introduction

This dissertation is devoted to the investigation of *Noncommutative Solvable Models in Quantum and Field Theories*. Quantum Field Theory (QFT) describes the properties of elementary particles in terms of relativistic quantum fields, while Particle Physics studies the fundamental constituents of matter and their interactions. Four fundamental interactions are today known:

In the *electromagnetic interaction*, the symmetry group is a unitary transformation called $U(1)$. Since there is a single generator, the force is mediated by a single particle, i.e. the photon, which is known to be massless. The electromagnetic force is due to the photon (γ) exchange. If a particle is massless and spin-1, it can only have two polarization states. Photons do not carry charge.

In the *weak interaction*, the gauge group is the group $SU(2)$ which has three generators. The three physical gauge bosons that mediate the weak force are W^+ (carrying $+1$ electric charge), W^- (with $+1$ electric charge), and Z (electrically neutral particle). The gauge bosons for the weak interaction are massive and possess three polarization states.

The electromagnetic and weak interactions have been unified into a single theoretical framework called electroweak theory.

In the *strong interaction*, the gauge group is $SU(3)$ which has eight generators. The gauge bosons corresponding to these generators are called gluons. Gluons mediate interactions between quarks and are therefore responsible for binding neutrons and protons together in the nucleus. A gluon is a massless spin-1 particle, and like the photon, has two polarization states. Gluons carry the charge of the strong force, and can interact among themselves, something that is not possible with photons since photons carry no charge. The theory that describes the strong force is called quantum chromodynamics.

The general theory of *gravitational interaction* is one of the most beautiful and successful theories in classical physics, called general relativity or Einstein theory. Einstein proposed the following principles to construct the general relativity. The first is that all laws in physics take the same forms in any coordinate system. The second principle states that there exists a coordinate system in which the effect of a gravitational field locally vanishes.[2] Any theory of gravity must be reduced to Newton's theory of gravity in the weak-field limit. In Newton's theory, the gravitational potential Φ satisfies the Poisson equation $\Delta\Phi = 4\pi G\rho$, where ρ is the mass density and G the Newton constant of gravity. The Einstein equation generalizes this classical result so that the principle of general relativity is satisfied. In general relativity, the gravitational potential is replaced by the components of the metric tensor. The Einstein equation is defined by $G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathcal{R}$. $Ric_{\mu\nu}$ is the Ricci tensor, $\mathrm{g}_{\mu\nu}$ the metric tensor and $\mathcal{R}$ the scalar curvature. Similarly, the mass density is replaced by a more general object, called the energy momentum

[2] An observer in a freely falling lift does not feel gravity until in crashes.

tensor (EMT) $T_{\mu\nu}$. The Einstein equation takes a very similar form $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. Let ∇_μ be the connection associated to metric $g_{\mu\nu}$, then the EMT is locally conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$. Before we leave general relativity, we should note that Einstein equation can be derived from a field theory, using the Hilbert action $S_H = \frac{1}{8\pi G}\int d^4x\,(\sqrt{-g}\mathcal{R} + 8\pi G L_M)$, where L_M is the Lagrangian associated to the matter field. To summarize we allow for non-flat metrics in general relativity, as mentioned above. Roughly, this means that the metric depends on position on the manifold, leading to curvature in contrary of the standard model.

It is now widely believed to offer a coherent mathematical framework for relativistic models like the standard $U(1)\times SU(2)\times SU(3)$ model. This model includes all the particles and interactions observed up to now, except for the gravity. Forces in nature are believed to result from the exchange of the gauge bosons. For each interaction, there is a field, and the gauge bosons are the quanta of the field. The number of gauge bosons that exist for a particular field is given by the number of generators of the field. The gauge bosons for the electromagnetic, weak, and strong forces are all spin-1. If the gavity is the gauge theory and therefore be quantized, the force-carrying particle (called the graviton) is a spin-2 particle. For a particular field, the generators come from the unitary group used to describe the symmetries of the field.

There exist many approaches to include gravity into quantum field theory. The ultimate step forward for quantum field theory is a unified theory known as *String theory*. This theory was originally proposed as for the strong interraction, but it fell out of favor when quantum chromodynamics was developed. The basic idea of the string theory is that the fundamental objects in the universe are not pointlike elementary particles, but are instead objects spread out in one dimension, called strings. Excitations of the string give the different particles we see in the universe. String theory is popular because it appears to be a completely unified theory. Quantum field theory unifies quantum mechanics and special relativity, and as result is able to describe interactions involving three of the four known forces. Gravity is left out. Currently gravity is best described by Einstein's general theory of relativity, a classical theory that not take quantum mechanics into account. Efforts to bring quantum theory into the gravitational realm or vice versa have met with some difficulties. One reason is that interactions at a point cause the theory to "blow up"; in other words you get calculations with infinite results. By proposing that the fundamental objects of the theory are strings rather than point particles, interactions disappear. In addition, a spin-2 state of the string naturally arises in string theory. It is known that the quantum of the gravitational field, if it exists, will be a massless spin-2 particle. Since this arises naturally in string theory, many people believe it is strong candidate for a unifield of all interactions. The intersting solution of string theory shows that space coordinates do not commute. Recent interest in noncommutative field theory (NCFT) is then strongly motivated by the discovery that string theory leads to NC geometry in certain limits.

To make spacetime NC, the commutative algebra of functions is usually replaced by a NC algebra generated by coordinates x^μ with commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\tilde{\Theta}^{\mu\nu}(x). \tag{1}$$

In the canonical case, this commutator is a constant, i.e. $\tilde{\Theta} = \Theta \in \mathbb{R}$. Gauge theory on this space was studied in great detail in the last few years, mainly due to its appearance in string theory. But if we think that noncommutativity is an effect of quantum gravity, the canonical case can only be the simplest example. Other, more complicated structures should be studied, especially structures that are related to curved backgrounds. But also in view of our second motivation, the canonical case proved to be disappointing: it doesn't cure the inimities of QFT, it rather adds new ones.

At the length scale $l_p = \sqrt{\frac{\hbar G}{c^3}} \simeq 1,6.10^{-35} meters$, which corresponds to the Planck energy $E_p = \frac{\hbar c}{l_p} = \sqrt{\frac{\hbar c^5}{G}} \simeq 1,210^{19} GeV$, the universal constants $c, \hbar$ and G appear naturally equivalent. Trying to measure smaller and smaller distances, we are forced to use test particles with more and more energy. This energy will affect the geometry of space itself creating black holes which finally become bigger than the distances we want to measure. Under the Planck length, distance looses its meaning. At these super-short distances, physical phenomena are believed to be nonlocal opposed to the locality of traditional geometrical theories of gravitation and quantum gauge field theories of particle physics. The solving of quantum field theories at this level implies *"unification of all interaction of nature"*. In the absence of a consistent formulation of quantum gravity, we do not know the exact nature of quantized space-time, but it is clear that the usual notion of a differentiable manifold should be replaced by something reflecting the quantum nature of space-time at very small distances.

By using a noncommutative (NC) structure of space-time at very small length scales l_p, one could also introduce an ultraviolet cutoff. There, the divergencies appearing in the quantization are UV-effects, and therefore related to small distances. The introduction of noncommutativity could work as a ultraviolet cut-off, making quantum field theory (QFT) finite. Even though the UV-divergencies are now well under control through the renormalization programme, they nevertheless suggest that space-time should change its nature at very small distances. The simplest NC model, namely $\phi^4_{\star 4}$, whose action is given by

$$S[\phi] = \int d^4x \, \left[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\mathrm{m}^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\right](x). \tag{2}$$

was found to be not renormalizable because of a surprising phenomenon, the so called UV/IR mixing. The UV/IR mixing of the noncommutative field theories was first discovered in [66] in the context of scalar field theory. There, it was noticed that regulating the integrals, as in the commutative theory, leads to a new kind of mixing of the ultraviolet (UV) and infrared (IR) regimes. That is, the field theory does not give the same result independently of the order in which the UV and IR limits are taken. Hence the name UV/IR mixing. To avoid the UV/IR mixing problem, several models which involve an oscillator like counter term have been put forward. On the one hand such models break translation invariance due to the explicit x-dependence of the action, but on the other hand they in general show a much better divergence behaviour at higher loops or are even (in the case of the scalar Grosse-Wulkenhaar (GW) model) proven to be renormalizable. In the following, we will present the GW model followed by three gauge models based on similar ideas.

Ideed, in 2004, the first renormalizable NC scalar field model in Euclidean space was introduced by H. Grosse and R. Wulkenhaar [37] (for a Minkowskian version, see reference [27]). Their trick was to add a harmonic oscillator-like term to the action as follows:

$$S[\phi] = \int d^4x \left[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\mathrm{m}^2}{2}\phi \star \phi + \frac{\Omega^2}{2}(\widetilde{x}\phi) \star (\widetilde{x}\phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\right] . \tag{3}$$

Here $\widetilde{x} = 2(\Theta^{-1}) \cdot x$ and ϕ is a real scalar field. The additional constant harmonic term Ω stands for the action UV/IR freedom implying renormalizability and the Lagrangian covariance under Langmann-Szabo duality [57]. With its explicit dependence in x^μ, the Lagrangian includes an interaction as an external harmonic source. Then, the system described by action (1.64) is possibly an open system. At the parameter limit $\Theta \to 0$, the singularity $(\Theta^{-1}) \to \infty$ invokes a divergence from the classical ϕ^4 scalar field. Furthermore, it is not invariant under space-time translation. Besides, at the parameter limit $\theta \to 0$, the model does not converge to the ordinary ϕ^4 scalar field theory due to the presence of the inverse matrix (Θ^{-1}), then causing a singularity. The $\star$-GW ϕ^4_4 theory is renormalizable at all orders in λ. This result has been now proved by various methods (see [73] and references therein). By exchanging $\tilde{x} \leftrightarrow p$ one can see that the GW action becomes covariant under Langmann-Szabo (LS) duality [57], i.e. covariant under the symmetry:

$$\mathcal{S}[\phi, m, \lambda, \Omega] \to \Omega^2 \mathcal{S}[\phi, \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]. \tag{4}$$

Note that (4) is invariant for $\Omega = 1$. The requirement of LS duality has been helpfull to construct other renormalizable actions for complex-valued scalar fields. However, more investigations are needed to clarify the actual role of the LS duality in the controle of the UV/IR-mixing and renormalizability.

The construction of a renormalizable gauge theory on NC space remains still unsolved and appears to be a quite challenging problem. The naive NC extension of the pure Yang-Mills action on Moyal space has UV/IR mixing which make its renormalizabilty quite unlikely unless it is suitably modified. Unfortunately, the harmonic solution proposed in (3) cannot be merely extended to gauge theories on Moyal spaces. Finding such a suitable extension would first amount to determine whether or not naive NC Yang-Mills action can be supplemented by additional gauge invariant terms providing a natural gauge theory counterpart of those harmonic terms. As shown in [32], the calculation performed within the x-space formalism singles out a class of gauge invariant actions given by

$$\mathcal{S} = \int d^4x \Big(\frac{\alpha}{4g^2} F_{\mu\nu} \star F^{\mu\nu} + \frac{\Omega'}{4g^2} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \frac{\kappa}{2} \mathcal{A}_\mu \star \mathcal{A}_\mu \Big) \tag{5}$$

where $\mathcal{A}_\mu$ denotes a specific gauge covariant tensorial form, linked with the existence of a canonical gauge invariant connection.

This dissertation is organized as follows. The first chapter addresses investigations on the energy-momentum tensors for both ordinary $\phi^4_{\star D}$ and renormalizable NC GW complex scalar field theories. Locally conserved non symmetric energy momentum tensors are obtained, using algebraic techniques. The properties inherited from the models are clarified. Nonlinear Euler-Lagrange equations of motion are derived and solved using a matrix base method.

In chapter 2, we deal with the Hamiltonian formalulation of the GW $\phi^4_{\star D}$ model and its generalization. The study is based on $D+1$ dimensional space-time formulation of D dimensional non-local theories. The analysis of constraints shows that the secondary constraints describe the Euler-Lagrange equations of motion.

In Chapter 3, we computate Noether currrents for the renormalizable GW $\phi^{\star 4}$ model subjected to a dynamical noncomutativity realized through a twisted Moyal product. The NC energy-momentum tensor (EMT), angular momentum tensor (AMT) and dilatation current (DC) are explicitly derived. The breaking of translational and rotational invariances has been avoided via a constraint equation. Then main properties of the NC gauge theory are investigated in a $2-$dimensional twisted Moyal plane, generated by the vector fields $X_a = e_a^\mu(x)\partial_\mu$; the dynamical effects are induced by a non trivial tensor $e_a^\mu(x)$. Finally, connections in such a NC space are defined. Symmetry analysis is performed and related NC action is proved to be invariant under defined NC gauge transformations. A locally conserved Noether current is explicitly computed. Both commuting and NC vector fields X_a are considered.

Chapter 4 first reports on a study of a harmonic oscillator (ho) in the twisted Moyal space, in a well defined matrix basis, generated by the vector fields $X_a = e_a^\mu(x)\partial_\mu = (\delta_a^\mu + \omega_{ab}^\mu x^b)\partial_\mu$, which induce a dynamical star product. The usual multiplication law can be hence reproduced in the ω_{ab}^μ null limit. The star actions of creation and annihilation functions are explicitly computed. The ho states are infinitely degenerate with energies depending on the coordinate functions. Then, the harmonic oscillator is studied in a complete NC phase space with non-vanished space-space and momentum-momentum coordinates.

Finally, there follow the concluding remarks and discussions.

Grosse-Wulkenhaar Model

The existence of symmetries and fundamental invariants in physics has a number of consequences. To mention a few, it restricts the class of dynamical models, simplifies calculations for generating solutions, and involves the conservation of physical quantities (charge, energy, momentum, etc.). In classical field theory, symmetries and continuous transformations fall generally into two categories: spacetime (geometric) transformations (Poincaré - Galilei, conformal group symmetries, etc.) and internal transformations (Lie group symmetries, mixing of fields etc.). One could further classify internal transformations as local gauge and global transformations according to that their generators are position dependent and not, respectively. More developments can be found in [56]. The implications of having a classical field system invariant under a set of transformations (finite dimensional connected continuous group) are governed by the so-called Noether theorem [2][33][56].

With the advent of noncommutative field theory (NCFT)[25], a series of papers has been produced dealing with noncommutative (NC) versions of Noether currents within the framework of field modules over Moyal algebra for different kinds of symmetries [1][19][28][36][65] and including models with interactions and NC spinor currents [40][84]. Let us recall that the Moyal algebra of functions [34] on the spacetime $\mathbb{R}^D$ (endowed with some Euclidean or Minkowskian metric $(\eta = \eta_{\mu\nu})$) is a subset of the Schwartz class of infinitely differentiable complex valued functions with rapid decay, mainly equipped with a multiplication called $\star$-Moyal product defined by

$$(f \star g)(x) = \mathrm{m}\left(e^{\frac{i}{2}\Theta^{\rho\sigma}\,\partial_\rho\otimes\partial_\sigma} f(x)\otimes g(x)\right) \quad x \in \mathbb{R}^D \tag{1.1}$$

m being the ordinary multiplication of functions and $\Theta = (\Theta^{\rho\sigma})$ the constant noncommutativity antisymmetric tensor. A salient feature of NCFT over the Moyal algebra is that the tensor $\Theta = (\Theta^{\rho\sigma})$ (1.1) is not a (Lorentz-Euclide) metric tensor in the sense that $\Theta^{\rho\sigma} \neq \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \Theta^{\mu\nu}$, $\Lambda^\rho{}_\mu$ being some spacetime rotation, implying that it explicitly breaks the rotation invariance. The particular issue of Lorentz invariance violation in NCFTs was discussed from different point of view and some breakthroughs were proposed considering whether the simplest algebra in which the Θ tensor is promoted to an operator so that Lorentz invariance is preserved [14], or some canonical formulation and realization of the NC space-space ($\Theta^{0i} = 0$) Moyal geometry with a NC deformed Lorentz symmetry under which, via well defined vector field transformations, both the $\star$-product and the interval length ds^2 remain invariant [13]. There were also different proposals to maintain Lorentz covariance of NC Lagrangian density involving the twisted Poincaré symmetry [15][53][84] and the κ-Poincaré quasi group whose elements are deformed Poincaré symmetries

[63]. Thus, Poincaré symmetries remain to be understood in NCFTs.

Nevertheless, some approaches in order to define a Noether theorem in theories over Moyal algebra have been developed. A first attempt to a definition of such a theorem maybe belong to Micu and Sheikh-Jabbari [65]. Their study was performed using formal variations of the NC action with applications to translation for the NC ϕ^4 scalar field theory and $U(1)$ local gauge symmetry for NC Dirac spinor field theory. They also conjectured the property that: *For any current* J^μ*, there are functions* f *and* g *such that* $\partial_\mu J^\mu = [f, g]_\star$ which involves that, in space-space Moyal noncommutativity fixed by $\Theta^{0i} = 0$, the charge $Q = \int d^3x \; J^0$ is conserved in time. Besides, a NC Noether procedure for translation and dilatation symmetries was studied by Gerhold *et al.* [28]. Here, the authors use Ward identity operators[1] (WIops) in order to define an exact definition of NC action invariance under a set of continuous transformations. Some further developments on translation symmetry of NC ϕ^4 scalar field theory and Yang-Mills fields are worked out by Abou-Zeid and Dorn [1] and by Grimstrup *et al.* [36]. Furthermore, NC $U(N)$ Noether currents for matrix gauge transformations were considered in [36]. Note that investigations on rotation symmetry are avoided in the above studies.

As a peculiar feature of NCFTs, in general, Noether currents fail to be locally conserved. More precisely, the Noether procedure applied to NC ϕ^4 scalar field theory for infinitesimal translations yields a symmetric not locally conserved energy momentum tensor (EMT). For a massless theory, adding the term $(1/6)(g_{\mu\nu} \cdot \partial^2 - \partial_\mu\partial_\nu)(\phi \star \phi)$ to the NC Lagrangian density allows to define a traceless EMT tensor but still not locally conserved. A computational procedure defined in [1] may be used to recover a locally conserved EMT, with as inheritance, a nonsymmetric part. Regarding NC Yang-Mills theory with spinor current under infinitesimal translations, the canonical EMT proves to be asymmetric, not locally conserved, not traceless and not gauge invariant [36][40]. In the case of pure NC gauge theory, Jackiw recipe could be applied in order to define a symmetric EMT which transforms covariantly under gauge transformations and which is locally covariantly conserved [36]. Further developments concerning broken dilatation symmetry and improvement procedures are available in Refs.[1][19][28][36].

So far, studies on the NC analogues of Noether currents and symmetries in NC spaces have been carried out only in the framework of the so-called 'naive'[2] NCFTs regardless to the additional harmonic terms of Grosse and Wulkenhaar [37] ensuring Langmann-Szabo duality [57] and being the key of a NC renormalizable theory [73][74][81] (for a review of NC renormalizable field theory see [74]). Moreover, an induced gauge theory coupled with a complex scalar field have led to the de Goursac *et al.* conjecture of the NC renormalizable Yang-Mills action [31]. It could be then natural to investigate all the above formalism around the questions of NC symmetries, Noether quantities and related properties in these new classes of theories which, undoubtedly, have given a new lease of life to NCFTs.

In this chapter, from a classification of NC infinitesimal Poincaré-Galilei and local gauge transformations, we recall a rigorous formulation (by Ben Geloun and Hounkonnou) of the NC analogue of Noether theorem on trivial bimodules and matrix field algebra over the Moyal algebra with the ambition to go as far as possible in developing a deformed differential calculus and algebraic con-

[1] In quantum electrodynamics, Ward operators are useful to determine the gauge invariance of the theory which is stated through the so-called Ward identity of relevance in the study of renormalizability. Often, in this situation, those operators take the name of Ward identity operators.

[2] We shall refer this theory to as ordinary NCFT and the quantities occurring in this class of NCFTs to as ordinary NC quantities.

cepts without invoking topological aspects. Given the above quoted infinitesimal transformation generators, NC Noether currents of NC scalar and gauge theories are computed as illustrations. These NC currents include, as original contributions, Grosse-Wulkenhaar harmonic terms. Discussions on broken rotation and dilatation symmetries follow. Using a particular WIop, we succeed in defining an analogue of angular momentum in the Moyal space, albeit this analogue appears only under broken rotation invariance. In the case of pure translations, a regularization procedure proves to be efficient in order to improve the EMTs for the local conservation order. The ordinary NC currents are recovered in a particular parameter limit where the Grosse-Wulkenhaar terms collapse whereas the classical currents ('classical' always refers to a commutative classical field theory) are obtained after taking the second limit $\Theta \to 0$.

1.1 Moyal algebra

NCFTs theories can be studied by replacing the NC operators acting in a Hilbert space by functions in phase space with the Moyal product. Then the canonical commutation relation between coordinates functions is given by

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\Theta^{\mu\nu}. \tag{1.2}$$

The Moyal star-product is noncommutative and associative, i.e.

$$f \star g \neq g \star f, \quad f \star (g \star h) = (f \star g) \star h, \quad \forall f, g, h \in \mathcal{C}^\infty(\mathrm{R}^D_\Theta). \tag{1.3}$$

Consider $E =: \{x^\mu, \mu \in [[1, D]]\}$ and $\mathbb{C}[[x^1, x^2, \cdots, x^D]]$, the free algebra generated by E. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[[x^1, x^2, \cdots, x^D]]$, generated by the elements $x^\mu \star x^\nu - x^\nu \star x^\mu - i\Theta^{\mu\nu}$. The Moyal Algebra $\mathcal{A}_\Theta$ is the quotient $\mathbb{C}[[x^1, x^2, \cdots, x^D]]/\mathcal{I}$. Each element in $\mathcal{A}_\Theta$ is a formal power series in the x^μ's for which the relation $[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}$ holds. Moyal algebra can be here also defined as the linear space of smooth and rapidly decreasing functions equipped with the NC star product given in []. By using the Fourier transform of functions $f, g \in \mathcal{S}(\mathrm{R}^D_\Theta)$, star-product (1.1) has the integral representation given by

$$\begin{aligned} (f \star g)(x) &= \frac{1}{(2\pi)^D} \int_{\mathrm{R}^{\mathrm{D}}_\Theta} d^D k \, d^D y \, f(x + \frac{1}{2}\Theta.k) \, g(x+y) e^{ik.y} \\ &= \frac{1}{\pi^D |det\Theta|} \int_{\mathrm{R}^{\mathrm{D}}_\Theta} d^D y \, d^D z \, f(x+y) \, g(x+z) e^{-2iy\Theta^{-1}z}. \end{aligned} \tag{1.4}$$

Using (1.4), we can show the associativity of the star-product. We also deduce that

$$\int d^D x \, (f \star g)(x) = \int d^D x \, (g \star f)(x) = \int d^D x \, f(x).g(x), \tag{1.5}$$

$$\int d^D x \, (f_1 \star f_2 \star \cdots \star f_n)(x) = \int d^D x \, (f_n \star f_1 \star \cdots \star f_{n-1})(x). \tag{1.6}$$

The cyclicity of the star-product implies that

$$< f \star g, h > = < f, g \star h > = < g, h \star f >, \quad \text{where} \quad < f, g > = \int_{\mathrm{R}^{\mathrm{D}}_\Theta} d^D x \, (f \star g)(x) \tag{1.7}$$

and allows to estend the Moyal algebra by duality into an algebra of tempered distributions. Let us consider the product of a tempered distribution with a Schwartz-class function. Let $T \in \mathcal{S}'(\mathrm{R}^{\mathrm{D}}_{\Theta})$ and $g \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta})$, we define $< T, h >=: T(h)$ and $< T^*, h >= \overline{< T, \bar{h} >}$. We define $T \star f$ and $f \star T$ for all $f \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta})$ by

$$< T \star f, h >=< T, f \star g >, \quad < f \star T, h >=< T, h \star f > . \tag{1.8}$$

We are now ready to define the linear space $\mathcal{M}$ as the intersection of two subspaces $\mathcal{M}_L$ and $\mathcal{M}_R$ of $\mathcal{S}'(\mathrm{R}^{\mathrm{D}}_{\Theta})$

Definition 1.1 *The Moyal multiplier algebra is defined by*

$$\mathcal{M} = \mathcal{M}_L \cap \mathcal{M}_R \tag{1.9}$$

where

$$\mathcal{M}_L = \{T \in \mathcal{S}'(\mathrm{R}^{\mathrm{D}}_{\Theta}); \forall \mathrm{f} \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta}), \mathrm{T} \star \mathrm{f} \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta})\} \tag{1.10}$$

$$\mathcal{M}_R = \{T \in \mathcal{S}'(\mathrm{R}^{\mathrm{D}}_{\Theta}); \forall \mathrm{f} \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta}), \mathrm{f} \star \mathrm{T} \in \mathcal{S}(\mathrm{R}^{\mathrm{D}}_{\Theta})\}. \tag{1.11}$$

On can show that $\mathcal{M}$ is an associative $*-$algebra. It contains, among others, the identity, polynomials, the δ distribution and its derivatives. Then the relation $[x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}$ often given as a definition of the Moyal space, holds in $\mathcal{M}$ but not in $\mathcal{A}_\Theta$. Other relevant properties of the star-product that hold on $\mathcal{M}$ are

$$\partial_\mu(f \star g) = \partial_\mu f \star g + f \star \partial_\mu g, \;\; (f \star g)^\dagger = g^\dagger \star f^\dagger, \;\; [x^\mu, f]_\star = i\Theta^{\mu\nu}\partial_\nu f \tag{1.12}$$

$$x^\mu \star f = x^\mu f + \frac{i}{2}\Theta^{\mu\nu}\partial_\nu f, \;\; x^\mu(f \star g) = (x^\mu f) \star g - \frac{i}{2}\Theta^{\mu\nu} f \star \partial_\nu g. \tag{1.13}$$

1.1.1 Module over Moyal algebra

The multiplier Moyal algebra $\mathcal{M}$ is unital involutive associative $\star$-algebra. It contains the plane wave functions. NCFTs over Moyal algebra of functions can be defined as field theories over module $\mathcal{H}$ over the NC algebra $\mathcal{M}$ or as matrix field theories with coefficients in $\mathcal{M}$. Connections defined on vector bundles in ordinary differential geometry are replaced, in NC geometry, by the generalizing concept of connection on projective modules. The sesquilinear maps

$$\nabla_\mu : \mathcal{H} \to \mathcal{H} \tag{1.14}$$

are called connections if they satisfy the differentiation chain rule

$$\nabla_\mu(f \star m) = (\partial_\mu f) \star m + f \star \nabla_\mu m, \quad \text{for} \quad f \in \mathcal{M}, \; m \in \mathcal{H} \tag{1.15}$$

assumed here to be a left module over $\mathcal{H}$, and if they are compactible with the Hermitian structure of $\mathcal{H}$ i.e.,

$$\partial_\mu \mathrm{h}(m_1, m_2) = \mathrm{h}(\nabla_\mu m_1, m_2) + \mathrm{h}(m_1, \nabla_\mu m_2). \tag{1.16}$$

Suppose that $m = \mathbf{1}$ the unit element of $\mathcal{M}$. Then

$$\nabla_\mu(f \star \mathbf{1}) = \partial_\mu f + f \star \nabla_\mu \mathbf{1} \quad \text{with} \quad \nabla_\mu \mathbf{1} =: iA_\mu. \tag{1.17}$$

We get then the connection ∇_μ in the anti-fundamental representation given by

$$\nabla_\mu(.) =: \partial_\mu(.) + i(.) \star A_\mu. \tag{1.18}$$

Then A_μ is called the gauge potential in the anti-fundamental representation. We can also defined the gauge connection in the fundamental representation using the right module over $\mathcal{M}$ by

$$\nabla_\mu(m \star f) = (\nabla_\mu m) \star f + m \star \partial_\mu f, \;\; \nabla_\mu(\mathbf{1}) =: -iA_\mu. \tag{1.19}$$

We now get

$$\nabla_\mu(.) =: \partial_\mu(.) - iA_\mu \star (.). \tag{1.20}$$

A_μ is called the gauge potential in the fundamental représentation. We can also defined the gauge connection in the adjoint representation given by

$$\nabla_\mu(m \star f \star m) = (\nabla_\mu m) \star f \star m + m \star \partial_\mu f \star m + m \star f \star \nabla_\mu f \tag{1.21}$$

and for $m = \mathbf{1}$ we get

$$\nabla_\mu(.) =: \partial_\mu(.) + i[(.), A_\mu]_\star. \tag{1.22}$$

Then A_μ is called the gauge potential in the adjoint representation. One can see that these three representations are equivalent. Recall that the sesquilinear map h satisfies the relation $\mathrm{h}(f_1, f_2) = f_1^\dagger \star f_2$ if we impose the condition $\mathcal{M} = \mathcal{H}$. This implies that the gauge connection is Hermitian i.e. $A_\mu^\dagger = A_\mu$. In the rest of this thesis, we notify that

$$\nabla_\mu = \partial_\mu(.) - iA_\mu \star (.) \tag{1.23}$$

$$\nabla_\mu^\natural = \partial_\mu(.) + i(.) \star A_\mu \tag{1.24}$$

$$\mathrm{D}_\mu = \partial_\mu(.) + i[(.), A_\mu]_\star \tag{1.25}$$

1.1.2 Gauge Transformation

The gauge transformation is the morphism of module $\mathcal{H}$ denoted by γ satisfying the relation

$$\gamma(m \star f) = \gamma(m) \star f, \;\; \forall m \in \mathcal{H}, \;\; \forall f \in \mathcal{M} \tag{1.26}$$

and preserving the hermitian structure h i.e.

$$\mathrm{h}\Big(\gamma(f), \gamma(g)\Big) = \mathrm{h}(f, g), \;\; \forall f, g \in \mathcal{M}. \tag{1.27}$$

If $m = \mathbf{1}$, $\gamma(f) = \gamma(\mathbf{1}) \star f$, and if $\mathcal{H} = \mathcal{M}$, then $\gamma(\mathbf{1})^\dagger \star \gamma(\mathbf{1}) = \mathbf{1}$. We now arrive at the conclusion that $\gamma(\mathbf{1}) \in \mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M}) =: U_\star(D/2)$ is the group of unitary transformations. Let us write $\gamma(\mathbf{1}) =: U$ and $U \star \phi =: \phi^U$, where $\phi \in \mathcal{M}$. Then we have

$$U^\dagger \star U = U \star U^\dagger = \mathbf{1}. \tag{1.28}$$

The connection ∇_μ is the covariant derivative by further satisfying the gauge transformation

$$\nabla_\mu^U(\phi) = U \star \nabla_\mu(U^{-1} \star \phi), \quad \text{or } \nabla_\mu^U(\phi^U) = U \star \nabla_\mu(\phi), \;\; \forall \phi \in \mathcal{M}. \tag{1.29}$$

Equation (1.29) implies that

$$\nabla_\mu^U = \partial_\mu - iA_\mu^U\star, \quad \text{and} \quad A_\mu^U = U\star A_\mu \star U^\dagger + iU\star\partial_\mu U^\dagger. \tag{1.30}$$

Given the connection ∇, the corresponding curvature is

$$F_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]_\star = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star \tag{1.31}$$

and its gauge transformation

$$F_{\mu\nu}^U = U\star F_{\mu\nu}\star U^\dagger. \tag{1.32}$$

Of course the NC version of the Maxwell action given by

$$\mathcal{S}_M = -\frac{1}{4\kappa^2}\int d^Dx\, F_{\mu\nu}\star F^{\mu\nu} = -\frac{1}{4\kappa^2}\int d^Dx\, F_{\mu\nu}.F^{\mu\nu} \tag{1.33}$$

is gauge invariant since, as explained above, integration acts as a trace for the star-product. Note that the gauge group in this case is $U_\star(1)$.

$$\begin{aligned}\int d^Dx\, F_{\mu\nu}^U \star F^{\mu\nu U} &= \int d^Dx\, U^\dagger\star F_{\mu\nu}\star U\star U^\dagger F^{\mu\nu}\star U \\ &= \int d^Dx\, F_{\mu\nu}\star F^{\mu\nu}.\end{aligned} \tag{1.34}$$

The elements of the $U_\star(1)$ group are the star-exponentials given by

$$U(x) = \exp_\star(i\alpha(x)) = 1 + i\alpha(x) - \frac{1}{2}\alpha(x)\star\alpha(x) + \cdots. \tag{1.35}$$

This relation implies that $U^{-1} = U^\dagger$. Gauge invariant coupling to matter fields can be easily defined. Consider for example Dirac fermion fields coupled to a gauge field. A gauge invariant massless situation can be defined by the action $\mathcal{S}_D$ written in the form

$$\mathcal{S}_D = \int d^Dx\, \bar\psi\star(i\gamma^\mu\nabla_\mu\psi) \tag{1.36}$$

where γ^μ are the Dirac matrices, ψ is the Dirac spinor, and $\bar\psi = \psi^\dagger\gamma^0$.

1.2 Ward Identity Operators and NC Noether Theorem

The subsequent developments, in the core of this work, aim at giving a formulation of the NC Noether theorem using the WIop method.

Consider a continuous set of transformations $x^\mu \to x'^\mu$ and a family of matrix fields $\phi_i = \{(\phi_i)_{AB}\}$ which transforms as $\phi_i \to \phi_i'$ with $i = 1, \dots, n$. The Jacobian of the transformation is nothing but $|\partial x'^\mu/\partial x^\nu| = 1 + \partial_\mu\delta x^\mu$.

Next, consider a NC action $S = \int d^Dx\ \mathrm{Tr}\ \mathcal{L}_\star(\phi_i, \partial_\mu\phi_i, x^\mu)$ which, under the above transformations, transforms formally as $S \to S'_\pm$ and admits the infinitesimal variations

$$(\delta S)_\pm = S'_\pm - S \;=\; \int d^Dx\ \mathrm{Tr}\left(\left|\frac{\partial x'}{\partial x}\right|\star\mathcal{L}'_\star\right)_\pm - \int d^Dx\ \mathrm{Tr}\ \mathcal{L}_\star$$

$$= \int d^D x \ \mathrm{Tr}\,[\,\delta\mathcal{L}_\star + (\partial_\mu \delta x^\mu \star \mathcal{L}_\star)_\pm\,]\,. \tag{1.37}$$

The derivation δ can be chosen symmetrized (implicit summation is used) with the expansion

$$\delta(\cdot) = \frac{1}{2}\Bigg\{\left[\delta\phi_{i,AB} \star \frac{\delta(\cdot)}{\delta\phi_{i,AB}} + \frac{\delta(\cdot)}{\delta\phi_{i,AB}} \star \delta\phi_{i,AB}\right] + \left[\delta x^\mu \star \frac{\partial(\cdot)}{\partial x^\mu} + \frac{\partial(\cdot)}{\partial x^\mu} \star \delta x^\mu\right]\Bigg\}. \tag{1.38}$$

From (1.38), evaluating $\delta\mathcal{L}_\star$, we obtain the action variation

$$\delta S = \int d^D x \ \mathrm{Tr}\ \left[\delta\phi_{i,AB} \star \frac{\delta\mathcal{L}_\star}{\delta\phi_{i,AB}} + \partial_\mu(\delta x^\mu \star \mathcal{L}_\star)\right]. \tag{1.39}$$

Assuming that the infinitesimal transformations $\delta\phi_{i,AB}$ are spanned by a family of parameters $\{\lambda_a\}$ such that $\delta\phi_{i,AB} = \delta\phi_{i,AB}(\lambda_a)$, the global canonical WIop for the transformation is given by

$$\{W_\star(\lambda_a)\}(\cdot) = \int d^D x \ \mathrm{Tr}\Bigg\{\frac{1}{2}\Bigg[\left(\delta\phi_{i,AB} \star \frac{\delta(\cdot)}{\delta\phi_{i,AB}} + \frac{\delta(\cdot)}{\delta\phi_{i,AB}} \star \delta\phi_{i,AB}\right) + \left(\delta x^\mu \star \frac{\partial(\cdot)}{\partial x^\mu} + \frac{\partial(\cdot)}{\partial x^\mu} \star \delta x^\mu\right)\Bigg]\Bigg\}. \tag{1.40}$$

Then, a NC analogue of Noether theorem can be formulated in this way:

Theorem 1.2 *(NC Noether theorem)*
If a NC action S is invariant under a set of transformations generated by a family of infinitesimal parameters $\{\lambda_a\}$, then

$$\{W_\star(\lambda_a)\}\, S = -\int d^D x \ \mathrm{Tr}\ \partial^\mu J_\mu(\lambda_a) = 0$$

and there exists a globally conserved vector current $J_{\mu,a}$ such that

$$\frac{\partial}{\partial\lambda_a}\{W_\star(\lambda_a)\}\, S = -\int d^D x \ \mathrm{Tr}\ \partial^\mu J_{\mu,a} = 0.$$

It is then natural to ask if it is valuable to consider this as a NC analogue of the classical Noether theorem, the local conservation of the Noether current being not ensured in the formulation.

Many facts invite us to consider this as actually the best of what we can expect from the NC action formulation. First, it is the NC method the closest of classical Noether procedure. Furthermore, as we will see hereafter, in NC space-space $\Theta^{0i} = 0$ geometry and for infinitesimal translation symmetry, an exactly conserved D-vector momentum P_μ may be exhibited. This shows the similarity between the classical and NC procedures. Besides, we will prove that, through some computational techniques, regularization procedures improve quantities to recover local conservation property. Finally, at the parameter limit $\Theta \to 0$ the globally conserved currents obtained through the NC WIop method are reduced to classical Noether currents.

1.3 Complex scalar Grosse-Wulkenhaar Model

Despite the enormous amount of investigations centered around the study of noncommutative field theories (see [25]-[57] and references therein for reviews), some of the most fundamental questions surrounding them have yet to be answered to full satisfaction. Regarding for instance peculiar features of noncommutative field theories (NCFT) in the Moyal $\star$-product description, the Noether theorem for translation symmetry generally leads to an energy momentum tensor (EMT) which is not locally conserved, not traceless in the massless situation and, not symmetric and not gauge invariant in gauge theories [1],[25],[28],[36].

The present work addresses comparative results on energy momentum tensors computed in both ordinary $\phi^4_{\star D}$ and renormalizable NC Grosse and Wulkenhaar complex scalar field theories.

In the first part we discuss the energy momentum tensor computation in complex $\phi^4_{\star D}$ NCFT. Second part deals with the investigation of the NC complex Grosse and Wulkenhaar model.

1.3.1 Energy momentum tensor in complex $\phi^4_{\star D}$ NC field theory

The action of a complex scalar field ϕ coupled with the interaction $(\phi\star\bar\phi)^2$ can be expressed in a D-dimensional Euclidean spacetime $\mathbb{R}^D_\Theta$ by

$$\begin{aligned} S_\star[\phi,\bar\phi] &= \int d^Dx\,\Big[\partial_\mu\phi\star\partial^\mu\bar\phi+m^2\phi\star\bar\phi \\ &+ \frac{\lambda_1}{4!}(\phi\star\bar\phi\star\phi\star\bar\phi)+\frac{\lambda_2}{4!}(\phi\star\bar\phi\star\bar\phi\star\phi)\Big], \end{aligned} \tag{1.41}$$

$\bar\phi$ being the conjugate of the field ϕ. λ_1 and λ_2 are real constants. In the sequel, we assume that $\lambda_1=\lambda_2=\lambda/2$. Then, (3) becomes

$$\begin{aligned} S_\star[\phi,\bar\phi] &= \int d^Dx\,\Big[\partial_\mu\phi\star\partial^\mu\bar\phi+m^2\phi\star\bar\phi \\ &+ \frac{\lambda}{2.4!}\Big(\phi\star\bar\phi\star\phi\star\bar\phi+\phi\star\bar\phi\star\bar\phi\star\phi\Big)\Big] \end{aligned} \tag{1.42}$$

from which the peculiar Euler Lagrange equations of motion can be readily derived by direct application of the variational principle, as follows:

$$\frac{\delta S_\star}{\delta\phi}=-\partial_\rho\partial^\rho\bar\phi+m^2\bar\phi+\frac{\lambda}{2.(4!)}(2\bar\phi\star\phi\star\bar\phi+\{\bar\phi\star\bar\phi,\phi\}_\star)=0, \tag{1.43}$$

and

$$\frac{\delta S_\star}{\delta\bar\phi}=-\partial_\rho\partial^\rho\phi+m^2\phi+\frac{\lambda}{2.(4!)}(2\phi\star\bar\phi\star\phi+\{\phi\star\phi,\bar\phi\}_\star)=0. \tag{1.44}$$

The global Ward identity operator for the model, defined as [7],[28]:

$$W^\Theta_\mu=\int d^Dx\,\frac{1}{2}\Big(\partial_\mu\phi\star\frac{\delta}{\delta\phi}+\frac{\delta}{\delta\phi}\star\partial_\mu\phi+\partial_\mu\bar\phi\star\frac{\delta}{\delta\bar\phi}+\frac{\delta}{\delta\bar\phi}\star\partial_\mu\bar\phi\Big) \tag{1.45}$$

acts on the noncommutative $S_\star$ to yield the canonical energy momentum tensor (EMT):

$$W^\Theta_\mu S_\star=0\Longrightarrow T_{\rho\mu}=\frac{1}{2}\{\partial_\mu\phi,\partial_\rho\bar\phi\}_\star+\frac{1}{2}\{\partial_\mu\bar\phi,\partial_\rho\phi\}_\star-g_{\rho\mu}\mathcal{L}_\star \tag{1.46}$$

where

$$\begin{aligned}\mathcal{L}_\star &= \frac{1}{2}\{\partial_\rho\phi,\partial^\rho\bar\phi\}_\star+\frac{m^2}{2}\{\phi,\bar\phi\}_\star+\frac{\lambda}{2.(4!)}\Big[\frac{1}{2}\{\phi,\bar\phi\star\phi\star\bar\phi\}_\star\\ &+ \frac{1}{4}\Big(\{\phi\star\bar\phi,\bar\phi\star\phi\}_\star+\{\phi\star\phi,\bar\phi\star\bar\phi\}_\star\Big)\Big]. \end{aligned}\tag{1.47}$$

$g_{\mu\nu}$ is the Euclidean metric. Remark that, instead of a symmetric and locally conserved energy momentum tensor occurring in the commutative spacetime, the $\Theta-$ deformed EMT is symmetric and nonlocally conserved. To get rid the theory of such a *pathology*, let us display the Coleman and Jackiw improvement procedure [16]. This requires for $D=4$ the computation of the EMT divergence:

$$\begin{aligned}\partial^\rho T_{\rho\mu} &= -\frac{\lambda}{4(4!)}\Big([\phi\star\bar\phi,\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi]_\star+\frac{1}{2}[\phi\star\bar\phi,\partial_\mu\bar\phi\star\phi\\ &- \bar\phi\star\partial_\mu\phi]_\star+\frac{1}{2}[\phi\star\phi,[\partial_\mu\bar\phi,\bar\phi]_\star]_\star\Big)+(\phi\leftrightarrow\bar\phi). \end{aligned}\tag{1.48}$$

Defining now the $\star'$-product

$$(f\star' g)(x):=f(x)\,\frac{\sin\big(\frac{1}{2}\overleftarrow{\partial_\mu}\,\Theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\big)}{\frac{1}{2}\overleftarrow{\partial_\mu}\,\Theta^{\mu\nu}\,\overrightarrow{\partial_\nu}}\,g(x)\;, \tag{1.49}$$

with the virtue that $-i[f,g]_\star=\Theta^{\mu\nu}\partial_\mu f\,\star'\,\partial_\nu g$ meaning that any $\star$-commutator is a divergence, and

$$[f,g]_\star=i\partial_\mu(\Theta^{\mu\nu}f\star'\partial_\nu g)=i\partial_\nu(\Theta^{\mu\nu}\partial_\mu f\star' g), \tag{1.50}$$

displaying the same technique as in [7], a correction term can be found to get a locally conserved albeit non symmetric EMT. Indeed, the relation (1.48) then leads to the expression

$$\begin{aligned}\partial^\rho T_{\rho\mu} &= -i\lambda\frac{\Theta^{\alpha\beta}}{4(4!)}g_{\rho\alpha}\partial^\rho\Big\{\phi\star\bar\phi\star'\partial_\beta\Big(\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi\Big)\\ &+ \frac{1}{2}\phi\star\bar\phi\star'\partial_\beta\Big(\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi\Big)+\frac{1}{2}\phi\star\phi\star'\partial_\beta\Big([\partial_\mu\bar\phi,\bar\phi]_\star\Big)\Big\}\\ &+ (\phi\leftrightarrow\bar\phi)=-\partial^\rho t_{\rho\mu} \end{aligned}\tag{1.51}$$

where

$$\begin{aligned}t_{\rho\mu} &= i\lambda\frac{\Theta^{\alpha\beta}}{4(4!)}g_{\rho\alpha}\Big\{\phi\star\bar\phi\star'\partial_\beta\Big(\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi\Big)\\ &+ \frac{1}{2}\phi\star\bar\phi\star'\partial_\beta\Big(\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi\Big)+\frac{1}{2}\phi\star\phi\star'\partial_\beta\Big([\partial_\mu\bar\phi,\bar\phi]_\star\Big)\Big\}\\ &+ (\phi\leftrightarrow\bar\phi) \end{aligned}\tag{1.52}$$

In the massless theory, one can then deduce an improved EMT in the form

$$T^I_{\rho\mu}=T^{m=o}_{\rho\mu}+\frac{1}{6}\Big(g_{\rho\mu}\Box-\partial_\rho\partial_\mu\Big)\{\phi,\bar\phi\}_\star \tag{1.53}$$

which is traceless, i. e. $tr(T^I_{\rho\mu}) = g^{\rho\mu}T^I_{\rho\mu} = T^{I\mu}_{\mu} = 0$ while the improved locally conserved EMT is given by

$$\hat{T}^I_{\rho\mu} = T^{m=o}_{\rho\mu} + t_{\rho\mu} + \frac{1}{6}\Big(g_{\rho\mu}\Box - \partial_\rho\partial_\mu\Big)\{\phi,\bar{\phi}\}_\star. \tag{1.54}$$

Let us mention that $tr(\hat{T}^I_{\rho\mu}) = tr(t_{\rho\mu})$. More precisely, for $D=4$

$$\begin{aligned} tr(\hat{T}^I_{\rho\mu}) = t^\mu_\mu = i\lambda\frac{\Theta^{\mu\beta}}{(4!)}\Big\{\phi\star\bar{\phi}\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar{\phi}-\phi\star\partial_\mu\bar{\phi}\Big)\\ +\frac{1}{2}\phi\star\bar{\phi}\star'\,\partial_\beta\Big(\partial_\mu\bar{\phi}\star\phi-\bar{\phi}\star\partial_\mu\phi\Big)+\frac{1}{2}\phi\star\phi\star'\,\partial_\beta\Big([\partial_\mu\bar{\phi},\bar{\phi}]_\star\Big)\Big\}\\ +(\phi\leftrightarrow\bar{\phi}). \end{aligned} \tag{1.55}$$

Further investigations in order to obtain a symmetric locally conserved EMT $\hat{T}^s_{\rho\mu}$ can be considered through the ordinary Belinfante trick. Let us define the tensor $\chi_{\sigma\rho\mu}$ such that

$$\hat{T}^s_{\rho\mu} \equiv \hat{T}_{\rho\mu} + \partial^\sigma\chi_{\sigma\rho\mu}, \qquad \chi_{\sigma\rho\mu} = -\chi_{\rho\sigma\mu}$$

The Belinfante problem requires $\hat{T}^s_{\rho\mu}$ to be symmetric:

$$\begin{aligned} \hat{T}^s_{\rho\mu} = \hat{T}^s_{\mu\rho} \qquad &\Longrightarrow \hat{T}_{\rho\mu} + \partial^\sigma\chi_{\sigma\rho\mu} = \hat{T}_{\mu\rho} + \partial^\sigma\chi_{\sigma\mu\rho}\\ &\Longrightarrow t_{\rho\mu} - t_{\mu\rho} = \partial^\sigma(\chi_{\sigma\mu\rho} - \chi_{\sigma\rho\mu}). \end{aligned} \tag{1.56}$$

By setting $\chi_{\sigma[\mu\rho]} =: \frac{1}{2}(\chi_{\sigma\mu\rho} - \chi_{\sigma\rho\mu})$, we obtain the Belinfante partial differential equation

$$\partial^\sigma\chi_{\sigma[\mu\rho]} = \frac{1}{2}(t_{\rho\mu} - t_{\mu\rho}) \tag{1.57}$$

or explicitly

$$\begin{aligned} \partial^\sigma\chi_{\sigma[\mu\rho]} &= i\lambda\frac{\Theta^{\alpha\beta}}{8(4!)}g_{\rho\alpha}\Big\{\phi\star\bar{\phi}\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar{\phi}-\phi\star\partial_\mu\bar{\phi}\Big)\\ &\quad+\frac{1}{2}\phi\star\bar{\phi}\star'\,\partial_\beta\Big(\partial_\mu\bar{\phi}\star\phi-\bar{\phi}\star\partial_\mu\phi\Big)\\ &+\quad\frac{1}{2}\phi\star\phi\star'\,\partial_\beta\Big([\partial_\mu\bar{\phi},\bar{\phi}]_\star\Big)+(\phi\leftrightarrow\bar{\phi})\Big\}\\ &-\quad(\mu\longleftrightarrow\rho). \end{aligned} \tag{1.58}$$

Finally, let us examine the term breaking the dilation symmetry. By considering the infinitesimal generator

$$\delta_\epsilon x^\mu = (1+\epsilon)x^\mu, \qquad \delta_\epsilon\phi = \epsilon(1 + x^\mu\star\partial_\mu)\phi$$

where ϵ is the dilatation parameter, we can express the dilation transformation in terms of the Ward functional differential operator

$$\begin{aligned} W^\Theta_D &= \frac{1}{2}\int d^Dx\Big[(1+x^\mu\star\partial_\mu)\phi\star\frac{\delta}{\delta\phi}+\frac{\delta}{\delta\phi}\star(1+x^\mu\star\partial_\mu)\phi\\ &+\quad(1+x^\mu\star\partial_\mu)\bar{\phi}\star\frac{\delta}{\delta\bar{\phi}}+\frac{\delta}{\delta\bar{\phi}}\star(1+x^\mu\star\partial_\mu)\bar{\phi}\Big] \end{aligned} \tag{1.59}$$

acting on the Euclidean action $S_\star[\phi,\bar\phi]$ in a massless field theory. As $x^\mu\star\partial_\mu\phi = x^\mu\partial_\mu\phi$, one can immediately check that

$$\int d^Dx\left[x^\mu\star\Big(\partial^\rho\frac{1}{2}\{\partial_\mu\phi,\partial_\rho\bar\phi\}_\star\Big)\right] = \int d^Dx\left[x^\mu\star\partial^\rho\Big(\partial_\mu\phi\star\partial_\rho\bar\phi\Big)\right] \tag{1.60}$$

and

$$\int d^Dx\left\{x^\mu\star\left[\partial_\mu\Big(\frac{1}{2}\{\partial_\rho\phi,\partial^\rho\bar\phi\}_\star\Big) - \partial^\rho\phi\star\partial_\mu\partial_\rho\bar\phi - \partial^\rho\bar\phi\star\partial_\mu\partial_\rho\phi\right] = 0.\right. \tag{1.61}$$

It follows that

$$\begin{aligned} W_D^\Theta S_\star[\phi,\bar\phi] &= -\int d^Dx\Big[\partial^\rho\Big(\frac{1}{2}\{x^\mu,\hat T^I_{\rho\mu}\}_\star\Big) - t^\mu_\mu \\ &- \frac{1}{2}\Big\{x^\mu,\frac{\lambda}{2.(4!)}\Big[\partial_\mu\phi\star\Big(2\bar\phi\star\phi\star\bar\phi+\{\phi,\bar\phi\star\bar\phi\}_\star\Big) \\ &+ \partial_\mu\bar\phi\Big(2\phi\star\bar\phi\star\phi+\{\bar\phi,\phi\star\phi\}_\star+\Big) - \frac{1}{2}\partial_\mu\{\phi,\bar\phi\star\phi\star\bar\phi\}_\star \\ &- \frac{1}{4}\partial_\mu\Big(\{\phi\star\bar\phi,\bar\phi\star\phi\}_\star+\{\phi\star\phi,\bar\phi\star\bar\phi\}_\star\Big]\Big\}_\star\Big] \end{aligned} \tag{1.62}$$

affording a breaking term given by

$$\begin{aligned} B &= -t^\mu_\mu - \frac{1}{2}\Big\{x^\mu,\frac{\lambda}{2.(4!)}\Big[\partial_\mu\phi\star\Big(2\bar\phi\star\phi\star\bar\phi+\{\bar\phi\star\bar\phi,\phi\}_\star\Big) \\ &\quad+\partial_\mu\bar\phi\Big(2\phi\star\bar\phi\star\phi+\{\bar\phi,\phi\star\phi\}_\star\Big) - \frac{1}{2}\partial_\mu\Big(\{\phi,\bar\phi\star\phi\star\bar\phi\}_\star\Big) \\ &\quad-\frac{1}{4}\partial_\mu\Big(\{\phi\star\bar\phi,\bar\phi\star\phi\}_\star+\{\phi\star\phi,\bar\phi\star\bar\phi\}_\star\Big)\Big]\Big\}_\star. \end{aligned} \tag{1.63}$$

1.3.2 Energy momentum tensor in NC complex Grosse-Wulkenhaar model

This section aims at prolonging the investigations started in [7, 8] by considering now the non-commutative (NC) complex Grosse-Wulkenhaar (GW) Lagrangian action as follows:

$$\begin{aligned} S^\Omega_\star[\phi,\bar\phi] &= \int d^Dx\Big[\partial_\mu\phi\star\partial^\mu\bar\phi + m^2\phi\star\bar\phi + \frac{\Omega^2}{2}(\tilde x_\mu\phi)\star(\tilde x^\mu\bar\phi) \\ &+ \frac{\lambda}{2.4!}(\phi\star\bar\phi\star\phi\star\bar\phi+\phi\star\bar\phi\star\bar\phi\star\phi)\Big], \end{aligned} \tag{1.64}$$

where $\widetilde{x} = 2(\Theta^{-1})\cdot x$, Θ breaks into diagonal blocks $\begin{pmatrix} 0 & \Theta \\ -\Theta & 0\end{pmatrix}$. ϕ is a complex scalar field (with rapid decay). The harmonic term Ω, ($\Omega\in[0,\sqrt{2}]$) ensures ultraviolet (UV)/infrared (IR) freedom for the action implying its renormalizability, and such that the Lagrangian action becomes covariant under Langmann-Szabo duality [57], i.e. covariant under the symmetry: $\tilde x_\mu \longleftrightarrow p_\mu \equiv \partial_\mu$. The Lagrangian density, depending explicitly on x^μ through the field ϕ interaction with a harmonic external source, does not describe a closed system. Furthermore, it is not invariant under space-time translation. Besides, at the parameter limit $\Theta\to 0$, the model does not converge to the ordinary ϕ^4 scalar field theory due to the presence of the inverse matrix (Θ^{-1}), then causing a singularity. The $\star$-Grosse-Wulkenhaar ϕ^4_D theory is renormalizable at all orders in λ. This result has been now proved by various methods (see [73] and references therein). It is a matter of algebra to recast the Grosse-Wulkenhaar harmonic term as

$$\begin{aligned}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\bar{\phi}) &= \frac{1}{4}\big(\tilde{x}_\mu\star\phi\star\tilde{x}^\mu\star\bar{\phi}+\tilde{x}_\mu\star\phi\star\bar{\phi}\star\tilde{x}^\mu\\ &+ \phi\star\tilde{x}_\mu\star\tilde{x}^\mu\star\bar{\phi}+\phi\star\tilde{x}_\mu\star\bar{\phi}\star\tilde{x}^\mu\big)\end{aligned}$$

and to re-express accordingly the action (1.64) so that it now entirely lies in the $\star$-algebra of fields with the advantage to be stable under formal $\star$-algebraic computations (such that the cyclicity of $\star$-factors under integral). One can then deduce a suitable NC complex GW Lagrangian density in the form

$$\begin{aligned}\mathcal{L}^\Omega_\star &= \mathcal{L}_\star+\frac{\Omega^2}{8}\Big[\tilde{x}\star\phi\star\tilde{x}\star\bar{\phi}+\tilde{x}\star\bar{\phi}\star\tilde{x}\star\phi+\frac{1}{2}\Big(\tilde{x}\star\bar{\phi}\star\phi\star\tilde{x}\\ &+\tilde{x}\star\phi\star\bar{\phi}\star\tilde{x}+\tilde{x}\star\tilde{x}\star\phi\star\bar{\phi}+\tilde{x}\star\tilde{x}\star\bar{\phi}\star\phi\Big)\Big]\end{aligned}\tag{1.65}$$

Besides, through the usual variational principle, the Euler-Lagrange equations of motion for ϕ and $\bar{\phi}$ take the form:

$$\begin{aligned}\frac{\delta S^\Omega_\star}{\delta\phi}=0\Leftrightarrow &- \partial_\rho\partial^\rho\bar{\phi}+m^2\bar{\phi}+\frac{\lambda}{2.4!}(2\bar{\phi}\star\phi\star\bar{\phi}+\{\bar{\phi}\star\bar{\phi},\phi\}_\star)\\ &+ \frac{\Omega^2}{8}(2\tilde{x}\star\bar{\phi}\star\tilde{x}+\{\bar{\phi},\tilde{x}\star\tilde{x}\}_\star)=0\end{aligned}\tag{1.66}$$

$$\begin{aligned}\frac{\delta S^\Omega_\star}{\delta\bar{\phi}}=0\Leftrightarrow &- \partial_\rho\partial^\rho\phi+m^2\phi+\frac{\lambda}{2.4!}(2\phi\star\bar{\phi}\star\phi+\{\phi\star\phi,\bar{\phi}\}_\star)\\ &+ \frac{\Omega^2}{8}(2\tilde{x}\star\phi\star\tilde{x}+\{\phi,\tilde{x}\star\tilde{x}\}_\star)=0,\end{aligned}$$

with the additional constraint relation $\delta S^\Omega_\star/\delta\tilde{x}_\rho=0$, i.e.

$$\frac{\Omega^2}{8}\Big(2\phi\star\tilde{x}^\rho\star\bar{\phi}+2\bar{\phi}\star\tilde{x}^\rho\star\phi+\{\phi\star\bar{\phi},\tilde{x}^\rho\}_\star+\{\bar{\phi}\star\phi,\tilde{x}^\rho\}_\star\Big)=0\tag{1.67}$$

stated to avoid translational invariance violation due to the presence of the coordinate $\widetilde{x}^\rho$. As expected, all equations carry contributions from both complex field and its conjugate counterpart, making the Lagrange equations of motion and the corresponding constraint relation more cumbersome than in the pure real scalar field background. Furthermore, the explicit appearance of the coordinate $\widetilde{x}^\mu$ in the Grosse and Wulkenhaar Lagrangian density implies no invariance under spacetime translation.

1.3.3 Regularization of the EMTs

The energy momentum of the theory can be now computed considering infinitesimal translations. Indeed, from infinitesimal translations, we can define the global canonical Ward identity operator

(WIop) for the NC complex GW model

$$\mathcal{W}_\mu^\Theta = \frac{1}{2}\int d^Dx\,(\partial_\mu\phi\star\frac{\delta}{\delta\phi}+\frac{\delta}{\delta\phi}\star\partial_\mu\phi+\partial_\mu\bar\phi\star\frac{\delta}{\delta\bar\phi}+\frac{\delta}{\delta\bar\phi}\star\partial_\mu\bar\phi + \partial_\mu\tilde{x}_\rho\star\frac{\delta}{\delta\tilde{x}_\rho}+\frac{\delta}{\delta\tilde{x}_\rho}\star\partial_\mu\tilde{x}_\rho) \tag{1.68}$$

such that its action on the Lagrangian density

$$\mathcal{W}_\mu^\Theta S_\star^\Omega \equiv -\int d^Dx\;\partial^\rho\,T^\Omega_{\rho\mu}=0 \tag{1.69}$$

yields the canonical energy momentum tensor (EMT)

$$T^\Omega_{\rho\mu}=\frac{1}{2}\{\partial_\mu\phi,\partial_\rho\bar\phi\}_\star+\frac{1}{2}\{\partial_\mu\bar\phi,\partial_\rho\phi\}_\star-g_{\rho\mu}\mathcal{L}^\Omega_\star, \tag{1.70}$$

where $g_{\rho\mu}$ is the Euclidean metric, $\mathcal{L}^\Omega_\star$ the NC Lagrangian, $\{(\cdot),(\cdot)\}_\star$ the $\star$-anticommutator. The EMT then conserves its form comparatively to the result of [7]. $T^\Omega_{\rho\mu}$ is symmetric, nonlocally conserved, and in massless theory, not traceless. Moreover, putting the mass term to zero, the usual improved tensor

$$T^{I,\Omega}_{\rho\mu} = T^{\Omega,m=o}_{\rho\mu}+\frac{1}{6}\Big(g_{\rho\mu}\Box-\partial_\rho\partial_\mu\Big)\{\phi,\bar\phi\}_\star$$

is not traceless too. Let us investigate now an improvement to (1.70) for the local conservation order. After some algebra, it can be deduced

$$\begin{aligned}\partial^\rho T^\Omega_{\rho\mu} &= \partial^\rho T_{\rho\mu}-\frac{\Omega^2}{16}\Big([\tilde{x}\star\tilde{x},\partial_\mu\phi\star\bar\phi+\partial_\mu\bar\phi\star\phi]_\star\\ &+ [\tilde{x}\star\bar\phi,\partial_\mu\phi\star\tilde{x}]_\star+[\tilde{x}\star\phi,\partial_\mu\bar\phi\star\tilde{x}]_\star\\ &+ [\tilde{x}\star\partial_\mu\phi,\bar\phi\star\tilde{x}]_\star+[\tilde{x}\star\partial_\mu\bar\phi,\phi\star\tilde{x}]_\star\Big)\\ &- \frac{\Omega^2}{8}[\tilde{x},\partial_\mu\phi\star\tilde{x}\star\bar\phi+\partial_\mu\bar\phi\star\tilde{x}\star\phi]_\star\\ &=: -\partial^\rho t^\Omega_{\rho\mu},\end{aligned} \tag{1.71}$$

where

$$\begin{aligned}\partial^\rho T_{\rho\mu} &= -\frac{\lambda}{4(4!)}\Big([\phi\star\bar\phi,\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi]_\star\\ &+ [\bar\phi\star\phi,\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi]_\star\\ &+ \frac{1}{2}[\phi\star\bar\phi,\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi]_\star\\ &+ \frac{1}{2}[\bar\phi\star\phi,\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi]_\star\\ &+ \frac{1}{2}[\phi\star\phi,[\partial_\mu\bar\phi,\bar\phi]_\star]_\star+\frac{1}{2}[\bar\phi\star\bar\phi,[\partial_\mu\phi,\phi]_\star]_\star\Big).\end{aligned} \tag{1.72}$$

$[(\cdot),(\cdot)]_\star$ is the Moyal $\star$-commutator. Thus the "Wulkenization" [73] process clearly governs the EMT improvement mechanism. A closer look on (1.71) shows that $T^\Omega_{\rho\mu}$ is globally conserved, as in NCFT the $\star$-commutators under integral cancel. For physical interpretation, let us consider the space-space NCFT determined by $\Theta^{0i}=0$. Then one can readily prove that there exists a conserved D-vector momentum P^{Ω}_{μ}, i.e. satisfying $\partial^0 P^{\Omega}_{\mu} = \partial^0 \int d^{D-1}x\, T^{\Omega}_{0\mu} = 0$. Such a vector conserved quantity is also observed in the *naive* (unrenormalizable) NC scalar field [28] as well as in the NC real GW model [7]. Displaying the same tedious algebraic apparatus as in [7] and [41], a correction term can be found to get a new locally conserved albeit non symmetric EMT $\hat{T}^{I,\Omega}_{\rho\mu} = T^{I,\Omega}_{\rho\mu} + t^{\Omega}_{\rho\mu}$, where

$$\begin{aligned}
t^{\Omega}_{\rho\mu} &= -i\lambda\frac{\Theta^{\alpha\beta}}{4(4!)}g_{\rho\alpha}\Big\{\phi\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi\Big)\\
&+ \bar\phi\star\phi\star'\,\partial_\beta\Big(\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi\Big)\\
&+ \frac{1}{2}\phi\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi\Big)\\
&+ \frac{1}{2}\bar\phi\star\phi\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi\Big)\\
&+ \frac{1}{2}\phi\star\phi\star'\,\partial_\beta\Big([\partial_\mu\bar\phi,\bar\phi]_\star\Big)+\frac{1}{2}\bar\phi\star\bar\phi\star'\,\partial_\beta\Big([\partial_\mu\phi,\phi]_\star\Big)\Big\}\\
&- i\frac{\Omega^2}{16}g_{\rho\alpha}\Theta^{\alpha\beta}\Big\{\tilde x\star\tilde x\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar\phi+\partial_\mu\bar\phi\star\phi\Big)\\
&+ \tilde x\star\phi\star'\,\partial_\beta\Big(\partial_\mu\bar\phi\star\tilde x\Big)+\tilde x\star\partial_\mu\phi\star'\,\partial_\beta\Big(\bar\phi\star\tilde x\Big)\\
&+ \tilde x\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\phi\star\tilde x\Big)+\tilde x\star\partial_\mu\bar\phi\star'\,\partial_\beta\Big(\phi\star\tilde x\Big)\Big\}\\
&- i\frac{\Omega^2}{8}g_{\rho\alpha}\Theta^{\alpha\beta}\tilde x\star'\,\partial_\beta\Big(\partial_\mu\phi\star\tilde x\star\bar\phi+\partial_\mu\bar\phi\star\tilde x\star\phi\Big),
\end{aligned}\tag{1.73}$$

with non-vanishing trace for m = 0, recalling that the theory is not scale invariant. Furthermore, one can work out a symmetric locally conserved EMT through the ordinary Belifante trick (see [7] and references therein), defining the tensor $\chi_{\sigma\rho\mu}$ such that

$$\hat{T}^{\Omega s}_{\rho\mu} = \hat{T}^{\Omega}_{\rho\mu} + \partial^\sigma\chi_{\sigma\rho\mu}, \qquad \chi_{\sigma\rho\mu} = -\chi_{\rho\sigma\mu}.$$

The underlying Belifante type partial differential equation

$$\hat{T}^{\Omega}_{\rho\mu} - \hat{T}^{\Omega}_{\mu\rho} = \partial^\sigma(\chi_{\sigma\mu\rho}-\chi_{\sigma\rho\mu}) =: \partial^\sigma\chi_{\sigma[\mu\rho]} \tag{1.74}$$

$$\begin{aligned}
\partial^\sigma\chi_{\sigma[\mu\rho]} &= -i\lambda\frac{\Theta^{\alpha\beta}}{8(4!)}g_{\rho\alpha}\Big\{\phi\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar\phi-\phi\star\partial_\mu\bar\phi\Big)\\
&+ \frac{1}{2}\phi\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\bar\phi\star\phi-\bar\phi\star\partial_\mu\phi\Big)\\
&+ \frac{1}{4}\phi\star\phi\star'\,\partial_\beta\Big(\partial_\mu\bar\phi\star\bar\phi\Big)+(\phi\leftrightarrow\bar\phi)\Big\}\\
&- i\frac{\Omega^2}{32}g_{\rho\alpha}\Theta^{\alpha\beta}\Big\{\tilde x\star\tilde x\star'\,\partial_\beta\Big(\partial_\mu\phi\star\bar\phi+\partial_\mu\bar\phi\star\phi\Big)\\
&+ \Big[\tilde x\star\bar\phi\star'\,\partial_\beta\Big(\partial_\mu\phi\star\tilde x\Big)+\tilde x\star\partial_\mu\phi\star'\,\partial_\beta\Big(\bar\phi\star\tilde x\Big)+(\phi\leftrightarrow\bar\phi)\Big]\Big\}\\
&- i\frac{\Omega^2}{16}g_{\rho\alpha}\Theta^{\alpha\beta}\tilde x\star'\,\partial_\beta\Big(\partial_\mu\phi\star\tilde x\star\bar\phi+\partial_\mu\bar\phi\star\tilde x\star\phi\Big)\\
&- (\mu\longleftrightarrow\rho)
\end{aligned}\tag{1.75}$$

is less comfortable than the one worked out by Abou-Zeid and Dorn [1]. The nonlocal conservation of the canonical massless EMT obviously induces a dilatation symmetry breaking. In addition, even if an improved locally conserved EMT is provided, the scale invariance is no longer valid and predictable since the evidence of a non-vanishing trace of this improved EMT. Both these arguments on dilatation symmetry breaking are valid for massless Grosse and Wulkenhaar model. Indeed, defining infinitesimal dilatation generators and the corresponding global symmetrized Wlop $W^{\Theta}_{D,\epsilon}$, respectively,

$$\begin{aligned} \delta_\epsilon \tilde{x}_\mu &= (1+\epsilon)\tilde{x}_\mu, \quad \delta_{1,\epsilon}\phi = \epsilon D_1\phi, \quad \delta_{2,\epsilon}\phi = \epsilon D_2\phi, \\ D_1(.) &= (1 + x^\mu \star \partial_\mu)(.), \quad D_2(.) = ((.) + \partial_\mu(.) \star x^\mu) \end{aligned} \tag{1.76}$$

$$\begin{aligned} W^{\Theta}_{D,\epsilon}(.) &= \int d^D x \Big\{ \frac{\epsilon}{4}\Big[D_1\phi \star \frac{\delta(.)}{\delta\phi} + \frac{\delta(.)}{\delta\phi} \star D_1\phi + D_2\phi \star \frac{\delta(.)}{\delta\phi} \\ &+ \frac{\delta(.)}{\delta\phi} \star D_2\phi + (\phi \longleftrightarrow \bar{\phi})\Big] + \frac{1}{2}\Big[\delta_\epsilon \tilde{x}_\rho \star \frac{\delta(.)}{\partial \tilde{x}_\rho} \\ &+ \frac{\delta(.)}{\delta \tilde{x}_\rho} \star \delta_\epsilon \tilde{x}_\rho\Big]\Big\} \end{aligned}$$

so that $\frac{\partial}{\partial\epsilon} W^{\Theta}_{D,\epsilon}(S^{\Omega}_\star) = -\int d^D x(\partial^\rho \mathcal{D}^{\Omega}_\rho + B^{\Omega}_\star)$ with $\mathcal{D}^{\Omega}_\rho$ the dilatation current given by $\mathcal{D}^{\Omega}_\rho = \frac{1}{2}\{x^\mu, \hat{T}^{I,\Omega}_{\rho\mu}\}_\star$, the breaking quantity $B^{\Omega}_\star$ reveals a dependence on both the non-vanishing trace of the local conservation improving tensor $t^{\Omega}_{\rho\mu}$ through $\hat{T}^{I,\Omega,\mu}_{\mu}$ and the GW term. Explicitly, one finally gets

$$\begin{aligned} B^{\Omega}_\star &= -\hat{T}^{I,\Omega,\mu}_{\mu} - \frac{1}{2}\{x^\mu, \frac{\lambda}{2.(4!)}[\partial_\mu\phi \star (2\bar{\phi} \star \phi \star \bar{\phi} + \{\bar{\phi} \star \bar{\phi}, \phi\}_\star) \\ &+ \partial_\mu\bar{\phi}(2\phi \star \bar{\phi} \star \phi + \{\bar{\phi}, \phi \star \phi\}_\star) - \frac{1}{2}\partial_\mu(\{\phi \star \bar{\phi} \star \phi, \bar{\phi}\}_\star) \\ &+ \frac{1}{4}\partial_\mu(\{\phi \star \bar{\phi}, \bar{\phi} \star \phi\}_\star + \{\phi \star \phi, \bar{\phi} \star \bar{\phi}\}_\star)] \\ &+ \frac{\Omega^2}{8}[-\partial_\mu(\{\tilde{x} \star \phi, \tilde{x} \star \bar{\phi}\}_\star + \frac{1}{2}\{\tilde{x} \star \{\bar{\phi}, \phi\}_\star, \tilde{x}\}_\star) \\ &+ \partial_\mu\phi \star (2\tilde{x} \star \bar{\phi} \star \tilde{x} + \{\bar{\phi}, \tilde{x} \star \tilde{x}\}_\star) \\ &+ \partial_\mu\bar{\phi} \star (2\tilde{x} \star \phi \star \tilde{x} + \{\phi, \tilde{x} \star \tilde{x}\}_\star)]\}_\star \end{aligned} \tag{1.77}$$

where, for $D = 4$,

$$\begin{aligned} \hat{T}^{I,\Omega,\mu}_{\mu} &\equiv tr(T^{I,\Omega}_{\rho\mu}) \\ &= -\frac{\Omega^2}{16}\Big\{2[\tilde{x}, \phi \star \tilde{x} \star \bar{\phi} + \bar{\phi} \star \tilde{x} \star \phi]_\star + [\tilde{x} \star \bar{\phi}, \phi \star \tilde{x}]_\star \\ &+ [\tilde{x} \star \phi, \bar{\phi} \star \tilde{x}]_\star + [\tilde{x} \star \tilde{x}, \{\phi, \bar{\phi}\}_\star]_\star + \{\bar{\phi}, \tilde{x} \star \tilde{x} \star \phi\}_\star \\ &+ \{\phi, \tilde{x} \star \tilde{x} \star \bar{\phi}\}_\star + 3\tilde{x} \star \{\phi, \bar{\phi}\}_\star \star \tilde{x} + 3\tilde{x} \star \tilde{x} \star \{\phi, \bar{\phi}\}_\star \\ &+ 8\tilde{x} \star (\phi \star \tilde{x} \star \bar{\phi} + \bar{\phi} \star \tilde{x} \star \phi)\Big\}. \end{aligned} \tag{1.78}$$

Physically admissible Lagrangians

In this subsection, let us list the 8 relevant Lagrangians, obtained by permutation symmetry, denoted by $\mathcal{L}^{\Omega}_{\star i}, \quad i = \overline{1,\, 8}$, that can be used to improve the energy momentum tensor by the formula: $\partial^\rho T^\Omega_{\rho\mu} = \partial^\rho t^\Omega_{\rho\mu}$. They are given with the expressions of corresponding $\partial^\rho T^\Omega_{\rho\mu}$.

- $$\begin{aligned}\mathcal{L}^\Omega_{\star 1} &= \mathcal{L}_\star + \frac{\Omega^2}{8}\Big[\tilde{x}\star\phi\star\tilde{x}\star\phi^\dagger + \tilde{x}\star\bar{\phi}\star\tilde{x}\star\phi + \frac{1}{2}\Big(\tilde{x}\star\bar{\phi}\star\phi\star\tilde{x}\\ &+\tilde{x}\star\phi\star\bar{\phi}\star\tilde{x} + \tilde{x}\star\tilde{x}\star\phi\star\bar{\phi} + \tilde{x}\star\tilde{x}\star\bar{\phi}\star\phi\Big)\Big]. \qquad (1.79)\end{aligned}$$

$$\begin{aligned}\partial^\rho T^\Omega_{\rho\mu} &= \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\tilde{x}\star\tilde{x}, \partial_\mu\phi\star\bar{\phi} + \partial_\mu\bar{\phi}\star\phi]_\star + [\tilde{x}\star\bar{\phi}, \partial_\mu\phi\star\tilde{x}]_\star\\ &+[\tilde{x}\star\phi, \partial_\mu\bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\phi, \bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\bar{\phi}, \phi\star\tilde{x}]_\star\Big)\\ &-\frac{\Omega^2}{8}[\tilde{x}, \partial_\mu\phi\star\tilde{x}\star\bar{\phi} + \partial_\mu\bar{\phi}\star\tilde{x}\star\phi]_\star. \qquad (1.80)\end{aligned}$$

- $$\begin{aligned}\mathcal{L}^\Omega_{\star 2} &= \mathcal{L}_\star + \frac{\Omega^2}{8}\Big[\phi\star\tilde{x}\star\bar{\phi}\star\tilde{x} + \bar{\phi}\star\tilde{x}\star\phi\star\tilde{x} + \frac{1}{2}\Big(\tilde{x}\star\bar{\phi}\star\phi\star\tilde{x}\\ &+\tilde{x}\star\phi\star\bar{\phi}\star\tilde{x} + \tilde{x}\star\tilde{x}\star\phi\star\bar{\phi} + \tilde{x}\star\tilde{x}\star\bar{\phi}\star\phi\Big)\Big]. \qquad (1.81)\end{aligned}$$

$$\begin{aligned}\partial^\rho T^\Omega_{\rho\mu} &= \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\tilde{x}\star\tilde{x}, \partial_\mu\phi\star\bar{\phi} + \partial_\mu\bar{\phi}\star\phi]_\star + [\tilde{x}\star\bar{\phi}, \partial_\mu\phi\star\tilde{x}]_\star\\ &+[\tilde{x}\star\phi, \partial_\mu\bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\phi, \bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\bar{\phi}, \phi\star\tilde{x}]_\star\Big)\\ &+\frac{\Omega^2}{8}[\tilde{x}, \partial_\mu\phi\star\tilde{x}\star\bar{\phi} + \partial_\mu\bar{\phi}\star\tilde{x}\star\phi]_\star. \qquad (1.82)\end{aligned}$$

- $$\begin{aligned}\mathcal{L}^\Omega_{\star 3} &= \mathcal{L}_\star + \frac{\Omega^2}{8}\Big[\tilde{x}\star\phi\star\tilde{x}\star\bar{\phi} + \tilde{x}\star\bar{\phi}\star\tilde{x}\star\phi + \frac{1}{2}\Big(\tilde{x}\star\bar{\phi}\star\phi\star\tilde{x}\\ &+\tilde{x}\star\phi\star\bar{\phi}\star\tilde{x} + \phi\star\bar{\phi}\star\tilde{x}\star\tilde{x} + \bar{\phi}\star\phi\star\tilde{x}\star\tilde{x}\Big)\Big]. \qquad (1.83)\end{aligned}$$

$$\begin{aligned}\partial^\rho T^\Omega_{\rho\mu} &= \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\bar{\phi}\star\partial_\mu\phi + \phi\star\partial_\mu\bar{\phi}, \tilde{x}\star\tilde{x}]_\star + [\tilde{x}\star\bar{\phi}, \partial_\mu\phi\star\tilde{x}]_\star\\ &+[\tilde{x}\star\phi, \partial_\mu\bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\phi, \bar{\phi}\star\tilde{x}]_\star + [\tilde{x}\star\partial_\mu\bar{\phi}, \phi\star\tilde{x}]_\star\Big)\\ &-\frac{\Omega^2}{8}[\tilde{x}, \partial_\mu\phi\star\tilde{x}\star\bar{\phi} + \partial_\mu\bar{\phi}\star\tilde{x}\star\phi]_\star \qquad (1.84)\end{aligned}$$

- $$\begin{aligned}\mathcal{L}^\Omega_{\star 4} &= \mathcal{L}_\star + \frac{\Omega^2}{8}\Big[\phi\star\tilde{x}\star\bar{\phi}\star\tilde{x} + \bar{\phi}\star\tilde{x}\star\phi\star\tilde{x} + \frac{1}{2}\Big(\tilde{x}\star\bar{\phi}\star\phi\star\tilde{x}\\ &+\tilde{x}\star\phi\star\bar{\phi}\star\tilde{x} + \phi\star\bar{\phi}\star\tilde{x}\star\tilde{x} + \bar{\phi}\star\phi\star\tilde{x}\star\tilde{x}\Big)\Big]. \qquad (1.85)\end{aligned}$$

$$\partial^\rho T^\Omega_{\rho\mu} = \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\bar\phi\star\partial_\mu\phi+\phi\star\partial_\mu\bar\phi,\tilde x\star\tilde x]_\star+[\tilde x\star\bar\phi,\partial_\mu\phi\star\tilde x]_\star$$
$$+[\tilde x\star\phi,\partial_\mu\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\phi,\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\bar\phi,\phi\star\tilde x]_\star\Big)$$
$$+\frac{\Omega^2}{8}[\tilde x,\partial_\mu\phi\star\tilde x\star\bar\phi+\partial_\mu\bar\phi\star\tilde x\star\phi]_\star \tag{1.86}$$

•

$$\mathcal{L}^\Omega_{\star 5} = \mathcal{L}_\star+\frac{\Omega^2}{8}\Big[\tilde x\star\phi\star\tilde x\star\bar\phi+\bar\phi\star\tilde x\star\phi\star\tilde x+\frac{1}{2}\Big(\tilde x\star\bar\phi\star\phi\star\tilde x$$
$$+\tilde x\star\phi\star\bar\phi\star\tilde x+\tilde x\star\tilde x\star\phi\star\bar\phi+\tilde x\star\tilde x\star\bar\phi\star\phi\Big)\Big]. \tag{1.87}$$

$$\partial^\rho T^\Omega_{\rho\mu} = \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\tilde x\star\tilde x,\partial_\mu\phi\star\bar\phi+\partial_\mu\bar\phi\star\phi]_\star+[\tilde x\star\bar\phi,\partial_\mu\phi\star\tilde x]_\star$$
$$+[\tilde x\star\phi,\partial_\mu\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\phi,\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\bar\phi,\phi\star\tilde x]_\star\Big)$$
$$-\frac{\Omega^2}{8}[\tilde x,\partial_\mu\phi\star\tilde x\star\bar\phi-\bar\phi\star\tilde x\star\partial_\mu\phi]_\star. \tag{1.88}$$

•

$$\mathcal{L}^\Omega_{\star 6} = \mathcal{L}_\star+\frac{\Omega^2}{8}\Big[\tilde x\star\phi\star\tilde x\star\bar\phi+\bar\phi\star\tilde x\star\phi\star\tilde x+\frac{1}{2}\Big(\tilde x\star\bar\phi\star\phi\star\tilde x$$
$$+\tilde x\star\phi\star\bar\phi\star\tilde x+\phi\star\bar\phi\star\tilde x\star\tilde x+\bar\phi\star\phi\star\tilde x\star\tilde x\Big)\Big]. \tag{1.89}$$

$$\partial^\rho T^\Omega_{\rho\mu} = \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\bar\phi\star\partial_\mu\phi+\phi\star\partial_\mu\bar\phi,\tilde x\star\tilde x]_\star+[\tilde x\star\bar\phi,\partial_\mu\phi\star\tilde x]_\star$$
$$+[\tilde x\star\phi,\partial_\mu\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\phi,\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\bar\phi,\phi\star\tilde x]_\star\Big)$$
$$-\frac{\Omega^2}{8}[\tilde x,\partial_\mu\phi\star\tilde x\star\bar\phi-\bar\phi\star\tilde x\star\partial_\mu\phi]_\star \tag{1.90}$$

•

$$\mathcal{L}^\Omega_{\star 7} = \mathcal{L}_\star+\frac{\Omega^2}{8}\Big[\phi\star\tilde x\star\bar\phi\star\tilde x+\tilde x\star\bar\phi\star\tilde x\star\phi+\frac{1}{2}\Big(\tilde x\star\bar\phi\star\phi\star\tilde x$$
$$+\tilde x\star\phi\star\bar\phi\star\tilde x+\tilde x\star\tilde x\star\phi\star\bar\phi+\tilde x\star\tilde x\star\bar\phi\star\phi\Big)\Big]. \tag{1.91}$$

$$\partial^\rho T^\Omega_{\rho\mu} = \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\tilde x\star\tilde x,\partial_\mu\phi\star\bar\phi+\partial_\mu\bar\phi\star\phi]_\star+[\tilde x\star\bar\phi,\partial_\mu\phi\star\tilde x]_\star$$
$$+[\tilde x\star\phi,\partial_\mu\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\phi,\bar\phi\star\tilde x]_\star+[\tilde x\star\partial_\mu\bar\phi,\phi\star\tilde x]_\star\Big)$$
$$-\frac{\Omega^2}{8}[\tilde x,\partial_\mu\bar\phi\star\tilde x\star\phi-\phi\star\tilde x\star\partial_\mu\bar\phi]_\star. \tag{1.92}$$

•

$$\mathcal{L}^\Omega_{\star 8} = \mathcal{L}_\star+\frac{\Omega^2}{8}\Big[\phi\star\tilde x\star\bar\phi\star\tilde x+\tilde x\star\bar\phi\star\tilde x\star\phi+\frac{1}{2}\Big(\tilde x\star\bar\phi\star\phi\star\tilde x$$
$$+\tilde x\star\phi\star\bar\phi\star\tilde x+\phi\star\bar\phi\star\tilde x\star\tilde x+\bar\phi\star\phi\star\tilde x\star\tilde x\Big)\Big]. \tag{1.93}$$

$$
\begin{aligned}
\partial^\rho T^\Omega_{\rho\mu} &= \partial^\rho T_{\rho\mu} - \frac{\Omega^2}{16}\Big([\bar\phi\star\partial_\mu\phi + \phi\star\partial_\mu\bar\phi, \tilde x\star\tilde x]_\star + [\tilde x\star\bar\phi, \partial_\mu\phi\star\tilde x]_\star \\
&\quad + [\tilde x\star\phi, \partial_\mu\bar\phi\star\tilde x]_\star + [\tilde x\star\partial_\mu\phi, \bar\phi\star\tilde x]_\star + [\tilde x\star\partial_\mu\bar\phi, \phi\star\tilde x]_\star\Big) \\
&\quad - \frac{\Omega^2}{8}[\tilde x, \partial_\mu\bar\phi\star\tilde x\star\phi - \phi\star\tilde x\star\partial_\mu\bar\phi]_\star.
\end{aligned} \tag{1.94}
$$

Hamiltonian formulation of the GW model and its generalization

This chapter is devoted to the construction of a Hamiltonian formulation of the Grosse-Wulkenhaar model, one of the very few renormalizable noncommutative theories. The Hamiltonian formulation of classical field theory, crucial in the quantization procedure, remains a task to be solved in the noncommutative field theories (NCFTs) widely developed in recent years [24]-[34] (and references therein). So far, all attempts to solve this problem have been made before the advent of the new class of renormalizable NCFTs built on the Grosse and Wulkenhaar (GW) ϕ^4 scalar field theory. See [30] and [61] (and references therein) for more details. This work aims at filling this gap, considering the class of renormalizable GW models treated with a method that generalizes previous construction [30]. The expression of the total Hamiltonian of the system is given. From a space-time Galilean transformation and imposing an additional constraint from the application of the Noether's theorem, the Noether currents are computed as a $U_\star(N)$ gauge currents from $U_\star(N)$ gauge transformations with finite translations, i.e. $g_\epsilon(x) = e^{-i\epsilon^\mu \Theta^{-1}_{\mu\nu} x^\nu} \in U_\star(N)$ such that $g_\epsilon(x) \star f(x) \star g^\dagger_\epsilon(x) = f(x+\epsilon)$. The rotation group of $\mathbb{R}^D$ can be considered as a particular concrete case.

2.1 Hamiltonian formulation of the NCFTs

In this section, we briefly review the Hamiltonian formulation of NCFTs recently developed by Gomis et al [30]-[61]. We then generalize this formulation by introducing a compact support $w_h(x)$.

2.1.1 Quick review of Hamiltonian formulation of NCFTs

This subsection, mainly based on [30] and [61] (and references therein), addresses a Hamiltonian formulation of field theories in a noncommutative space-time. This formulation involves two time coordinates t and λ, and the dynamics in this space is described in such a way that the evolution is local with respect to one of the times. The non-local Lagrangian at time t, $L^{non}(t)$, depends not only on variables at time t but also on ones at different times. In other words, it depends on an infinite number of time derivatives of the position $q_i(t)$. The analogue of the tangent bundle for Lagrangians depending on positions and velocities is now infinite dimensional and can be represented as the space of all possible trajectories. The action is given by

$$S[q] = \int dt\, L^{non}(t) = \int dt\, L([q(t+\lambda)]). \tag{2.1}$$

The functional variational principle can be applied to the action (2.1) to produce the Euler-Lagrange (EL) equation of motion as follows:

$$\frac{\delta S[q]}{\delta q(t)} = \int dt' \, \frac{\delta L^{non}(t')}{\delta q(t)} = 0 \tag{2.2}$$

which must be understood as a functional relation to be fulfilled by physical trajectories. The latter are not obtained as evolution of some given initial conditions. Since the equation of motion is of infinite degree in time derivatives, one should give as initial conditions the value of all these derivatives at some initial time. In other words, we should give the whole trajectory (or part of it) as the initial condition. Let $J = \{q(\lambda), \lambda \in \mathbb{R}\}$ be the space of all possible trajectories. Then the EL equation of motion (2.2) is a Lagrangian constraint defining the subspace $J_R \subset J$ of physical trajectories. In $1+1$ dimensional field theory, we introduce new dynamical variables $\mathcal{Q}(t,\lambda)$ such that

$$\mathcal{Q}(t,\lambda) = q(t+\lambda) = T_t q(\lambda) \tag{2.3}$$

where T_t is the time evolution operator for a given initial trajectory $q(\lambda)$. t is the evolution parameter and λ is a continuous parameter indexing the degrees of freedom.

Definition 2.1 *If we denote by $\mathcal{P}(t,\lambda)$ the canonical momentum of $\mathcal{Q}(t,\lambda)$, and*

$$\mathcal{Q}'(t,\lambda) =: \partial_\lambda \mathcal{Q}(t,\lambda), \;\; \dot{\mathcal{Q}}(t,\lambda) =: \partial_t \mathcal{Q}(t,\lambda), \tag{2.4}$$

then the Hamiltonian is defined as

$$H(t,[\mathcal{Q},\mathcal{P}]) =: \int d\lambda\, \mathcal{P}(t,\lambda)\mathcal{Q}'(t,\lambda) - \widetilde{L}(t,[\mathcal{Q}]) \tag{2.5}$$

where $\widetilde{L}(t,[\mathcal{Q}])$ is a functional defined by $\widetilde{L}(t,[\mathcal{Q}]) = \int d\lambda\, \delta(\lambda)\mathcal{L}(t,\lambda)$.

The density $\mathcal{L}(t,\lambda)$ is constructed from the original non-local Lagrangian density $L^{non}(t)$ by replacing $q(t)$ by $\mathcal{Q}(t,\lambda)$, the t-derivatives of $q(t)$ by λ-derivatives of $\mathcal{Q}(t,\lambda)$ and $q(t+\rho)$ by $\mathcal{Q}(t,\lambda+\rho)$. In this construction of the Hamiltonian, λ inherits the signature of the original time t and is a time-like coordinate. $\mathcal{L}(t,\lambda)$ is local in t and is non-local in λ. H depends linearly on $\mathcal{P}(t,\lambda)$ but does not depend on $\dot{\mathcal{Q}}(t,\lambda)$.

The first and second Hamilton equations can be written as:

$$\dot{\mathcal{Q}}(t,\lambda) = \mathcal{Q}'(t,\lambda), \;\; \dot{\mathcal{P}}(t,\lambda) = \mathcal{P}'(t,\lambda) + \frac{\delta \widetilde{L}(t,[\mathcal{Q}])}{\delta \mathcal{Q}(t,\lambda)}. \tag{2.6}$$

Their solutions are related to those of the EL equations of motion of the original non-local Lagrangian L^{non} if we impose a constraint on the momentum

$$\varphi(t,\lambda) \equiv \mathcal{P}(t,\lambda) - \int d\sigma\, \frac{\epsilon(\lambda)-\epsilon(\sigma)}{2} \frac{\delta \mathcal{L}(t,\sigma)}{\delta \mathcal{Q}(t,\lambda)} \approx 0. \tag{2.7}$$

Here $\epsilon(\lambda)$ is the sign distribution. The symbols "$\equiv$" and "$\approx$ " stand for "strong" and " weak" equalities, respectively. Further constraints are generated by requiring the stability of the primary ones. In the first step, we obtain:

$$\dot{\varphi}(t,\lambda) \equiv \varphi'(t,\lambda) + \delta(\lambda)\int d\sigma\,\frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}(t,0)} \approx 0 \tag{2.8}$$

or simply

$$\dot{\varphi}_0(t,\lambda) \equiv \delta(\lambda)\int d\sigma\,\frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}(t,\lambda)} \approx 0 \tag{2.9}$$

which reduces to the EL equation of motion. Repeating this, we get an infinite set of Hamiltonian constraints. So doing, we are able to describe the original non-local Lagrangian system as a $1+1$ dimensional local (in one of the times) Hamiltonian system, governed by the Hamiltonian H and a set of constraints. Note that this formalism can be viewed as a generalization of the Ostrogradski construction in the case of infinite order derivative theories.

2.1.2 A generalization in $1+1$ dimensional field theory

In this subsection, we aim at enlarging the class of Hamiltonians that can be constructed in the framework of the above mentioned formalism. The corresponding system of Hamilton equations and the constraints are deduced. This generalization will be used in the next section to construct an infinite set of Hamiltonians compatible with the GW model.

Consider a parameter $h \in]0,1[, \quad x,\ y \in \mathbb{R}^n$ and define

$$|x-y| = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}, \quad \omega_h(x-y) = \frac{\omega(\frac{x-y}{h})}{h^n} \tag{2.10}$$

where

$$\omega(u) = \begin{cases} c\cdot\exp(\frac{1}{|u|^2-1}) & |u|<1 \\ 0 & |u|\geq 1 \end{cases}, \quad c = \Big[\int_{|u|<1}\exp(\frac{1}{|u|^2-1})du\Big]^{-1}. \tag{2.11}$$

Then we consider the family of Hamiltonians

$$\mathcal{H}_h(t,[\mathcal{Q}_h,\mathcal{P}_h]) = \int d\lambda \mathcal{P}_h(t,\lambda)\mathcal{Q}'_h(t,\lambda) - L_h(t,[\mathcal{Q}_h]) \tag{2.12}$$

where the quantities $\mathcal{P}_h(t,\lambda)$, $\mathcal{Q}_h(t,\lambda)$, $L_h(t,[\mathcal{Q}_h])$ are defined as follows:

$$\mathcal{P}_h(t,\lambda) = \int_{\mathbb{R}^2} dy\,\omega_h(x-y)\mathcal{P}(y),\ \ \mathcal{Q}_h(t,\lambda) = \int_{\mathbb{R}^2} dy\,\omega_h(x-y)\mathcal{Q}(y) \tag{2.13}$$

$$L_h(t,[\mathcal{Q}_h]) = \int_{\mathbb{R}^2} dy\,\omega_h(x-y)L(t',[Q]) \tag{2.14}$$

$x=(t,\lambda)$ and $y=(t',\lambda')$, $\mathcal{Q}'_h(t,\lambda) = \partial_\lambda\mathcal{Q}_h(t,\lambda)$ and $\int_{\mathbb{R}^n} dy\,\omega_h(x-y) = 1$.

Lemma: *Let $L_h, \quad h \in]0,1[$, define a class of differentiable functionals with compact support $\omega_h(x)$, i.e. for $|x| < M, \quad L_h(x) \neq 0$, and $L_h(x) = 0$ otherwise, where M is a positive number:*

$$L_h(x) = \int_{\mathbb{R}^n} dy\omega_h(x-y)L(y).$$

If the functional L is summable on $x \in \mathcal{D} \subset \mathbb{R}^n$, then

$$\int_{\mathcal{D}} |L_h(x) - L(x)| dx \to 0 \text{ for } h \to 0, \tag{2.15}$$

i.e. L_h converges in average to L for $h \to 0$ where $\mathcal{D}$ is a bounded measurable set.

Proof: For $x \in \mathcal{D}$, we have

$$L_h(x) - L(x) = \int_{\mathbb{R}^n} \omega_h(x-y)[L(y) - L(x)]\mathsf{d}y \tag{2.16}$$

where

$$L(x) = \int_{\mathbb{R}^n} \omega_h(x-y)L(x)\mathsf{d}y = \frac{1}{h^n}\int_{|x-y|\leq h} \omega\Big(\frac{x-y}{h}\Big)L(x)dy.$$

Then

$$|L_h(x) - L(x)| \leq \tfrac{c_1}{h^n}\textstyle\int_{|x-y|\leq h} |L(y) - L(x)|dy, \tag{2.17}$$

$$\text{with} \qquad c_1 = \max_{|x-y|\leq h} \omega\Big(\tfrac{x-y}{h}\Big). \tag{2.18}$$

Using Fubini's theorem we can get

$$\begin{aligned}\int_{\mathcal{D}} |L_h(x) - L(x)| dx &\leq \frac{c_1}{h^n}\int_{\mathcal{D}}\int_{|x-y|\leq h} |L(y) - L(x)|dydx \\ &\leq \frac{c_1}{h^n}\int_{|y'|\leq h} \mathsf{d}y' \int_{\mathcal{D}} |L(x+y') - L(x)|\mathsf{d}x.\end{aligned} \tag{2.19}$$

By the mean-continuity property, i.e. for all small $\epsilon > 0$ there exists a small $\delta > 0$ such that

$$\int_{\mathcal{D}} |L(x+y') - L(x)|\mathsf{d}x \leq \epsilon, \text{ for } |y'| \leq \delta. \tag{2.20}$$

Then

$$\begin{aligned}&\int_{\mathcal{D}} |L_h(x) - L(x)|\mathsf{d}x \leq \frac{c_1\epsilon}{h^n}\int_{|y'|\leq h} dy' \Rightarrow \int_{\mathcal{D}} |L_h(x) - L(x)|\mathsf{d}x \leq c_1\epsilon\int_{|z|\leq 1} \mathsf{d}z \\ &\Rightarrow \int_{\mathcal{D}} |L_h(x) - L(x)|\mathsf{d}x \leq c_1 \cdot c_2\epsilon\end{aligned}$$

where c_2 is the volume of the unit sphere in $\mathbb{R}^n$. □

Consider now the Lagrangian density $\mathcal{L}_h(t,\lambda)$ defined by

$$\mathcal{L}_h(t,[\mathcal{Q}_h]) = \int_{\mathbb{R}^2} dy\ \omega(x-y)\mathcal{L}_h(y). \tag{2.21}$$

Following [30], the density $\mathcal{L}_h(t,\lambda)$ is constructed from $\mathcal{L}_h^{non}(t) = L([q_h(t+\lambda)])$ by replacing $q_h(t)$ by $\mathcal{Q}_h(t,\lambda)$, the t-derivatives of $q_h(t)$ by λ-derivatives of $\mathcal{Q}_h(t,\lambda)$ and $q_h(t+\rho)$ by $\mathcal{Q}_h(t,\lambda+\rho)$. In this construction of the Hamiltonian (4.134), λ inherits the signature of the original time t and is a time-like coordinate [30]. Furthermore the symmetry of the original Lagrangian is realized canonically in the enlarged space. Note that $\mathcal{L}_h(t,\lambda)$ is local in t and nonlocal in λ. Defining then a time evolution operator T_t for a given initial trajectory $q(t)$ as follows

$$T_t : q(\lambda) \mapsto q(t+\lambda), \tag{2.22}$$

we introduce a family of new dynamical variables $\mathcal{Q}_h(t,\lambda)$, for $0 < h < 1$ as:

$$\mathcal{Q}_h(t,\lambda) = q_h(t+\lambda) =: T_t\Big(q_h(\lambda)\Big). \tag{2.23}$$

t is the "evolution" parameter and λ is a continuous parameter indexing the degrees of freedom. In differential form, condition (2.23) reads:

$$\frac{\partial \mathcal{Q}_h}{\partial t}(t,\lambda) = \frac{\partial \mathcal{Q}_h}{\partial \lambda}(t,\lambda). \tag{2.24}$$

The generalized fundamental Poisson bracket turns to be by a straightforward computation using (2.13),

$$\{\mathcal{Q}_h(t,\lambda), \mathcal{P}_h(t,\lambda')\} = \omega_h(\lambda-\lambda'). \tag{2.25}$$

The relation (2.24) which well yields, in the limit case $h \to 0$, the usual local canonical commutation $\{\mathcal{Q}(t,\lambda), \mathcal{P}(t,\lambda')\} = \delta(\lambda-\lambda')$ defines a family of first Hamilton equations for (4.134). The corresponding family of second Hamilton equations can be written as follows:

$$\dot{\mathcal{P}}_h(t,\alpha) \;=\; \mathcal{P}'_h(t,\alpha) + \frac{\partial L_h(t,[\mathcal{Q}_h])}{\partial \mathcal{Q}_h(t,\alpha)} \tag{2.26}$$

where $\mathcal{P}_h(t,\lambda)\omega_h(\lambda-\alpha)\Big|_{\lambda=-M}^{M} = 0$ ($\mathcal{P}_h$ with compact support). $\dot{\mathcal{P}}_h(t,\alpha)$ and $\mathcal{P}'_h(t,\alpha)$ are defined in (2.4). Now integrating the second Hamilton equations yields

$$\Gamma_h(t,\lambda,[\mathcal{Q}_h,\mathcal{P}_h]) \equiv \mathcal{P}_h(t,\lambda) - \int d\sigma \frac{\delta \mathcal{L}_h(t,\sigma)}{\delta \mathcal{Q}_h(t,\lambda)} \cdot \frac{\epsilon(\lambda)-\epsilon(\sigma)}{2} \approx 0. \tag{2.27}$$

The stability of primary constraints implies the secondary constraints given by

$$\Xi_h \equiv \int d\lambda \, \frac{\delta \mathcal{L}_h(t,\lambda)}{\delta \mathcal{Q}_h(t,0)} \approx 0. \tag{2.28}$$

2.2 Hamiltonian formulation of the Grosse-Wulkenhaar model

We now consider the transformation of the D canonical field variables into the $D+1$ ones:

$$x^\mu = (t, x^i) \longmapsto X^\mu = (t, x^0, x^i) = (t, \bar{x}^i), \;\; \phi(x) \longmapsto \mathcal{Q}(t,\bar{x})$$

and $\tilde{x} \longmapsto \tilde{X} = (t, \tilde{\bar{x}}) = (t, 2(\Theta^{-1})\bar{x})$. In this case, the GW Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}^{non}(t,\bar{x}) \;&=\; \frac{1}{2}\partial_\mu \mathcal{Q}(t,\bar{x}) \star \partial^\mu \mathcal{Q}(t,\bar{x}) + \frac{\Omega^2}{2}\Big(\tilde{\bar{x}}_\mu \mathcal{Q}(t,\bar{x})\Big) \star \Big(\tilde{\bar{x}}^\mu \mathcal{Q}(t,\bar{x})\Big) \\ &+\; \frac{m^2}{2}\mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) + \frac{\lambda}{4!}\mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}) \star \mathcal{Q}(t,\bar{x}).\end{aligned} \tag{2.29}$$

Substituting $\phi(x)$ by $\mathcal{Q}_h(t, x^0, x^i)$, we get the family of Lagrangian densities

$$\begin{aligned}\mathcal{L}^{non}_h(t,\bar{x}) &= \frac{1}{2}\partial_\mu\mathcal{Q}_h(t,\bar{x})\star\partial^\mu\mathcal{Q}_h(t,\bar{x})+\frac{m^2}{2}\mathcal{Q}_h(t,\bar{x})\star\mathcal{Q}_h(t,\bar{x})\\ &+ \frac{\Omega^2}{2}\Big(\tilde{\bar{x}}_\mu\mathcal{Q}_h(t,\bar{x})\Big)\star\Big(\tilde{\bar{x}}^\mu\mathcal{Q}_h(t,\bar{x})\Big)\\ &+ \frac{\lambda}{4!}\mathcal{Q}_h(t,\bar{x})\star\mathcal{Q}_h(t,\bar{x})\star\mathcal{Q}_h(t,\bar{x})\star\mathcal{Q}_h(t,\bar{x}). \end{aligned} \tag{2.30}$$

Setting

$$\begin{aligned}\Upsilon_h(t,\bar{x}) : &= \int d^D\bar{x}'\frac{\delta\mathcal{L}_h(t,\bar{x}')}{\delta\mathcal{Q}_h(t,\bar{x})}\cdot\frac{\epsilon(\bar{x}^0)-\epsilon(\bar{x}'^0)}{2}\\ &= -\delta(\bar{x}^0)\partial_{\bar{x}^0}\mathcal{Q}_h(t,\bar{x})+\frac{\lambda}{4!}\int d^Dy_1d^Dy_2d^Dy_3d^D\bar{x}'\Big(\frac{\epsilon(\bar{x}^0)-\epsilon(\bar{x}'^0)}{2}\Big)\\ &\quad\times\mathcal{Q}_h(t,y_1)\mathcal{Q}_h(t,y_2)\mathcal{Q}_h(t,y_3)\Phi(x,y_1,y_2,y_3,y_4)\\ &+ \frac{\Omega^2}{8}\int d^Dy_1d^Dy_2d^Dy_3d^D\bar{x}'\,\tilde{y}_1\tilde{y}_2\mathcal{Q}_h(t,y_3)\frac{\epsilon(\bar{x}^0)-\epsilon(\bar{x}'^0)}{2}\\ &\quad\times\Psi(x,y_1,y_2,y_3,y_4) \end{aligned} \tag{2.31}$$

with

$$\begin{aligned}\Phi(x,y_1,y_2,y_3,y_4) &= \Big[K(\bar{x}-\bar{x}',y_1-\bar{x}',y_2-\bar{x}',y_3-\bar{x}')\\ &+ K(y_1-\bar{x}',\bar{x}-\bar{x}',y_2-\bar{x}',y_3-\bar{x}')\\ &+ K(y_1-\bar{x}',y_2-\bar{x}',\bar{x}-\bar{x}',y_3-\bar{x}')\\ &+ K(y_1-\bar{x}',y_2-\bar{x}',y_3-\bar{x}',\bar{x}-\bar{x}')\Big] \end{aligned} \tag{2.32}$$

and

$$\begin{aligned}\Psi(x,y_1,y_2,y_3,y_4) &= \Big[K(\bar{x}'-y_1,\bar{x}'-\bar{x},\bar{x}'-y_2,\bar{x}'-y_3)\\ &+ K(\bar{x}'-\bar{x},\bar{x}'-y_1,\bar{x}'-y_3,\bar{x}'-y_2)\\ &+ K(\bar{x}'-y_1,\bar{x}'-y_3,\bar{x}'-\bar{x},\bar{x}'-y_2)\\ &+ K(\bar{x}'-y_2,\bar{x}'-\bar{x},\bar{x}'-y_1,\bar{x}'-y_3)\\ &+ K(\bar{x}'-y_1,\bar{x}'-y_3,\bar{x}'-y_2,\bar{x}'-\bar{x})\\ &+ K(\bar{x}'-y_3,\bar{x}'-y_1,\bar{x}'-\bar{x},\bar{x}'-y_2)\\ &+ K(\bar{x}'-y_1,\bar{x}'-\bar{x},\bar{x}'-y_3,\bar{x}'-y_2)\\ &+ K(\bar{x}'-y_2,\bar{x}'-y_3,\bar{x}'-y_1,\bar{x}'-\bar{x})\Big], \end{aligned} \tag{2.33}$$

we can define the symmetric Kernel K of four star products by

$$K(x-x_1,x-x_2,x-x_3,x-x_4)=e^{-ix\wedge\sum_{i=1}^4(-1)^{i+1}x_i}e^{-i\varphi_4} \tag{2.34}$$

where $\varphi_4=\sum_{i<j=1}^4(-1)^{i+j+1}x_i\wedge x_j$, $x\wedge y=2x\Theta^{-1}y$. Expanding this kernel expression, we get

$$K(x-x_1,x-x_2,x-x_3,x-x_4)=\exp\Big\{-i\Big[(x-x_1)\wedge(x-x_2)$$

$$-(x-x_1)\wedge(x-x_3)+(x-x_1)\wedge(x-x_4)+(x-x_2)\wedge(x-x_3)$$
$$-(x-x_2)\wedge(x-x_4)+(x-x_3)\wedge(x-x_4)\Big]\Big\}. \tag{2.35}$$

The primary constraints can be reexpressed by the formula

$$\Gamma_h(t,\bar{x})\equiv\mathcal{P}_h(t,\bar{x})-\Upsilon_h(t,\bar{x})\approx 0 \tag{2.36}$$

characterizing the family of primary constraints for the class of GW models defined by the parameter h. The family of secondary constraints can be obtained in the same way. The previous lemma guarantees the convergence:

$$\mathrm{h}\to 0\Rightarrow\mathcal{P}_h\to\mathcal{P};\qquad\mathcal{Q}_h\to\mathcal{Q};\qquad\Gamma_h\to\Gamma \tag{2.37}$$

as well as the limit of the family (2.36) of primary constraints:

$$\Gamma=\lim_{h\to 0}\Gamma_h=\mathcal{P}(t,\bar{x})-\Upsilon(t,\bar{x})\approx 0.$$

The secondary constraints appear as the equation of motion of the field $\mathcal{Q}_h$, i.e.

$$\Xi_h(t,\bar{x})\approx 0. \tag{2.38}$$

The total Hamiltonian can be then defined as

$$\mathcal{H}_h^T(t,[\mathcal{Q}_h,\mathcal{P}_h])=\mathcal{H}_h(t,[\mathcal{Q}_h,\mathcal{P}_h])+\Lambda^1(t,\bar{x})\star\Gamma_h(t,\bar{x})$$
$$+\Lambda^2(t,\bar{x})\star\Xi_h(t,\bar{x}), \tag{2.39}$$

where $\Lambda^i(t,\bar{x}), i=1,2$ are Lagrange multipliers. The corresponding field theory action $\mathcal{S}_h^T(t,\bar{x})$

$$\begin{aligned}\mathcal{S}_h^T(t,\bar{x}) &= \int dtd^D\bar{x}\Big(\mathcal{L}_h(t,\bar{x})+\Lambda^1(t,\bar{x})\star\Gamma_h(t,\bar{x})+\Lambda^2(t,\bar{x})\star\Xi_h(t,\bar{x})\Big)\\ &= \int dtd^D\bar{x}\,\mathcal{L}_h^T(t,\bar{x}),\qquad \Lambda^i(t,\bar{x})\in T^*J\end{aligned} \tag{2.40}$$

generates the Euler-Lagrange equation of motion

$$\begin{aligned}\frac{\delta\mathcal{S}_h^T(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')} &= \int dtd^D\bar{x}\Big(\frac{\delta\mathcal{L}_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}+\Lambda^1(t,\bar{x})\star\frac{\delta\Gamma_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\\ &+ \frac{\delta\Lambda^1(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\star\Gamma_h(t,\bar{x})+\Lambda^2(t,\bar{x})\star\frac{\delta\Xi_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\\ &+ \frac{\delta\Lambda^2(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\star\Xi_h(t,\bar{x})\Big)=0\end{aligned} \tag{2.41}$$

which gives

$$\frac{\delta\mathcal{L}_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}+\Lambda^1(t,\bar{x})\star\frac{\delta\Gamma_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}+\Lambda^2(t,\bar{x})\star\frac{\delta\Xi_h(t,\bar{x})}{\delta\mathcal{Q}_h(t,\bar{x}')}\Big)\approx 0 \tag{2.42}$$

where the constraints equations (2.36) and (2.38) have been taken into account. If we perform the following set of infinitesimal transformations of simply connected continuous arbitrary group

G:

$$\bar{x} \longmapsto \bar{x}' = \bar{x} + \frac{1}{2}\Big(\varpi_a \star \frac{\delta \bar{x}}{\delta \varpi_a} + \frac{\delta \bar{x}}{\delta \varpi_a} \star \varpi_a\Big), \quad (a = 1, 2, \cdots)$$
$$\mathcal{Q}_h(t,\bar{x}) \longmapsto \mathcal{Q}_h^t(t,\bar{x}') = \mathcal{Q}_h(t,\bar{x}) + \frac{1}{2}\Big(\varpi_a \star \frac{\delta \mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta \varpi_a} + \frac{\delta \mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta \varpi_a} \star \varpi_a\Big), \tag{2.43}$$

where $\mathcal{F}(\mathcal{Q}_h(t,\bar{x}))$ is a transformation of fields $\mathcal{Q}_h(t,\bar{x})$ and $\{\varpi_a(\bar{x})\}$ defines a family of infinitesimal parameters of this group, then the transformation $\mathcal{Q}_h^t(t,\bar{x}')$ of fields $\mathcal{Q}_h(t,\bar{x}')$ at a same point $\bar{x}'$ can be expressed through the generators G_μ^a as:

$$\mathcal{Q}_h^t(t,\bar{x}') = \Big(1 - \frac{i}{2}\{\varpi_a, G_\mu^a\}_\star + \mathbf{O}(\varpi^2)\Big) \star \mathcal{Q}_h(t,\bar{x}') \;=\; e_\star^{-\frac{i}{2}\{\varpi_a, G_\mu^a\}_\star} \star \mathcal{Q}_h(t,\bar{x}')$$
$$=: \; g \star \mathcal{Q}_h(t,\bar{x}'), \tag{2.44}$$

with

$$e_\star^{i\alpha} = 1 + i\alpha + \frac{i^2}{2!}\alpha \star \alpha + \frac{i^3}{3!}\alpha \star \alpha \star \alpha + \ldots; \alpha \in C^\infty(\mathbb{R}) \tag{2.45}$$

and

$$\begin{aligned} \mathcal{Q}_h(t,\bar{x}') &= \mathcal{Q}_h\Big(t, \bar{x} + \frac{1}{2}(\varpi_a \star \frac{\delta \bar{x}}{\delta \varpi_a} + \frac{\delta \bar{x}}{\delta \varpi_a} \star \varpi_a)\Big) \\ &= \mathcal{Q}_h(t,\bar{x}) + \frac{1}{2}(\varpi_a \star \frac{\delta \bar{x}^\mu}{\delta \varpi_a} + \frac{\delta \bar{x}^\mu}{\delta \varpi_a} \star \varpi_a) \star \partial_\mu \mathcal{Q}_h(t,\bar{x}) + \mathbf{O}(\varpi^2) \\ &= \mathcal{Q}_h(t,\bar{x}) + \frac{1}{2}\{\varpi_a, \frac{\delta \bar{x}^\mu}{\delta \varpi_a}\}_\star \star \varpi_a) \star \partial_\mu \mathcal{Q}_h(t,\bar{x}) + \mathbf{O}(\varpi^2). \end{aligned} \tag{2.46}$$

Evidently, the group element $g = e_\star^{-\frac{i}{2}\{\varpi_a, G_\mu^a\}_\star} \in U_\star(N)$, where $U_\star(N)$ is the NC transformation of unitary gauge group. Using (2.43) and (2.46) leads to the noncommutative generators G_μ^a are determined by the relation:

$$\begin{aligned} \frac{i}{2}\Big\{\varpi_a, G_\mu^a\Big\}_\star \star \mathcal{Q}_h(t,\bar{x}) &= \frac{1}{2}\Big\{\frac{\delta \bar{x}^\mu}{\delta \varpi_a}, \varpi_a\Big\}_\star \star \partial_\mu \mathcal{Q}_h(t,\bar{x}) \\ &\quad -\frac{1}{2}\Big\{\varpi_a, \frac{\delta \mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta \varpi_a}\Big\}_\star. \end{aligned} \tag{2.47}$$

One can check that the action of $U_\star(N)$ on the coordinates well lies on the Moyal algebra $\mathcal{A}_\Theta$,

$$x^\mu \in \mathcal{A}_\Theta, \;\; g \star x^\mu = g x^\mu - \frac{1}{4}\Theta^{\mu\rho}\partial_\rho\{\varpi_a, G_\mu^a\}_\star \in \mathcal{A}_\Theta \tag{2.48}$$

Let us now write the nonlocal Lagrangians (4.77) in the following form:

$$\mathcal{L}_h(t,\bar{x}) = \mathcal{L}_h^\star(\mathcal{Q}_h(t,\bar{x}), \partial_\mu \mathcal{Q}_h(t,\bar{x}), \bar{x}) \tag{2.49}$$
$$= \mathcal{L}_h\Big(\mathcal{Q}_h(t,\bar{x}), \partial_\mu \mathcal{Q}_h(t,\bar{x}), \partial_\mu\partial_\nu \mathcal{Q}_h(t,\bar{x}), \cdots, \bar{x}; \Theta^{\alpha\beta}\Big). \tag{2.50}$$

Remark that in equation (2.49), all products are the star ones and the EL equation of motion can be written in a similar form as in the usual commutative field theories:

$$\frac{\partial \mathcal{L}_h^\star}{\partial \mathcal{Q}_h} - \partial_\mu \frac{\partial \mathcal{L}_h^\star}{\partial \partial_\mu \mathcal{Q}_h} = 0. \tag{2.51}$$

Theorem 2.2 *Assume the quantity* $\mathcal{B}(\varpi) = \frac{1}{4}\Big\{\zeta(\varpi, \bar{x}), \partial_\mu\Big(\{\partial_\nu \mathcal{Q}_h(t,\bar{x}), \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})}\}_\star\Big)\Big\}_\star$*, where* $\zeta(\varpi, f) = \frac{1}{2}\Big(\varpi_a \star \frac{\delta f}{\delta \varpi_a} + \frac{\delta f}{\delta \varpi_a} \star \varpi_a\Big)$*, is vanished, i.e* $\mathcal{B}(\varpi) = 0$*. Then, the GW model is invariant under the infinitesimal transformation (2.43) and the conserved tensor is given by*

$$\mathcal{J}^a_\mu = \frac{1}{4}\Big\{\Big\{\varpi_a, \frac{\delta \bar{x}^\nu}{\delta \varpi_a}\Big\}_\star, \mathcal{T}_{\mu\nu}\Big\}_\star - \frac{1}{4}\Big\{\Big\{\varpi_a, \frac{\delta \mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta \varpi_a}\Big\}_\star, \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})}\Big\}_\star. \tag{2.52}$$

Proof: Setting $\zeta(\varpi, f) = \frac{1}{2}\Big(\varpi_a \star \frac{\delta f}{\delta \varpi_a} + \frac{\delta f}{\delta \varpi_a} \star \varpi_a\Big)$, then $\partial^t_\mu = \Big(\delta^\nu_\mu - \partial_\mu \zeta(\varpi, \bar{x})\Big)\partial_\nu$ and we can deduce the identity $d^D\bar{x}' = [1 + \partial_\mu\zeta(\varpi,\bar{x}) + \mathbf{O}(\varpi^2)]\mathrm{d}^D\bar{x}$. Using the relation (2.49), a direct evaluation of $\delta\mathcal{S}$ yields

$$\begin{aligned}
\delta\mathcal{S} &= \mathcal{S}^t - \mathcal{S} \\
&= \int dt d\bar{x}'\, \mathcal{L}^\star_h(\mathcal{Q}^t_h(t,\bar{x}'), \partial^t_\mu \mathcal{Q}^t_h(t,\bar{x}'), \bar{x}') - \int dt d\bar{x}\, \mathcal{L}^\star_h(\mathcal{Q}_h(t,\bar{x}), \partial_\mu \mathcal{Q}_h(t,\bar{x}), \bar{x}) \\
&= \int dt d^D\bar{x}\, \Big[(1 + \partial_\mu \zeta(\varpi,\bar{x})) \star \mathcal{L}_h(t,\bar{x}) + \zeta(\varpi, \mathcal{F}) \star \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial \mathcal{Q}_h(t,\bar{x})} \\
&\quad + \zeta(\varpi,\bar{x}) \star \partial_\mu(\mathcal{L}_h(t,\bar{x})) + \partial_\mu(\zeta(\varpi,\mathcal{F})) \star \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})} \\
&\quad - \partial_\mu\zeta(\varpi,\bar{x})\partial_\nu \mathcal{Q}_h(t,\bar{x}) \star \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})} + \mathbf{O}(\varpi^2)\Big] - \int dt d\bar{x}\, \mathcal{L}_h(t,\bar{x}) \\
&= \int dt d\bar{x}\, \Big[- \partial^\mu \mathcal{J}^a_\mu + \zeta(\varpi,\mathcal{F}) \star \Big(\frac{\partial \mathcal{L}_h}{\partial \mathcal{Q}_h} - \partial_\mu \frac{\partial \mathcal{L}_h}{\partial\partial_\mu \mathcal{Q}_h}\Big) + \mathcal{B}(\varpi)\Big].
\end{aligned} \tag{2.53}$$

In this relation, the first term is a *divergence* term defining the NC tensor $\mathcal{J}^a_\mu$ expressed as follows:

$$\mathcal{J}^a_\mu = \frac{1}{4}\Big\{\Big\{\varpi_a, \frac{\delta \bar{x}^\nu}{\delta \varpi_a}\Big\}_\star, \mathcal{T}_{\mu\nu}\Big\}_\star - \frac{1}{4}\Big\{\Big\{\varpi_a, \frac{\delta \mathcal{F}(\mathcal{Q}_h(t,\bar{x}))}{\delta \varpi_a}\Big\}_\star, \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})}\Big\}_\star. \tag{2.54}$$

The second term contains the EL equation of motion while the last term, usually called the breaking term, is given by the relation

$$\mathcal{B}(\varpi) = \frac{1}{4}\Big\{\zeta(\varpi,\bar{x}), \partial_\mu\Big(\{\partial_\nu \mathcal{Q}_h(t,\bar{x}), \frac{\partial \mathcal{L}_h(t,\bar{x})}{\partial\partial_\mu \mathcal{Q}_h(t,\bar{x})}\}_\star\Big)\Big\}_\star. \tag{2.55}$$

$\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor computed in [7], defined with non-local variables $\mathcal{Q}_h(t,\bar{x})$:

$$\mathcal{T}_{\rho\mu} = \frac{1}{2}\{\partial_\rho \mathcal{Q}_h(t,\bar{x}), \partial_\mu \mathcal{Q}_h(t,\bar{x})\}_\star - \mathrm{g}_{\rho\mu}\mathcal{L}^\star_h. \tag{2.56}$$

Then translational invariance violation, engendered by the appearance of the coordinate $\tilde{\bar{x}}^\mu$, can be avoided by imposing the constraint $\mathcal{B}(\varpi) \approx 0$. □

It is worth noticing that if $\varpi_a \star (\delta x^\mu / \delta \varpi_a)$ is a constant parameter and $\mathcal{F}$ is trivial, then the current (2.54) is reduced to the NC energy momentum tensor (2.56). If $\varpi_a \star (\delta x^\mu/\delta\varpi_a)$ is defined as $\varpi^{\mu\nu}_a x_\nu$, where $\varpi^{\mu\nu}_a$ is the Lorentz tensor and $\varpi_a \star (\delta\mathcal{F}/\delta\varpi_a) = -\varpi^{\mu\nu}_a x_\nu \partial_\mu \mathcal{Q}_h(t,\bar{x})$, then the current (2.54) is reduced to the angular momentum tensor. The current $\mathcal{J}^a_\mu$ is not symmetric, nonlocally conserved, and in massless theory, not traceless.

Twisted Field Theories

3.1 Generalities

It is generally believed that the picture of spacetime as a manifold M locally modelled on the flat Minkowski space should break down at very short distances of the order of the Planck length $l_p = (G\hbar/c^3)^{1/2}$. Limitations in the possible accuracy of localization of spacetime events should in fact be a feature of a quantum theory incorporating gravitation. The obtaining of a better understanding of physics at short distances and the cure of the problems occuring when trying to quantize gravity should lead to change the nature of spacetime in a fundamental way. This could be realized by implementing the noncommutativity through the coordinates which satisfy the commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = i\tilde{\Theta}^{\mu\nu}(\hat{x}) \neq 0$. In general, the function $\tilde{\Theta}^{\mu\nu}(\hat{x})$ is unknown, but, for physical reasons, should vanish at large distances where we experience the commutative world and may be determined by experiments [24] and [71]. The $\Theta-$deformation case which may at very short distances provide a reasonable approximation for $\tilde{\Theta}^{\mu\nu}(\hat{x})$ is described by the commutation relation $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$. The algebra of functions of such noncommuting coordinates can be represented by the algebra of functions on ordinary spacetime, equipped with a noncommutative $\star-$product. For a constant antisymmetric matrix $\Theta^{\mu\nu}$, this can be represented by the Groenewold-Moyal product (1.1). This product can be generalized under the form

$$(f \star g)(x) = \mathrm{m}\Big\{e^{i\frac{\Theta^{ab}}{2}X_a\otimes X_b} f(x) \otimes g(x)\Big\} \tag{3.1}$$

where $X_a = e^\mu_a(x)\partial_\mu$ are vector fields. The commutation relation of coordinates then becomes $[x^\mu, x^\nu]_\star = i\Theta^{ab}e^\mu_a(x)e^\nu_b(x) := i\widetilde{\Theta}^{\mu\nu}(x)$ engendering a twisted scalar field theory where e^μ_a, and hence the $\star-$product itself, appear dynamical. We interpret that the spacetime is not neccessarily flat. But insted we have just been considering physics in orthogonal non-coordinate bases of the spacetime manifold. The orthogonal basis vectors for the tangent spaces of the spacetime manifold are X_a and the basis one-forms for the cotangent spaces are $X^{*a} = e^a_\mu dx^\mu$. where ∂_μ and dx^μ are the coordinate bases for the tangent and cotangent spaces of the spacetime manifold and have conventionally the inverse of the vierbein $e^a_\mu = (e^\mu_a)^{-1}$. In general, the non-coordinate base has a non vanishing Lie bracket and satisfies

$$[X_a, X_b] = e^c_\nu\Big[e^\mu_a\partial_\mu e^\nu_b - e^\mu_b\partial_\mu e^\nu_a\Big]X_c = C^c_{ab}X_c. \tag{3.2}$$

The particular condition $[X_a, X_b] = 0$, (i. e. the vector fields are commuting), implies constraints on e^μ_a, namely $e^\nu_{[a}\partial_\nu e^\mu_{b]} = 0$, that can be solved off-shell in terms of D scalar fields ϕ^a, (see [3] and

[4]). Supposing that the square matrix e_a^μ has an inverse e_μ^a everywhere, so that the X_a are linearly independent, then the above condition becomes $\partial_{[\mu} e_{\nu]}^a = 0$ which is satisfied by $e_\nu^a = \partial_\nu \phi^a$. Since $X_a \phi^b = \delta_a^b$, the field ϕ^b can be seen as new coordinates along the X_a directions. Besides, the Leibniz rule extends to the commuting fields X_a as follows: $X_a(f \star g) = (X_a f) \star g + f \star (X_a g)$. The metric tensor of the spacetime manifold is defined by the orthonormal basis vectors X_a with respect to the metric g in all point of the spacetime

$$\mathrm{g}(X_a, X_b) = e_a^\mu . e_b^\nu \mathrm{g}_{\mu\nu} = \delta_{ab}. \tag{3.3}$$

Hence the spacetime is not anymore flat. Furtheremore, expanding the dynamical $\star$-product (3.1) of two functions as follows:

$$\begin{aligned} f \star g &= fg + \frac{i}{2}\Theta^{ab} X_a f X_b g \\ &+ \frac{1}{2!}\Big(\frac{i}{2}\Big)^2 \Theta^{a_1 b_1}\Theta^{a_2 b_2}(X_{a_1}X_{a_2} f)(X_{b_1}X_{b_2} g) + \cdots \\ &\equiv e^{\Delta}(f,g) \end{aligned} \tag{3.4}$$

where powers of the bilinear operator Δ are defined as

$$\Delta(f,g) = \frac{i}{2}\Theta^{ab}(X_a f)(X_b g) \qquad \Delta^0(f,g) = fg$$
$$\Delta^n(f,g) = \Big(\frac{i}{2}\Big)^n \Theta^{a_1 b_1}\cdots\Theta^{a_n b_n}(X_{a_1}\cdots X_{a_n} f)(X_{b_1}\cdots X_{b_n} g) \tag{3.5}$$

one can deduce the following rules (straightforwardly generalizing the usual Moyal product identities):

$$f \star g = fg + X_a T(\Delta)(f, \widetilde{X}^a g) \tag{3.6}$$
$$[f,g]_\star = f \star g - g \star f = 2 X_a S(\Delta)(f, \widetilde{X}^a g) \tag{3.7}$$
$$\{f,g\}_\star = f \star g + g \star f = 2fg + 2 X_a R(\Delta)(f, \widetilde{X}^a g) \tag{3.8}$$

where

$$\begin{aligned} T(\Delta) &= \frac{e^{\Delta} - 1}{\Delta} \qquad S(\Delta) = \frac{sinh(\Delta)}{\Delta} \\ R(\Delta) &= \frac{cosh(\Delta) - 1}{\Delta} \text{ and } \widetilde{X}^a = \frac{i}{2}\Theta^{ab} X_b. \end{aligned} \tag{3.9}$$

$S(\Delta)(.,\widetilde{X}.)$ is a bilinear antisymmetric operator and

$$T(\Delta)(f, \widetilde{X}^a g) - T(\Delta)(g, \widetilde{X}^a f) = 2S(\Delta)(f, \widetilde{X}^a g). \tag{3.10}$$

The Moyal product (3.1) correspond to the twist $\mathcal{F} = \exp\Big(-\frac{i}{2}\Theta^{ab} X_a \otimes X_b\Big)$. The question arises whether we can deform the symmetry in such a way that it acts consistently on the deformed space i.e. leaves the deformed space invariant and such it is reduced to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras. Hopf algebras coming from a Lie algebra are also called quantum groups. Quantum group symmetries lead to new features of field theories on NC space.

3.2 Twisted Grosse-Wulkenhaar Model

Recently, Paolo Aschieri *et al* [3] introduced a so-called dynamical noncommutativity to investigate Noether currents in an ordinary nonrenormalizable twisted $\phi^{\star 4}$ theory. This work addresses questions of the applicability of such a formalism on the new class of renormalizable NC field theories (NCRFT) built on the Grosse and Wulkenhaar (GW) $\phi^{\star 4}$ scalar field model defined in Euclidean space-time by the action functional [37]

$$S_\star^\Omega[\phi] = \int \mathsf{d}^D x \,\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big) \qquad (3.11)$$

where $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and $S_\star^\Omega[\phi]$ is covariant under Langmann-Szabo duality [57]. Ω and λ are dimensionless parameters.

We derive the field equations of motion and provide with the explicit computation of relevant physical quantities such as the noncommutative energy momentum tensor (NC EMT), the angular momentum tensor (AMT) and the dilatation current (DC). Furthermore, we proceed to the symmetry analysis including the translation, rotation and dilatation transformations and compute the conserved currents.

3.2.1 Twisted Grosse-Wulkenhaar model: Noether currents

The integral in (3.11), defined with the dynamical Moyal $\star-$product (3.1), is not cyclic; even with suitable boundary conditions at infinity,

$$\int \mathsf{d}^D x\,(f\star g) \neq \int \mathsf{d}^D x\,(g\star f). \qquad (3.12)$$

Using now the measure ed^Dx where $e = det(e_\mu^a)$, a cyclic integral can be defined so that, up to boundary terms:

$$\int e\mathsf{d}^D x\,(f\star g) = \int e\mathsf{d}^D x(fg) = \int e\mathsf{d}^D x\,(g\star f). \qquad (3.13)$$

Therefore, the NC GW Lagrangian action (3.11) can be rewritten by means of a cyclic integral as follows:

$$\begin{aligned} \mathcal{S}_\star^\Omega[\phi] &=: \int ed^Dx\,\Big(\mathcal{L}_\star^\Omega\star e^{-1}\Big) \\ &=: \int e\mathsf{d}^D x\,\Big\{\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \\ &\quad + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{1}{2}\partial_\mu\phi_a\star\partial^\mu\phi^a\Big\}\star e^{-1} \end{aligned} \qquad (3.14)$$

where $e = \mathsf{det}e_\mu^a$. From (3.14) the peculiar Euler Lagrange equations of motion can be readily derived by direct application of the variational principle and the use of formulas of derivatives and variations given by [3]

$$\begin{aligned} &\delta_{\phi^c}e_a^\mu = -e_a^\nu e_b^\mu\delta_{\phi^c}e_\nu^b = -e_a^\nu e_b^\mu\partial_\nu\delta\phi^b = -e_b^\mu X_a(\delta\phi^b) \qquad \partial_\mu e = eX_a(\partial_\mu\phi^a) \\ &\delta_{\phi^c}X_a = \delta_{\phi^c}(e_a^\mu\partial_\mu) = -e_b^\mu X_a(\delta\phi^b)\partial_\mu = -X_a(\delta\phi^b)X_b \\ &\delta_{\phi^c}e = eX_a(\delta\phi^a) \qquad \delta_{\phi^c}e^{-1} = -e^{-1}X_a(\delta\phi^a) \qquad eX_a(f) = \partial_\mu(ee_a^\mu f). \end{aligned} \qquad (3.15)$$

To compute δ_{ϕ^c} variations, it turns out that the following identity:

$$\delta_{\phi^c}(f\star g) = -(\delta\phi^c X_c f)\star g - f\star(\delta\phi^c X_c g) + \delta\phi^c X_c(f\star g) \tag{3.16}$$

where the functions f and g do not depend on ϕ^c, is useful. By induction, one can immediately prove that (3.16) holds for $\star$-products of an arbitrary number of factors:

$$\begin{aligned}\delta_{\phi^c}(f\star g\star\cdots\star h) &= -(\delta\phi^c X_c f)\star g\star\cdots\star h\\ &\quad -f\star(\delta\phi^c X_c g)\star\cdots\star h\\ &\quad -\cdots - f\star g\star\cdots\star(\delta\phi^c X_c h)\\ &\quad +\delta\phi^c X_c(f\star g\star\cdots\star h).\end{aligned} \tag{3.17}$$

Equations of motion for ϕ and related currents

The action (3.14) can be viewed as the sum of four actions pertaining to different terms as follows:

$$\begin{aligned}\mathcal{S}_\star^{\Omega,0}[\phi] &= \frac{1}{2}\int e\mathsf{d}^D x\,\Big(\partial_\mu\phi\star\partial^\mu\phi\star+\partial_\mu\phi_a\star\partial^\mu\phi^a\Big)\star e^{-1}\\ \mathcal{S}_\star^{\Omega,m^2}[\phi] &= \frac{m^2}{2}\int e\mathsf{d}^D x\,(\phi\star\phi\star e^{-1})\\ \mathcal{S}_\star^{\Omega,\lambda}[\phi] &= \frac{\lambda}{4!}\int e\mathsf{d}^D x\,(\phi\star\phi\star\phi\star\phi\star e^{-1})\\ \mathcal{S}_\star^{\Omega,\mathsf{har}}[\phi] &= \frac{\Omega^2}{2}\int e\mathsf{d}^D x\,(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)\star e^{-1}.\end{aligned}$$

The variations of these quantities with respect to the field ϕ, using the cyclicity property of the integral, give the following relations:

$$\begin{aligned}\delta_\phi(\mathcal{S}_\star^{\Omega,0}) &= \int \mathsf{d}^D x\,\Big\{-\delta\phi.\partial_\sigma\Big(\frac{e}{2}\{\partial^\sigma\phi,e^{-1}\}_\star\Big)+\partial_\sigma\Big[\frac{e\delta\phi}{2}.\{\partial^\sigma\phi,e^{-1}\}_\star\\ &\quad +ee^\sigma_b T(\Delta)\Big(\delta\partial_\mu\phi,\frac{\widetilde{X}^b}{2}\{\partial^\mu\phi,e^{-1}\}_\star\Big)\\ &\quad +ee^\sigma_b S(\Delta)\Big(\partial_\mu\phi,\widetilde{X}^b(\partial^\mu\delta\phi\star e^{-1})\Big)\Big]\Big\}\end{aligned} \tag{3.18}$$

$$\begin{aligned}\delta_\phi(\mathcal{S}_\star^{\Omega,m^2}) &= \int \mathsf{d}^D x\,\Big\{\delta\phi.(\frac{em^2}{2}\{\phi,e^{-1}\}_\star)\\ &\quad +\partial_\sigma\Big[\frac{m^2}{2}ee^\sigma_b T(\Delta)\Big(\delta\phi,\widetilde{X}^b\{\phi,e^{-1}\}_\star\Big)\\ &\quad +m^2 ee^\sigma_b S(\Delta)\Big(\phi,\widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big]\Big\}\end{aligned} \tag{3.19}$$

$$\begin{aligned}\delta_\phi(\mathcal{S}_\star^{\Omega,\lambda}) &= \frac{\lambda}{4!}\int \mathsf{d}^D x\,\Big\{\delta\phi\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\\ &\quad +\partial_\sigma\Big[ee^\sigma_b T(\Delta)\Big(\delta\phi,\widetilde{X}^b\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\Big)\\ &\quad +2ee^\sigma_b S(\Delta)\Big(\phi,\widetilde{X}^b(\delta\phi\star\phi\star\phi\star e^{-1})\Big)\end{aligned}$$

$$+2ee_b^\sigma S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b(\delta\phi\star\phi\star e^{-1})\Big)$$
$$+2ee_b^\sigma S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big]\Big\} \tag{3.20}$$

$$\begin{aligned}\delta_\phi(\mathcal{S}_\star^{\Omega,\mathsf{har}}) &= \frac{\Omega^2}{8}\int \mathsf{d}^D x\,\Big\{e\delta\phi\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star \\ &+\partial_\sigma\Big[ee_b^\sigma T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star\Big) \\ &+2ee_b^\sigma S(\Delta)\Big(\tilde{x}, \widetilde{X}^b(\delta\phi\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big) \\ &+2ee_b^\sigma S(\Delta)\Big(\{\tilde{x},\phi\star\tilde{x}\}_\star, X^b(\delta\phi\star e^{-1})\Big) \\ &+2ee_b^\sigma S(\Delta)\Big(\{\phi,\tilde{x}\}_\star, \widetilde{X}^b(\delta\phi\star\tilde{x}\star e^{-1})\Big)\Big]\Big\}. \end{aligned}\tag{3.21}$$

Summing all these four variations and factoring out $\delta\phi$ from the resulting expression and grouping the surface terms, source of the current hereafter denoted by $\mathcal{K}^\sigma$, we can write the GW action variation with respect to the field ϕ into the global form

$$\delta_\phi\mathcal{S}_\star^\Omega = \int \mathsf{d}^D x\,\Big(\delta\phi\mathcal{E}_\phi + \partial_\sigma\mathcal{K}^\sigma\Big) \tag{3.22}$$

from which we deduce the equation of motion of the field ϕ as:

$$\begin{aligned}\mathcal{E}_\phi &= -\frac{1}{2}\partial_\sigma\Big(e\{\partial^\sigma\phi, e^{-1}\}_\star\Big) + \frac{m^2}{2}e\{\phi,e^{-1}\}_\star + \frac{\lambda}{4!}e\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star \\ &+ \frac{\Omega^2}{8}e\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star = 0.\end{aligned}\tag{3.23}$$

In the commutative limit $\Theta\to 0$, the equation (3.23) becomes the usual ϕ^4 field equation of motion

$$\Box\phi - m^2\phi - \frac{\lambda}{3!}\phi^3 = 0. \tag{3.24}$$

The current $\mathcal{K}^\sigma$ results from the combination of following contributions:

$$\mathcal{K}^\sigma = \mathcal{K}^\sigma(0) + \mathcal{K}^\sigma(m^2) + \mathcal{K}^\sigma(\lambda) + \mathcal{K}^\sigma(\Omega^2) \tag{3.25}$$

induced, respectively, by:

1. the velocity term contribution

$$\begin{aligned}\mathcal{K}^\sigma(0) &= \frac{e\delta\phi}{2}.\{\partial^\sigma\phi, e^{-1}\}_\star + ee_b^\sigma\Big[T(\Delta)\Big(\delta\partial_\mu\phi, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\ &+S(\Delta)\Big(\partial_\mu\phi, \widetilde{X}^b(\partial^\mu\delta\phi\star e^{-1})\Big)\Big]\end{aligned}\tag{3.26}$$

2. the mass term

$$\mathcal{K}^\sigma(m^2) = ee_b^\sigma\Big[\frac{m^2}{2}T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\phi,e^{-1}\}_\star\Big) + m^2 S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big] \tag{3.27}$$

3. the $\phi^{\star 4}$ interaction

$$\begin{aligned}\mathcal{K}^\sigma(\lambda) &= ee^\sigma_b\Big[\frac{\lambda}{4!}T(\Delta)\Big(\delta\phi, \widetilde{X}^b\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\Big)\\ &+\frac{\lambda}{12}S(\Delta)\Big(\phi,\widetilde{X}^b(\delta\phi\star\phi\star\phi\star e^{-1})\Big)\\ &+\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi,\widetilde{X}^b(\delta\phi\star\phi\star e^{-1})\Big)\\ &+\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi\star\phi,\widetilde{X}^b(\delta\phi\star e^{-1})\Big)\Big] \qquad (3.28)\end{aligned}$$

4. and the GW harmonic interaction

$$\begin{aligned}\mathcal{K}^\sigma(\Omega^2) &= ee^\sigma_b\Big[\frac{\Omega^2}{8}T(\Delta)\Big(\delta\phi,\widetilde{X}^b\{\tilde{x},\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star\Big)\\ &+\frac{\Omega^2}{4}S(\Delta)\Big(\tilde{x},\widetilde{X}^b(\delta\phi\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\\ &+\frac{\Omega^2}{4}S(\Delta)\Big(\{\tilde{x},\phi\star\tilde{x}\}_\star, X^b(\delta\phi\star e^{-1})\Big)\\ &+\frac{\Omega^2}{4}S(\Delta)\Big(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b(\delta\phi\star\tilde{x}\star e^{-1})\Big)\Big]. \qquad (3.29)\end{aligned}$$

Equations of motion for ϕ^c and related currents

The ϕ^c variation of the action (3.14)

$$\begin{aligned}\delta_{\phi^c}S &= \delta_{\phi^c}\{\int \mathsf{d}^Dx\, e\Big[\mathcal{L}^\Omega_\star\star e^{-1}\Big]\}\\ &= \int \mathsf{d}^Dx\,\Big[(\delta_{\phi^c}e)\mathcal{L}^\Omega_\star\star e^{-1}+e(\delta_{\phi^c}(\mathcal{L}^\Omega_\star\star e^{-1}))\Big] \qquad (3.30)\end{aligned}$$

where the ordinary Leibniz rule is used when the variation δ_{ϕ^c} acts on the pointwise product, can be considered as a sum of two terms A and B. The term A is given by:

$$\begin{aligned}A &= \int \mathsf{d}^Dx\,(\delta_{\phi^c}e)[\mathcal{L}^\Omega_\star\star e^{-1}]\\ &= \int \mathsf{d}^Dx\,\Big[-e\delta\phi^aX_a(\mathcal{L}^\Omega_\star\star e^{-1})+\partial_\rho\Big(ee_a^{\ \rho}\delta\phi^a(\mathcal{L}^\Omega_\star\star e^{-1})\Big)\Big] \qquad (3.31)\end{aligned}$$

where $\delta_{\phi^c}e = eX_a(\delta\phi^a)$, while the second term B encompasses contributions from the velocity, the mass, the ϕ^4 interaction and the harmonic potential denoted by B_0, B_{m^2}, B_λ, B_{har}, respectively.

The mass term, B_{m^2}, depends on the $\star$ product and e^{-1} as follows:

$$\begin{aligned}B_{m^2} &= \frac{m^2}{2}\int \mathsf{d}^Dx\, e\delta_{\phi^c}(\phi\star\phi\star e^{-1})\\ &= \frac{m^2}{2}\int \mathsf{d}^Dx\, e\Big(-(\delta\phi^aX_a\phi)\star\phi\star e^{-1}-\phi\star(\delta\phi^aX_a\phi)\star e^{-1}\\ &-\phi\star\phi\star(\delta\phi^aX_ae^{-1})+\delta\phi^aX_a(\phi\star\phi\star e^{-1})+\phi\star\phi\star(\delta_{\phi^c}e^{-1})\Big). \qquad (3.32)\end{aligned}$$

Noting that $\delta_{\phi^c} e^{-1} = -e^{-1}(X_a \delta\phi^a)$, we obtain

$$\begin{aligned} B_{m^2} &= \frac{m^2}{2}\int \mathsf{d}^D x\, e\{\delta\phi^a X_a(\phi\star\phi\star e^{-1}) - \phi\star\phi\star X_a(\delta\phi^a e^{-1}) \\ &\quad -(\delta\phi^a X_a\phi)\star\phi\star e^{-1} - \phi\star(\delta\phi^a X_a\phi)\star e^{-1}\}. \end{aligned} \tag{3.33}$$

Adding and substracting $(\delta\phi^a X_a\phi)\star e^{-1}\star\phi$ from (3.33) enable to combine the terms under the integral in the following way:

$$\begin{aligned} B_{m^2} &= \frac{m^2}{2}\int \mathsf{d}^D x\, \{e\delta\phi^a\Big(X_a(\phi\star\phi\star e^{-1}) + e^{-1}X_a(\phi\star\phi) \\ &\quad -(X_a\phi)\{\phi, e^{-1}\}_\star\Big) + eX_b\Big(-\phi\star\phi\star(\delta\phi^b e^{-1}) \\ &\quad +T(\Delta)[X_a(\phi\star\phi), \widetilde{X}^b(\delta\phi^a e^{-1})] - T(\Delta)[\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi \overset{\star}{,} e^{-1}\}] \\ &\quad +2S(\Delta)[\delta\phi^a(X_a\phi)\star e^{-1}, \widetilde{X}^b\phi]\Big)\} \end{aligned} \tag{3.34}$$

where the terms proportional to $e\delta\phi^a$,

$$\frac{m^2}{2} e\delta\phi^a\Big(X_a(\phi\star\phi\star e^{-1}) + e^{-1}X_a(\phi\star\phi) - (X_a\phi)\{\phi, e^{-1}\}_\star\Big) \tag{3.35}$$

contribute to the equation of motion, while those proportional to eX_b,

$$\begin{aligned} &-\frac{m^2}{2}\partial_\mu\Big\{ee^\mu_b\Big(-\phi\star\phi\star(\delta\phi^b e^{-1}) + T(\Delta)[X_a(\phi\star\phi), \widetilde{X}^b(\delta\phi^a e^{-1})] \\ &-T(\Delta)[\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi, e^{-1}\}_\star] + 2S(\Delta)[\delta\phi^a(X_a\phi)\star e^{-1}, \widetilde{X}^b\phi]\Big)\Big\} \end{aligned} \tag{3.36}$$

are surface terms giving the current $\mathcal{J}^\sigma$ to be defined later. The remaining contributions can be computed in analogous way to give the relations:

$$\begin{aligned} B_0 &= \frac{1}{2}\int \mathsf{d}^D x\,\Big\{e\delta\phi^c\Big[X_c(\partial_\mu\phi\star\partial^\mu\phi\star e^{-1}) + e^{-1}X_c(\partial_\mu\phi\star\partial^\mu\phi) \\ &\quad -X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star\Big] + \partial_\sigma\Big[-ee^\sigma_b\Big(\partial_\mu\phi\star\partial^\mu\phi\star(\delta\phi^b e^{-1})\Big) \\ &\quad -ee^\sigma_b T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\ &\quad -2ee^\sigma_b S(\Delta)\Big(\partial^\mu\phi, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi)\star e^{-1})\Big) \\ &\quad +ee^\sigma_b T(\Delta)\Big(X_c(\partial_\mu\phi\star\partial^\mu\phi), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big]\Big\} \\ &\quad +\frac{1}{2}\int \mathsf{d}^D x\,\Big\{e\delta\phi^c\Big[X_c\Big(\partial_\mu\phi_a\star\partial^\mu\phi^a\star e^{-1}\Big) + e^{-1}X_c(\partial_\mu\phi_a\star\partial^\mu\phi^a) \\ &\quad -X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star - \frac{2}{e}\partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big)\Big] \\ &\quad +\partial_\sigma\Big[-ee^\sigma_b\Big(\partial_\mu\phi_a\star\partial^\mu\phi^a\star(\delta\phi^b e^{-1})\Big) + e\delta\phi^a\{\partial^\sigma\phi_a, e^{-1}\}_\star \\ &\quad -ee^\sigma_b T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\Big) \\ &\quad -2ee^\sigma_b S(\Delta)\Big(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})\Big) \end{aligned}$$

$$
\begin{aligned}
&+2ee_b^\sigma S(\Delta)\Big(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1})\Big)\\
&+2ee_b^\sigma T(\Delta)\Big(\delta\partial_\mu\phi_a, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi^a, e^{-1}\}_\star\Big)\\
&+ee_b^\sigma T(\Delta)\Big(X_c(\partial_\mu\phi_a\star\partial^\mu\phi^a), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big]\Big\} \qquad (3.37)
\end{aligned}
$$

$$
\begin{aligned}
B_\lambda \;=\; &\frac{\lambda}{4!}\int \mathsf{d}^D x\,\Big\{e\delta\phi^a\Big(X_a(\phi\star\phi\star\phi\star\phi\star e^{-1}) + e^{-1}X_a(\phi\star\phi\star\phi\star\phi)\\
&-X_a\phi.\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\Big) + \partial_\sigma\Big[-ee_b^\sigma(\phi\star\phi\star\phi\star\phi\star\delta\phi^b e^{-1})\\
&-ee_b^\sigma T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\Big)\\
&-2ee_b^\sigma S(\Delta)\Big(\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star\phi\star e^{-1})\Big)\\
&-2ee_b^\sigma S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star e^{-1})\Big)\\
&-2ee_b^\sigma S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star e^{-1})\Big)\\
&+ee_b^\sigma T(\Delta)\Big(X_c(\phi\star\phi\star\phi\star\phi), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big]\Big\} \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
B_{\mathsf{har}} \;=\; &\frac{\Omega^2}{2}\int \mathsf{d}^D x\,\Big\{e\delta\phi^c\Big(X_c((\tilde{x}\phi)\star(\tilde{x}\phi)\star e^{-1}) + e^{-1}X_c((\tilde{x}\phi)\star(\tilde{x}\phi))\\
&-\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star - (X_c\phi)\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star\Big)\\
&+\partial_\sigma\Big[-ee_b^\sigma\Big((\tilde{x}\phi)\star(\tilde{x}\phi)\star(\delta\phi^b e^{-1})\Big)\\
&-ee_b^\sigma T(\Delta)\Big(\delta\phi^c X_c(\tilde{x}\phi), \widetilde{X}^b\{\tilde{x}\phi, e^{-1}\}_\star\Big)\\
&-2ee_b^\sigma S(\Delta)\Big(\tilde{x}\phi, \widetilde{X}^b((\delta\phi^c X_c(\tilde{x}\phi))\star e^{-1})\Big)\\
&+ee_b^\sigma T(\Delta)\Big(X_c((\tilde{x}\phi)\star(\tilde{x}\phi)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big]\Big\}. \qquad (3.39)
\end{aligned}
$$

Summing now all the contributions and rearranging, after tedious algebraic transformations, in terms of two components representing the $\delta\phi^c$ factor and the current surface counterpart, the GW action ϕ^c-variation takes the form

$$
\begin{aligned}
\delta_{\phi^c}\mathcal{S}_\star^\Omega &= \int \mathsf{d}^D x\,\Big(\delta\phi^c\mathcal{E}_{(\phi,\phi^c)} + \partial_\sigma\mathcal{J}^\sigma\Big) \qquad (3.40)\\
&= \int \mathsf{d}^D x\,\Big(-\delta\phi^c X_c\phi\mathcal{E}_\phi - \delta\phi^c\mathcal{E}_{\phi^c} + \partial_\sigma\mathcal{J}^\sigma\Big) \qquad (3.41)
\end{aligned}
$$

from which we get the following field equation:

$$
\begin{aligned}
\mathcal{E}_{(\phi,\phi^c)} \;=\; &e\Big[\frac{1}{e}X_c(\mathcal{L}_\star^\Omega) - (X_c\phi)\Big(\frac{m^2}{2}\{\phi,e^{-1}\}_\star + \frac{\lambda}{4!}\{\phi\star\phi,\{\phi,e^{-1}\}_\star\}_\star\\
&+\frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi,e^{-1}\}_\star\Big) - \frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi,e^{-1}\}_\star
\end{aligned}
$$

$$
-\frac{1}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star - \frac{1}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star
-\frac{1}{e}\partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big] = 0. \tag{3.42}
$$

Using the identities $\tilde{x}_\mu \star \phi = \tilde{x}_\mu\phi + i\partial_\mu\phi$ and $\phi \star \tilde{x}_\mu = \tilde{x}_\mu\phi - i\partial_\mu\phi$ implying $\tilde{x}\phi = \frac{1}{2}\{\tilde{x}, \phi\}_\star$, we can deduce that $\frac{\Omega^2}{2}\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star$, and the equation of motion can be reexpressed as

$$
\begin{aligned}
\mathcal{E}_{(\phi,\phi^c)} &= -X_c\phi\mathcal{E}_\phi + X_c\mathcal{L}_\star^\Omega - \frac{1}{2}X_c\phi\partial_\mu\Big(e\{\partial^\mu\phi, e^{-1}\}_\star\Big) \\
&\quad -e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star - \frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star \\
&\quad -\frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star - \partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big) \\
&= -X_c\phi\mathcal{E}_\phi - \mathcal{E}_{\phi^c} = 0
\end{aligned} \tag{3.43}
$$

where

$$
\begin{aligned}
\mathcal{E}_{\phi^c} &= -X_c\mathcal{L}_\star^\Omega + \frac{1}{2}X_c\phi\partial_\mu\Big(e\{\partial^\mu\phi, e^{-1}\}_\star\Big) + e\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star \\
&\quad +\frac{e}{2}X_c\partial_\mu\phi.\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{2}X_c\partial_\mu\phi_a.\{\partial^\mu\phi^a, e^{-1}\}_\star \\
&\quad +\partial_\mu\Big(\frac{e}{2}\{\partial^\mu\phi_c, e^{-1}\}_\star\Big)
\end{aligned} \tag{3.44}
$$

with

$$
\frac{\Omega^2}{2}\phi X_c\tilde{x}.\{\tilde{x}\phi, e^{-1}\}_\star = \frac{\Omega^2}{8}X_c\tilde{x}.\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star.
$$

One can immediately show that, as expected from [3], when ϕ is on shell (i.e. $\mathcal{E}_\phi = 0$), the ϕ^c field equation of motion simply reduces to $\mathcal{E}_{\phi^c} = 0$, and in the commutative limit, we get $\Box\phi^c = 0$ as it should. Besides, the field equations (3.23) and (3.44) are trivially satisfied by the solution $\phi = 0$, $e^a_\mu = \partial_\mu\phi^a = \delta^a_\mu$ corresponding to the usual Moyal product. The field ϕ acts as a source for the noncommutativity field ϕ^c.

In the same vein, the current $\mathcal{J}^\sigma$ is given by

$$
\mathcal{J}^\sigma = \mathcal{J}^\sigma(0) + \mathcal{J}^\sigma(m^2) + \mathcal{J}^\sigma(\lambda) + \mathcal{J}^\sigma(\Omega^2) \tag{3.45}
$$

where the contributions engendered by the velocity term, the mass term, the $\phi^{\star 4}$ interaction and the GW harmonic interaction source are, respectively, expressed as

$$
\begin{aligned}
\mathcal{J}^\sigma(0) &= \frac{1}{2}e\delta\phi^a\{\partial^\sigma\phi_a, e^{-1}\}_\star \\
&\quad +ee^\sigma_b\Big\{\frac{1}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\Big) \\
&\quad -2S(\Delta)\Big(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a) \star e^{-1})\Big) \\
&\quad +2S(\Delta)\Big(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a \star e^{-1})\Big) \\
&\quad +2T(\Delta)\Big(\delta\partial_\mu\phi_a, \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi^a, e^{-1}\}_\star\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star\Big)\\
&-2S(\Delta)\Big(\partial^\mu\phi, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi)\star e^{-1})\Big)\Big]\\
&-\mathcal{L}^\Omega_\star(0)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(0)\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(0)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
\mathcal{J}^\sigma(m^2) \;=\;& ee^\sigma_b\Big\{\frac{m^2}{2}\Big[-T(\Delta)\Big(\delta\phi^a(X_a\phi), \widetilde{X}^b\{\phi, e^{-1}\}\Big)\\
&+2S(\Delta)\Big(\delta\phi^a(X_a\phi)\star e^{-1}, \widetilde{X}^b\phi\Big)\Big]\\
&-\mathcal{L}^\Omega_\star(m^2)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(m^2)\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(m^2)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
\mathcal{J}^\sigma(\lambda) \;=\;& ee^\sigma_b\Big\{\frac{\lambda}{4!}\Big[-T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star\Big)\\
&-2S(\Delta)\Big(\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star\phi\star e^{-1})\Big)\\
&-2S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star\phi\star e^{-1})\Big)\\
&-2S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b((\delta\phi^c X_c\phi)\star e^{-1})\Big)\Big]\\
&-\mathcal{L}^\Omega_\star(\lambda)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(\lambda)\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(\lambda)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
\mathcal{J}^\sigma(\Omega^2) \;=\;& ee^\sigma_b\Big\{\frac{\Omega^2}{2}\Big[-T(\Delta)\Big(\delta\phi^c X_c(\tilde{x}\phi), \widetilde{X}^b\{\tilde{x}\phi, e^{-1}\}_\star\Big)\\
&-2S(\Delta)\Big(\tilde{x}\phi, \widetilde{X}^b((\delta\phi^c X_c(\tilde{x}\phi))\star e^{-1})\Big)\Big]\\
&-\mathcal{L}^\Omega_\star(\Omega^2)\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star(\Omega^2)\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star(\Omega^2)), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\Big\}
\end{aligned} \tag{3.49}
$$

where

$$
\begin{aligned}
&\mathcal{L}^\Omega_\star(0) = \frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{1}{2}\partial_\mu\phi_a\star\partial^\mu\phi^a \qquad \mathcal{L}^\Omega_\star(m^2) = \frac{m^2}{2}\phi\star\phi\\
&\mathcal{L}^\Omega_\star(\lambda) = \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \qquad \mathcal{L}^\Omega_\star(\Omega^2) = \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi).
\end{aligned} \tag{3.50}
$$

3.2.2 Symmetries and conserved currents

Let us now deal with the symmetry analysis and deduce the conserved currents. Performing a functional variation of the fields and a coordinate transformation

$$
\phi'(x) = \phi(x) + \delta\phi(x) \qquad \phi'^c(x) = \phi^c(x) + \delta\phi^c(x) \qquad x'^\mu = x^\mu + \epsilon^\mu \tag{3.51}
$$

and using $\mathrm{d}^D x' = [1 + \partial_\mu\epsilon^\mu + \mathbf{O}(\epsilon^2)]\mathrm{d}^D x$ lead to the following variation of the action, to first order in $\delta\phi(x), \delta\phi^c(x), \tilde{x}$ and ϵ^μ:

$$
\delta\mathcal{S}^\Omega_\star \;=\; \int e\mathrm{d}^D x\Big\{\Big|\frac{\partial x'}{\partial x}\Big|\star(\mathcal{L}'^\Omega_\star\star e^{-1})\Big\} - \int e\mathrm{d}^D x\,(\mathcal{L}^\Omega_\star\star e^{-1})
$$

$$\begin{aligned}
&= \int \mathsf{d}^D x \Big\{ \delta\Big((\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) + \partial_\mu \epsilon^\mu \star \Big((\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) \Big\} \\
&= \int \mathsf{d}^D x \, \Big\{ \delta_\phi \Big((\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) + \delta_{\phi^c} \Big((\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) \\
&\quad + \delta_{\tilde{x}} \Big(\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) + \epsilon^\mu \star \partial_\mu [(\mathcal{L}^\Omega_\star \star e^{-1}) e] + \partial_\mu \epsilon^\mu \star (\mathcal{L}^\Omega_\star \star e^{-1}) e \Big\}. \qquad (3.52)
\end{aligned}$$

On shell, and integrated on a submanifold $M \subset \mathbb{R}^D$ with fields non vanishing at the boundary (so that the total derivative terms do not disappear), we get:

$$\delta \mathcal{S}^\Omega_\star = \int_M \mathsf{d}^D x \, \partial_\sigma \Big[\mathcal{K}^\sigma + \mathcal{J}^\sigma + \mathcal{R}^\sigma + \epsilon^\sigma \star \Big((\mathcal{L}^\Omega_\star \star e^{-1}) e \Big) \Big] \qquad (3.53)$$

encompassing different contributions explicated below. In the computations, we decompose the GW harmonic term as follows [7]

$$(\tilde{x}\phi) \star (\tilde{x}\phi) = \frac{1}{4} (\tilde{x} \star \phi \star \tilde{x} \star \phi + \tilde{x} \star \phi \star \phi \star \tilde{x} + \phi \star \tilde{x} \star \tilde{x} \star \phi + \phi \star \tilde{x} \star \phi \star \tilde{x}) \qquad (3.54)$$

in order to get the NC Lagrangian entirely lain in the $\star-$ algebra of fields with the advantage to be stable under formal $\star-$ algebraic computations (such that the cyclicity of $\star-$ factors under integral). By first performing the ϕ^c variation of the harmonic term in the GW action (3.14), using the right hand side of (3.54), and then identifying the result with (3.39) one can infer the identity

$$\begin{aligned}
&\frac{\Omega^2}{2} \Big\{ -T(\Delta) \Big(\delta\phi^c X_c(\tilde{x}\phi), \widetilde{X}^b \{\tilde{x}\phi, e^{-1}\}_\star \Big) \Big\} + \\
&\frac{\Omega^2}{2} \Big\{ -2S(\Delta) \Big(\tilde{x}\phi, \widetilde{X}^b ((\delta\phi^c X_c(\tilde{x}\phi)) \star e^{-1}) \Big) \Big\} = \\
&\frac{\Omega^2}{8} \Big\{ -T(\Delta) \Big(\delta\phi^c X_c \phi, \widetilde{X}^b (\{\tilde{x}, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star) \Big) \\
&-T(\Delta) \Big(\delta\phi^c X_c \tilde{x}, \widetilde{X}^b (\{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star) \Big) \Big\} \\
+ \; &\frac{\Omega^2}{4} \Big\{ -S(\Delta) \Big(\tilde{x}, \widetilde{X}^b ((\delta\phi^c X_c \phi) \star \{\phi, \tilde{x}\}_\star \star e^{-1}) \Big) \\
&-S(\Delta) \Big(\{\tilde{x}, \phi \star \tilde{x}\}_\star, \widetilde{X}^b ((\delta\phi^c X_c \phi) \star e^{-1}) \Big) \\
&-S(\Delta) \Big(\{\phi, \tilde{x}\}_\star, \widetilde{X}^b ((\delta\phi^c X_c \phi) \star \tilde{x} \star e^{-1}) \Big) \\
&-S(\Delta) \Big(\phi, \widetilde{X}^b ((\delta\phi^c X_c \tilde{x}) \star \{\phi, \tilde{x}\}_\star \star e^{-1}) \Big) \\
&-S(\Delta) \Big(\{\phi, \tilde{x}\}_\star, \widetilde{X}^b ((\delta\phi^c X_c \tilde{x}) \star \phi \star e^{-1}) \Big) \\
&-S(\Delta) \Big(\{\phi, \tilde{x} \star \phi\}_\star, \widetilde{X}^b ((\delta\phi^c X_c \tilde{x}) \star e^{-1}) \Big) \Big\}. \qquad (3.55)
\end{aligned}$$

Then, from $\tilde{x}$ variation of the action (3.14) expressed as

$$\delta_{\tilde{x}} \mathcal{S}^\Omega_\star = \int d^D x \; (\delta\tilde{x} \, e \frac{\Omega^2}{8} \{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star + \partial_\sigma \mathcal{R}^\sigma) \qquad (3.56)$$

we deduce the current term

$$\mathcal{R}^\sigma \;\; = \;\; \frac{\Omega^2}{8} e e^\sigma_b \Big\{ T(\Delta) \Big(\delta\tilde{x}, \widetilde{X}^b \{\phi, \{e^{-1}, \{\tilde{x}, \phi\}_\star\}_\star\}_\star \Big)$$

$$
\begin{aligned}
&+2S(\Delta)\Big(\{\tilde{x},\phi\}_\star, \widetilde{X}^b(\delta\tilde{x}\star\phi\star e^{-1})\Big)\\
&+2S(\Delta)\Big(\{\phi,\tilde{x}\star\phi\}_\star, \widetilde{X}^b(\delta\tilde{x}\star e^{-1})\Big)\\
&+2S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\tilde{x}\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\Big\}. \qquad (3.57)
\end{aligned}
$$

On the other side, using the identity $\delta\phi^c X_c\partial_\mu\phi = \partial_\mu(\delta\phi^c X_c\phi) - \partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi$ and (3.55), and collecting different terms in appropriate way, the current $\mathcal{J}^\sigma$ (3.45) can be now written as

$$
\begin{aligned}
\mathcal{J}^\sigma &= \mathcal{K}^\sigma(\delta\phi\to -\delta\phi^c X_c\phi) + \mathcal{R}^\sigma(\delta\tilde{x}\to -\delta\phi^c X_c\tilde{x})\\
&+\frac{e\delta\phi^c}{2}X_c\phi.\{\partial^\sigma\phi, e^{-1}\}_\star + \frac{e\delta\phi^c}{2}.\{\partial^\sigma\phi_c, e^{-1}\}_\star\\
&+ee_b^\sigma\Big\{-\mathcal{L}^\Omega_\star\star(\delta\phi^b e^{-1}) + \delta\phi^b(\mathcal{L}^\Omega_\star\star e^{-1})\\
&+T(\Delta)\Big(X_c(\mathcal{L}^\Omega_\star), \widetilde{X}^b(\delta\phi^c e^{-1})\Big)\\
&+\frac{1}{2}T(\Delta)\Big(\partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi, \widetilde{X}^b\{\partial^\mu\phi, e^{-1}\}_\star\Big)\\
&+S(\Delta)\Big(\partial_\mu\phi, \widetilde{X}^b((\partial_\mu(\delta\phi^c e_c^\rho)\partial_\rho\phi)\star e^{-1})\Big)\Big\}\\
&+\frac{1}{2}ee_b^\sigma\Big\{-T(\Delta)\Big(\delta\phi^c X_c\partial_\mu\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\Big)\\
&-2S(\Delta)\Big(\partial^\mu\phi_a, \widetilde{X}^b((\delta\phi^c X_c\partial_\mu\phi^a)\star e^{-1})\Big)\\
&+2S(\Delta)\Big(\partial_\mu\phi_a, \widetilde{X}^b(\partial^\mu\delta\phi^a\star e^{-1})\Big)\\
&+T(\Delta)\Big(\partial_\mu\delta\phi_a, \widetilde{X}^b\{\partial^\mu\phi^a, e^{-1}\}_\star\Big)\Big\}, \qquad (3.58)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{K}^\sigma(-\delta\phi^c X_c\phi) &\equiv \mathcal{K}^\sigma(\delta\phi\to -\delta\phi^c X_c\phi)\\
&= -\frac{e\delta\phi^c X_c\phi}{2}.\{\partial^\sigma\phi, e^{-1}\}_\star\\
&-ee_b^\sigma\Big[T(\Delta)\Big(\partial_\mu(\delta\phi^c X_c\phi), \frac{\widetilde{X}^b}{2}\{\partial^\mu\phi, e^{-1}\}_\star\Big)\\
&+S(\Delta)\Big(\partial_\mu\phi, \widetilde{X}^b(\partial^\mu(\delta\phi^c X_c\phi)\star e^{-1})\Big)\\
&+\frac{m^2}{2}T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi, e^{-1}\}_\star\Big)\\
&+m^2 S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star e^{-1})\Big)\\
&+\frac{\lambda}{4!}T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\phi\star\phi, \{\phi, e^{-1}\}_\star\}_\star\Big)\\
&+\frac{\lambda}{12}S(\Delta)\Big(\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star\phi\star\phi\star e^{-1})\Big)\\
&+\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star\phi\star e^{-1})\Big)\\
&+\frac{\lambda}{12}S(\Delta)\Big(\phi\star\phi\star\phi, \widetilde{X}^b(\delta\phi^c X_c\phi\star e^{-1})\Big)\\
&+\frac{\Omega^2}{8}T(\Delta)\Big(\delta\phi^c X_c\phi, \widetilde{X}^b\{\tilde{x}, \{e^{-1}, \{\tilde{x},\phi\}_\star\}_\star\}_\star\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Omega^2}{4}S(\Delta)\Big(\tilde{x},\widetilde{X}^b(\delta\phi^c X_c\phi\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\\
&+\frac{\Omega^2}{4}S(\Delta)\Big(\{\tilde{x},\phi\star\tilde{x}\}_\star,X^b(\delta\phi^c X_c\phi\star e^{-1})\Big)\\
&+\frac{\Omega^2}{4}S(\Delta)\Big(\{\phi,\tilde{x}\}_\star,\widetilde{X}^b(\delta\phi^c X_c\phi\star\tilde{x}\star e^{-1})\Big)\Big]
\end{aligned}
\tag{3.59}
$$

and

$$
\begin{aligned}
\mathcal{R}^\sigma(-\delta\phi^c X_c\tilde{x}) &\equiv \mathcal{R}^\sigma(\delta\tilde{x}\to-\delta\phi^c X_c\tilde{x})\\
&= -\frac{\Omega^2}{8}ee^\sigma_b\Big\{T(\Delta)\Big(\delta\phi^c X_c\tilde{x},\widetilde{X}^b\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star\Big)\\
&\quad+2S(\Delta)\Big(\{\tilde{x},\phi\}_\star,\widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star\phi\star e^{-1})\Big)\\
&\quad+2S(\Delta)\Big(\{\phi,\tilde{x}\star\phi\}_\star,\widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star e^{-1})\Big)\\
&\quad+2S(\Delta)\Big(\phi,\widetilde{X}^b(\delta\phi^c X_c\tilde{x}\star\{\tilde{x},\phi\}_\star\star e^{-1})\Big)\Big\}.
\end{aligned}
\tag{3.60}
$$

$\mathcal{K}^\sigma$ keeps the previous defined expression in (3.25). In contrary to the result in [3] for ordinary $\phi^4_\star$ theory, the twisted GW action is not invariant under global translation. Now imposing the constraint $\mathcal{E}_{\tilde{x}} = \frac{\delta\mathcal{S}^\Omega_\star}{\delta\tilde{x}} = 0$ giving

$$
e\frac{\Omega^2}{8}\{\phi,\{e^{-1},\{\tilde{x},\phi\}_\star\}_\star\}_\star = 0 \tag{3.61}
$$

coupled to the transformations

$$
\delta\phi = -\epsilon^\nu\partial_\nu\phi \qquad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c \qquad \epsilon^\nu = \text{constant} \tag{3.62}
$$

that we subtitute into (3.130) and taking into account $e^a_\nu = \partial_\nu\phi^a$, we infer from the relation

$$
0 = \delta\mathcal{S}^\Omega_\star = \int_M \mathrm{d}^D x\, \epsilon^\nu\partial_\mu\mathcal{T}^\mu_\nu \tag{3.63}
$$

the EMT

$$
\begin{aligned}
\mathcal{T}^\mu_\nu &= -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi,e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c,e^{-1}\}_\star\\
&\quad+ee^\mu_b\Big\{\mathcal{L}^\Omega_\star\star(e^{-1}\partial_\nu\phi^b)+T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star,\widetilde{X}^b(e^{-1}\partial_\nu\phi^c)\Big)\\
&\quad+\Omega^2\Theta^{-1}_{\gamma\nu}\Big[S(\Delta)\Big(\{\tilde{x}^\gamma,\phi\}_\star,\widetilde{X}^b(\phi\star e^{-1})\Big)\\
&\quad+S(\Delta)\Big(\{\phi,\tilde{x}^\gamma\star\phi\}_\star,\widetilde{X}^b(e^{-1})\Big)\\
&\quad+S(\Delta)\Big(\phi,\widetilde{X}^b\{\tilde{x}^\gamma,\phi\}_\star\star e^{-1}\Big)\Big]\Big\}.
\end{aligned}
\tag{3.64}
$$

This tensor is neither symmetric nor locally conserved. In the case of standard Moyal product, it reduces to the NC EMT computed in [7] and its regularization can be worked out in the same way as done in that work. Similarly, using the transformation

$$
\delta\phi = -\epsilon^\nu\partial_\nu\phi = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi \qquad \delta\phi^c = -\epsilon^\nu\partial_\nu\phi^c = -\epsilon^{\nu\rho}x_\rho\partial_\nu\phi^c \qquad \epsilon^\nu = \epsilon^{\nu\rho}x_\rho \tag{3.65}
$$

where $\epsilon^{\nu\rho}$ is an infinitesimal constant skew symmetric Lorentz tensor, and $\epsilon^{\nu\rho}x_{[\nu}\partial_{\rho]}\phi = -2\epsilon^{\nu\rho}x_\rho\partial_\nu\phi$ substituted into (3.130) yields

$$0 = \delta\mathcal{S}^\Omega_\star = \int_M \mathrm{d}^D x\, \epsilon^{\nu\rho}\partial_\mu\mathcal{M}^\mu_{\nu\rho} \tag{3.66}$$

which affords the AMT as

$$\begin{aligned}
\mathcal{M}^\mu_{\nu\rho} \;=\;& \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^\mu\phi, e^{-1}\}_\star + \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_c\{\partial^\mu\phi^c, e^{-1}\}_\star \\
& -\frac{ee^\mu_b}{2}\Big(\mathcal{L}^\Omega_\star \star (e^{-1}x_{[\nu}\partial_{\rho]}\phi^b)\Big) \\
& +\frac{ee^\mu_b}{2}\Big\{T(\Delta)\Big(X_c\mathcal{L}^\Omega_\star, \widetilde{X}^b(e^{-1}x_{[\nu}\partial_{\rho]}\phi^c)\Big) \\
& -T(\Delta)\Big(\partial_{[\nu}\phi, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi, e^{-1}\}_\star)\Big) \\
& -T(\Delta)\Big(\partial_{[\nu}\phi^d, \frac{1}{2}\widetilde{X}^b(\{\partial_{\rho]}\phi_d, e^{-1}\}_\star)\Big) \\
& +S(\Delta)\Big(\partial_{[\nu}\phi, \widetilde{X}^b(\partial_{\rho]}\phi \star e^{-1})\Big) \\
& +S(\Delta)\Big(\partial_{[\nu}\phi_d, \widetilde{X}^b(\partial_{\rho]}\phi^d \star e^{-1})\Big) \\
& -\frac{\Omega^2}{4}\Theta^{-1}_{\gamma[\nu}\Big[T(\Delta)\Big(x_{\rho]}, \widetilde{X}^b(\{\phi, \{e^{-1}, \{\tilde{x}^\gamma, \phi\}_\star\}_\star\}_\star)\Big) \\
& +2S(\Delta)\Big(\{\tilde{x}^\gamma, \phi\}_\star, \widetilde{X}^b(x_{\rho]} \star \phi \star e^{-1})\Big) \\
& +2S(\Delta)\Big(\{\phi, \tilde{x}^\gamma \star \phi\}_\star, \widetilde{X}^b(x_{\rho]} \star e^{-1})\Big) \\
& +2S(\Delta)\Big(\phi, \widetilde{X}^b(x_{\rho]} \star \{\tilde{x}, \phi\}_\star \star e^{-1})\Big)\Big]\Big\}.
\end{aligned} \tag{3.67}$$

This angular momentum tensor is not conserved, in contrary to the result obtained for the non renormalizable twisted $\phi^{\star 4}$ model studied in [3]. This analysis is compatible with the previous GW model investigation [8]. One recovers the canonical angular momentum tensor of the decoupled fields in the commutative limit. Defining now the dilatation transformation by

$$x \to x' = \epsilon x \qquad \phi(x) \to \phi'(x') = \phi'(\epsilon x) = \epsilon^{-\Delta}\phi(x) \tag{3.68}$$

where ϵ is a constant number, and Δ is the scale dimension of the field ϕ, we note that the GW action is invariant over dilatation symmetry if $\Delta = 0$ and $\epsilon = \pm 1$, implying

$$x' = x \qquad \phi'(x) = \phi(x) \qquad \text{or} \qquad x' = -x \qquad \phi'(-x) = \phi(x) \tag{3.69}$$

which is nothing but a parity transformation of the space-time inducing a conserved current:

$$\mathcal{D}^\mu = \mathcal{R}^\mu(\delta\tilde{x} \to -2\tilde{x}) - 2x^\mu(\mathcal{L}^\Omega_\star \star e^{-1})e. \tag{3.70}$$

Finally, the EMT, AMT and DC can be computed under the well defined field values at the boundary, i.e. $\int e\mathrm{d}^D x\, X_b S(\Delta)(f, \widetilde{X}^b g) = 0$, to give simplified expressions. In this case, there follow

$$\mathcal{T}^\mu_\nu \;=\; -\frac{e}{2}(\partial_\nu\phi)\{\partial^\mu\phi, e^{-1}\}_\star - \frac{e}{2}(\partial_\nu\phi_c)\{\partial^\mu\phi^c, e^{-1}\}_\star$$

$$+ee^{\mu}_{b}\Big\{\mathcal{L}^{\Omega}_{\star}\star(e^{-1}\partial_{\nu}\phi^{b})+T(\Delta)\Big(X_{c}\mathcal{L}^{\Omega}_{\star},\widetilde{X}^{b}(e^{-1}\partial_{\nu}\phi^{c})\Big)\Big\} \tag{3.71}$$

and

$$\begin{aligned}\mathcal{M}^{\mu}_{\nu\rho} &= \frac{e}{4}x_{[\nu}\partial_{\rho]}\phi\{\partial^{\mu}\phi,e^{-1}\}_{\star}+\frac{e}{4}x_{[\nu}\partial_{\rho]}\phi_{c}\{\partial^{\mu}\phi^{c},e^{-1}\}_{\star}\\ &-\frac{ee^{\mu}_{b}}{2}\Big(\mathcal{L}^{\Omega}_{\star}\star(e^{-1}x_{[\nu}\partial_{\rho]}\phi^{b})\Big)\\ &+\frac{ee^{\mu}_{b}}{2}\Big\{T(\Delta)\Big(X_{c}\mathcal{L}^{\Omega}_{\star},\widetilde{X}^{b}(e^{-1}x_{[\nu}\partial_{\rho]}\phi^{c})\Big)\\ &-T(\Delta)\Big(\partial_{[\nu}\phi,\frac{1}{2}\widetilde{X}^{b}(\{\partial_{\rho]}\phi,e^{-1}\}_{\star})\Big)\\ &-T(\Delta)\Big(\partial_{[\nu}\phi^{d},\frac{1}{2}\widetilde{X}^{b}(\{\partial_{\rho]}\phi_{d},e^{-1}\}_{\star})\Big)\\ &-\frac{\Omega^{2}}{4}\Theta^{-1}_{\gamma[\nu}T(\Delta)\Big(x_{\rho]},\widetilde{X}^{b}(\{\phi,\{e^{-1},\{\tilde{x}^{\gamma},\phi\}_{\star}\}_{\star}\}_{\star})\Big)\Big\}\end{aligned} \tag{3.72}$$

and the current of dilatation symmetry expressed as

$$\mathcal{D}^{\mu} = -\Omega^{2}ee^{\mu}_{b}T(\Delta)\Big(\tilde{x},\widetilde{X}^{b}\{\phi,\{e^{-1},\{\tilde{x},\phi\}_{\star}\}_{\star}\}_{\star}\Big)-2x^{\mu}(\mathcal{L}^{\Omega}_{\star}\star e^{-1})e. \tag{3.73}$$

3.3 Twisted Yang-Mills Model

As already mentioned, the construction of noncommutative field theories in a nontrivial background metric generally implies a non-constant deformation matrix $\widetilde{\Theta}^{\mu\nu}=\widetilde{\Theta}^{\mu\nu}(x)$. There naturally results the difficulty of finding a suitable explicit closed Moyal type formula and consequently, defining a noncommutative product becomes rather complicated.The situation is simpler when one deals with the Moyal space $\mathbb{R}^{D}_{\Theta}$, i.e. the deformed $D-$dimensional space endowed with a constant Moyal $\star-$bracket of coordinate functions $[x^{\mu},x^{\nu}]_{\star}=i\Theta^{\mu\nu}$. In this case the star product (see [76]-[19] and reference therein) is defined by

$$(f\star g)(x)=m\Big\{e^{i\frac{\Theta^{\mu\nu}}{2}\partial_{\mu}\otimes\partial_{\nu}}f(x)\otimes g(x)\Big\}\quad x\in\mathbb{R}^{D}_{\Theta}\quad\forall f,g\in C^{\infty}(\mathbb{R}^{D}_{\Theta}) \tag{3.74}$$

m is the ordinary multiplication of functions, i.e. $m(f\otimes g)=f.g$. In the coordinate basis, this space is generated by the usual commuting vector field $\partial_{\mu}=:\frac{\partial}{\partial x^{\mu}}\in T_{x}\mathbb{R}^{D}_{\Theta}$, the tangent space of $\mathbb{R}^{D}_{\Theta}$, conferring to Moyal space the properties of a flat space.

On the contrary, in the context of a dynamical noncommutative field theory, the vector field can be generalized to take the form $X_{a}=e^{\mu}_{a}(x)\partial_{\mu}$, where $e^{\mu}_{a}(x)$ is a tensor depending on the coordinate functions in the complex general linear matrix group of order D, $\mathsf{GL}(D,\mathbb{C})$. The star product then takes the form

$$(f\star g)(x)=m\Big\{e^{i\frac{\Theta^{ab}}{2}X_{a}\otimes X_{b}}f(x)\otimes g(x)\Big\}\quad x\in\mathbb{R}^{D}_{\tilde{\Theta}}\quad\forall f,g\in C^{\infty}(\mathbb{R}^{D}_{\tilde{\Theta}}) \tag{3.75}$$

and the vielbeins are given by the infinitesimal affine transformation as

$$e^{\mu}_{a}(x)=\delta^{\mu}_{a}+\omega^{\mu}_{ab}x^{b}, \tag{3.76}$$

where $\omega_{ab} \in \mathsf{GL}(D, \mathbb{C})$. Using (3.76), the non vanishing Lie bracket peculiar to the non-coordinate base [44]

$$[X_a, X_b] = e^c_\nu\Big[e^\mu_a\partial_\mu e^\nu_b - e^\mu_b\partial_\mu e^\nu_a\Big]X_c = C^c_{ab}X_c \tag{3.77}$$

is here simply reduced to

$$[X_a, X_b] = \omega^\mu_{ba}\partial_\mu - \omega^\mu_{ab}\partial_\mu = -2\omega^\mu_{ab}\partial_\mu. \tag{3.78}$$

Besides, the dynamical star product (3.75) can now expressed as

$$(f \star g)(x) = m\Big[\exp\Big(\frac{i}{2}\theta e^{-1}\epsilon^{\mu\nu}\partial_\mu \otimes \partial_\nu\Big)(f \otimes g)(x)\Big] \tag{3.79}$$

where $e^{-1} =: det(e^\mu_a) = 1 + \omega^1_{12}x^2 - \omega^2_{12}x^1$; $\epsilon^{\mu\nu}$ is the symplectic tensor in two dimensions, $(D = 2)$, i.e $\epsilon^{12} = -\epsilon^{21} = 1,\ \epsilon^{11} = \epsilon^{22} = 0$.

The coordinate function commutation relation becomes $[x^\mu, x^\nu]_\star = i\widetilde{\Theta}^{\mu\nu} = i(\Theta^{\mu\nu} - \Theta^{a[\mu}\omega^{\nu]}_{ab}x^b)$ which can be reduced to the usual Moyal space relation, as expected, by setting $\omega^\mu_{ab} = [0]$. One can check that the Jacobi identity is also well satisfied, i.e.

$$[x^\mu, [x^\nu, x^\rho]_\star]_\star + [x^\rho, [x^\mu, x^\nu]_\star]_\star + [x^\nu, [x^\rho, x^\mu]_\star]_\star = \Theta^{b\mu}\Theta^{d[\nu}\omega^{\rho]}_{bd} = 0 \tag{3.80}$$

conferring a Lie algebra structure to the defined twisted Moyal space. This identity ensures the associativity of the star-product (3.75) and implies that

$$\widetilde{\Theta}^{\sigma\rho}\partial_\rho\widetilde{\Theta}^{\mu\nu} + \widetilde{\Theta}^{\nu\rho}\partial_\rho\widetilde{\Theta}^{\sigma\mu} + \widetilde{\Theta}^{\mu\rho}\partial_\rho\widetilde{\Theta}^{\nu\sigma} = 0. \tag{3.81}$$

Remark that with the relation (3.78), the requirement that ω_{ab} is a symmetric tensor trivially ensures the associativity of the star product. In the interesting particular case addressed in this work, the associativity of the star product (3.75) can be shown even for the non symmetric tensor ω_{ab}.

Proof of associativity: *Notwithstanding the condition $[X_a, X_b] \neq 0$, i.e. ω^μ_{ab} is skew-symmetric, the twisted $\star-$product defined in (3.75) remains noncommutative and associative. Indeed, using the twisted star-product*

$$(f \star g)(x) = m\Big[\exp\Big(\frac{i}{2}\theta e^{-1}\epsilon^{\mu\nu}\partial_\mu \otimes \partial_\nu\Big)(f \otimes g)(x)\Big], \tag{3.82}$$

one can see that

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x}e^{-\frac{i}{2}\theta e^{-1}k\epsilon q}. \tag{3.83}$$

The Fourier transform of $f, g \in \mathcal{S}(\mathbb{R}^2_{\tilde{\Theta}})$ can be written as

$$\tilde{f}(k) = \int d^2x\ e^{-ikx}f(x), \quad \tilde{g}(q) = \int d^2x\ e^{-iqx}g(x) \tag{3.84}$$

with the function inverse transform given by

$$f(x) = \int d^2k\ e^{ikx}\tilde{f}(k), \quad g(x) = \int d^2q\ e^{iqx}\tilde{g}(q). \tag{3.85}$$

We can redefine the twisted star-product of two Schwartz functions f, g as:

$$\begin{aligned}(f\star g)(x) &= \int d^2kd^2q\ \tilde{f}(k)\tilde{g}(q)e^{ikx}\star e^{iqx}\\ &= \int d^2kd^2q\ \tilde{f}(k)\tilde{g}(q)e^{i(k+q)x}e^{-\frac{i}{2}\theta e^{-1}k\epsilon q}.\end{aligned} \tag{3.86}$$

Then, we have

$$\begin{aligned}\Big((f\star g)\star h\Big)(x) &= \Big[\int d^2kd^2q\ \tilde{f}(k)\tilde{g}(q)e^{-\frac{i}{2}\theta e^{-1}k\epsilon q}e^{i(k+q)x}\Big]\star\Big[\int d^2p\ e^{ipx}\tilde{h}(p)\Big]\\ &= \int d^2kd^2qd^2p\ \tilde{f}(k)\tilde{g}(q)\tilde{h}(p)\Big(e^{-\frac{i}{2}\theta e^{-1}k\epsilon q}e^{i(k+q)x}\Big)\star e^{ipx}\end{aligned} \tag{3.87}$$

Recalling that $e^{-1}=1+\omega_\mu x^\mu$, we get

$$\begin{aligned}\Big((f\star g)\star h\Big)(x) &= \int d^2kd^2qd^2p\ \tilde{f}(k)\tilde{g}(q)\tilde{h}(p)e^{-\frac{i}{2}\Big(\theta k\epsilon q-\frac{1}{2}\theta^2\omega(k\epsilon q)\epsilon p\Big)}e^{-\frac{i}{2}\theta e^{-1}(k+q)\epsilon p}\\ &\times\ e^{i\Big(k+q+p-\frac{1}{2}\theta\omega k\epsilon q\Big)x}\\ &= \int d^2kd^2qd^2p\ \tilde{f}(k)\tilde{g}(q)\tilde{h}(p)e^{-\frac{i}{2}\Big(\theta k\epsilon q+\theta(k+q)\epsilon p-\frac{1}{2}\theta^2\omega(k\epsilon q)\epsilon p\Big)}\\ &\times\ e^{i\Big(k+q+p-\frac{i}{2}\theta\omega(k+q)\epsilon p-\frac{1}{2}\theta\omega k\epsilon q\Big)x}\end{aligned} \tag{3.88}$$

In the other side,

$$\begin{aligned}\Big(f\star(g\star h)\Big)(x) &= \int d^2kd^2qd^2p\ \tilde{f}(k)\tilde{g}(q)\tilde{h}(p)e^{ikx}\star\Big(e^{-\frac{i}{2}\theta e^{-1}q\epsilon p}e^{i(q+p)x}\Big)\\ &= \int d^2kd^2qd^2p\ \tilde{f}(k)\tilde{g}(q)\tilde{h}(p)e^{-\frac{i}{2}\Big(\theta q\epsilon p+\theta k\epsilon(q+p)-\frac{1}{2}\theta^2\omega(k\epsilon q)\epsilon p\Big)}\\ &\times\ e^{i\Big(k+q+p-\frac{1}{2}\theta\omega q\epsilon p-\frac{1}{2}\theta\omega k\epsilon(q+p)\Big)x}\end{aligned} \tag{3.89}$$

A straightforward expansion shows that (3.88) and (3.89) coincide. There results the conclusion that the used twisted star-product (3.75) is well associative.□

From the particular condition $[X_a, X_b]=0$, (i. e. the vector fields are commuting), there result constraints on e_a^μ, namely $e^\mu_{[a}\partial_\mu e^\nu_{b]}=0$, that can be solved off-shell in terms of D scalar fields ϕ^a. Supposing that the square matrix e^μ_a has an inverse e^a_μ everywhere so that the X_a are linearly independent, then the above condition becomes $\partial_{[\mu}e^a_{\nu]}=0$ which is satisfied by $e^a_\nu=\partial_\nu\phi^a$. Since $X_a\phi^b=\delta^b_a$, the field ϕ^b can be viewed as new coordinates along the X_a directions. The metric g on $\mathbb{R}^D_{\tilde{\Theta}}$ can be chosen to be $\mathrm{g}(X_a, X_b)=e^\mu_a e^\nu_b \mathrm{g}_{\mu\nu}=\delta_{ab}$. See [3]-[43] for more details. In the whole work, we deal with Euclidean signature and $D=2$.

In this section we provide the general properties of gauge theory in twisted Moyal space, and define related connections. The tensor ω^μ_{ab} is an infinitesimal tensor, skew-symmetric in the indexes a and b. We study the symmetry of pure gauge theory and show that the related NC action is invariant under $U_\star(1)$ gauge transformation. Besides, we compute the resulting Noether current. Finally, we investigate the properties of the model for commuting vector fields X_a.

3.3.1 Connections and gauge transformation

Consider $E = \{x^\mu, \mu \in [[1,2]]\}$ and $\mathbb{C}[[x^1,x^2]]$, the free algebra generated by E. Let $\mathcal{I}$ be the ideal of $\mathbb{C}[[x^1,x^2]]$, engendered by the elements $x^\mu \star x^\nu - x^\nu \star x^\mu - i\tilde{\Theta}^{\mu\nu}$. The twisted Moyal Algebra $\mathcal{A}_{\tilde{\Theta}}$ is the quotient $\mathbb{C}[[x^1,x^2]]/\mathcal{I}$. Each element in $\mathcal{A}_{\tilde{\Theta}}$ is a formal power series in the x^μ's for which the relation $[x^\mu, x^\nu]_\star = i\tilde{\Theta}^{\mu\nu}$ holds. Moyal algebra can be here also defined as the linear space of smooth and rapidly decreasing functions equipped with the NC star product given in (3.75). The gauge symmetries on this noncommutative space can be realized in their enveloping algebra. However, there is an isomorphism mapping the noncommutative functions algebra $\mathcal{A}_{\tilde{\Theta}}$ into the commutative one, equipped with an additional noncommutative $\star-$product [76]. We consider the following infinitesimal affine transformation

$$e_a^\mu(x) = \delta_a^\mu + \omega_{ab}^\mu x^b, \quad \omega_{ab}^\mu =: -\omega_{ba}^\mu, \text{ and } |\omega^\mu| << 1. \tag{3.90}$$

For $D = 2$, e_a^μ and Θ^{ab} can be expressed as follows:

$$(e)_a^\mu = \begin{pmatrix} 1+\omega_{12}^1 x^2 & \omega_{12}^2 x^2 \\ -\omega_{12}^1 x^1 & 1-\omega_{12}^2 x^1 \end{pmatrix} \quad \text{and} \quad (\Theta)^{ab} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \theta(\epsilon)^{ab} \tag{3.91}$$

where $\epsilon^{12} = -\epsilon^{21} = 1,\ \epsilon^{11} = \epsilon^{22} = 0$. There follow the relations

$$e^{-1} =: det(e_a^\mu) = 1 + \omega_{12}^1 x^2 - \omega_{12}^2 x^1 \tag{3.92}$$

$$e =: det(e_\mu^a) = 1 - \omega_{12}^1 x^2 + \omega_{12}^2 x^1. \tag{3.93}$$

The tensor $\widetilde{\Theta}^{\mu\nu}$ can be written as [44]

$$(\widetilde{\Theta})^{\mu\nu} = (\Theta)^{\mu\nu} - (\Theta^{a[\mu}\omega_{ab}^{\nu]})x^b = \begin{pmatrix} 0 & \theta e^{-1} \\ -\theta e^{-1} & 0 \end{pmatrix}. \tag{3.94}$$

Let us now define the space-time ($M \subseteq \mathbb{R}^2_{\tilde{\Theta}}$) metric as

$$\mathrm{g}_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab} = \begin{pmatrix} 1-2\omega_{12}^1 x^2 & \omega_{12}^1 x^1 - \omega_{12}^2 x^2 \\ \omega_{12}^1 x^1 - \omega_{12}^2 x^2 & 1+2\omega_{12}^2 x^1 \end{pmatrix}, \tag{3.95}$$

$$\text{where } \ e_\mu^a = \begin{pmatrix} 1-\omega_{12}^1 x^2 & -\omega_{12}^2 x^2 \\ \omega_{12}^1 x^1 & 1+\omega_{12}^2 x^1 \end{pmatrix} \tag{3.96}$$

and its inverse as

$$\mathrm{g}^{\mu\nu} = e_a^\mu e_b^\nu \delta^{ab} = \begin{pmatrix} 1+2\omega_{12}^1 x^2 & \omega_{12}^2 x^2 - \omega_{12}^1 x^1 \\ \omega_{12}^2 x^2 - \omega_{12}^1 x^1 & 1-2\omega_{12}^2 x^1 \end{pmatrix}, \tag{3.97}$$

$$\text{where } \ e_a^\mu = \begin{pmatrix} 1+\omega_{12}^1 x^2 & \omega_{12}^2 x^2 \\ -\omega_{12}^1 x^1 & 1-\omega_{12}^2 x^1 \end{pmatrix} \tag{3.98}$$

with $\mathrm{g} = -det(\mathrm{g}_{\mu\nu})$. Noncommutative field theory over Moyal algebra of functions can be defined as field theories over module $\mathcal{H}$ on the noncommutative algebra $\mathcal{A}_{\tilde{\Theta}}$ or as matrix theories with coefficients in $\mathcal{A}_{\tilde{\Theta}}$. In the following, we restrict the study of field modules to rank trivial bi-modules $\mathcal{H}$ over $\mathcal{A}_{\tilde{\Theta}}$ with a Hilbert space structure defined by the scalar product

$$< a, b >=: \int ed^2x\ Tr(a^\dagger \star b) \star e^{-1};\ \ a, b \in \mathcal{A}_{\tilde{\Theta}}. \tag{3.99}$$

Provided this framework, the notion of connection defined on vector bundles in ordinary differential geometry is replaced, in NC geometry, by the generalized concept of connection on the projective modules as follows.

Definition 3.1 *The sesquilinear maps $\nabla_\mu : \mathcal{H} \to \mathcal{H}$ are called connections if they satisfy the differentiation chain rule*

$$\nabla_\mu(m \star f) = m \star \partial_\mu(f) + \nabla_\mu(m) \star f, \quad \text{for } f \in \mathcal{A}_{\tilde{\Theta}} \text{ and } m \in \mathcal{H} \tag{3.100}$$

(assumed here to be a right module over $\mathcal{A}_{\tilde{\Theta}}$), and if they are compatible with the Hermitian structure of $\mathcal{H}$ defined as $\mathrm{h}(f,g) = f^\dagger \star g$, i.e.

$$\partial_\mu \mathrm{h}(m_1, m_2) = \mathrm{h}(\nabla_\mu m_1, m_2) + \mathrm{h}(m_1, \nabla_\mu m_2). \tag{3.101}$$

In the sequel, we can identify $\mathcal{A}_{\tilde{\Theta}}$ with $\mathcal{H}$.

Definition 3.2 *Denoting by $\mathbf{1}$ the unit element of $\mathcal{A}_{\tilde{\Theta}}$, we define the gauge potential by $\nabla_\mu \mathbf{1} = -iA_\mu$. Then the connection can be explicitly written as*

$$\nabla_\mu(.) = \partial_\mu(.) - iA_\mu \star (.) \tag{3.102}$$

A_μ is called the gauge potential in the fundamental representation.

Note that the left module can be used to define the connection in the anti-fundamental representation by $\nabla_\mu(.) = \partial_\mu(.) + i(.) \star A_\mu$. In the same vein, the module can be used to define the connection on the adjoint representation by $\nabla_\mu(.) = \partial_\mu(.) - i[A_\mu, (.)]_\star$. Here, we adopt the fundamental representation. Now, we define the gauge transformation as a morphism of module, denoted by γ, satisfying the relation

$$\gamma(m \star f) = \gamma(m) \star f \quad \text{for } f \in \mathcal{A}_{\tilde{\Theta}} \text{ and } m \in \mathcal{H} \tag{3.103}$$

and preserving the Hermitian structure h, i.e.

$$\mathrm{h}(\gamma(f), \gamma(g)) = \mathrm{h}(f, g) \quad \text{for } f, g \in \mathcal{A}_{\tilde{\Theta}}. \tag{3.104}$$

For $f = g = 1$, one can prove that $\gamma(1) \in U(\mathcal{A}_{\tilde{\Theta}})$, the group of unitary elements of $\mathcal{A}_{\tilde{\Theta}}$, i.e. $\gamma^\dagger(1) \star \gamma(1) = 1$.

Note that the Jacobi identity is covariantly written in the form

$$\widetilde{\Theta}^{\sigma\rho} \nabla_\rho \widetilde{\Theta}^{\mu\nu} + \widetilde{\Theta}^{\nu\rho} \nabla_\rho \widetilde{\Theta}^{\sigma\mu} + \widetilde{\Theta}^{\mu\rho} \nabla_\rho \widetilde{\Theta}^{\nu\sigma} = 0. \tag{3.105}$$

This equation is evidently satisfied whenever the following condition holds

$$\nabla_\rho \widetilde{\Theta}^{\mu\nu} = 0, \tag{3.106}$$

which is very simple to handle in two-dimensional space. Indeed, in $D = 2$ the most general $\widetilde{\Theta}$ can be written in the form

$$\widetilde{\Theta}^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{-\mathrm{g}(x)}} \theta(x^1, x^2), \tag{3.107}$$

where $\theta(x^1, x^2)$ is a constant, simply denoted by θ. Then

$$e^{-1} = 1/\sqrt{-\mathrm{g}(x)} \quad \text{or} \quad e = \sqrt{-\mathrm{g}(x)}. \tag{3.108}$$

To end this section, let us mention that the integral $\int d^D x \, f \star g$, defined with the dynamical Moyal $\star-$product (3.75), is not cyclic, even with suitable boundary conditions at infinity, i.e.

$$\int \mathsf{d}^D x \, (f \star g) \neq \int \mathsf{d}^D x \, (g \star f). \tag{3.109}$$

Using now the measure $ed^D x$ where $e = det(e^a_\mu)$, a cyclic integral can be defined so that, up to boundary terms:

$$\int e\mathsf{d}^D x \, (f \star g) = \int e\mathsf{d}^D x (fg) = \int e\mathsf{d}^D x \, (g \star f). \tag{3.110}$$

In flat space $\sqrt{-\mathrm{g}(x)} = 1$.

3.3.2 Dynamical pure gauge theory

We consider a field ψ, element of the algebra $\mathcal{A}_{\tilde{\Theta}}$, $(\psi \in \mathcal{A}_{\tilde{\Theta}})$, and its infinitesimal gauge variation $\delta\psi$ such that, under an infinitesimal gauge transformation $\alpha(x)$, the relation $\delta_\alpha \psi(x) = i\alpha(x) \star \psi(x)$ is obeyed. The covariant coordinates are defined as

$$X^\mu = x^\mu + A^\mu, \quad A^\mu \in \mathcal{A}_{\tilde{\Theta}} \tag{3.111}$$

A^μ is called the gauge potential and satisfies the relation $\delta_\alpha A^\mu = \tilde{\Theta}^{\mu\rho}\partial_\rho \alpha + i[\alpha, A^\mu]_\star$. One can check that

$$[\alpha(x), \tilde{\Theta}^{\mu\sigma}]_\star = i\Theta^{a[\mu}\omega^{\sigma]}_{ac}\Theta^{c\rho}\partial_\rho\alpha(x), \quad \text{and} \quad \tilde{\Theta}^{-1}_{\mu\sigma}\delta_\alpha\tilde{\Theta}^{\mu\sigma} = 2\omega^a_{ac}\delta x^c. \tag{3.112}$$

From the last two equations and the definition of A_σ such that $A^\mu = \tilde{\Theta}^{\mu\sigma}A_\sigma$, we derived the gauge variation

$$\delta_\alpha A_\sigma = \partial_\sigma\alpha(x) + i[\alpha(x), A_\sigma]_\star + 2\omega^a_{ac}\Big(\Theta^{c\rho}\partial_\rho\alpha(x) - \delta x^c\Big)A_\sigma \tag{3.113}$$

There result two contributions in the expression of $\delta_\alpha A_\sigma$: the first contribution consisting of the first two terms of the ordinary Moyal product [64], and the second one given by the last term pertaining to the twisted effects of the theory. Of course, when $\omega = 0$, we recover the usual Moyal result.

The NC gauge tensor $T^{\mu\nu} \in \mathcal{A}_{\tilde{\Theta}}$ is defined by $T^{\mu\nu} = [X^\mu, X^\nu]_\star - i\tilde{\Theta}^{\mu\nu}$ and satisfies the properties

$$\delta_\alpha T^{\mu\nu} = i[\alpha(x), T^{\mu\nu}]_\star. \tag{3.114}$$

It is then convenient to use the relation $T^{\mu\nu} = i\tilde{\Theta}^{\mu\sigma}\tilde{\Theta}^{\nu\lambda}F_{\sigma\lambda}$ to derive the twisted Faraday tensor $F_{\sigma\lambda}$ as

$$F_{\sigma\lambda} \;=\; \partial_\sigma A_\lambda - \partial_\lambda A_\sigma - i[A_\sigma, A_\lambda]_\star - \Theta^{-1}_{\nu\lambda}\Theta^{a[\nu}\omega^{\lambda]}_{a\sigma}A_\lambda + \Theta^{-1}_{\mu\sigma}\Theta^{a[\mu}\omega^{\sigma]}_{a\lambda}A_\sigma$$

$$-\Big(\Theta^{-1}_{\mu\sigma}\Theta^{a[\mu}\omega^{\sigma]}_{ac}\Theta^{c\rho}\partial_\rho A_\lambda\Big)A_\sigma + \Big(\Theta^{-1}_{\nu\lambda}\Theta^{a[\nu}\omega^{\lambda]}_{ac}\Theta^{c\rho}\partial_\rho A_\sigma\Big)A_\lambda \tag{3.115}$$

$$= \partial_\sigma A_\lambda - \partial_\lambda A_\sigma - i[A_\sigma, A_\lambda]_\star + 2\omega^a_{a[\sigma}A_{\lambda]} - 2\omega^a_{ac}\Theta^{c\rho}(\partial_\rho A_{[\sigma})A_{\lambda]}. \tag{3.116}$$

Canceling the "*twisted*" contributions involved in the last two terms on the right hand side of this relations, we turn back to usual Moyal product result.

We then arrive at the expression of the dynamical NC pure gauge action defined as

$$\mathcal{S}_{YM} = -\frac{1}{4\kappa^2}\int ed^2x\,\Big(F_{\mu\nu}\star F^{\mu\nu}\star e^{-1}\Big) \tag{3.117}$$

where $e =: det(e^a_\mu)$.

Define also the gauge transformation U by

$$\begin{aligned} U &= e^{i\alpha}_\star = \sum_{k=0}^{\infty}\frac{i^k(\alpha)^k_\star}{k!} \\ &= 1 + i\alpha + (i^2/2!)\alpha\star\alpha + (i^3/3!)\alpha\star\alpha\star\alpha + \cdots;\;\; \alpha\in C^\infty(\mathbb{R}) \end{aligned} \tag{3.118}$$

$U_\star(1)$ is NC gauge group generated by elements $U \in U_\star(1)$. The infinitesimal gauge transformation $U = 1 + i\alpha(x)$ defined in the noncommutative Moyal space is the same as the ordinary infinitesimal gauge transformation in commutative space. Making the gauge transformation of tensor $F_{\mu\nu}$ into $F^U_{\mu\nu} = U\star F_{\mu\nu}\star U^\dagger$, then the transformed gauge action yields:

$$\begin{aligned} \mathcal{S}^U_{YM} &= -\frac{1}{4\kappa^2}\int ed^2x\,\Big(U\star F_{\mu\nu}\star U^\dagger\star U\star F^{\mu\nu}\star U^\dagger\star e^{-1}\Big) \\ &= -\frac{1}{4\kappa^2}\int ed^2x\,\Big(U\star F_{\mu\nu}\star F^{\mu\nu}\star U^\dagger\star e^{-1}\Big) \\ &= -\frac{1}{4\kappa^2}\int ed^2x\,\Big(F_{\mu\nu}\star F^{\mu\nu}\star U^\dagger\star e^{-1}\star U \\ &\quad +2S(\Delta)(U, \widetilde{X}^a(F_{\mu\nu}\star F^{\mu\nu}\star U^\dagger\star e^{-1}))\Big) \end{aligned} \tag{3.119}$$

Proposition 3.3 *Provided the stubborn requirement that the surface terms be vanished, the action (3.117) is invariant under the global gauge transformation, i.e. setting* $\alpha = \alpha_0 = c^{ste}$ *in (3.118).*

Proof: From the infinitesimal gauge transformation $U(\alpha) = 1 + i\alpha(x)$, its conjugate given by $U^\dagger(\alpha) = 1 - i\alpha(x)$ and the definition $e^{-1} := 1 + \omega_\mu x^\mu$, where $\omega_1 = \omega^1_{12}$ and $\omega_2 = -\omega^2_{12}$, we have

$$U^\dagger\star e^{-1} = e^{-1}(1 - i\alpha(x)) + \frac{1}{2}\widetilde{\Theta}^{\mu\nu}\partial_\mu\alpha(x)\omega_\nu$$

$$\Rightarrow U^\dagger\star e^{-1}\star U = e^{-1} + \widetilde{\Theta}^{\mu\nu}\omega_\nu\partial_\mu\alpha(x) = e^{-1} + \Theta^{\mu\nu}\omega_\nu\partial_\mu\alpha(x).$$

$$U^\dagger\star e^{-1}\star U = e^{-1} \Rightarrow \Theta^{\mu\nu}\omega_\nu\partial_\mu\alpha(x) = 0 \Rightarrow \alpha(x) = \alpha_0 = c^{ste} \quad \square$$

Furthermore, the following statement is true.

Proposition 3.4 *Provided the same requirement of vanishing condition of the surface terms, (i) the action (3.117) is invariant under the noncommutative group of unitary transformations $U_\star(1)$ defined by the parameter $\alpha = \alpha_0 + \epsilon_\sigma x^\sigma$, where ϵ_σ is an infinitesimal parameter and α_0 an arbitrary constant, and (ii) there exists an isomorphism between the NC gauge group induced by (3.113) and $U_\star(1)$ group.*

Proof: The part (i) is immediate from the previous proof. (ii) Imposing the condition $\alpha = \alpha_0 + \epsilon_\lambda x^\lambda$, the NC gauge transformation (3.113) is reduced to the form:

$$\delta_\alpha A_\sigma = \partial_\sigma \alpha - \epsilon_\mu \Theta^{\mu\rho}\partial_\rho A_\sigma = \partial_\rho\Big(\delta^\rho_\sigma \frac{\alpha}{2} - \epsilon_\mu\Theta^{\mu\rho}A_\sigma\Big) = \partial\Lambda \tag{3.120}$$

giving rise to the isomorphism

$$f : \partial\Lambda \to e_\star^{i\Lambda} = 1 + i\Big(\frac{\alpha}{2} - \epsilon_\mu\Theta^{\mu\rho}A_\sigma\Big) \tag{3.121}$$

mapping the NC gauge group (3.113) into the $U_\star(1)$ group. $\square$

Therefore, the $U_\star(1)$ group can be considered as the invariance NC gauge group for the Yang-Mills action defined in (3.117). Note that setting $\epsilon_\mu = 0$ we recover the global gauge transformation of the usual gauge field theory.

The A_μ variation of the action (3.117) is given by

$$\delta_A \mathcal{S}_{YM} = -\frac{1}{4\kappa^2}\int d^2x\ (\delta A_\beta \mathcal{E}_A + \partial_\beta J^\beta) \tag{3.122}$$

where the equation of motion of the field A is provided by

$$\begin{aligned}\frac{\delta\mathcal{S}_{YM}}{\delta A_\beta} = \mathcal{E}_A \ =\ & -\partial_\mu(e\{F^{\mu\beta}, e^{-1}\}_\star) + \partial_\nu(e\{F^{\beta\nu}, e^{-1}\}_\star)\\ & -ie[A_\nu, \{F^{\beta\nu}, e^{-1}\}_\star]_\star + ie[A_\mu, \{F^{\mu\beta}, e^{-1}\}_\star]_\star\\ & -2\omega^a_{ac}\Theta^{c\rho}\Big(\partial_\rho(-eA_\nu\{F^{\beta\nu}, e^{-1}\}_\star + eA_\mu\{F^{\mu\beta}, e^{-1}\}_\star)\Big)\\ & -2e\omega^a_{ac}\Theta^{c\rho}\Big(\partial_\rho A_\mu\{F^{\mu\beta}, e^{-1}\}_\star - \partial_\rho A_\nu\{F^{\beta\nu}, e^{-1}\}_\star\Big)\\ & +2e\Big(\omega^a_{a\mu}\{F^{\mu\beta}, e^{-1}\}_\star - \omega^a_{a\nu}\{F^{\beta\nu}, e^{-1}\}_\star\Big) = 0\end{aligned} \tag{3.123}$$

and the current J^β by

$$\begin{aligned}J^\beta \ =\ & -\frac{1}{4\kappa^2}\Big[e\delta A_\nu\{F^{\beta\nu}, e^{-1}\}_\star - e\delta A_\mu\{F^{\mu\beta}, e^{-1}\}_\star\\ & -iee^\beta_a\Big(T(\Delta)(\delta A_{[\mu}, \widetilde{X}^a[A_{\nu]}, \{F^{\mu\nu}, e^{-1}\}_\star]_\star)\\ & +T(\Delta)([\delta A_{[\mu}, A_{\nu]}]_\star, \widetilde{X}^a\{F^{\mu\nu}, e^{-1}\}_\star)\\ & +2S(\Delta)(A_{[\mu}, \widetilde{X}^a(\delta A_{\nu]}\{F^{\mu\nu}, e^{-1}\}_\star))\Big)\\ & -2\omega^a_{ac}\Theta^{c\beta}e\delta A_{[\mu}A_{\nu]}\{F^{\mu\nu}, e^{-1}\}_\star\\ & +ee^\beta_a\Big(T(\Delta)(\delta F^{\mu\nu}, \widetilde{X}^a\{F^{\mu\nu}, e^{-1}\}_\star)\\ & +2S(\Delta)(F^{\mu\nu}, \widetilde{X}^a\delta F^{\mu\nu}\star e^{-1})\Big)\Big]\end{aligned} \tag{3.124}$$

where $[\delta A_{[\mu}, A_{\nu]}]_\star = \delta A_\mu \star A_\nu - A_\nu \star \delta A_\mu - \delta A_\nu \star A_\mu + A_\mu \star \delta A_\nu$. Using the property that $F^{\mu\nu} = -F^{\nu\mu}$ and the fact that the surface terms are canceled, the equation of motion $\mathcal{E}_A = 0$ and the current J^β can be re-expressed, respectively, as

$$\begin{aligned}\frac{\delta \mathcal{S}_{YM}}{\delta A_\beta} = \mathcal{E}_A \quad &= \quad -2\partial_\mu(e\{F^{\mu\beta}, e^{-1}\}_\star) - 4\omega^a_{ac}\Theta^{c\rho}\partial_\rho(eA_\mu\{F^{\mu\beta}, e^{-1}\}_\star) \\ &\quad -4e\omega^a_{ac}\Theta^{c\rho}\partial_\rho A_\mu\{F^{\mu\beta}, e^{-1}\}_\star + 4e\omega^a_{a\mu}\{F^{\mu\beta}, e^{-1}\}_\star = 0 \qquad (3.125)\end{aligned}$$

and

$$J^\beta = \frac{1}{2\kappa^2}\Big(e\delta A_\mu\{F^{\mu\beta}, e^{-1}\}_\star + \omega^a_{ac}\Theta^{c\beta} e\delta A_{[\mu}A_{\nu]}\{F^{\mu\nu}, e^{-1}\}_\star\Big). \qquad (3.126)$$

Let us now deal with the symmetry analysis and deduce the conserved currents. Performing the following functional variation of fields and coordinate transformation

$$A'_\mu(x) = A_\mu(x) + \delta A_\mu(x), \qquad x'^\mu = x^\mu + \epsilon^\mu, \quad \epsilon^\mu = \delta x^\mu = 0 \qquad (3.127)$$

and using $\mathsf{d}^2 x' = [1 + \partial_\mu\epsilon^\mu + \mathbf{O}(\epsilon^2)]\mathsf{d}^2 x = \mathsf{d}^2 x$ lead to the following variation of the action, to first order in $\delta A_\mu(x)$ and $\delta\phi^c(x)$:

$$\begin{aligned}\delta\mathcal{S}_{YM} \quad &= \quad \int e\mathsf{d}^2 x\Big\{\Big|\frac{\partial x'}{\partial x}\Big| \star (\mathcal{L}'_{YM} \star e^{-1})\Big\} - \int e\mathsf{d}^2 x\,(\mathcal{L}_{YM} \star e^{-1}) \\ &= \quad \int \mathsf{d}^2 x\;\delta\Big((\mathcal{L}_{YM} \star e^{-1})e\Big) = \int \mathsf{d}^2 x\,\Big\{\delta_{A_\mu}\Big((\mathcal{L}_{YM} \star e^{-1})e\Big)\Big\} \qquad (3.128)\end{aligned}$$

where

$$\mathcal{L}_{YM} = -\frac{1}{4\kappa^2}F^{\mu\nu} \star F_{\mu\nu} \quad \text{and} \quad \mathcal{L}'_{YM} = -\frac{1}{4\kappa^2}F^{\mu\nu}_U \star F^U_{\mu\nu}. \qquad (3.129)$$

On shell, and integrated on a submanifold $M \subset \mathbb{R}^2$ with fields non vanishing at the boundary (so that the total derivative terms do not disappear), we get:

$$\delta\mathcal{S}_{YM} = \int_M \mathsf{d}^2 x\;\partial_\sigma\mathcal{J}^\sigma = 0. \qquad (3.130)$$

Proposition 3.5 *The noncommutative Noether current $\mathcal{J}^\sigma$ is locally conserved.*

Proof: The analysis of the local properties of this tensor requires the useful formulas

$$\delta_\alpha A_\mu = \epsilon_\mu(1 + \Theta^{\rho\sigma}\partial_\rho A_\sigma), \;\; \omega^a_{ac} = -\omega_c, \;\; \partial_\beta e = -\omega_\beta, \;\; \{F^{\mu\nu}, e^{-1}\}_\star = 2e^{-1}F^{\mu\nu}. \qquad (3.131)$$

A straightforward computation gives

$$J^\beta = \frac{\epsilon_\mu}{\kappa^2}\Big(1 + \Theta^{\rho\sigma}\partial_\rho A_\sigma\Big)F^{\mu\beta} \Rightarrow \partial_\beta J^\beta = \frac{\epsilon_\mu}{\kappa^2}\Big(1 + \Theta^{\rho\sigma}\partial_\rho A_\sigma\Big)\partial_\beta F^{\mu\beta}. \qquad (3.132)$$

The equation of motion (3.125) can be simply re-expressed in the form

$$\partial_\mu F^{\mu\beta} = 2\omega_\mu F^{\mu\beta} - 4\omega_c\Theta^{c\rho}(\partial_\rho A_\mu)F^{\mu\beta} - 2\omega_c\Theta^{c\rho}A_\mu\partial_\rho F^{\mu\beta}. \qquad (3.133)$$

Now using the fact that $\epsilon\omega = 0$ yields the result. □

Remark 3.6 .

- *The equation of motion (3.133) is reduced to $\partial_\mu F^{\mu\beta} = 0$ in ordinary Moyal plane.*
- *The action of gauge theory covariantly coupled with the matter fields defined by*

$$\mathcal{S} = \mathcal{S}_{YM} + \mathcal{S}_M \tag{3.134}$$

where

$$\begin{aligned}\mathcal{S}_M &= \int_{\mathbb{R}^2} ed^2x\Big[\bar{\psi}(x)\Big(-i\Gamma^\mu\nabla_\mu + m\Big)\psi(x) + \lambda_1(\bar{\psi}\star\psi\star\bar{\psi}\star\psi)(x) \\ &\quad +\lambda_2(\bar{\psi}\star\bar{\psi}\star\psi\star\psi)(x)\Big]\star e^{-1},\end{aligned} \tag{3.135}$$

is also invariant under global gauge transformation ($\delta\psi = i\alpha_0\psi,\ \ \delta\bar{\psi} = -i\alpha_0\bar{\psi}$). The current can be also easily deduced in the same manner as above.

3.3.3 Case of commuting vector fields

Consider the non coordinates base $e^\mu_a = \delta^\mu_a + \omega^\mu_{ab}x^b$ and the symmetric tensor (between the index a and b) ω^μ_{ab}. Then, the twisted star product is naturally associative since

$$[X_a, X_b] = \omega^\mu_{ba}\partial_\mu - \omega^\mu_{ab}\partial_\mu = 0. \tag{3.136}$$

The matrix representation of e^μ_a is given by

$$(e)^\mu_a = \begin{pmatrix} 1+\omega^1_{11}x^1+\omega^1_{12}x^2 & \omega^2_{11}x^1+\omega^2_{12}x^2 \\ \omega^1_{12}x^1+\omega^1_{22}x^2 & 1+\omega^2_{12}x^1+\omega^2_{22}x^2 \end{pmatrix} \tag{3.137}$$

and

$$(e)^a_\mu = \begin{pmatrix} 1-\omega^1_{11}x^1-\omega^1_{12}x^2 & -\omega^2_{11}x^1-\omega^2_{12}x^2 \\ -\omega^1_{12}x^1-\omega^1_{22}x^2 & 1-\omega^2_{12}x^1-\omega^2_{22}x^2 \end{pmatrix}. \tag{3.138}$$

Further,

$$\begin{aligned} e^{-1} &= det(e^\mu_a) = 1 + (\omega^1_{11}+\omega^2_{12})x^1 + (\omega^2_{22}+\omega^1_{12})x^2 \\ e &= det(e^a_\mu) = 1 - (\omega^1_{11}+\omega^2_{12})x^1 - (\omega^2_{22}+\omega^1_{12})x^2., \end{aligned} \tag{3.139}$$

The noncommutative tensor is provided by $(\widetilde{\Theta})^{\mu\nu} = \theta e^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Besides, the matrix e^a_μ can be written as $e^a_\mu = \delta^a_\mu + \omega^{ab}_\mu x_b$, where $\omega^{ab}_\mu = -\omega^\mu_{ab}$. Finally, the solution of the field equation $e^a_\mu = \partial_\mu\phi^a$ is well given by $\phi^a = x^a + \frac{1}{2}\omega^{ab}_\mu x_b\, x^\nu\delta^\mu_\nu$ as deduced in [44]. The ϕ^c variation of the action can be easily computed and the resulting equation of motion is

$$\frac{\delta\mathcal{S}_{YM}}{\delta\phi^c} = \mathcal{E}_{\phi^c,A} = e^{-1}X_a(F_{\mu\nu}\star F^{\mu\nu}) - (X_aF^{\mu\nu})\{F_{\mu\nu}, e^{-1}\}_\star = 0 \tag{3.140}$$

This variation generates the current

$$\mathcal{K}^\beta = -\frac{ee^\beta_b}{4\kappa^2}\Big[(-F_{\mu\nu}\star F^{\mu\nu}\star\delta\phi^b e^{-1}) + T(\Delta)\Big(X_a(F_{\mu\nu}\star F^{\mu\nu}), \widetilde{X}^b(\delta\phi^a e^{-1})\Big)$$

$$-T(\Delta)\Big(\delta\phi^a(X_aF_{\mu\nu}),\widetilde{X}^b\{F^{\mu\nu},e^{-1}\}_\star\Big)+\delta\phi^b(F_{\mu\nu}\star F^{\mu\nu}\star e^{-1})\Big]$$
$$+2S(\Delta)\Big(\delta\phi^a(X_aF_{\mu\nu})\star e^{-1},\widetilde{X}^bF^{\mu\nu}\Big). \tag{3.141}$$

Performing the transformation $\phi'^c(x)=\phi^c(x)+\delta\phi^c(x)$ where $\delta\phi^c(x)=i\alpha\star\phi^c(x)$, with $\alpha=\alpha_0$ or $\alpha=\alpha_1$, the variation of the action yields the result:

$$\begin{aligned}\delta\mathcal{S}_{YM} &= \int \mathsf{d}^2x\;\delta\Big((\mathcal{L}_{YM}\star e^{-1})e\Big)\\ &= \int \mathsf{d}^2x\,\Big\{\delta_{A_\mu}\Big((\mathcal{L}_{YM}\star e^{-1})e\Big)+\delta_{\phi^c}\Big((\mathcal{L}_{YM}\star e^{-1})e\Big)\Big\}\\ &= \int_M \mathsf{d}^2x\,\partial_\sigma\Big(\mathcal{J}^\sigma+\mathcal{K}^\sigma\Big)=0 \end{aligned}\tag{3.142}$$

Then $\mathcal{J}^\sigma$ can be computed in the same way as for symmetric ω^μ_{ab}. See relation (3.124). The gauge invariance of the YM action furnishes the current $\mathcal{J}^{'\sigma}=\mathcal{J}^\sigma+\mathcal{K}^\sigma$. Under vanishing condition of the surface terms, $\mathcal{J}^{'\sigma}$ is locally conserved on shell.

Noncommutative Solvable Quantum Models

Quantum mechanics, and its extension, quantum field theory, are counted amongst the greatest scientific advances of the 20th century. This is not only due to the unprecedented success of quantum theory in explaining and predicting experimental results, but also maybe more than anything, quantum theory represented a conceptual revolution, challenging the deterministic paradigm of classical physics. In quantum mechanics, a system is allowed to be in a superposition of different states simultaneously, and only upon the intervention of measurement is it forced to take on a definite state. Any prediction of experimental outcomes is therefore statistical in nature.

Despite its accomplishments, quantum mechanics appears to have one obvious limitation: it is formulated exclusively to describe point particles, regarded as entities without spatial extent or structure. Of course, any particle looks like a point from sufficiently far, and at low densities we would expect finite-size effects to play a minor role. However, at high densities and energies we would expect such effects to be significant.

In this chapter, we first review the formalism of noncommutative quantum mechanics, where noncommutative quantum mechanics is formulated as a quantum system on the Hilbert space of Hilbert-Schmidt operators acting on classical configuration space. Then we investigate some relevant solvable models.

4.1 Relation between configuration space and Fock space

Let us specialize to two-dimensional spacetime with NC coordinates $\hat{x}^1, \hat{x}^2$. In this case the canonical commutation relation can be written as $[\hat{x}^1, \hat{x}^2] = i\Theta$ or defining $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{a} = \frac{\hat{x}^1 + i\hat{x}^2}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{\hat{x}^1 - i\hat{x}^2}{\sqrt{2}} \quad \text{or} \quad \hat{b} = \frac{\hat{a}}{\sqrt{\Theta}}, \quad \hat{b}^\dagger = \frac{\hat{a}^\dagger}{\sqrt{\Theta}}. \tag{4.1}$$

Then the operator $\hat{a}$ and $\hat{a}^\dagger$ realize the algebra of annihilation and creation operators usually introduced in the process of second quantization. One can then consider a Fock space with a basis $|n>$ $(n \in \mathbb{N})$ provided by the eigenfunctions of the number operator $\hat{N} = \hat{b}^\dagger \hat{b}$ such that $\hat{N}|n>= n|n>$, with

$$\hat{b}|n>= \sqrt{n}|n-1>, \quad \hat{b}^\dagger|n>= \sqrt{n+1}|n+1> \ \text{ or} \tag{4.2}$$

$$\hat{a}|n>= \sqrt{\Theta n}|n-1>, \quad \hat{a}^\dagger|n>= \sqrt{\Theta(n+1)}|n+1> \tag{4.3}$$

and the vacuum state $|0>$ defined so that $\hat{a}|0>=0=\hat{b}|0>$. For $\Theta \simeq 0$ (not equal to 0), one can write

$$\hat{N}=\frac{(\hat{x}^1)^2+(\hat{x}^2)^2}{2\Theta}=\frac{\hat{r}^2}{2\Theta} \tag{4.4}$$

so that configuration space at infinity can be connected with $n\to\infty$ in Fock space. Let us now derive a precise connection, known as the Weyl connection, between the Moyal product of fields in configuration space and the product of operators in Fock space. To this end, we consider a field $f(x^1,x^2)$ in configuration space and take its Fourier transform

$$\tilde{f}(k,\bar{k})=\int d^2x\, f(x^1,x^2)\, e^{i(x^1k_1+x^2k_2)}. \tag{4.5}$$

Then, define an operator, acting in Fock space, associated to f, by

$$W_f(\hat{b},\hat{b}^\dagger)=\frac{1}{(2\pi\sqrt{\Theta})^2}\int d^2k\, \tilde{f}(k,\bar{k})e^{-i(\bar{k}\hat{b}+k\hat{b}^\dagger)} \tag{4.6}$$

In term of the complex coordinates at this point, one can already verify that

$$\int d^2z\, f(z,\bar{z})=2\pi\Theta\, trW_f \quad z=\frac{x^1+ix^2}{\sqrt{2}} \tag{4.7}$$

and tr means the Fock space trace of operator W_f. To see this, we start from (4.6) and write

$$<n|W_f|n>=\frac{1}{4\pi^2\Theta}\int d^2k\, \tilde{f}(k,\bar{k})<n|e^{-i(\bar{k}\hat{b}+k\hat{b}^\dagger)}|n>=W_{nn} \tag{4.8}$$

which, using the Baker-Campbell-Haussdorf formula, can be arranged as

$$W_{nn}=\frac{1}{4\pi^2\Theta}\int d^2k\, \tilde{f}(k,\bar{k})e^{\frac{k\bar{k}}{2}}<n|e^{-ik\hat{b}^\dagger}\, e^{-i\bar{k}\hat{b}}|n>. \tag{4.9}$$

Then, use the Schwinger formula

$$<n|e^{-ik\hat{b}^\dagger}\, e^{-i\bar{k}\hat{b}}|n>=L_n(k\bar{k}). \tag{4.10}$$

A derivation of this formulae, as well as a number of the resulting properties of the Laguerre polynomials, is given in [14]. One gets

$$trW_f=\sum_n W_{nn}=\frac{1}{4\pi^2\Theta}\int d^2k\, \tilde{f}(k,\bar{k})e^{\frac{k\bar{k}}{2}}\sum_n L_n(k\bar{k}). \tag{4.11}$$

Then using the identity

$$\sum_n t^nL_n(x)=\frac{1}{1-t}e^{\frac{-tx}{1-t}} \quad \sum_n L_n(k\bar{k})=2\pi\delta(k)\delta(\bar{k}) \tag{4.12}$$

leads to

$$trW_f=\frac{1}{2\pi\Theta}\tilde{f}(0,0)=\frac{1}{2\pi\Theta}\int d^2zf(z,\bar{z}). \tag{4.13}$$

The natural basis to use in Fock space in order to expand operators W consists of the elementary operators $|m><n|$,

$$W_f = \sum_{mn} (W_f)_{mn} |m><n|. \tag{4.14}$$

The basis operators can be in turn expressed in terms of $\hat{b}^\dagger$ and $\hat{b}$ in the form

$$|m><n| =: \frac{\hat{b}^{\dagger m}}{\sqrt{m!}} e^{-\hat{b}^\dagger \hat{b}} \frac{\hat{b}^n}{\sqrt{n!}} : \tag{4.15}$$

where : : denotes normal ordering. That identity (4.15) holds can be seen just by verifying that, when acting on kets $|p>$'s and on bras $|q>$'s, both sides give the same answer. Expression (4.6) gives a symmetric ordered operator. We can write an analogous formula but for a normal ordered operator just by using the Baker-Campbell-Hausdorff relation. One has, starting from (4.6)

$$\begin{aligned} : W_f(\hat{b}, \hat{b}^\dagger) : &= \frac{1}{4\pi^2\Theta} \int d^2k \tilde{f}_N(k, \bar{k}) : e^{-i(\bar{k}\hat{b} + k\hat{b}^\dagger)} : \\ &= \frac{1}{4\pi^2\Theta} \int d^2k \tilde{f}_N(k, \bar{k}) e^{-i(\bar{k}\hat{b} + k\hat{b}^\dagger)} e^{k^2/4}. \end{aligned} \tag{4.16}$$

Note that we use the subscipt N associated to the normal-ordered expression. Consider the operator $W_n = |n><n|$. For simplicity, we temporarily put $\Theta = 1$. Using representation (4.15), we have

$$W_n =: \frac{\hat{b}^{\dagger n}}{\sqrt{n!}} e^{-\hat{b}^\dagger \hat{b}} \frac{\hat{b}^n}{\sqrt{n!}} := \frac{1}{4\pi^2} \int d^2k \, \tilde{g}^n_N(k, \bar{k}) : e^{-i(\bar{k}\hat{b} + k\hat{b}^\dagger)} : \tag{4.17}$$

with

$$\tilde{g}^n_N(k, \bar{k}) = \frac{1}{n!} \int d^2z \, e^{i(k\bar{z} + \bar{k}z)} \bar{z}^n e^{-\bar{z}z} z^n. \tag{4.18}$$

We can use at this point an integral representation for the Laguerre polynomials. See the very useful book on coherent states by A. Perelemov, or that recently published by J-P. Gazeau. In fact, one has:

$$L_n(k^2/2) = \frac{1}{2\pi n!} e^{k^2/2} \int d^2z \, |z|^{2n} e^{-i(\bar{k}\hat{b} + k\hat{b}^\dagger)} \tag{4.19}$$

and use the second line in the equation (4.16) to write

$$|n><n| = \frac{1}{2\pi} \int d^2k \, e^{-k^2/4} L_n(k^2/2) e^{-i(\bar{k}\hat{b} + k\hat{b}^\dagger)}. \tag{4.20}$$

The function $g^n(x)$ that corresponds to the operator W_n can be copied from (4.20)

$$\begin{aligned} g^n(x) &= \frac{1}{2\pi} \int d^2k \, e^{-k^2/4} L_n(k^2/2) e^{-ikx} \\ &= 2(-1)^n e^{-r^2} L_n(2r^2) \end{aligned} \tag{4.21}$$

where $r^2 = (x^1)^2 + (x^2)^2$. Re-introducing Θ we then have the connection

$$|n><n| \to 2(-1)^n e^{-r^2/\Theta} L_n(2r^2/\Theta). \tag{4.22}$$

Now, we are ready to present the most significant formula in this section,

$$\underbrace{W_f W_g}_{\text{operator product}} = W \underbrace{f \star g}_{\text{star product}} . \tag{4.23}$$

It shows that the star product of fields in configuration space as defined in (1.1) becomes a simple operator product in Fock space. In this way, one can either work using Moyal products or operator products and pass from one language to the other just by Weyl (anti)transforming the results. In order to prove (4.23) we start from the l.h.s. and use (4.16) to write

$$\begin{aligned} W_f.W_g &= \frac{1}{16\pi^4\Theta}\int d^2k \int d^2k'\, \tilde{f}(k,\bar{k})\tilde{g}(k',\bar{k}')e^{-\frac{(k^2+k'^2)}{4}}e^{-i(\bar{k}\hat{b}+k\hat{b}^\dagger)}e^{-i(\bar{k}'\hat{b}+k'\hat{b}^\dagger)} \\ &= \frac{1}{16\pi^4\Theta}\int d^2k \int d^2k'\, \tilde{f}(k,\bar{k})\tilde{g}(k',\bar{k}')e^{-\frac{(k^2+k'^2)}{4}}e^{-i[(\bar{k}+\bar{k}')\hat{b}+(k+k')\hat{b}^\dagger]} \\ &\quad \times e^{\frac{i}{2}(k\bar{k}'-k'\bar{k})}. \end{aligned} \tag{4.24}$$

We now proceed to the change of variable $k+k'=p$, $(k-k')/2=q$ and analogously for "bar" variables. Then (4.24) becomes

$$\begin{aligned} W_f.W_g &= \frac{1}{4\pi^2}\int d^2p\, e^{-i(\bar{p}\hat{b}+p\hat{b}^\dagger)}\Big[\frac{1}{4\pi^2}\int d^2q\, e^{\frac{i}{2}(q\bar{p}-p\bar{q})}\,\tilde{f}(q+p/2,\bar{q}+\bar{p}/2) \\ &\quad \times\tilde{g}(-q+p/2,-\bar{q}+\bar{p}/2)\Big] \end{aligned} \tag{4.25}$$

Now, the factor in square brackets is nothing but the Fourier transform of $f \star g$ for NC $\mathbb{R}^2$ (and with $\Theta = 1$). Hence, we end with

$$W_f.W_g = \frac{1}{4\pi^2}\int d^2p\, e^{-i(\bar{p}\hat{b}+p\hat{b}^\dagger)}\widetilde{f\star g}(p,\bar{p}) = W_{f\star g}. \tag{4.26}$$

So, we have established the announced connection between operator multiplication and the star product of functions.

It is easy to identify the operation that corresponds to differentiation in Fock space. Indeed, starting from

$$[\hat{b}^\dagger, \hat{b}^n] = -n\hat{b}^{n-1} \tag{4.27}$$

we see that, for any holomorphic function $f(\hat{b})$ written in the form

$$f(\hat{b}) = \sum c_n\hat{b}^n, \tag{4.28}$$

one can define a differentiation operation through the formula

$$\frac{\partial f}{\partial\hat{b}} = -[\hat{b}^\dagger, f(\hat{b})] \tag{4.29}$$

and analogously for any $f(\hat{b}^\dagger)$. Then, differentiation of a field $\phi(z,\bar{z})$ becomes, in operator language,

$$\partial_z\phi(z,\bar{z}) \to -\frac{1}{\sqrt{\Theta}}[\hat{b}^\dagger, W_\phi], \qquad \partial_{\bar{z}}\phi(z,\bar{z}) \to -\frac{1}{\sqrt{\Theta}}[\hat{b}, W_\phi] \tag{4.30}$$

Then the action of NCFT can be written in operator language. Note that for $z = re^{i\varphi}$

$$|0><0| = 2e^{-r^2/\Theta} \tag{4.31}$$

Using the formula

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n+1><n| = \frac{\sqrt{2\Theta}}{r}\Big(1-e^{-r^2/\Theta}\Big)e^{-i\varphi} \tag{4.32}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} |n><n+1| = \frac{\sqrt{2\Theta}}{r}\Big(1-e^{-r^2/\Theta}\Big)e^{i\varphi} \tag{4.33}$$

we can deduce:

$$|n><n+l| = 2(-1)^n \sqrt{\frac{n!}{(n+l)!}} \Big(\frac{2r^2}{\Theta}\Big)^{l/2} e^{-r^2/\Theta} e^{il\varphi} L_n^l\Big(2r^2/\Theta\Big). \tag{4.34}$$

In the commutative limit, we find

$$\frac{1}{\Theta} e^{-r^2/\Theta} \overset{\Theta\to 0}{\widehat{\longrightarrow}} \pi\delta(x^1)\delta(x^2) \tag{4.35}$$

4.1.1 Harmonic oscillator basis on Moyal space: Algebraic formulation

Following the matrix base method [34], we represent the elements of the $D-$dimensional Moyal algebra $\mathcal{M}$ in a matrix base. Let $b^{(D)}_{kl}(x)$ be eigenfunctions of the harmonic oscillator

$$H = \sum_{l=1}^{\frac{D}{2}} \frac{1}{2}\Big(x^2_{2l-1} + x^2_{2l}\Big), \text{ for } l = 0, 1 \cdots \frac{D}{2}. \tag{4.36}$$

Properties 4.1 *Let* $b^{(D)}_{00} = 2^{D/2}e^{-2H/\Theta}$, *then we have the property* $b^{(D)}_{00} \star b^{(D)}_{00} = b^{(D)}_{00}$.

Prove: To prove this formula, let us write the Moyal star product as

$$(f \star g)(x) = \int d^D y \frac{d^D k}{(2\pi)^D}\ f(x + \frac{1}{2}\Theta k) g(x+y) e^{iky} \tag{4.37}$$

and take $D = 2$. Then $(\Theta) = \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix}$ and $b^{(2)}_{00} = 2e^{\frac{-|x|^2}{\Theta}}$; $|x|^2 = (x^1)^2 + (x^2)^2$.

$$b^{(2)}_{00} \star b^{(2)}_{00} = 4 \int d^2 y \frac{d^2 k}{(2\pi)^2}\ e^{-\frac{1}{\Theta}|x+\frac{1}{2}\Theta k|^2} e^{-\frac{1}{\Theta}|x+y|^2} e^{iky}$$

$$|x + \frac{1}{2}\Theta k|^2 = (x^1 + \frac{1}{2}\Theta k_2)^2 + (x^2 - \frac{1}{2}\Theta k_1)^2 = |x|^2 + \frac{1}{4}\Theta^2|k|^2 + \Theta x \wedge k$$

where $x \wedge k = x^1 k_2 - x^2 k_1$. We have also

$$|x+y|^2 = |x|^2 + |y|^2 + 2xy$$

Then

$$b_{00}^{(2)} \star b_{00}^{(2)} = \frac{4e^{\frac{-2|x|^2}{\Theta}}}{(2\pi)^2} \int d^2k d^2y \; e^{-\frac{1}{4}\Theta|k|^2 - x\wedge k - \frac{1}{\Theta}|y|^2 - \frac{2}{\Theta}xy + iky}.$$

Remark that

$$-x \wedge k - \frac{2}{\Theta}xy = (y,k)\begin{pmatrix} -\frac{2}{\Theta}x \\ \wedge x \end{pmatrix} = (y,k)b$$

and

$$-\frac{1}{4}\Theta|k|^2 - \frac{1}{\Theta}|y|^2 - iky = -\frac{1}{2}(y,k)\begin{pmatrix} \frac{2}{\Theta} & -i \\ -i & \frac{\Theta}{2} \end{pmatrix}\begin{pmatrix} y \\ k \end{pmatrix}$$

$X = (y,k)$ is a vector on $\mathbb{R}^2_\Theta \times \mathbb{R}^2$. In the same consideration $\begin{pmatrix} \frac{2}{\Theta} & -i \\ -i & \frac{\Theta}{2} \end{pmatrix} = A$ is 4×4 matrix given by $A = \begin{pmatrix} \frac{2}{\Theta}I_2 & -iI_2 \\ -iI_2 & \frac{\Theta}{2}I_2 \end{pmatrix}$. I_2 is 2×2 identity matrix.

$$b_{00}^{(2)} \star b_{00}^{(2)} = \frac{4e^{\frac{-2|x|^2}{\Theta}}}{(2\pi)^2} \int d^4X \; e^{-\frac{1}{2}<X,AX>+<b,X>}.$$

Then using the generalized gaussian integral

$$\int d^Dx \; e^{-\left(\frac{1}{2}<X,AX>+<b,X>+c\right)} = (2\pi)^{D/2} e^{\frac{1}{2}<b,A^{-1}b>-c}(detA)^{-1/2} \tag{4.38}$$

we get

$$b_{00}^{(2)} \star b_{00}^{(2)} = \frac{4e^{\frac{-2|x|^2}{\Theta}}}{(2\pi)^2}(2\pi)^2 e^{\frac{1}{2}<-b,A^{-1}(-b)>}\frac{1}{2}$$

$\frac{1}{2} < -b, A^{-1}(-b) >= \frac{|x|^2}{\Theta} \Rightarrow b_{00}^{(2)} \star b_{00}^{(2)} = b_{00}^{(2)}$. This result can be generalized to D dimensions.□

Definition 4.2 *Defining the annihilation and creation operators as*

$$a_l = \frac{x_{2l-1} + ix_{2l}}{\sqrt{2}}, \text{ and } \bar{a}_l = \frac{x_{2l-1} - ix_{2l}}{\sqrt{2}}, \tag{4.39}$$

the elements of matrix basis on Moyal space are given by

$$b_{kl}^{(D)} = \frac{\bar{a}_\star^k \star b_{00}^{(D)} \star a_\star^l}{\sqrt{k!l!\Theta^{|k|+|l|}}} \tag{4.40}$$

where $a = \sum_{i=1}^{D/2} a_i$*, and* $\bar{a} = \sum_{i=1}^{D/2} \bar{a}_i$.

Properties 4.3 *We have*

$$a \star b_{kl}^{(D)} = \sqrt{|k|\Theta}b_{k-1,l}^{(D)}, \quad b_{kl}^{(D)} \star a = \sqrt{\Theta(|l+1|)}b_{k,l+1}^{(D)}, \tag{4.41}$$

$$\bar{a} \star b_{kl}^{(D)} = \sqrt{\Theta(|k+1|)}b_{k+1,l}^{(D)}, \quad b_{kl}^{(D)} \star \bar{a} = \sqrt{|l|\Theta}b_{k,l-1}^{(D)} \tag{4.42}$$

$$H \star b_{kl}^{(D)} = \Theta(|k| + \frac{1}{2})b_{kl}^{(D)}, \quad b_{kl}^{(D)} \star H = \Theta(|l| + \frac{1}{2})b_{kl}^{(D)}, \tag{4.43}$$

where $k, l \in \mathbb{N}^{D/2}$ *and* $|k| = \sum_{i=1}^{D/2} k_i$.

Prove For simplicity, let us restrict the proof to $D = 2-$dimensional Moyal space. We get

$$f \star a = \Big(a - \frac{\theta}{2}\frac{\partial}{\partial \bar{a}}\Big)f, \quad f \star \bar{a} = \Big(\bar{a} + \frac{\theta}{2}\frac{\partial}{\partial a}\Big)f \tag{4.44}$$

and

$$a \star f = \Big(a + \frac{\theta}{2}\frac{\partial}{\partial \bar{a}}\Big)f, \quad \bar{a} \star f = \Big(\bar{a} - \frac{\theta}{2}\frac{\partial}{\partial a}\Big)f. \tag{4.45}$$

Then

$$\begin{aligned} b^{(2)}_{mn} \star a &= \frac{\bar{a}^m_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} \star a = \frac{\bar{a}^m_\star \star b^{(2)}_{00} \star a^{n+1}_\star}{\sqrt{m!n!\theta^{m+n}}} \\ &= \frac{\bar{a}^m_\star \star b^{(2)}_{00} \star a^{n+1}_\star}{\sqrt{m!(n+1)!\theta^{m+n+1}}}\sqrt{\theta(n+1)} = \sqrt{\theta(n+1)}b^{(2)}_{m,n+1} \end{aligned} \tag{4.46}$$

$$\begin{aligned} \bar{a} \star b^{(2)}_{mn} &= \bar{a} \star \frac{\bar{a}^m_\star \star f_0 \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} = \frac{\bar{a}^{m+1}_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} \\ &= \frac{\bar{a}^{m+1}_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{(m+1)!n!\theta^{m+n+1}}}\sqrt{\theta(m+1)} = \sqrt{\theta(m+1)}b^{(2)}_{m+1,n}. \end{aligned} \tag{4.47}$$

Note that

$$\begin{aligned} a \star b^{(2)}_{mn} &= ab^{(2)}_{mn} + \frac{\theta}{2}\frac{\partial b^{(2)}_{mn}}{\partial \bar{a}} = ab^{(2)}_{mn} + \frac{\theta}{2}\Big[m\frac{\bar{a}^{m-1}_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} + \frac{\bar{a}^m_\star \star (\frac{-2a}{\theta}b^{(2)}_{00}) \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}}\Big] \\ &= ab^{(2)}_{mn} + \frac{m\theta}{2}\frac{\bar{a}^{m-1}_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{(m-1)!n!\theta^{m-1+n}}}\frac{1}{\sqrt{m\theta}} - \frac{\bar{a}^m_\star \star (ab^{(2)}_{00}) \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} \\ &= ab^{(2)}_{mn} + \frac{\sqrt{m\theta}}{2}b^{(2)}_{m-1,n} - \frac{\bar{a}^m_\star \star (ab^{(2)}_{00}) \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}}. \end{aligned} \tag{4.48}$$

We now prove that

$$\frac{\bar{a}^m_\star \star (ab^{(2)}_{00}) \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} = ab^{(2)}_{mn} - \frac{\sqrt{m\theta}}{2}b^{(2)}_{m-1,n}. \tag{4.49}$$

Applying $\bar{a}^m_\star$ to $ab^{(2)}_{00}$ $m = 1, 2, \cdots$, we have the result

$$\bar{a}^m_\star \star (ab^{(2)}_{00}) = 2^m a\bar{a}^m b^{(2)}_{00} - m2^{m-2}\bar{a}^{m-1}\theta b^{(2)}_{00}. \tag{4.50}$$

Remark that $\bar{a}^m b^{(2)}_{00} = \frac{1}{2^m}\bar{a}^m_\star \star b^{(2)}_{00}$. Then $\bar{a}^m_\star \star (ab^{(2)}_{00})$ can be written as

$$\bar{a}^m_\star \star (ab^{(2)}_{00}) = a.\bar{a}^m_\star \star b^{(2)}_{00} - \frac{m\theta}{2}\bar{a}^{m-1}_\star \star b^{(2)}_{00} \tag{4.51}$$

and

$$\frac{\bar{a}^m_\star \star (ab^{(2)}_{00}) \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} = a\frac{\bar{a}^m_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} - \frac{m\theta}{2}\frac{\bar{a}^{m-1}_\star \star b^{(2)}_{00} \star a^n_\star}{\sqrt{m!n!\theta^{m+n}}} = ab^{(2)}_{mn} - \frac{\sqrt{m\theta}}{2}b^{(2)}_{m-1,n}. \tag{4.52}$$

Finally $a \star b^{(2)}_{mn} = \sqrt{m\theta}b^{(2)}_{m-1,n}$.

Anagously,

$$b^{(2)}_{mn}\star\overline{a}=\overline{a}b^{(2)}_{mn}\frac{\sqrt{n\theta}}{2}b^{(2)}_{m,n-1}-\frac{\overline{a}^m_\star\star(\overline{a}b^{(2)}_{00})\star a^n_\star}{\sqrt{m!n!\theta^{m+n}}}. \tag{4.53}$$

In the same vein

$$(\overline{a}b^{(2)}_{00})\star a^n_\star=2^na^n\overline{a}b^{(2)}_{00}-n2^{n-2}a^{n-1}\theta b^{(2)}_{00}. \tag{4.54}$$

Using the identity $b^{(2)}_{00}a^n=\frac{1}{2^n}b^{(2)}_{00}\star a^n_\star$ we have

$$(\overline{a}b^{(2)}_{00})\star a^n_\star=\overline{a}b^{(2)}_{00}\star a^n_\star-\frac{n\theta}{2}b^{(2)}_{00}\star a^{n-1}\Longrightarrow\frac{\overline{a}^m_\star\star(\overline{a}b^{(2)}_{00})\star a^n_\star}{\sqrt{m!n!\theta^{m+n}}}=\overline{a}b^{(2)}_{mn}-\frac{\sqrt{n\theta}}{2}b^{(2)}_{m,n-1}. \tag{4.55}$$

Finally $b^{(2)}_{mn}\star\overline{a}=\sqrt{n\theta}b^{(2)}_{m,n-1}$. □

For $D=2$, $b^{(2)}_{kl}=f_{kl}$ which can be expanded in polar coordinates, ($x_1=rcos(\varphi)$, $x_2=rsin(\varphi)$), to give

$$f_{kl}=2(-1)^k\sqrt{\frac{k!}{l!}}e^{i(l-k)\varphi}\Big(\frac{2r^2}{\Theta}\Big)^{\frac{l-k}{2}}L^{l-k}_k\Big(\frac{2r^2}{\Theta}\Big)e^{-\frac{r^2}{\Theta}} \tag{4.56}$$

where the $L^k_n(x)$ are the associated Laguerre polynomials. The generalization to higher dimensions is straightforward. In particular, for $D=4$, we get $k=(k_1,k_2)$, $l=(l_1,l_2)$ and

$$b^{(4)}_{kl}(x)=f_{k_1,l_1}(x_1,x_2)f_{k_2,l_2}(x_3,x_4).$$

More generally, the following properties are satisfied:

$$(b^{(D)}_{kl}\star b^{(D)}_{k'l'})(x) = \delta_{lk'}b^{(D)}_{kl'}(x), \tag{4.57}$$

$$\int d^Dx\ b^{(D)}_{kl}(x) = (2\pi\Theta)^{D/2}\delta_{kl}, \tag{4.58}$$

$$(b^{(D)}_{kl})^\dagger = b^{(D)}_{lk}. \tag{4.59}$$

The existence of an isomorphism between the unital involutive Moyal algebra and a sub-algebra of the unital involutive algebra of complex infinite-dimensional matrices allows to define, for all $g\in\mathcal{M}$, a unique matrix (g_{kl}) given by

$$g_{kl}=\frac{1}{(2\pi\Theta)^{D/2}}\int d^Dx\ g(x)b^{(D)}_{kl},$$

satisfying

$$\forall x\in\mathbb{R}^D\quad g(x)=\sum_{k,l\in\mathbb{N}^{D/2}}g_{kl}\,b^{(D)}_{kl}(x).$$

We have

$$[\tilde{x}_\mu,\phi]_\star=2i\partial_\mu\phi\Rightarrow\partial_\mu\partial^\mu\phi=-\frac{1}{4}[\tilde{x}_\mu,[\tilde{x}_\mu,\phi]_\star]_\star.$$

Setting $\phi(x)=b^{(D)}_{kl}(x)$,

$$\tilde{x}\star\tilde{x}\star b^{(D)}_{kl}=-\frac{8}{\Theta^2}H\star b^{(D)}_{kl}=-\frac{8}{\Theta}(|k|+\frac{1}{2})b^{(D)}_{kl}, \tag{4.60}$$

with

$$H = \sum_{l=1}^{\frac{D}{2}} \frac{1}{2}\Big(x_{2l-1}^2 + x_{2l}^2\Big), \text{ for } l = 0, 1 \cdots \frac{D}{2},$$

and

$$\tilde{x} \star b_{kl}^{(D)} \star \tilde{x} = -\frac{4}{\Theta^2}(a \star b_{kl}^{(D)} \star \bar{a} + \bar{a} \star b_{kl}^{(D)} \star a) \tag{4.61}$$

$$= -\frac{4}{\Theta}\Big(\sqrt{|k||l|}b_{k-1,l-1}^{(D)} + \sqrt{|k+1||l+1|}b_{k+1,l+1}^{(D)}\Big). \tag{4.62}$$

4.2 Harmonic oscillator in twisted Moyal plane

Recently [3], a formulation of dynamical noncommutativity, which allows for a consistent interpretation of position measurement and the solution of the problem of a noncommutative well, has been put forward. In their approach, the authors required that the vector fields X_a commute, ensuring the associativity of the star product (3.1). This work addresses a study of a harmonic oscillator properties in the twisted Moyal plane. The associativity property does not play any specific role in the formulation of a non coordinate base [72] as used here and therefore is not required. Besides, it is worth noticing that such hypothesis made in [3] (consisting to assume the associativity of the star product or, equivalently, the commutativity of the fields X_a), does not change the results of our investigations.

Furthermore, the use of non coordinate base leads to consider a curve geometry, more general and richer than a flat one. This could be also of some importance in the study of quantum gravity. Using appropriate matrix basis and deforming the issue of a twisted product, we solve the resulting eigenvalue problem to find the states and the energy spectrum of the harmonic oscillator Hamiltonian. These states are infinitely degenerate.

As a prelude to the construction of a matrix basis appropriate for this study, let us set up main algebraic relations pertaining to twisted noncommutative coordinate transformations.

4.2.1 Useful relations

We consider the following infinitesimal affine transformation

$$e_a^\mu(x) = \delta_a^\mu + \omega_{ab}^\mu x^b, \quad \omega_{ab}^\mu =: -\omega_{ba}^\mu, \text{ and } |\omega^\mu| << 1. \tag{4.63}$$

In the sequel, we restrict the discussion to $D = 2$, where e_a^μ and Θ^{ab} can be expressed as follows:

$$(e)_a^\mu = \begin{pmatrix} 1 + \omega_{12}^1 x^2 & \omega_{12}^2 x^2 \\ -\omega_{12}^1 x^1 & 1 - \omega_{12}^2 x^1 \end{pmatrix} \quad \text{and} \quad (\Theta)^{ab} = \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix} = \Theta(\epsilon)^{ab} \tag{4.64}$$

where $\epsilon^{12} = -\epsilon^{21} = 1,\ \epsilon^{11} = \epsilon^{22} = 0$. There follow the relations

$$e^{-1} =: det(e_a^\mu) = 1 + \omega_{12}^1 x^2 - \omega_{12}^2 x^1 \tag{4.65}$$

$$e =: det(e_\mu^a) = 1 - \omega_{12}^1 x^2 + \omega_{12}^2 x^1. \tag{4.66}$$

Before going further in the development, let us immediately recall that all results of our investigation remain also valid even when the considered vector fields commute.

The $\star-$product of two Schwartz functions on $\mathbb{R}^2_\Theta$ can be written under the form

$$(f \star g)(x) = \mathrm{m}\Big[\exp\Big(\frac{i}{2}\Theta e^{-1} J^{\mu\nu}\partial_\mu \otimes \partial_\nu\Big)(f \otimes g)(x)\Big] \tag{4.67}$$

where $\mu, \nu = 1, 2$ and $\partial_\mu =: \frac{\partial}{\partial x^\mu}$. Using the twisted star product (4.67) one can see that

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x} e^{-\frac{i}{2}\Theta e^{-1} kJq}. \tag{4.68}$$

The Fourier transform of $f, g \in \mathcal{S}(\mathbb{R}^2_\Theta)$ can be written as

$$\tilde{f}(k) = \int d^2x\ e^{-ikx} f(x), \quad \tilde{g}(q) = \int d^2x\ e^{-iqx} g(x) \tag{4.69}$$

with the function inverse transform given by

$$f(x) = \frac{1}{(2\pi)^2}\int d^2k\ e^{ikx}\tilde{f}(k), \quad g(x) = \frac{1}{(2\pi)^2}\int d^2q\ e^{iqx}\tilde{g}(q). \tag{4.70}$$

We can then redefine the twisted star product of two Schwartz functions f, g as:

$$\begin{aligned}(f \star g)(x) &= \frac{1}{(2\pi)^4}\int d^2kd^2q\ \tilde{f}(k)\tilde{g}(q)e^{ikx} \star e^{iqx} \\ &= \frac{1}{(2\pi)^4}\int d^2kd^2q\ \tilde{f}(k)\tilde{g}(q)e^{i(k+q)x}e^{-\frac{i}{2}\Theta e^{-1}kJq} \\ &= \frac{1}{(2\pi)^4}\int d^2kd^2q\int d^2yd^2z\ f(y)g(z) \\ &\quad \times e^{ik(x-y-\frac{1}{2}\Theta e^{-1}Jq)}e^{iq(x-z)}\end{aligned} \tag{4.71}$$

Using the identity

$$\int d^2k\ e^{ik(x-y-\frac{1}{2}\Theta e^{-1}Jq)} = (2\pi)^2\delta^{(2)}(x - y - \frac{1}{2}\Theta e^{-1}Jq) \tag{4.72}$$

and the variable change q to $q' = \frac{1}{2}\Theta e^{-1}Jq$, we arrive at the adapted form for the proof of the next Proposition 4.4:

$$\begin{aligned}(f \star g)(x) &= \Big(\frac{e}{\pi\Theta}\Big)^2\int d^2yd^2z\ f(y)g(z)e^{-\frac{2ei}{\Theta}(x-y)J(x-z)} \\ &= \Big(\frac{e}{\pi\Theta}\Big)^2\int d^2yd^2z\ f(x-y)g(x-z)e^{\frac{-2ei}{\Theta}yJz} \\ &= \int d^2z\frac{d^2t}{(2\pi)^2}\ f(x - \frac{1}{2}\Theta e^{-1}t)g(x-z)e^{-itJz}.\end{aligned} \tag{4.73}$$

Proposition 4.4 *If f and g are two Schwartz functions on $\mathbb{R}^2_\Theta$, then $f \star g$ is also a Schwartz function on $\mathbb{R}^2_\Theta$.*

Proof: It is immediate by induction on the formula (4.73) using integration by parts. □
The tensor $\widetilde{\Theta}^{\mu\nu}$ can be explicit as

$$(\widetilde{\Theta})^{\mu\nu} = (\Theta)^{\mu\nu} - (\Theta^{a[\mu}\omega^{\nu]}_{ab})x^b = \begin{pmatrix} 0 & \Theta e^{-1} \\ -\Theta e^{-1} & 0 \end{pmatrix}. \tag{4.74}$$

The twisted Moyal product of fields generates some basic properties like the Jacobi identity

$$[x^\mu,[x^\nu,x^\rho]_\star]_\star+[x^\rho,[x^\mu,x^\nu]_\star]_\star+[x^\nu,[x^\rho,x^\mu]_\star]_\star=\Theta^{b\mu}\Theta^{d[\nu}\omega^{\rho]}_{bd}=0 \tag{4.75}$$

conferring a Lie algebra structure to the defined twisted Moyal space, and

$$x^\mu\star f=x^\mu f+\frac{i}{2}\Theta^{ab}e^\mu_a e^\rho_b\partial_\rho f \ \text{ and } \ f\star x^\mu=x^\mu f-\frac{i}{2}\Theta^{ab}e^\mu_a e^\rho_b\partial_\rho f. \tag{4.76}$$

The star brackets (anticommutator and commutator) of x^μ and f can be immediately deduced as follows: $\{x^\mu,f\}_\star=2x^\mu f,\quad [x^\mu,f]_\star=i\Theta^{ab}e^\mu_a e^\rho_b\partial_\rho f$. The relations (4.76) can be detailed for $x^\mu,\ \mu=1,2$ as:

$$x^1\star f=x^1f+\frac{i}{2}\Theta e^{-1}\partial_2 f \qquad f\star x^1=x^1f-\frac{i}{2}\Theta e^{-1}\partial_2 f \tag{4.77}$$

$$x^2\star f=x^2f-\frac{i}{2}\Theta e^{-1}\partial_1 f \qquad f\star x^2=x^2f+\frac{i}{2}\Theta e^{-1}\partial_1 f \tag{4.78}$$

giving rise to the creation and annihilation functions

$$a=\frac{x^1+ix^2}{\sqrt{2}} \qquad \bar a=\frac{x^1-ix^2}{\sqrt{2}} \tag{4.79}$$

with the commutation relation $[a,\bar a]_\star=\Theta e^{-1}$. It then becomes a matter of algebra to use the transformations of the vector fields ∂_1 and ∂_2 into $\partial_a=:\frac{\partial}{\partial a}$ and $\partial_{\bar a}=:\frac{\partial}{\partial \bar a}$ and vice-versa to infer $e^{-1}=1-a\omega-\bar a\bar\omega$ and $e=1+a\omega+\bar a\bar\omega$, where

$$\omega=:\frac{\omega^2_{12}+i\omega^1_{12}}{\sqrt{2}} \quad \text{and} \quad \bar\omega=:\frac{\omega^2_{12}-i\omega^1_{12}}{\sqrt{2}} \tag{4.80}$$

leading to useful relations

$$\frac{\partial e^{-1}}{\partial a}=-\omega,\quad \frac{\partial e^{-1}}{\partial \bar a}=-\bar\omega \text{ and for } k\in\mathbb{Z},\quad \omega e^k=\omega,\quad \bar\omega e^k=\bar\omega. \tag{4.81}$$

Expressing the twisted $\star-$product (4.67) in terms of vectors fields ∂_a and $\partial_{\bar a}$ as

$$\begin{aligned}(f\star g)(a,\bar a) \;=\; \mathrm{m}\Big[&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!(n-k)!}(\frac{1}{2}\Theta e^{-1})^n\\ &\times(\partial_a\otimes\partial_{\bar a})^k(\partial_{\bar a}\otimes\partial_a)^{n-k}(f\otimes g)(a,\bar a)\Big]\end{aligned} \tag{4.82}$$

and using equations (4.77) and (4.78) (or independently (4.82)) yield

$$a\star f=\Big(a+\frac{\Theta e^{-1}}{2}\frac{\partial}{\partial\bar a}\Big)f \qquad \bar a\star f=\Big(\bar a-\frac{\Theta e^{-1}}{2}\frac{\partial}{\partial a}\Big)f \tag{4.83}$$

$$f\star a=\Big(a-\frac{\Theta e^{-1}}{2}\frac{\partial}{\partial\bar a}\Big)f \qquad f\star\bar a=\Big(\bar a+\frac{\Theta e^{-1}}{2}\frac{\partial}{\partial a}\Big)f. \tag{4.84}$$

Provided the above definitions, we can now introduce the notions of right and left harmonic oscillator states denoted by f^R_{m0} and f^L_{0n}, respectively.

4.2.2 The right and left states

Let $f^R_{00} \in L^2(\mathbb{R}^2_\Theta)$ be the ho "*right*" fundamental state such that

$$a \star f^R_{00} =: 0 \quad \text{with} \quad f^R_{00} = 2e^{-\frac{2a\bar{a}}{\Theta e^{-1}}(1-\frac{1}{2}\bar{a}\bar{\omega})}. \tag{4.85}$$

Then, f^R_{00} solves the eigenvalue problem $H\star f^R_{00} = \mathcal{E}^R_{00} f^R_{00}$ with the corresponding right fundamental eigenvalue $\mathcal{E}^R_{00} = \frac{\Theta}{2}(1-2\bar{a}\bar{\omega})$ of the self-adjoint unbounded twisted ho Hamiltonian operator

$$\begin{aligned} H\star(.) &= \; :\bar{a}a\star(.) = \left[\bar{a}\star a + \frac{\Theta e^{-1}}{2}\right]\star(.) = \left[a\star\bar{a} - \frac{\Theta e^{-1}}{2}\right]\star(.) \\ &= \frac{1}{2}\left[(x^1)^2 + (x^2)^2 + \left(i\Theta e^{-1}x^1 - \frac{\Theta^2}{4}\omega^1_{12}\right)\partial_2 \right. \\ &- \left.\left(i\Theta e^{-1}x^2 - \frac{\Theta^2}{4}\omega^2_{12}\right)\partial_1 - \frac{\Theta^2}{4}e^{-2}(\partial_1^2+\partial_2^2)\right] \equiv \frac{1}{2}\mu_1 \end{aligned} \tag{4.86}$$

defined in the domain

$$\mathcal{D}(H\star) = \left\{f \in L^2(\mathbb{R}^2_\Theta) \mid f, f_{x^1}, f_{x^2} \in \mathcal{AC}_{loc}(\mathbb{R}^2_\Theta); \frac{\mu_1}{2} f \in L^2(\mathbb{R}^2_\Theta)\right\}. \tag{4.87}$$

$\mathcal{AC}_{loc}(\mathbb{R}^2_\Theta)$ denotes the set of the locally absolutely continuous functions on $\mathbb{R}^2_\Theta$. Similarly, the fundamental left state $f^L_{00} \in L^2(\mathbb{R}^2_\Theta)$ defined such that

$$f^L_{00}\star\bar{a} =: 0 \quad \text{with} \quad f^L_{00} = 2e^{-\frac{2a\bar{a}}{\Theta e^{-1}}(1-\frac{1}{2}a\omega)} \tag{4.88}$$

solves the eigenvalue problem $f^L_{00}\star H = \mathcal{E}^L_{00} f^L_{00}$ with the left fundamental eigenvalue $\mathcal{E}^L_{00} = \frac{\Theta}{2}(1-2a\omega)$ of the self-adjoint unbounded twisted ho Hamiltonian operator

$$\begin{aligned} (.)\star H &= \; :(.)\star\bar{a}a = (.)\star\left[\bar{a}\star a + \frac{\Theta e^{-1}}{2}\right] = (.)\star\left[a\star\bar{a} - \frac{\Theta e^{-1}}{2}\right] \\ &= \frac{1}{2}\left[(x^1)^2 + (x^2)^2 - \left(i\Theta e^{-1}x^1 - \frac{\Theta^2}{4}\omega^1_{12}\right)\partial_2 \right. \\ &+ \left.\left(i\Theta e^{-1}x^2 - \frac{\Theta^2}{4}\omega^2_{12}\right)\partial_1 - \frac{\Theta^2}{4}e^{-2}(\partial_1^2+\partial_2^2)\right] \equiv \frac{1}{2}\mu_2 \end{aligned} \tag{4.89}$$

defined in the domain

$$\mathcal{D}(\star H) = \left\{f \in L^2(\mathbb{R}^2_\Theta) \mid f, f_{x^1}, f_{x^2} \in \mathcal{AC}_{loc}(\mathbb{R}^2_\Theta); \frac{\mu_2}{2} f \in L^2(\mathbb{R}^2_\Theta)\right\}. \tag{4.90}$$

Then, the other states follow from the next statement.

Proposition 4.5 *The vectors $f^R_{m0} \in L^2(\mathbb{R}^2_\Theta)$ given for any $m \in \mathbb{N}$ by*

$$f^R_{m0} = \frac{1}{\sqrt{m!\Theta^m}}\left[2^m\bar{a}^m(1 + \frac{ma\omega}{2} - \frac{m\bar{a}\bar{\omega}}{4}) - \frac{U_m\Theta\omega\bar{a}^{m-1}}{2}\right]f^R_{00} \tag{4.91}$$

solve the eigenvalue problem $H\star f^R_{m0} = \mathcal{E}^R_{m0} f^R_{m0}$ with

$$\mathcal{E}^R_{m0} = \frac{\Theta}{2}\left[2m+1-ma\omega-(3m+2)\bar{a}\bar{\omega} - \frac{m^2\Theta\omega}{4\bar{a}} + \frac{\Theta\omega U_m}{2^m\bar{a}}\right],\; m\in\mathbb{N} \tag{4.92}$$

where

$$U_m = (m-1)2^{m-2} + \sum_{k=0}^{m-3}(k+1)2^{k+1}, \quad m\geq 3, \qquad U_{i\leq 1} = 0,\, U_2 = 1. \tag{4.93}$$

Proof: The results are immediate by induction, performing similar analysis as in [34] to construct the right states f^R_{m0} such that $\bar{a}\star f^R_{m0}=\sqrt{\Theta(m+1)}f^R_{m+1,0}$. The identity

$$\partial^k_a f^R_{00}=\Big[k(k-1)\omega\Big(-\frac{2\bar{a}}{\Theta}\Big)^{k-1}+\Big(-\frac{2\bar{a}}{\Theta e^{-1}}\Big)^k(1+ka\omega-\frac{k\bar{a}\bar{\omega}}{2})\Big]f^R_{00}, \tag{4.94}$$

$$\partial^k_{\bar{a}} f^R_{00}=\Big[\frac{k(k-1)}{2}\bar{\omega}\Big(-\frac{2a}{\Theta}\Big)^{k-1}+\Big(-\frac{2a}{\Theta e^{-1}}\Big)^k\Big]f^R_{00} \tag{4.95}$$

$$\begin{aligned}\partial^k_a\Big(-\frac{2a}{\Theta e^{-1}}\Big)^l &= kl\omega\frac{l!}{(l-k+1)!}\Big(-\frac{2}{\Theta}\Big)^{k-1}\Big(-\frac{2a}{\Theta}\Big)^{l-k+1}\\ &+\frac{l!}{(l-k)!}\Big(-\frac{2}{\Theta}\Big)^{k}\Big(-\frac{2a}{\Theta}\Big)^{l-k}e^l\end{aligned} \tag{4.96}$$

$$\begin{aligned}\partial^k_{\bar{a}}\Big(-\frac{2\bar{a}}{\Theta e^{-1}}\Big)^l &= kl\bar{\omega}\frac{l!}{(l-k+1)!}\Big(-\frac{2}{\Theta}\Big)^{k-1}\Big(-\frac{2\bar{a}}{\Theta}\Big)^{l-k+1}\\ &+\frac{l!}{(l-k)!}\Big(-\frac{2}{\Theta}\Big)^{k}\Big(-\frac{2\bar{a}}{\Theta}\Big)^{l-k}e^l.\end{aligned} \tag{4.97}$$

are useful. □

Similarly, the study of the ho left states provides the following result.

Proposition 4.6 *The vectors $f^L_{0n}\in L^2(\mathbb{R}^2_\Theta)$ given for any $n\in\mathbb{N}$ by*

$$f^L_{0n}=\frac{1}{\sqrt{n!\Theta^n}}\Big[2^na^n(1+\frac{n\bar{a}\bar{\omega}}{2}-\frac{na\omega}{4})-\frac{U_n\Theta\bar{\omega}a^{n-1}}{2}\Big]f^L_{00} \tag{4.98}$$

solve the eigenvalue problem $f^L_{0n}\star H=\mathcal{E}^L_{0n}f^L_{0n}$ with

$$\mathcal{E}^L_{0n}=\frac{\Theta}{2}\Big[2n+1-n\bar{a}\bar{\omega}-(3n+2)a\omega-\frac{n^2\Theta\bar{\omega}}{4a}+\frac{\Theta\bar{\omega}U_n}{2^na}\Big],\ n\in\mathbb{N}. \tag{4.99}$$

Proof: It uses the same procedure as previously, but with the construction of the left states f^L_{0n} such that $f^L_{0n}\star a=\sqrt{\Theta(n+1)}f^L_{0,n+1}$. The identity

$$\partial^k_{\bar{a}} f^L_{00}=\Big[k(k-1)\bar{\omega}\Big(-\frac{2a}{\Theta}\Big)^{k-1}+\Big(-\frac{2a}{\Theta e^{-1}}\Big)^k(1+k\bar{a}\bar{\omega}-\frac{ka\omega}{2})\Big]f^L_{00}, \tag{4.100}$$

$$\partial^k_a f^L_{00}=\Big[\frac{k(k-1)}{2}\omega\Big(-\frac{2\bar{a}}{\Theta}\Big)^{k-1}+\Big(-\frac{2\bar{a}}{\Theta e^{-1}}\Big)^k\Big]f^L_{00}. \tag{4.101}$$

are also useful. □

Besides $\lim_{\omega,\bar{\omega}\to 0}\mathcal{E}^R_{m0}=\Theta\Big(m+\frac{1}{2}\Big)$ and $\lim_{\omega,\bar{\omega}\to 0}\mathcal{E}^L_{0n}=\Theta\Big(n+\frac{1}{2}\Big)$ corresponding to the usual Moyal $\star-$product spectrum of the harmonic oscillator Hamiltonian H.

All these results show that the ho right and left states as well as their respective energy spectrum are expressible in terms of the space deformation constant Θ and of an additional piece inherent to the nature of the induced infinitesimal transformation through the parameter ω and its conjugate. Besides, a noteworthy feature of these states is the following.

Proposition 4.7 *The right and left fundamental states defined by* $f_{00}^{(m)R} =: a^{m+1}\star f_{m0}^R$ *and* $f_{00}^{(n)L} =: f_{0n}^L \star \bar{a}^{n+1}$ *are given by the following expressions:*

$$f_{00}^{(m)R} = -\frac{\sqrt{m!\Theta^{m+2}}}{8}\sum_{j=1}^{m}\frac{(m+4j+1)}{(m-j)!}\bar{\omega}f_{00}^R \quad \text{and} \tag{4.102}$$

$$\begin{aligned} f_{00}^{(n)L} &= f_{00}^{(n)R}(\bar{a}\leftrightarrow a, \bar{\omega}\leftrightarrow\omega) \\ &= -\frac{\sqrt{n!\Theta^{n+2}}}{8}\sum_{j=1}^{n}\frac{(n+4j+1)}{(n-j)!}\omega f_{00}^L \end{aligned} \tag{4.103}$$

which, in the usually Moyal product case, are reduced to 0*. Besides, the twisted harmonic oscillator states* f_{m0}^R *and* f_{m0}^L *are degenerate with respect to the rules*

$$a^{m+2}\star f_{m0}^R = 0 \;\; \text{and} \;\; f_{0m}^L \star \bar{a}^{m+2} = 0 \;\; \forall\, m \geq 1. \tag{4.104}$$

Prove To prove the proposition we derive the right and left $\star-$actions of the creation and annihilation functions onto the ho states.

$$\begin{aligned}
a\star f_{m0}^R &= \frac{m2^{m-1}\bar{a}^{m-1}}{\sqrt{m!\Theta^{m-2}}}(1+\frac{m-2}{2}a\omega - \frac{m+5}{4}\bar{a}\bar{\omega})f_{00}^R \\
&- \frac{(m-1)\Theta\omega U_m\bar{a}^{m-2}}{4\sqrt{m!\Theta^{m-2}}}f_{00}^R \\
a^2\star f_{m0}^R &= \frac{m(m-1)2^{m-2}\bar{a}^{m-2}}{\sqrt{m!\Theta^{m-4}}}\Big[1+\frac{m-4}{2}a\omega \\
&- \frac{(m+9)(m-1)+m+5}{4(m-1)}\bar{a}\bar{\omega}\Big]f_{00}^R \\
&- \frac{(m-1)(m-2)\Theta\omega U_m\bar{a}^{m-3}}{8\sqrt{m!\Theta^{m-4}}}f_{00}^R \\
&\vdots \\
a^k\star f_{m0}^R &= \frac{m(m-1)\cdots(m-k+1)2^{m-k}\bar{a}^{m-k}}{\sqrt{m!\Theta^{m-2k}}}\Big[1+\frac{m-2k}{2}a\omega \\
&- \frac{\bar{a}\bar{\omega}}{4}(m-k)!\sum_{j=1}^{k}\frac{(m+4j+1)}{(m-j)!}\Big]f_{00}^R \\
&- \frac{(m-1)(m-2)\cdots(m-k)\Theta\omega U_m\bar{a}^{m-k-1}}{2^{k+1}\sqrt{m!\Theta^{m-2k}}}f_{00}^R \\
&\quad \text{where } k\leq m \\
&\vdots \\
a^m\star f_{m0}^R &= \frac{m!}{\sqrt{m!\Theta^{-m}}}\Big[1-\frac{ma\omega}{2}-\frac{\bar{a}\bar{\omega}}{4}\sum_{j=1}^{m}\frac{(m+4j+1)}{(m-j)!}\Big]f_{00}^R
\end{aligned}$$

$$a^{m+1}\star f_{m0}^R = -\frac{\sqrt{m!\Theta^{m+2}}\bar{\omega}}{8}\sum_{j=1}^{m}\frac{(m+4j+1)}{(m-j)!}f_{00}^R \propto f_{00}^R \tag{4.105}$$

$$a^{m+2}\star f_{m0}^R = 0. \tag{4.106}$$

Similarly, $f^L_{0n}\star\bar a$, $f^L_{0n}\star\bar a^2,\cdots f^L_{0n}\star\bar a^n$, $f^L_{0n}\star\bar a^{n+1}$ can be computed as

$$\begin{aligned}
f^L_{0n}\star\bar a &= \frac{n2^{n-1}a^{n-1}}{\sqrt{n!\Theta^{n-2}}}(1+\frac{n-2}{2}\bar a\bar\omega-\frac{n+5}{4}a\omega)f^L_{00}\\
&- \frac{(n-1)\Theta\bar\omega U_n a^{n-2}}{4\sqrt{n!\Theta^{n-2}}}f^L_{00}\\
f^L_{0n}\star\bar a^2 &= \frac{n(n-1)2^{n-2}a^{n-2}}{\sqrt{n!\Theta^{n-4}}}\Big[1+\frac{n-4}{2}\bar a\bar\omega\\
&- \frac{(n+9)(n-1)+n+5}{4(n-1)}a\omega\Big]f^L_{00}\\
&- \frac{(n-1)(n-2)\Theta\bar\omega U_n a^{n-3}}{8\sqrt{n!\Theta^{n-4}}}f^L_{00}\\
&\vdots\\
f^L_{0n}\star\bar a^k &= \frac{n(n-1)\cdots(n-k+1)2^{n-k}a^{n-k}}{\sqrt{n!\Theta^{n-2k}}}\Big[1+\frac{n-2k}{2}\bar a\bar\omega\\
&- \frac{a\omega}{4}(n-k)!\sum_{j=1}^{k}\frac{(n+4j+1)}{(n-j)!}\Big]f^L_{00}\\
&- \frac{(n-1)(n-2)\cdots(n-k)\Theta\bar\omega U_n a^{n-k-1}}{2^{k+1}\sqrt{n!\Theta^{n-2k}}}f^L_{00};\\
&\quad \text{where } k\le n\\
&\vdots\\
f^L_{0n}\star\bar a^n &= \frac{n!}{\sqrt{n!\Theta^{-n}}}\Big[1-\frac{n\bar a\bar\omega}{2}-\frac{a\omega}{4}\sum_{j=1}^{n}\frac{(n+4j+1)}{(n-j)!}\Big]f^L_{00}\\
f^L_{0n}\star\bar a^{n+1} &= -\frac{\sqrt{n!\Theta^{n+2}}\omega}{8}\sum_{j=1}^{n}\frac{(n+4j+1)}{(n-j)!}f^L_{00}\propto f^L_{00} \qquad (4.107)\\
f^L_{0n}\star\bar a^{n+2} &= 0. \qquad (4.108)
\end{aligned}$$

□

Remark 4.8 *1. f^R_{00} and f^L_{00} are the twisted fundamental states restoring, in the limit of ordinary Moyal space, the fundamental state given by $2e^{-\frac{2a\bar a}{\Theta}}$. For the analysis purpose, we call f^R_{00} and f^L_{00} the normal twisted fundamental states.*

2. *The states f^R_{m0} correspond to twisted right $m+1$ particles states, reducing, in the usual case, to right m particles states, while the states f^L_{0n} represent the twisted left $n+1$ particles states.*
3. *There are an infinite number of twisted right $m-k$ particles states and an infinite number of twisted left $n-k$ particles states given by $a^{k+1}\star f^R_{m0}$ and $f^L_{0n}\star\bar a^{k+1}$, respectively.*

4.2.3 Matrix basis of the theory

The usual construction of a matrix basis [34] exploits the $\star-$multiplication of f^R_{m0} with f^L_{0n}, i.e.

$$L^2(\mathbb{R}^2_\Theta)\ni b^{(2)}_{mn} = :\chi(\omega,\bar\omega,\Theta,m,n)f^R_{m0}\star f^L_{0n}$$

$$= \chi(\omega,\bar\omega,\Theta,m,n)\frac{(\bar a^m\star f_{00}^R)\star(f_{00}^L\star a^n)}{\sqrt{m!n!\Theta^{m+n}}}. \quad (4.109)$$

Without loss of generality, we set the normalization constant $\chi(\omega,\bar\omega,\Theta,m,n)=:1$ by convention. The corresponding eigenvalue problems are given by

$$H\star b^{(2)}_{mn}=\mathcal{E}^R_{m0}b^{(2)}_{mn} \quad \text{and} \quad b^{(2)}_{mn}\star H=\mathcal{E}^L_{0n}b^{(2)}_{mn} \quad (4.110)$$

while the $\star-$actions of the annihilation and creation functions a and $\bar a$ are reproduced as follows: $\bar a\star b^{(2)}_{mn}=\sqrt{\Theta(m+1)}b^{(2)}_{m+1,n}$ and $b^{(2)}_{mn}\star a=\sqrt{\Theta(n+1)}b^{(2)}_{m,n+1}$, with the basis fundamental state $b^{(2)}_{00}=f^R_{00}\star f^L_{00}$ satisfying the expected requirements $a\star b^{(2)}_{00}=0,\quad b^{(2)}_{00}\star\bar a=0$. Given the $(1,1)-$particles states defined by $L^2(\mathbb{R}^2_\Theta)\ni\Lambda^{1,1}_{mn}=:(a^m\star f^R_{m0})\star(f^L_{0n}\star\bar a^n)$, their twisted spectrums can be computed from the eigenvalue problems $H\star\Lambda^{1,1}_{mn}=\mathcal{E}^R_{\Lambda^{1,1}_{m0}}\Lambda^{1,1}_{mn}$ and $\Lambda^{1,1}_{mn}\star H=\mathcal{E}^L_{\Lambda^{1,1}_{0n}}\Lambda^{1,1}_{mn}$ to get, depending on the right and left Hamiltonian $\star-$actions,

$$\mathcal{E}^R_{\Lambda^{1,1}_{m0}} = \frac{\Theta}{2}\Big[1-\frac{\bar a\bar\omega}{2}\Big(\sum_{j=1}^m\frac{m+4j+1}{(m-j)!}+4\Big)\Big],\ m>0 \quad (4.111)$$

$$\begin{aligned}\mathcal{E}^L_{\Lambda^{1,1}_{0n}} &= \mathcal{E}^R_{\Lambda^{1,1}_{n0}}(\bar a\leftrightarrow a,\bar\omega\leftrightarrow\omega)\\ &= \frac{\Theta}{2}\Big[1-\frac{a\omega}{2}\Big(\sum_{j=1}^n\frac{n+4j+1}{(n-j)!}+4\Big)\Big],\quad n>0. \end{aligned}\quad (4.112)$$

For $\omega=0$ and $\bar\omega=0$, these energies are reduced to the usual Moyal space matrix basis right and left fundamental energies. As needed, the Wick rotation can be used to ensure the real value of the energy. In the same vein, one can define the single twisted $(m-k+1)$ right particles states by $a^k\star f^R_{m0}=:\Lambda^{m-k+1}_{m0}\in L^2(\mathbb{R}^2_\Theta)$ corresponding to the energy values obtained from the right Hamiltonian $\star-$action by

$$\begin{aligned}\mathcal{E}^R_{\Lambda^{m-k+1}_{m0}} &= \frac{\Theta}{2}\Big\{2m-2k+1-(m-k)a\omega\\ &+ \frac{\bar a\bar\omega}{2}\Big[(m-k-1)(m-k)!\sum_{j=1}^k\frac{m+4j+1}{(m-j)!}\\ &- (m-k)(m+4k+6)-4\Big]-\frac{(m-k)(m-2k)\Theta\omega}{4\bar a}\\ &+ \frac{(m-k+1)(m-k)\Theta\omega U_m}{m2^{m+1}\bar a}\Big\}\quad m\geq k>0. \end{aligned}\quad (4.113)$$

By analogy, the single twisted $(n-l+1)$ left particles states $f^L_{0n}\star\bar a^l=:\Lambda^{n-l+1}_{0n}\in L^2(\mathbb{R}^2_\Theta)$ are associated with the left action energy values

$$\begin{aligned}\mathcal{E}^L_{\Lambda^{n-l+1}_{0n}} &= \mathcal{E}^R_{\Lambda^{n-l+1}_{n0}}\Big(\bar\omega\leftrightarrow\omega,\bar a\leftrightarrow a\Big)\\ &= \frac{\Theta}{2}\Big\{2n-2l+1-(n-l)\bar a\bar\omega\\ &+ \frac{a\omega}{2}\Big[(n-l-1)(n-l)!\sum_{j=1}^l\frac{n+4j+1}{(n-j)!}\\ &- (n-l)(n+4l+6)-4\Big]-\frac{(n-l)(n-2l)\Theta\bar\omega}{4a}\end{aligned}$$

$$+ \quad \frac{(n-l+1)(n-l)\Theta\bar{\omega}U_n}{n2^{n+1}a}\Big\}, \quad n \geq l > 0. \tag{4.114}$$

Proposition 4.9 *The energy spectrums (4.113) and (4.114) of the* mixed *twisted* $(m-k+1)$ *right and* $(n-l+1)$ *left particles states* $L^2(\mathbb{R}^2_\Theta) \ni \Lambda^{m-k+1,n-l+1}_{mn} =: (a^k \star f^R_{m0}) \star (f^L_{0n} \star \bar{a}^l)$ *solve the following respective eigenvalue problems:*

$$H \star \Lambda^{m-k+1,n-l+1}_{mn} = \mathcal{E}^R_{\Lambda^{m-k+1}_{m0}} \Lambda^{m-k+1,n-l+1}_{mn} \tag{4.115}$$

$$\Lambda^{m-k+1,n-l+1}_{mn} \star H = \mathcal{E}^L_{\Lambda^{n-l+1}_{0n}} \Lambda^{m-k+1,n-l+1}_{mn}. \tag{4.116}$$

We readily recover the spectrums (4.111) and (4.112) by replacing $m = k$ and $n = l$ in the relations (4.113) and (4.114). Of course, in the limit regime, these energies also well reproduce the ordinary Moyal plane $(m-k)$ right and $(n-l)$ left particles energies:

$$\lim_{\omega,\bar{\omega}\to 0} \mathcal{E}^R_{\Lambda^{m-k+1}_{m0}} = \frac{\Theta}{2}\Big[2(m-k)+1\Big] \tag{4.117}$$

$$\lim_{\omega,\bar{\omega}\to 0} \mathcal{E}^L_{\Lambda^{n-l+1}_{0n}} = \frac{\Theta}{2}\Big[2(n-l)+1\Big] \tag{4.118}$$

respectively.

4.2.4 Case of commuting vector fields

Consider the coordinates base $e^\mu_a = \delta^\mu_a + \omega^\mu_{ab}x^b$ and the symmetric tensor (between the index a and b) ω^μ_{ab}. Then, the twisted star product is associative and

$$[X_a, X_b] = \omega^\mu_{ba}\partial_\mu - \omega^\mu_{ab}\partial_\mu = 0. \tag{4.119}$$

The matrix representation of e^μ_a is given by

$$(e)^\mu_a = \begin{pmatrix} 1+\omega^1_{11}x^1+\omega^1_{12}x^2 & \omega^2_{11}x^1+\omega^2_{12}x^2 \\ \omega^1_{12}x^1+\omega^1_{22}x^2 & 1+\omega^2_{12}x^1+\omega^2_{22}x^2 \end{pmatrix} \tag{4.120}$$

and

$$(e)^a_\mu = \begin{pmatrix} 1-\omega^1_{11}x^1-\omega^1_{12}x^2 & -\omega^2_{11}x^1-\omega^2_{12}x^2 \\ -\omega^1_{12}x^1-\omega^1_{22}x^2 & 1-\omega^2_{12}x^1-\omega^2_{22}x^2 \end{pmatrix}. \tag{4.121}$$

Then,

$$\begin{aligned} e^{-1} &= det(e^\mu_a) = 1+(\omega^1_{11}+\omega^2_{12})x^1+(\omega^2_{22}+\omega^1_{12})x^2 = 1+a\omega+\bar{a}\bar{\omega} \\ e &= det(e^a_\mu) = 1-(\omega^1_{11}+\omega^2_{12})x^1-(\omega^2_{22}+\omega^1_{12})x^2 = 1-a\omega-\bar{a}\bar{\omega}, \end{aligned} \tag{4.122}$$

where $\omega = \frac{1}{\sqrt{2}}\Big((\omega^1_{11}+\omega^2_{12}) - i(\omega^2_{22}+\omega^1_{12})\Big)$ and $a = \frac{1}{\sqrt{2}}(x^1+ix^2)$. The noncommutative tensor is given by $(\widetilde{\Theta})^{\mu\nu} = \Theta e^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, we can immediately perform the computation and check that the eigenvalue problem of the harmonic oscillator yields the same solution as for the case of the skew symmetric tensor ω^μ_{ab}, but now defining ω and $\bar{\omega}$ as

$$\omega = \frac{1}{\sqrt{2}}\Big((\omega^1_{11}+\omega^2_{12}) - i(\omega^2_{22}+\omega^1_{12})\Big), \;\; \bar{\omega} = \frac{1}{\sqrt{2}}\Big((\omega^1_{11}+\omega^2_{12}) + i(\omega^2_{22}+\omega^1_{12})\Big). \tag{4.123}$$

Besides, the matrix e^a_μ can be written as $e^a_\mu = \delta^a_\mu + \omega^{ab}_\mu x_b$, where $\omega^{ab}_\mu = -\omega^\mu_{ab}$. Finally, the solution of the field equation $e^a_\mu = \partial_\mu \phi^a$ is well given by $\phi^a = x^a + \frac{1}{2}\omega^{ab}_\mu x_b x^\nu \delta^\mu_\nu$. From there follow all results obtained in the core of this work.

4.3 Harmonic oscillator in noncommutative phase space

In recent years, there is an increasing interest in the application of noncommutative (NC) geometry to physical problems [17] in solid-state and particle physics [76], mainly motivated by the idea of a strong connection of noncommutativity with field and string theories. Besides, the evidence coming from the latter and other approaches to the issues of quantum gravity suggests that attempts to unify gravity and quantum mechanics could ultimately lead to a non-commutative geometry of spacetime. The phase space of ordinary quantum mechanics is a well-known example of noncommuting space [86]. The momenta of a system in the presence of a magnetic field are noncommuting operators as well. Since the noncommutativity between spatial and time coordinates may lead to some problems with unitarity and causality, usually only spatial noncommutativity is considered. Besides, so far quantum theory on the NC space has been extensively studied, the main approach is based on the Weyl-Moyal correspondence which amounts to replacing the usual product by a $\star-$product in the NC space. Therefore, deformation quantization has special significance in the study of physical systems on the NC space. Moreover, the problem of quantum mechanics on NC spaces can be understood in the framework of deformation quantization [6]-[39]. In the same vein, some works on harmonic oscillators (ho) in the NC space from the point of view of deformation quantization have been reported in [38]-[55] and references therein. In this dissertation, we consider different representations of a harmonic oscillator in a general full noncommutative phase space with both the spatial and momentum coordinates being noncommutative. Indeed, noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the noncommutative coordinates. This work continues the investigations stated in [38],[11] and [12] devoted to the study of a quantum exactly solvable D-dimensional NC oscillator with quasi-harmonic behaviour. We intend to extend previous results presenting a similar analysis to the quantum version of the two-dimensional NC system with non-vanishing momentum components. For additional details in the motivation, see [38]. The physical model resembles to the Landau problem in NC quantum mechanics extensively studied in the literature. See [69] and [50] and references therein for more details. Broadly put, it is worth noticing that the description of a system of a two-dimensional harmonic oscillator in a full NC phase space is equivalent to that of the model of a two-dimensional ho in a constant magnetic field in some NC space.

Consider a 4D general NC phase space. The coordinates of position and momentum, $x = (x^1, x^2)$ and $p = (p^1, p^2)$, modelling the classical system of a two-dimensional ho maps into their respective quantum operators $\hat{x}$ and $\hat{p}$ giving rise to the Hamiltonian operator

$$\hat{H} = \frac{1}{2}\Big(\hat{p}_\mu \hat{p}^\mu + \hat{x}_\mu \hat{x}^\mu\Big) \tag{4.124}$$

with commutation relations

$$[\hat{x}^\mu, \hat{p}^\nu] = i\hbar_{eff}\delta^{\mu\nu}, \ \ [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \ \ [\hat{p}^\mu, \hat{p}^\nu] = i\bar{\Theta}^{\mu\nu}, \quad \mu, \nu = 1, 2 \tag{4.125}$$

where $\Theta^{\mu\nu}$ and $\bar{\Theta}^{\mu\nu}$ are skew-symmetric tensors carrying the dimensions of (length)^2 and (momentum)^2, respectively. The effective Planck constant is given by

$$\hbar_{eff} = \hbar\Big(1 + \frac{\Theta^{\mu\nu}\bar{\Theta}^{\mu\nu}}{4D\hbar^2}\Big), \tag{4.126}$$

where $D = 2$ is the dimension of the NC space. One can readily check that one can rewrite the operators $\hat{x}^\mu$ and $\hat{p}^\nu$ as

$$\hat{p}^\mu = \hat{\pi}^\mu + \frac{1}{2\hbar}\bar{\Theta}^{\mu\nu}\hat{q}_\nu, \quad \hat{x}^\mu = \hat{q}^\mu - \frac{1}{2\hbar}\Theta^{\mu\nu}\hat{\pi}_\nu \tag{4.127}$$

in terms of $\hat{\pi}^\mu$ and $\hat{q}^\nu$ that obey the standard Weyl-Heisemberg algebra

$$[\hat{q}^\mu, \hat{\pi}^\nu] = i\hbar\delta^{\mu\nu}; \quad [\hat{q}^\mu, \hat{q}^\nu] = 0 = [\hat{\pi}^\mu, \hat{\pi}^\nu]. \tag{4.128}$$

In the deformation quantization theory of a classical system in the noncommutative space, one treats (x, p) and their functions as classical quantities, but replaces the ordinary product between these functions by the following generalized $\star-$product

$$\begin{aligned} \star &= \star_{\hbar_{eff}} \star_\Theta \star_{\bar{\Theta}} \\ &= \exp\Big[\frac{i\hbar_{eff}}{2}\Big(\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{p^\mu} - \overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{x^\mu}\Big) + \frac{i\Theta^{\mu\nu}}{2}\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{x^\nu} \\ &\quad + \frac{i\bar{\Theta}^{\mu\nu}}{2}\overleftarrow{\partial}_{p^\mu}\overrightarrow{\partial}_{p^\nu}\Big]. \end{aligned} \tag{4.129}$$

The variables x^μ, p^μ on the NC phase space satisfy the following commutation relations similar to (4.130)

$$[x^\mu, p^\nu]_\star = i\hbar_{eff}\delta^{\mu\nu}, \ [x^\mu, x^\nu]_\star = i\Theta^{\mu\nu}, \ [p^\mu, p^\nu]_\star = i\bar{\Theta}^{\mu\nu} \quad \mu, \nu = 1, 2 \tag{4.130}$$

with the following uncertainty relations

$$\begin{aligned} \Delta x^1\Delta x^2 \geqslant \frac{\Theta}{2} &\qquad \Delta p^1\Delta p^2 \geqslant \frac{\bar{\Theta}}{2} \\ \Delta x^1\Delta p^1 \geqslant \frac{\hbar_{eff}}{2} &\quad \Delta x^2\Delta p^2 \geqslant \frac{\hbar_{eff}}{2}. \end{aligned} \tag{4.131}$$

The first two uncertainty relations show that measurements of positions and momenta in both directions x^1 and x^2 are not independent. Taking into account the fact that Θ and $\bar{\Theta}$ have dimensions of (length)^2 and (momentum)^2 respectively, then $\sqrt{\Theta}$ and $\sqrt{\bar{\Theta}}$ define fundamental scales of length and momentum which characterize the minimum uncertainties possible to achieve in measuring these quantities. One expects these fundamental scales to be related to the scale of the underlying field theory (possible the string scale), and thus to appear as small corrections at the low-energy level or quantum mechanics. Commonly, the time evolution function for a time-independent Hamiltonian H of a system is described by the $\star-$exponential function denoted here by $e_\star^{(.)}$:

$$e_\star^{\frac{Ht}{i\hbar_{eff}}} := \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{t}{i\hbar_{eff}}\Big)^n \overbrace{H \star H \star \cdots \star H}^{\text{n times}}, \tag{4.132}$$

which is the solution of the following time-dependent Schrodinger equation

$$\begin{aligned}
i\hbar_{eff}\frac{d}{dx}e_\star^{\frac{Ht}{i\hbar_{eff}}} &= H(x,p)\star e_\star^{\frac{Ht}{i\hbar_{eff}}}\\
&= H\Big(x^\mu+\frac{i\hbar_{eff}}{2}\partial_{p^\mu}+\frac{i\Theta^{\mu\rho}}{2}\partial_{x^\rho},p^\nu-\frac{i\hbar_{eff}}{2}\partial_{x^\nu}\\
&\quad+\frac{i\bar\Theta^{\mu\sigma}}{2}\partial_{x^\sigma}\Big)e_\star^{\frac{Ht}{i\hbar_{eff}}}. \qquad (4.133)
\end{aligned}$$

There corresponds the generalized $\star-$eigenvalue time-independent Schrodinger equation:

$$H\star\mathcal{W}_n=\mathcal{W}_n\star H=\mathcal{E}_n\mathcal{W}_n \qquad (4.134)$$

where $\mathcal{W}_n$ and $\mathcal{E}_n$ stand for the Wigner function and the corresponding energy eigenvalue of the system. The Fourier-Dirichlet expansion for the time-evolution function defined as

$$e_\star^{\frac{Ht}{i\hbar_{eff}}}=\sum_{n=0}^{\infty}e^{\frac{-i\mathcal{E}_nt}{\hbar_{eff}}}\mathcal{W}_n \qquad (4.135)$$

links the Wigner function to the $\star-$exponential function.

Provided the above, the operators on a NC Hilbert space can be represented by the functions on a NC phase space, where the operator product is replaced by relevant star-product. The algebra of functions of such noncommuting coordinates can be replaced by the algebra of functions on ordinary spacetime, equipped with a NC star-product. So, considering the transformations (4.127) and leaving out the operator symbol$\hat{}$, we arrive at (q,π) phase space and the commutation relation changes into (4.128), with the star-product defined in the following way.

Definition 4.10 *Let $C^\infty(\mathbb{R}^4)$ be the space of smooth functions $f:\mathbb{R}^4\to\mathbb{C}$. For $f,g\in C^\infty(\mathbb{R}^4)$, the formal star product is defined by*

$$f\star g=f\exp\Big[\frac{i\hbar}{2}\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu\Big]g. \qquad (4.136)$$

Here the smooth functions f and g depend on the real variables q^1, q^2, π^1 and π^2, and

$$\begin{aligned}
\overleftarrow{\partial}_\mu J^{\mu\nu}\overrightarrow{\partial}_\nu &= \Big(\frac{\overleftarrow{\partial}}{\partial q^1},\frac{\overleftarrow{\partial}}{\partial \pi^1},\frac{\overleftarrow{\partial}}{\partial q^2},\frac{\overleftarrow{\partial}}{\partial \pi^2}\Big)\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}\frac{\overrightarrow{\partial}}{\partial q^1}\\ \frac{\overrightarrow{\partial}}{\partial \pi^1}\\ \frac{\overrightarrow{\partial}}{\partial q^2}\\ \frac{\overrightarrow{\partial}}{\partial \pi^2}\end{pmatrix}\\
&= \frac{\overleftarrow{\partial}}{\partial q^1}\frac{\overrightarrow{\partial}}{\partial \pi^1}-\frac{\overleftarrow{\partial}}{\partial \pi^1}\frac{\overrightarrow{\partial}}{\partial q^1}+\frac{\overleftarrow{\partial}}{\partial q^2}\frac{\overrightarrow{\partial}}{\partial \pi^2}-\frac{\overleftarrow{\partial}}{\partial \pi^2}\frac{\overrightarrow{\partial}}{\partial q^2}. \qquad (4.137)
\end{aligned}$$

Therefore, the star product $f\star g$ represents a deformation of the classical product fg. This deformation depends on the Planck constant $\hbar$. In term of physics, the difference $f\star g-fg$ describes quantum fluctuation depending on $\hbar$. For the present case,

$$q^\mu\star\pi^\nu-q^\mu\pi^\nu=\tfrac{i\hbar}{2}\delta^{\mu\nu},\quad \pi^\nu\star q^\mu-\pi^\nu q^\mu=-\tfrac{i\hbar}{2}\delta^{\mu\nu}.\quad \text{Hence}$$
$$[q^\mu,\pi^\nu]_\star=i\hbar\delta^{\mu\nu}. \qquad (4.138)$$

Building now, in the standard manner, the creation and annihilation operators of ho system as

$$a_l = \frac{q^l + i\pi^l}{\sqrt{2}} \qquad \bar{a}_l = \frac{q^l - i\pi^l}{\sqrt{2}} \quad l = 1,2 \tag{4.139}$$

and using the polar coordinates such that

$$q^l = \rho_l \cos\varphi_l, \qquad \pi^l = \rho_l \sin\varphi_l, \tag{4.140}$$

we solve the right and left eigenvalue equations

$$\begin{aligned} a_l \star f_{mn} = \sqrt{m\hbar} f_{m-1,n} \quad \bar{a}_l \star f_{mn} = \sqrt{(m+1)\hbar} f_{m+1,n} \\ f_{mn} \star a_l = \sqrt{(n+1)\hbar} f_{m,n+1} \quad f_{mn} \star \bar{a}_l = \sqrt{n\hbar} f_{m,n-1} \end{aligned} \tag{4.141}$$

to find the eigenfunctions f_{mn} as

$$f_{mn} \equiv 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i(n-m)\varphi_l} \Big(\frac{2\rho_l^2}{\hbar}\Big)^{\frac{n-m}{2}} L_m^{n-m}\Big(\frac{2\rho_l^2}{\hbar}\Big) e^{-\frac{\rho_l^2}{\hbar}}, \quad m, n \in \mathbb{N} \tag{4.142}$$

with

$$f_{00} = 2e^{-\rho_l^2/\hbar}. \tag{4.143}$$

L_m^{n-m} are the generalized Laguerre polynomials defined for $n = 0, 1, 2, \cdots, \ \alpha > 1$, by

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}) = \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}. \tag{4.144}$$

Then the states defined by $b^{(4)}_{mn} = f_{m_1n_1} f_{m_2n_2}$, where $m = (m_1, m_2)$, $n = (n_1, n_2)$, $m_1, m_2, n_1, n_2 \in \mathbb{N}$, exactly solve the right and left eigenvalue problems of the Hamiltonian $H_0 = \sum_{l=1}^2 \bar{a}_l a_l$ as

$$H_0 \star b^{(4)}_{mn} = \hbar(|m|+1) b^{(4)}_{mn} \quad \text{and} \quad b^{(4)}_{mn} \star H_0 = \hbar(|n|+1) b^{(4)}_{mn} \tag{4.145}$$

where $|m| = m_1 + m_2$.

Now, consider the Hamiltonian (4.124) and use the relation (4.128) to re-express it with the help of variables q and π as follows:

$$H = H_0 + H_L + H_q(\bar{\Theta}) + H_\pi(\Theta) \tag{4.146}$$

where

$$H_0 = \frac{1}{2}\Big((q^1)^2 + (q^2)^2 + (\pi^1)^2 + (\pi^2)^2\Big) \tag{4.147}$$

$$H_L = -\frac{\Theta + \bar{\Theta}}{2\hbar} \overrightarrow{q} \wedge \overrightarrow{\pi} \qquad \overrightarrow{q} \wedge \overrightarrow{\pi} = q^1\pi_2 - q^2\pi_1 \tag{4.148}$$

and

$$H_q(\bar{\Theta}) = \frac{\bar{\Theta}^2}{8\hbar^2}\Big((q^1)^2 + (q^2)^2\Big) \quad H_\pi(\Theta) = \frac{\Theta^2}{8\hbar^2}\Big((\pi^1)^2 + (\pi^2)^2\Big). \tag{4.149}$$

It is a matter of computation to verify that the Hamiltonians H_0 and H_L $\star-$commute. Idem for the Hamiltonians H_L and $H_I = H_q(\bar{\Theta}) + H_\pi(\Theta)$. Therefore, the Hamiltonians of family $\{H_0, H_L\}$, (respectively $\{H_L, H_I\}$) can be simultaneously measured.

Proposition 4.11 *In the case $\Theta = -\bar{\Theta}$, the Hamiltonian H can be expressed as*

$$H = \Big(1 + \frac{\Theta^2}{4\hbar^2}\Big) H_0 \tag{4.150}$$

and the states $b^{(4)}_{mn}$ solve the right and left eigenvalue problems of H as

$$H \star b^{(4)}_{mn} = \mathcal{E}^R_{m0} b^{(4)}_{mn} \qquad \mathcal{E}^R_{m0} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|m| + 1) \tag{4.151}$$

and

$$b^{(4)}_{mn} \star H = \mathcal{E}^L_{0n} b^{(4)}_{mn} \qquad \mathcal{E}^L_{0n} = \hbar\Big(1 + \frac{\Theta^2}{4\hbar^2}\Big)(|n| + 1) \tag{4.152}$$

where $m = (m_1, m_2)$, $n = (n_1, n_2)$ $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $|m| = m_1 + m_2$.

In the general $(q, \pi)-$representation, the problem to solve is equivalent to that of the two-dimensional Landau problem in a symmetric gauge on a noncommutative space. Indeed, the Hamiltonian H can be re-transcribed as

$$H = \frac{\alpha^2}{2}\Big((q^1)^2 + (q^2)^2\Big) + \frac{\beta^2}{2}\Big((\pi^1)^2 + (\pi^2)^2\Big) - \gamma \overrightarrow{q} \wedge \overrightarrow{\pi} =: H_0^\natural + H_L \tag{4.153}$$

where

$$\alpha^2 = 1 + \frac{\bar{\Theta}^2}{4\hbar^2}, \qquad \beta^2 = 1 + \frac{\Theta^2}{4\hbar^2}, \qquad \gamma = \frac{\Theta + \bar{\Theta}}{2\hbar} \tag{4.154}$$

Remark that the Hamiltonian terms $H_0^\natural$ and H_L commute. Therefore, the eigenvectors of $\{H_0^\natural, H_L\}$ are automatically eigenvectors of H. As matter of convenience, to solve the Schrödinger eigen-equation, let us choose the polar coordinates

$$q^1 = \rho\cos\varphi \qquad q^2 = \rho\sin\varphi \tag{4.155}$$

and assume the variable separability to write

$$\tilde{f}(\rho, \varphi) = \xi(\rho) e^{ik\varphi}, \quad k = 0, \pm 1, \pm 2, \cdots \tag{4.156}$$

Then, from the static Schrödinger equation on NC space, $H \star \tilde{f}(\rho, \varphi) = \mathcal{E}\tilde{f}(\rho, \varphi)$, we deduce the radial equation as follows:

$$\Big[-\frac{\hbar^2\beta^2}{2}\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\Big) + \frac{\alpha^2}{2}\rho^2 - \gamma\hbar k\Big]\xi(\rho, \varphi) = \mathcal{E}\xi(\rho, \varphi) \tag{4.157}$$

yielding the spectrum of H under the form

$$\mathcal{E} = \hbar\frac{\alpha^2}{\beta^2}(n + 1) - \hbar\gamma k, \quad n = 0, 1, 2, \cdots \tag{4.158}$$

with

$$\xi(\rho, \varphi) \propto e^{-\frac{\alpha}{\hbar\beta}\rho^2} H_n\left(\frac{\alpha}{\hbar\beta}\rho^2\right). \tag{4.159}$$

The last term of the energy spectrum $\mathcal{E}$ falls down when $\gamma = 0$, i.e. $\Theta = -\bar{\Theta}$. In this case, $\alpha^2 = \beta^2$ and we recover the discrete spectrum of the usual two-dimensional harmonic oscillator as expected. The results obtained here can be reduced to specific expressions reported in the literature [38] for particular cases. Besides, the formalism displayed in this work permits to avoid the appearence of infinite degeneracy of states observed when $\hbar_{eff}^2 - \Theta\bar{\Theta} = 0$ in [69] where the phase space is divided into two phases based on the following conditions on the deformation parameters:

- Phase I for $\hbar_{eff}^2 - \Theta\bar{\Theta} > 0$
- Phase II for $\hbar_{eff}^2 - \Theta\bar{\Theta} < 0$.

Thanks to the linear transformations $(x, p) \to (q, \pi)$ performed in the present work, the real energy conditions shown in [50] also fall down.

4.3.1 Another representation

The star-product associated with the deformation (4.130) is given by

$$\star = e^{\frac{i}{2}\Sigma^{\mu\nu}\partial_\mu\otimes\partial_\nu}, \quad \text{where} \quad \Sigma^{\mu\nu} =: \begin{pmatrix} 0 & \hbar_{eff} & \Theta & 0 \\ -\hbar_{eff} & 0 & 0 & \bar{\Theta} \\ -\Theta & 0 & 0 & \hbar_{eff} \\ 0 & -\bar{\Theta} & -\hbar_{eff} & 0 \end{pmatrix} \tag{4.160}$$

while the operator ∂_μ is given by $\partial_\mu = (\partial_1, \partial_2, \partial_3, \partial_4) =: (\partial_{x^1}, \partial_{p^1}, \partial_{x^2}, \partial_{p^2})$. Consider the vector $X^\mu = (X^1, X^2, X^3, X^4) =: (x^1, p^1, x^2, p^2)$. Then the star action between X^μ and the function f acting in the phase space is given by

$$X^\mu \star f = X^\mu f + \frac{i}{2}\Sigma^{\mu\rho}\partial_\rho f, \quad f \star X^\mu = X^\mu f - \frac{i}{2}\Sigma^{\mu\rho}\partial_\rho f. \tag{4.161}$$

We verify that the nonvanishing commutation relations between coordinates and momentums are given by $[x^1, x^2]_\star = i\Theta$, $[p^1, p^2]_\star = i\bar{\Theta}$, $[x^1, p^1]_\star = i\hbar_{eff} = [x^2, p^2]_\star$. Let us now define the creation and annihilation operators as

$$a_1 = \frac{1}{\sqrt{2}}(x^1 + ip^1), \quad \bar{a}_1 = \frac{1}{\sqrt{2}}(x^1 - ip^1) \tag{4.162}$$

$$a_2 = \frac{1}{\sqrt{2}}(x^2 + ip^2), \quad \bar{a}_2 = \frac{1}{\sqrt{2}}(x^2 - ip^2). \tag{4.163}$$

Then, the star-product between the creation and annihilation operators and the function f acting in the NC phase space yields

$$a_1 \star f = a_1 f + \frac{1}{2}\hbar_{eff}\partial_{\bar{a}_1} f + \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{a_2} + (\Theta + \bar{\Theta})\partial_{\bar{a}_2}\Big) f \tag{4.164}$$

$$a_2 \star f = a_2 f + \frac{1}{2}\hbar_{eff}\partial_{\bar{a}_2} f - \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{a_1} + (\Theta + \bar{\Theta})\partial_{\bar{a}_1}\Big) f \tag{4.165}$$

$$f \star a_1 = a_1 f - \frac{1}{2}\hbar_{eff}\partial_{\bar{a}_1} f - \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{a_2} + (\Theta + \bar{\Theta})\partial_{\bar{a}_2}\Big) f \tag{4.166}$$

$$f \star a_2 = a_2 f - \frac{1}{2}\hbar_{eff}\partial_{\bar{a}_2} f + \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{a_1} + (\Theta + \bar{\Theta})\partial_{\bar{a}_1}\Big) f. \tag{4.167}$$

In the same way, if we use the relation $\overline{f \star g} = \bar{g} \star \bar{f}$,

$$\bar{a}_1 \star f = \bar{a}_1 f - \frac{1}{2}\hbar_{eff}\partial_{a_1} f + \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{\bar{a}_2} + (\Theta + \bar{\Theta})\partial_{a_2}\Big) f \tag{4.168}$$

$$\bar{a}_2 \star f = \bar{a}_2 f - \frac{1}{2}\hbar_{eff}\partial_{a_2} f - \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{\bar{a}_1} + (\Theta + \bar{\Theta})\partial_{a_1}\Big) f \tag{4.169}$$

$$f \star \bar{a}_1 = \bar{a}_1 f + \frac{1}{2}\hbar_{eff}\partial_{a_1} f - \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{\bar{a}_2} + (\Theta + \bar{\Theta})\partial_{a_2}\Big) f \tag{4.170}$$

$$f \star \bar{a}_2 = \bar{a}_2 f + \frac{1}{2}\hbar_{eff}\partial_{a_2} f + \frac{i}{4}\Big((\Theta - \bar{\Theta})\partial_{\bar{a}_1} + (\Theta + \bar{\Theta})\partial_{a_1}\Big) f. \tag{4.171}$$

We have also the commutation relations

$$\begin{array}{c} [a_1, a_2]_\star = \frac{i}{2}(\Theta - \bar{\Theta}), \quad [\bar{a}_1, \bar{a}_2]_\star = \frac{i}{2}(\Theta - \bar{\Theta}), \\ [a_1, \bar{a}_2]_\star = \frac{i}{2}(\Theta + \bar{\Theta}), \quad [a_2, \bar{a}_1]_\star = -\frac{i}{2}(\Theta + \bar{\Theta}), \\ [a_1, \bar{a}_1]_\star = \hbar_{eff}, \;\; [a_2, \bar{a}_2]_\star = \hbar_{eff} \end{array} \tag{4.172}$$

showing that the annihilation and creation operators defined in (4.162) and (4.163) are not appropriate for solving the considered eigenvalue problem of the harmonic oscillator.

The determinant of the skew-symmetric matrix Σ, $det\Sigma = (\hbar_{eff}^2 - \Theta\bar{\Theta})^2$, is positive. The critical point deduced from $det\Sigma = 0$ divides the space of the parameters into two phases:

- Phase I for $\hbar_{eff}^2 - \Theta\bar{\Theta} > 0$
- Phase II for $\hbar_{eff}^2 - \Theta\bar{\Theta} < 0$.

The critical point from $\hbar_{eff}^2 - \Theta\bar{\Theta} = 0$ corresponds to the reduction of dimensions in phase space and to infinite degeneracy of states, and is related to the NC Landau problem. It is obvious that the Hamiltonian 4.124 is invariant under rotation in the plane. The angular momentum, being the rotation generator, takes the form

$$L_{nc} = \frac{1}{\hbar_{eff}^2 - \Theta\bar{\Theta}}\Big[x^1p^2 - x^2p^1 + \frac{\bar{\Theta}}{2}\Big((x^1)^2 + (x^2)^2\Big) + \frac{\Theta}{2}\Big((p^1)^2 + (p^2)^2\Big)\Big]. \tag{4.173}$$

We observe that it acquires Θ-dependent corrections compared with the commutative case. The algebra $[X^\mu, X^\nu]_\star = i\Sigma^{\mu\nu}$ has many possible realizations. The minimal one in terms of two independent sets of canonical coordinates and momenta $(\bar{x}^\mu, \bar{p}^\mu)$ satisfying standard Heisenberg commutation relations would be

$$\begin{array}{ll} x^1 = \bar{x}^1, & p^1 = \bar{p}^1 + \bar{\Theta}\bar{x}^2 \\ x^2 = \bar{x}^2 + \Theta\bar{p}^1 & p^2 = \bar{p}^2. \end{array} \tag{4.174}$$

Because of this, the cases $\hbar_{eff}^2 - \Theta\bar{\Theta} < 0$ and $\hbar_{eff}^2 - \Theta\bar{\Theta} > 0$ should be treated differently.

- Consider first the case $\hbar_{eff}^2 - \Theta\bar{\Theta} < 0$ and take $\hbar_{eff} = 1$ for normalization condition. In this case, we can define

$$\begin{array}{ll} x^1 = \zeta\bar{x}^1, & p^1 = \frac{1}{\zeta}\bar{p}^1 + \varsigma\bar{x}^2 \\ x^2 = \zeta\bar{p}^1 & p^2 = \frac{1}{\zeta}\bar{x}^1 - \varsigma\bar{p}^2 \end{array} \tag{4.175}$$

where $\zeta^2 = \Theta$ and $\varsigma^2 = 1/\Theta - \bar{\Theta}$. $\bar{x}^\mu$ and $\bar{p}^\mu$ form a set of canonical variables, i.e. $[\bar{x}^\mu, \bar{p}^\nu]_\star = i\delta^{\mu\nu}$. The Hamiltonian for the oscillator with the magnetic field (4.124) is given by

$$H = \frac{1}{2}\Big[\Big(\zeta^2 + \frac{1}{\zeta^2}\Big)\Big((\bar{x}^1)^2 + (\bar{p}^1)^2\Big) + \varsigma^2\Big((\bar{x}^2)^2 + (\bar{p}^2)^2\Big) + \frac{2\varsigma}{\zeta}(\bar{x}^1\bar{p}^2 + \bar{x}^2\bar{p}^1)\Big] \quad (4.176)$$

We can now make a Bogolyubov transformation on this by expressing $\bar{x}^\mu, \bar{p}^\mu$ in terms of a canonical set Q^μ, P^μ by writing

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{p}^1 \\ \bar{p}^2 \end{pmatrix} = \cosh\lambda \begin{pmatrix} Q^1 \\ Q^2 \\ P^1 \\ P^2 \end{pmatrix} + \sinh\lambda \begin{pmatrix} P^1 \\ P^2 \\ Q^1 \\ Q^2 \end{pmatrix}. \quad (4.177)$$

Choosing

$$\tanh 2\lambda = -\frac{2\varsigma\zeta}{1 + \zeta^4 + \varsigma^2\zeta^2} \quad (4.178)$$

the Hamiltonian (4.176) becomes

$$H = \frac{1}{2}\Big[\Omega_+\Big((P^1)^2 + (Q^1)^2\Big) + \Omega_-\Big((P^2)^2 + (Q^2)^2\Big)\Big] \quad (4.179)$$

where

$$\Omega_\pm = \frac{1}{2}\sqrt{(\Theta - \bar{\Theta})^2 + 4} \pm \frac{1}{2}(\Theta + \bar{\Theta}). \quad (4.180)$$

The equation (4.179) shows that the spectrum is given by that of two harmonic oscillators of frequencies Ω_+ and Ω_-, i.e.

$$E_{mn} = \Omega_+(m + \frac{1}{2}) + \Omega_-(n + \frac{1}{2}). \quad (4.181)$$

- The case of $\hbar_{eff}^2 - \Theta\bar{\Theta} > 0$ can be treated in a similar way. With $\varsigma^2 = \bar{\Theta} - 1/\Theta$, we can write

$$\begin{array}{ll} x^1 = \zeta\bar{x}^1, & p^1 = \frac{1}{\zeta}\bar{p}^1 + \varsigma\bar{x}^2 \\ x^2 = \zeta\bar{p}^1 & p^2 = -\frac{1}{\zeta}\bar{x}^1 + \varsigma\bar{p}^2 \end{array} \quad (4.182)$$

The Hamiltonian (4.124) become

$$H = \frac{1}{2}\Big[\Big(\zeta^2 + \frac{1}{\zeta^2}\Big)\Big((\bar{x}^1)^2 + (\bar{p}^1)^2\Big) + \varsigma^2\Big((\bar{x}^2)^2 + (\bar{p}^2)^2\Big) + \frac{2\varsigma}{\zeta}(\bar{x}^2\bar{p}^1 - \bar{x}^1\bar{p}^2)\Big]. \quad (4.183)$$

The required Bogolyubov transformation is

$$\begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{p}^1 \\ \bar{p}^2 \end{pmatrix} = \cosh\lambda \begin{pmatrix} Q^1 \\ Q^2 \\ P^1 \\ P^2 \end{pmatrix} + \sinh\lambda \begin{pmatrix} P^2 \\ P^1 \\ -Q^2 \\ -Q^1 \end{pmatrix}. \quad (4.184)$$

The required choise of λ is given by

$$\tanh 2\lambda = \frac{2\varsigma\zeta}{1+\zeta^4-\varsigma^2\zeta^2} \tag{4.185}$$

H can then be written as (4.179) with

$$\Omega_\pm = \pm\frac{1}{2}\sqrt{(\Theta-\bar{\Theta})^2+4}+\frac{1}{2}(\Theta+\bar{\Theta}). \tag{4.186}$$

We again have two oscillators of frequencies $\Omega_\pm$.

We see from the above results that there is a critical value of the magnetic field or $\bar{\Theta}$ given by $\Theta\bar{\Theta} = 1$. Ω_- vanishes upon approaching this value from either side. The Hamiltonian is independent of P^2, Q^2. Thus the number of states for fixed energy will become unbounded, since all the states generated by P^2, Q^2 are now degenerate. This large degeneracy can also be seen from a semiclassical estimate of the number of states for fixed energy. Remark that the eigenvalues of the general matrix $i\Sigma$, are

$$\text{(phase II)} \quad \Omega_\pm = \frac{1}{2}\sqrt{(\Theta-\bar{\Theta})^2+4\hbar_{eff}^2}\pm\frac{1}{2}(\Theta+\bar{\Theta}) \tag{4.187}$$

$$\text{(phase I)} \quad \Omega_\pm = \pm\frac{1}{2}\sqrt{(\Theta-\bar{\Theta})^2+4\hbar_{eff}^2}+\frac{1}{2}(\Theta+\bar{\Theta}). \tag{4.188}$$

Then the pair (H,Σ) defines a system with given energy spectrum and corresponding energy eigenfunctions.

Concluding remarks and discussions

In this dissertation work, we have obtained the following results:

In Chapter I, we have provided a comparative study of ordinary complex scalar $\phi^4_{\star D}$ and complex GW model. In this context, EMTs have been explicitly computed and improved to satisfy known physics based properties in line with the Wilson and Jackiw techniques. In particular, the obtaining of a symmetric locally conserved EMTs has required the resolution of a nonlinear Belinfante type partial differential equation. For the complex GW model, the Euler-Lagrange equations of motion have been exactly solved for $\Omega = \sqrt{2}$ using the well known matrix base method. In all these theories, the dilatation symmetry has been broken and the breaking terms have been discussed. As expected, all computed physical quantities for ordinary complex $\phi^4_{\star D}$ NCFT are easily recovered from the results obtained for the complex GW NCFT by setting $\Omega = 0$.

We have also provided, in Chapter II, a generalization of the Hamiltonian formulation developed by Gomis $et\ al$ [30], which has been applied to the renormalizable Grosse-Wulkenhaar $\phi^4_{\star D}$ model. The Euler-Lagrange equation of motion has been derived in this case. The constraints and NC currents have been investigated and analyzed. This study reveals that:

1. It is possible to study the original D dimensional non-local GW Lagrangian system describing the renormalizable GW model as a $D+1$ dimensional local (in one of the times) Hamiltonian system, governed by the Hamiltonian (2.39) and a set of constraints.
2. Examples of Hamiltonian symmetry generators of class of the renormalizable GW model working in a $D+1$ dimensional space can be given.
3. As expected from previous investigations on NC Noether currents, the tensor $\mathcal{J}^a_\mu$ (2.54) is not symmetric, nonlocally conserved, and, in massless theory, not traceless.
4. A characteristic feature of the Hamiltonian formalism for non-local theories is that it contains the Euler-Lagrange equations as Hamiltonian constraints. The Euler-Lagrange equation of motion is a constraint in the space of trajectories.

The EL equation of motion in $D+1$ dimensions can be also solved using the matrix base formalism. In that case, the matrix elements can be written as:

$$\mathcal{B}^{(D+1)}_{h,kl}(t,\bar{x}) = \int dt'\, \omega_h(t-t') e^{t\frac{d}{d\bar{x}}}\Big(b^{(D)}_{kl}(\bar{x})\Big). \tag{4.189}$$

where $e^{t\frac{d}{d\bar{x}}}$ can be taken as the evolution operator T_t (translation operator). The fields $\mathcal{Q}_h(t,\bar{x})$ can be reexpressed as follows:

$$\mathcal{Q}_h(t,\bar{x}) = \sum_{k,l} C_{kl}\mathcal{B}^{(D+1)}_{h,kl}(t,\bar{x}). \tag{4.190}$$

Then, the formalism developed in [34] can be applied step by step. Further, the same matrix base method can be adapted to formulate the NC tensors $\mathcal{J}^a_\mu$. Unfortunately, such a computation is too tedious and gives rise to cumbersome expressions that are irrelevant for this work. Moreover, their interpretation needs more investigations whose results will be in the core of a forthcoming work.

In chapter III, we have implemented the dynamical noncommutativity introduced by Aschieri $et\ al$ [3] in the new class of renormalizable NC field theories (NCRFT) built on the Grosse and Wulkenhaar (GW) $\phi^{\star 4}$ scalar field model defined in Euclidean space. The corresponding equations of motion and Noether currents have been studied and explicitly computed taking into account different contributions from velocity term, mass term, $\phi^{\star 4}$ interaction and GW harmonic interaction term. When $e^a_\mu = \delta^a_\mu$ the $\star-$product between any two functions reduces to the Moyal product, as already observed in [3]. The field ϕ acts as a source for the noncommutativity field ϕ^c. Our investigation has showed that the twisted GW action is not invariant under global translation. Such an undesirable feature has been got round by imposing a constraint on the Lagrangian action, which is nothing but the equation of motion governing the GW harmonic term. The previous works [3]-[28] have pointed out that the ordinary ϕ^4-theory leads to nonlocally conserved and symmetric EMT and AMT while the twisted non renormalizable ϕ^4-theory restores the local conservation of these tensors. Contrarily, both ordinary GW [7, 8] and twisted GW models provide nonlocally conserved and nonsymmetric EMT, AMT and DC due to the presence of the harmonic term Ω. Fortunately, as shown in [7], all these physical quantities can be subjected to well known Jackiw and Wilson regularization procedures to acquire the local conservation property. We have also defined the twisted connections in noncommutative spaces and discussed NC gauge transformations. Then, the YM action, twisted in the dynamical Moyal space, has been proved to be invariant under $U_\star(1)$ local gauge transformation with the parameter $\alpha_1 = \alpha_0 + \epsilon_\mu x^\mu$, where ϵ_μ is an infinitesimal parameter and α_0 a constant. The NC gauge invariant currents are explicitly computed. These currents are locally conserved. Finally, it is worthy to mention that the approach developed here can be extended to investigate twisted gauge theory in finite $D-$dimensional Moyal space. The only technical difficulty resides in the fact that the choice of ω could not be arbitrarily made. For this reason, the canonical form of e^μ_a given by $\delta^\mu_a + \omega^\mu_{ab} x^b$ seems to be natural, except for the trivial case when $\omega = 0$.

We have investigated, in Chapter IV, the main properties of harmonic oscillator in the framework of a dynamical noncommutativity realized through a twisted Moyal product. The construction of the appropriate matrix basis has introduced an x-dependence in the definition of the star product. We have computed the states and energies of the twisted harmonic oscillator. The degeneracy states and their energy have been explicitly derived. All examined quantities easily acquire good physical properties when ω^2_{12} and x^1 are transformed into $i\omega^2_{12}$ and ix^1, respectively. Furthermore, the ordinary Moyal space tools are well recovered as expected by setting $\omega = 0$ and $\bar{\omega} = 0$. Working in the NC configuration space, explicit spectrums of harmonic oscillator with non-vanished momentum-momentum bracket are derived. Using algebraic method, we have computed the spectrums. It should be pointed out that, in order to maintain the Bose-Einstein statistics, the model parameters Θ and $\bar{\Theta}$ must satisfy the relation $\Theta^2 - \bar{\Theta}^2 = 0$. Therefore, the parameters Θ and $\bar{\Theta}$ reflect the intrinsic noncommutativity between positions and momenta, respectively, (as a Planck constant encodes the noncommutativity of position and momentum), which should be independent on the concrete physical model.

Bibliography

[1] Abou-Zeid, M. and Dorn, H.: 2001, 'Comments on the energy momentum tensor in noncommutative field theories', *Phys. Lett. B* **Vol. no. 514**, pp. 193-188.

[2] Antoine J.-P.: 2006 'Representation Theory in Classical and Quantum Physics', In: Proceedings of the Fourth International Workshop on Contemporary Problems in Mathematical Physics, Govaerts J., Hounkonnou M. and Mzesane A. (Eds), World Scientific, Singapore, pp. 3–64.

[3] Aschieri, P. Castellani, L. and Dimitrijevic, M.: 2008 'Dynamical Noncommutativity and Noether Theorem in Twisted $\phi^{\star 4}$ Theory' *Lett Math Phys.* **Vol. no. 85**, pp. 39-53.*[e-print hep-th/0106048]*

[4] Aschieri, P. Dimitrijevic, M. Meyer, F. and Wess, J.: 2005 'Noncommutative geometry and gravity' *[e-print hep-th/0510059]*.

[5] Atkinson, D. and Hounkonnou, M. N.: 2001 'Quantum mechanics, a self contained course' *Rinton Press, New Jersey*.

[6] Bayen, F. Flato, M. Fronsdal, A Lichnerowicz. Sternheimer, D.:1978 'Deformation theory and quantization. I. Deformations of symplectic structures', *Ann. Physics* **111,** 61.

[7] Ben Geloun J, and Hounkonnou M.N.: 2007 'Energy-Momentum Tensors in Renormalizable Noncommutative Scalar Field Theory' *Phys Lett B* **653** 343-345.

[8] Ben Geloun J and Hounkonnou, M N.: 2007 'Noncommutative Noether Theorem' *AIP Procceding* **956** 55-60.

[9] Ben Geloun J, Hounkonnou, M N and Massamba F.: 2007 'Moyal algebra: relevant properties, projective limits and applications in noncommutative field theory' *SUT Journal of Mathematics* **44** 55-88.

[10] J. Ben Geloun,: 2007 'Nonlinear Models in Noncommutative Geometry' *Phd thesis* ICMPA-MPA/2007.

[11] Binqsheng, L. and Sicong, J.: 2009 'Deformed squeezed states in noncommutative phase space' *[e-print math-th/0902.377]*

[12] Binqsheng, L. Sicong, J. and Taihua, H.: 2009 'Deformation quantization for coupled harmonic oscillators on a general noncommutative space' *[e-print math-th/0902.369]*

[13] Calmet X.:2005 'Space-Time Symmetries of Noncommutative Spaces', *Phys. Rev.* **D71** 085012, *e-print hep-th/0411147]*.

[14] Carlson C., Carone C. and Zobin N.: 2002 'Noncommutative Gauge Theory without Lorentz Violation', *Phys. Rev.* **D66** (2002) 075001, *e-print [hep-th/0206035]*.

[15] Chaichian, M., Kulish, P.P., Nishijima, K. and Tureanu, A.: 2004, 'On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and Its Implications on Noncommutative QFT', *Phys. Lett. B* **Vol. no. 604**, pp. 98–102;

Koch, F. and Tsouchnika, E.: 2005, 'Construction of Θ-Poincaré algebras and their invariants on M_Θ', *Nucl. Phys. B* **Vol. no. 717**, pp. 387–403;

Lukierski, J., Nowicki, A. and Ruegg, H.: 1992, 'Deformed Kinematics and Addition Law For Deformed Velocities', *Phys. Lett. B* **Vol. no. 293**, pp. 344–352.

[16] Coleman, S. and Jackiw, R.: 1971, 'Why dilatation generators do not generate dilatations?', *Annals Phys* **Vol. no. 67** pp. 552.

[17] Connes, A.: 1994 'Noncommutative Geometry' *Academic Press Inc.,* San Diego at http:/www.alainconnes.org/downloads.html

[18] Curtright, T. Fairlie, D. and Zachos, C.:1998 'Features of time-independent wigner functions', *Physical Review D*, **58**, 1-14.

[19] Das, A. and Frenkel, J.: 2003, 'On the energy momentum tensor in noncommutative field theories', *Phys. Rev. D* **Vol. no. 67**.

[20] Dello Sbarba, L. and Hussin, V.: 2005 'Group of Theoretical Methods in Physics' *Proceeding of the XXV International Colloqium on Group Theoretical Methods in Physics*, Institute of Physics Conferences Series, Vol. 185, Pogosyan, Eds. G. S. Vincent, L. E. and Wolf, K. B. (IOP, Bristol, 2005).

[21] Dirac, P.A.M.: 1925 'The Fundamental Equations of Quantum Mechanics' *Proc. Roy. Soc. Lond. A* **Vol. no. 109**, pp. 642.

[22] Dirac, P.A.M.: 1926 'On Quantum Algebra' *Proc. Cambridge Phil. Soc.* **Vol. no. 23**, pp. 412.

[23] Dirac, P.A.M.:2001 'Lectures on Quantum Mechanics' (Dover Books on Physics).

[24] Doplicher, S. Fredenhagen, K. and E.Roberts, J.: 1995, ' The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields', *Comm. Math. Phys.* **Vol. no. 172**, pp. 187-220.

[25] Douglaslas, M.R. and Nekrasov, N.A.: 2001 'Noncommutative Field Theory' *Rev. Mod. Phys* **Vol. no. 73**, pp. 977-1029. *[e-print hep-th/0106048]*

[26] Fefferman, C and Seco, L.: 1992 'Eigenvalues and eigenfunctions of ordirary differential operators' *Advances in Mathematics*, **95**, pp. 145-305

[27] Fischer, A. and Szabo, R. J.: 2009 'Duality covariant quantum field theory on noncommutative Minkowski space', *JHEP 02 031, [arXiv:0810.1195]*.

[28] Gerhold A, Grimstrup J, Grosse H, Popp L, Schweda M and Wulkenhaar R.: 2001 'The energy momentum tensor on noncommutative spaces - Some pedagogical comments' *[e-print hep-th/0012112]*.

[29] Gomis J. and Mehen T.: 2000 'Space-Time Noncommutative Field Theories and Unitarity', *Nucl. Phys.* **B591** 265–276, *[e-print hep-th/0005129]*.

[30] Gomis, J. Kammimoura, K. Llosa, J.: 2001 'Hamiltonian formalism for space-time non-commutative theories', Phys. Rev. D **63** , 045003; *hep-th/0006235*.

[31] de Goursac, A., Tanasa, A. and Wallet, J.C.: 2008, 'Vacuum configurations for renormalizable noncommutative scalar models', *Eur. Phys. J.C* **Vol. no. 53,459**.

[32] de Goursac A., Wallet J.-C. and Wulkenhaar R.: 2007 'Noncommutative Induced Gauge Theory' *European Physical Journal* **C4** (2007) 977–987, *[e-print hep-th/0703075]*.

[33] Govaerts J., *Hamiltonian Quantization and Constrained Dynamics*, Leuven University Press, Louvain, 1991.

[34] Gracia-Bondía, J.M. and Várilly, J.C.: 1988 'Algebras of Distributions Suitable for Phase-Space Quantum Mechanics I' *J. Math. Phys.* **Vol. no. 29**, pp. 869-879.

[35] Gracia-Bondía, J.M. and Várilly, J.C.:1988, 'Algebras of distributions suitable for phase-space quantum mécanics II. Topologies on the Moyal algebra', *J. Math. Phys* **Vol. no. 29**.

[36] Grimstrup, J.M., Kloiböck, B., Popp,L., Putz, V., Schweda, M. and Wickenhauser, M.: 2004 'The energy momentum tensor in noncommutative gauge field models', *Int. J. Mod. Phys. A* **Vol. no. 19**, pp. 5615-5624.

[37] Grosse, H. and Wulkenhaar, R.: 2005, 'Renormalization of ϕ^4-theory on Non Commutative R^4 in the matrix base', *Comm. Math. Phys.* **Vol. no. 256**, pp. 305–374.

[38] Hatzinkitas, A. and Smyrnakis, I.: 2001 ' The noncommutative harmonic oscillator in more than one dimensions' *[hep-th/0006235]*.

[39] Hirshfeld, A. C. and Henselder, P.:2002 'Deformation quantization in the teaching of quantum mechanics', *American Journal of Physics*, **70** (5) :537-547, May.

[40] Hounkonnou, M.N., Massamba, F. and Ben Geloun, J.: 2005 '2-dimensional noncommutative field theory on the light cone', *JGSP* **Vol. no. 4**, pp. 38-46.

[41] Hounkonnou, M,N. and Ousmane Samary, D.: 2008, 'Noncommutative complex Grosse-Wulkenhaar model', *American Institut of Physics: Proc* **Vol. no. 1079**, pp. 82-88.

[42] Hounkonnou, M,N. and Ousmane Samary, D.: 2009, 'Energy momentum tensor in noncommutative complex scalar field theories', *Copromath Proc* **Vol. no.** , pp. -.

[43] Hounkonnou, M.N. and Ousmane Samary, D.: 2010 'Twisted Grosse-Wulkenhaar $\phi^{\star 4}$ model: dynamical noncommutativity and Noether currents', *J. Phys. A: Math. and Theor.* **43** 155202.

[44] Hounkonnou, M.N. and Ousmane Samary, D.: 2010 'Harmonic oscillator in twisted Moyal plane: eigenvalue problem and relevant properties', *J. Math. Phys.***51** 102108 *[e-print math-ph/1008.1325]*.

[45] Hounkonnou, M.N. and Ousmane Samary, D.: 2011 'Twisted Yang-Mills field theory: connections and Noether currents' *J. Phys. A: Math. and Theor.* **44** 315401.

[46] Hounkonnou, M.N. and Ousmane Samary, D.: 2011 'of harmonic oscillator in a general noncommutative phase space: Spectrum in relevant representations' *American Institut of Physics: Proc* in press.

[47] Hounkonnou, M.N. and Ousmane Samary, D.: 2011 'Hamiltonian formulation of the Grosse-Wulkenhaar $\phi^4_{\star D}$ model', *J. Math. Phys.,[e-print math-ph/1008.1325],submitted*.

[48] Hubbard, J. 1963 *Proc. Roy. Soc.* **276** 238

Lieb, E.H.:1995 *XI Int. Cong. MP, Int. Press,* p.392. (cond-mat/9311033).

[49] Jackiw R.: 1978 'Gauge-Covariant Conformal Transformations', *Phys. Rev. Lett.* **41** 1635–1638.

[50] Jonke, L. and Meljanac, S.: 2003 ' Representations of noncommutative quantum mechanics and symmetries' *[e-print hep-th/0210042]*.

[51] Kempf, A. Mangano, G. and Mann, R.B.: 1995 'Hilbert Space Representation of the Minimal Length Uncertainty Relation' *J. Phys.* D **Vol. no. 52**, pp. 1108.

[52] Kim, B.J. Koh, H. Rotenberg, E. Oh, S.-J. Eisaki, H. Motoyama, N. Uchida, S. Tohyama, T. Maekawa, S. Shen, Z.-X. and Kim, C.:2006 *Nature Physics* **2** 397.

[53] Koch F. and Tsouchnika E.: 2005 'Construction of Θ-Poincaré Algebras and Their Invariants on M_Θ', *Nucl. Phys.* **B717** 387–403, *[e-print hep-th/0409012]*.

[54] Kontsevich, M.: 1999 'Deformation quantization of Poisson manifolds,I', *[e-print hep-th/9707040]*

[55] Land, M.: 2009 'Harmonic oscillator states with non-integer orbital angular momentum' *[e-print math-th/0902.1757]*

[56] Landau L. and Lifchitz E.: 1989 'Théorie des Champs', Fourth Edition, Editions Mir, Moscou.

Ryder L.: 1991 'Quantum Field Theory', Seventh Edition, Cambridge Univ. Press, Cambridge, 1991.

[57] Langmann E and Szabo R J 2002, 'Duality in Scalar Field Theory on Noncommutative Phase Spaces' *Phys Lett B.* **533** 168-177.

[58] Laughling, R.B.:1983 *Phys. Rev. Lett.* **50** 1395.

[59] Lee, P.A. Nagaosa, N. and Wen, X.G.:2006, *Rev. Mod. Phys*, **78** 17 (2006) 17; Dagotta, E.:1994 *Rev. Mod. Phys* **66** 000763.

[60] Lerda, A.:1993 'Anyons: Quantum mechanics of particles with fractional spin' (Lecture notes in Physics), *Springer Verlag, Berlin*.

[61] Llosa, J. and Vives, J.: 1994 'Hamiltonian formalism for nonlocal Lagrangians', J. Math. Phys. **35** 2856.

[62] Liu H. and Michelson J.: 2001 '$\star$-Trek: The One Loop N=4 Noncommutative SYM action', *Nucl. Phys*. **B614** 279–304, *[e-print hep-th/0008205]*.

[63] Lukierski J., Nowicki A. and Ruegg H.: 1992 *Phys. Lett*. **B293** 344–352.

[64] Madore J, Schraml S, Schupp P, Wess J.: 2000 'Gauge Theory on Noncommutative Spaces' *[e-print hep-th/0001203]*.

[65] Micu A. and Sheikh-Jabbari M.:2001 'Noncommutative ϕ^4 at Two Loops', *JHEP* **00** 025, *[e-print hep-th/0008057]*.

[66] Minwalla, S. van Raamsdonk, M. and Seiberg,N.:200O *JHEP* 0002 020 *[e-print hep-th/9912072.]*

[67] Nagaosa, N.:1994 'Quantum Field Theory in Strongly Correlated Electronic Systems' **9** *Springer Verlag, Berlin*.

[68] Nagatou, K. Nakao, M. and Wakayama, M.: 2OO2 'Verified numerical computations of eigenvalue problems for non-commutative harmonic oscillators' *Numerical Functional Analysis and Optimization*, **23**, pp. 633-650.

[69] Nair, V. P. and Polychronakos, A. P.:2001 'Quantum Mechanics on the Noncommutative Plane and Sphere', *[e-print hep-th/0011172]*.

[70] Niemi, A. Hounkonnou, M.N. and Ousmane Samary, D.: 2011 'Spin charge separation in the Schrodinger equation' , reprint.

[71] Meyer, F.: 2005 'A Gravity Theory on Noncommutative Spaces' *International Europhysics Conference on High Energy Physics*.

[72] Nakahara, M.: 2003 'Geometry, Topology and Physics' (Institute of Physics Publishing, Bristol and Philadelphia, Second Edition).

[73] Rivasseau, V.: 2007 'Noncommutative renormalization', *Seminaire Poincaré X Espace Quantique, Inst. Henri Poincaré, Paris*, pp. 15-95.

[74] Rivasseau V., Vignes-Tourneret F. and Wulkenhaar R.: 2006 'Renormalization of Noncommutative ϕ^4-theory by Multi-Scale Analysis', *Commun. Math. Phys.* **262** 565–594, *[e-print hep-th/0501036]*;

Gurau R., Magnen J., Rivasseau V. and Vignes-Tourneret F.: 2006 'Renormalization of Noncommutative ϕ^4_4 Field Theory in Direct Space', *Commun. Math. Phys.* **267** 515–542, *[e-print hep-th/0612271]*.

[75] Sayipjamal, Dulat. and Li, Kang.: 2008 'Landau problem in noncommutative quantum mechanics' *Chinese Physics C.* **Vol. no. 32, No. 2**, pp. 92-95.

[76] Seiberg N. and Witten E.:1999 'String Theory and Noncommutative Geometry', *JHEP*, **09** 032, *[e-print hep-th/9908142]*.

[77] Scholtz, F. G. Chakraborty, B. Gangopadhyay, S. and Govaerts,J.: 2005 *J. Phys. A: Math. Gen.* **38**, 9849-9858 (2005); [e-print `cond-mat/0509331`];

F. G. Scholtz, B. Chakraborty, S. Gangopadhyay and A. Ghosh Hazra, Phys. Rev. D **71**, 085005 (2005); [e-print `hep-th/0502143`].

[78] Szabo, R.J.: 2003 'Quantum Field Theory on Noncommutative Spaces' *Phys. Rept.* **Vol. no. 378**, pp. 207-299.*[e-print hep-th/0109162]*

[79] Twareque Ali, S. Bagarello, F. and Honnouvo, G.: .... ' Modular structures on trace class operators and applications to Landau levels'

[80] Vakarchuk, I.O.: 2005 'On Dirac Theory in the Space with Deformed Heisenberg Algebra: Exact Solutions' *J. Phys.* A: *Math. Gen.* **Vol. no. 38**, pp. 7567-7576.

[81] Vignes-Tourneret F.: 2007 'Renormalization of The Orientable Non-Commutative Gross-Neveu Model', *Ann. H. Poincaré*, **3** Vol.8 427–474, *[e-print math-ph/0606069]*.

[82] Villanueva V., Nieto J., Ruiz L. and Sylvas J.: 2005 'Hamiltonian Noether Theorem for Gauge Systems and Two Time Physics', *J. Phys. A: Math. Gen.* **38** 7183–7195, *[e-print hep-th/0503093]*.

[83] Wei, Gao-Feng. Long, Chao-Yun. Long, Zheng-Wen. and Quin Shui-Jie: 2008 'Exact solution to two-dimensional isotropic charged harmonic oscillator in uniform magnetic field in non-commutative phase space' *Chinese Physics C.* **Vol. no. 32, No. 4**, pp. 247-250.

[84] Wess J.: 2004 'Deformed Coordinate Spaces, Derivatives', *[e-print hep-th/0408080]*; : 2006 'Differential Calculus and Gauge Transformations on a Deformed Space',: 2006 *[e-print hep-th/0607251]*; :2006 'Deformed Gauge Theories', *[e-print hep-th/0608135]*.

[85] Wess, J.: 2004 'Deformed coordinate spaces, derivatives' *unpublished, [e-print hep-th/0408080]*.

[86] Weyl, H.: 1928 'Quantenmechanik und Gruppentheorie, *Z. Physik* **46**, 1; 1931 'The theory of groups and quantum mechanics', *Dover, New-York*, translated from Gruppentheorie und Quantenmechanik, *Hirzel Verlag, Leipzig 1928*.

[87] Wigner, E. P.: 1932 'Quantum corrections for thermodynamic equilibrum', *Phys. Rev.* **40**, 749.