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CENTRE DE RECHERCHE ET DE FORMATION DOCTORALE EN SCIENCES TECHNOLOGIE ET GEOSCIENCES
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THE UNIVERSITY OF YAOUNDE I
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POSTGRADUATE SCHOOL OF SCIENCE, TECHNOLOGY AND GEOSCIENCES
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RESEARCH AND TRAINING UNIT FOR DOCTORATE IN MATHEMATICS, INFORMATIC, BIOINFORMATICS AND APPLICATION

**DEPARTEMENT DE MATHEMATIQUE**
*DEPARTMENT OF MATHEMATICS*

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# Weighted estimates for operators associated to the Bergman-Besov kernels and Hardy Dirichlet spaces

**THESIS**

Submitted to the graduate school in fulfilment of the requirements for the degree of Doctor of Philosophy/Doctorat PhD in Mathematics

**Option :** Classical Analysis

By

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# ATTESTATION DE CORRECTION DU DOCTORAT/Ph.D

Nous soussignés, Pr. **AYISSI Raoul Domingo**, Président du jury, Pr. **NOUNDJEU Pierre**, Examinateur du jury de la thèse de Doctorat/Ph.D, présenté par **KEUMO NGUEKENG Adriel Raoul** , Matricule **10V0214**, intitulé : « **Weighted Estimates for operator associated to the Bergman-Besov kernel and weighted composition operator on Hardy space of Dirichlet series**» et soutenu en vue de l'obtention du diplôme de **Doctorat/Ph.D en Mathématiques**, attestons que cette thèse a été réalisée dans le cadre des exigences de l'Université de Yaoundé I, conformément aux normes académiques en vigueur, et que les corrections et recommandations demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

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# DECLARATION OF ORIGINALITY

This is to certify that to my knowledge, the content of this thesis is the product of my work. I confirm that this work has been done after the registration for the degree of PhD of Mathematics at the University of Yaounde I. 1 attest that I have not used inadmissible help of a third parties to produce this work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due to reference to the literature. I also attest that this work has not been previously included in a thesis or a dissertation submitted to this or any other institution for a degree or other purposes.

Date: January 18th 2024

KEUMO NGUEKENG Adriel Raoul

# ♣ Dédicace ♣

***A mon feu père KEUMO, qui n'a pas eu le temps de voir ce travail, à ma mère DJOUKOU ESTHER, à ma tante NGUETSA BRIGITTE, à la famille NOUMEYE.***

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# ♣ Résumé ♣

Nous caractérisons les poids pour lesquels les opérateurs intégraux standard induits par les noyaux de Bergman-Besov sont bornés entre deux classes de Lebesgue sur la boule unité de $\mathbb{C}^N$ en termes de conditions du type Békollè -Bonami sur les poids. Pour ce faire, nous utilisons la stratégie de preuve introduite par Békollè [11].

Nous considérons les espaces de Hardy de series de Dirichlet $\mathcal{H}^p$ et nous caractérisons les multiplicateurs c'est-à-dire les opérateurs $m : f \longrightarrow mf$ qui sont bornés de $\mathcal{H}^p$ à $\mathcal{H}^q$. Nous caractérisons également les fonctions $\phi$ pour lesquelles $f \longrightarrow f \circ \phi$ est bornée de $\mathcal{H}^p$ dans $\mathcal{H}^q$. Plus généralement, nous étudions l' opérateur de composition à poids borné $uC_\phi : f \longrightarrow u(f \circ \phi)$ de $\mathcal{H}^p$ dans $\mathcal{H}^q$.

# ♣ Abstract ♣

We characterize the weights for which we have the boundedness of standard weighted integral operators induced by the Bergman-Besov kernels acting between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$ in terms of Békollè - Bonami type condition on the weights. To accomplish this we employ the proof strategy originated by Békollè [11].

We consider the Hardy spaces of Dirichlet series $\mathcal{H}^p$ and characterize multiplier i.e such a operators $m : f \longrightarrow mf$ which are bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$ and we characterize functions $\phi$ so that $f \longrightarrow f \circ \phi$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$. More generally, we study the boundedness of weighted composition operator $uC_\phi : f \longrightarrow u(f \circ \phi)$ from $\mathcal{H}^p$ to $\mathcal{H}^q$.

# ♣ Contents ♣

# ♣ General introduction ♣

This thesis has two parts consisting of two chapters each. Part one is about weighted estimates for operators associated to the Bergman-Besov kernels and the second part is about multipliers and composition operators on Hardy Dirichlet spaces.

## 0.1 Introduction of part I

Weighted inequalities appeared almost simultaneously with the birth of singular integrals that stimulated their development. In particular, a natural question is the characterization of positive functions $\omega$ for which a singular integral maps $L^p(\omega d\mu)$ to itself. A famous example of a singular integral is the Bergman projection, whose boundedness problem, solved elsewhere by Békollè and Bonami, is historically linked to the duality problem for Bergman spaces.

The inner product and the norm in $\mathbb{C}^N$ are $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_N\overline{w_N}$ and $|z| = \sqrt{\langle z, z\rangle}$. We let $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ where $q > -1$ and $\mu$ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$, and $c_q$ is the normalized constant, that is $\mu_q(\mathbb{B}) = 1$. Take $(c_q = \frac{\Gamma(n+q+1)}{N!\Gamma(q+1)})$. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$. For $a > -1$, it is a well-known result of Békollè and Bonami that the Bergman projection $T_a$ defined by

$$T_a f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1-\langle z, x\rangle)^{N+1+a}} d\mu_a(x)$$

is bounded on $L^p(\omega d\mu_a)$ if and only if the weight $\omega$ belongs to the so-called Békollè - Bonami class [11]. The Bergman projection can be extended to all $a$ less than or equal to $-1$. Therefore a natural question is whether the Békollè - Bonami result can be generalized. To this setting, in this paper we work with more general operators than the extended Bergman projection, and more generally we characterize weights for which we have the boundedness between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$.

We set $L^p_q := L^p(d\mu_q)$, the Lebesgue space on $\mathbb{B}$ relative to $\mu_q$ with $1 \le p \le \infty$. Let $H(\mathbb{B})$ denote the space of holomorphic functions in the unit ball $\mathbb{B}$. For $q > -1$, a function $f \in H(\mathbb{B})$ belongs to the weighted Bergman space $A^p_q$ whenever $f \in L^p(d\mu_q)$. The norm $\|f\|_{A^p_q}$ is simply the $L^p_q$ norm of $f$.

Besov spaces extend weighted Bergman spaces to all $q$. To define them, we first take a radial differential operator $D^t_s$ of order $t$ for any $s, t \in \mathbb{R}$ defined on $H(\mathbb{B})$. Let $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which $f_k$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree $k$. We define, for $s, t \in \mathbb{R}$

$$D^t_s f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

where

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if} \quad a > -(N+1), \\ \frac{k!}{(1-N-a)_k} & \text{if} \quad a \leq -(N+1). \end{cases}$$

Consider the linear transformation $I^t_s$ defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) = (1-|z|^2)^t D^t_s f(z).$$

We say that a function $f \in H(\mathbb{B})$ belongs to the Besov space $B^p_q$ whenever $I^t_s f \in L^p_q$ for some $s, t$ satisfying

$$\begin{cases} q + pt > -1 & \text{if } 1 \leq p < \infty \\ t > 0 & \text{if } p = \infty. \end{cases}$$

It is well known [53] that the $L^p_q$-norm, $\|I^t_s f\|_{L^p_q}$, of any one of the functions $I^t_s f$ is an equivalent norm for $\|f\|_{B^p_q}$, the norm of $f$ in $B^p_q$. When $q > -1$ we have $A^p_q = B^p_q$. The space $B^2_q$ is a Hilbert space with reproducing kernel $K_q$ (see [53] or [9, Theorem 1.9] or [87]) defined by

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} = \sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, & \text{if } q > -(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z,w\rangle^k, & \text{if } q \leq -(N+1), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(u)_v$ is the Pochhammer symbol defined by $(u)_v = \frac{\Gamma(u+v)}{\Gamma(u)}$ with $\Gamma$ the gamma function. Namely, for a number $s$ satisfying $q+1 < p(s+1)$, if $t$ satisfies $q + pt > -1$ then for $f \in B^2_q$ (see [53, Theorem 1.2])

$$(P_s \circ I^t_s) f = \frac{N!}{(1+s+t)_N} f,$$

where

$$P_s f(z) = \int_{\mathbb{B}} K_s(z,w) f(w) (1-|w|^2)^s d\mu(w),$$

is the extended Bergman projection ($s$ may be smaller than or equal to $-1$).

For $a, b, s, t \in \mathbb{R}$ the operators that we are interested in are defined by (reproducing) Bergman-Besov kernels. For $f \in L^p(d\mu_q)$ we define

$$\begin{aligned} T^q_{a,b}f(z) &:= T_{a,b}f(z) = \int_{\mathbb{B}} K_a(z,w)f(w)(1-|w|^2)^{b-q}d\mu_q(w), \\ S^q_{a,b}f(z) &:= S_{a,b}f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|(1-|w|^2)^{b-q}d\mu_q(w), \\ P^q_{s,t}f(z) &:= P_{s,t}f(z) = (1-|z|^2)^t \int_{\mathbb{B}} K_{s+t}(z,w)f(w)(1-|w|^2)^{s-q}d\mu_q(w). \end{aligned}$$

Throughout the paper $b > -1$ and $s > -1$ because we want our operator to be well defined (see for example Lemma 3.4.1). Note that

$$P_{s,t}f(z) = (1-|z|^2)^t T_{s+t,s}f(z) \tag{0.1}$$

and

$$T_{a,b}f(z) = (1-|z|^2)^{b-a} P_{b,a-b}f(z). \tag{0.2}$$

Our main motivation comes from the operators $P_{s,0}$, and $P^+_{s,N+1+s}$, which are the Bergman projection and Berezin transform respectively, where $P^+_{s,N+1+s}f(z) = (1-|z|^2)^t S_{s+N+1+s,s}f(z)$. The operators $P_{s,t}$, $T_{a,b}$ and $S_{a,b}$ are important in the study of function-theoretic operator theory, see for example [88] when $q = -N-1$.

The boundedness of the operators $T^q_{a,b}$ was already studied by Kaptanoglu and Ureyen [54] in the cases where the operators $T^q_{a,b}$ act from $L^p_q$ to $L^P_Q$, with $q \in \mathbb{R}, 1 \leq p, P \leq \infty, Q > -1$.

**Theorem 0.1.1.** *[54, Theorem 1.2] Let $a, b, q, Q \in \mathbb{R}$, $1 \leq p \leq P \leq \infty$, and assume $Q > -1$ when $P < \infty$. Then, the following three conditions are equivalent.*

1. $T_{a,b} : L^p_q \to L^P_Q$;
2. $S_{a,b} : L^p_q \to L^P_Q$;
3. (a) $\frac{1+q}{p} < 1+b$ *and* $a \leq b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \leq P < \infty$;

   (b) $\frac{1+q}{p} \leq 1+b$ *and* $a \leq b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 = p \leq P \leq \infty$, *but at least one inequality must be strict;*

   (c) $\frac{1+q}{p} < 1+b$ *and* $a < b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \leq P = \infty$.

This result is useful for our work, especially for the case $p = P$ and $q = Q$, to investigate the case where these operators $P_{s,t}$ and $T_{a,b}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$. Here the weak Lebesgue space, $L^{1,\infty}_q$, is the space of measurable functions $f$ for which there exists $A > 0$ such that for all $\lambda > 0$, $\lambda\mu_q\{z \in \mathbb{B} : |f(z)| > \lambda\} \leq A$. Our main result in this direction is the following.

**Theorem 0.1.2.** *In the case $q = s$, $s + 2t > -1$ and $s + t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$ and not from $L^1_q$ to $L^1_q$.*

In this paper we are mainly interested on the weighted estimates for the operators $T_{a,b}$ or $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Here and throughout the paper $\omega$ is a locally integrable positive function called a weight. It follows from (0.1) and (0.2) that it will be enough to study weighted estimates for one of these two operators since the $(L^p(\omega d\mu_q), L^p(\omega d\mu_Q))$ inequality for the family $T_{a,b}$ is equivalent to the $(L^p(\omega d\mu_q), L^p(\omega d\mu_{Q+pt}))$ inequality for the family $P_{s,t}$ and conversely. In the special case of the Bergman projection $T_{a,0}$, Békollè [11] obtained the characterisation of the weights $\omega$ in terms of the Békollè -Bonami condition.

Let $d$ be the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z| - |w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle\right| & z, w \in \overline{\mathbb{B}}^* \\ |z| + |w| & z = 0 \, or \, w = 0. \end{cases}$$

We will consider pseudo-balls in $\mathbb{B}$, $B(z,r)$ as points $w$ of $\mathbb{B}$ so $d(z,w) < r$ and we say that $B(z,r)$ touches the boundary of $\mathbb{B}$ if $r > 1 - |z|$, we then denote by $B \cap \partial\mathbb{B} \neq \varnothing$

**Definition 0.1.3** (Békollè - Bonami class)**.** *Let $a > -1$. Let $\omega$ be a weight on $\mathbb{B}$. We say that $\omega d\mu_a$ belongs to $(B^a_p)$, $1 < p < \infty$, if there is a constant $B^a_p(\omega)$ such that for every ball B (with respect to the pseudo-distance $d$) of $\mathbb{B}$ that intersects the boundary of $\mathbb{B}$, we have*

$$\left(\frac{1}{\mu_a(B)} \int_B \omega(z) d\mu_a(z)\right) \left(\frac{1}{\mu_a(B)} \int_B \omega^{\frac{-1}{p-1}}(z) d\mu_a(z)\right)^{p-1} \leq B^a_p(\omega).$$

For $a > -1$, let

$$T_a f(z) := T_{a,0} f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1 - \langle z, x \rangle)^{N+1+a}} d\mu_a(x)$$

be the Bergman projection. Békollè showed in [11] that

**Theorem 0.1.4.** *Let $\omega$ be a weight on $\mathbb{B}$. The operator $T_a$, $a > -1$, is well defined and continuous on $L^p(\omega d\mu_a)$, $1 < p < \infty$, if and only if $\omega d\mu_a \in (B^a_p)$.*

The results we obtain depend upon the values of $s + t$, $q$ and $Q$. In the case $s + t < -(N+1)$ we have the following two main results.

**Theorem 0.1.5.** *In the case $s + t < -(N+1)$, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

**Theorem 0.1.6.** *Let $\omega$ be a weight on $\mathbb{B}$. In the case $s + t < -(N+1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

*Moreover*

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right) \left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

In order to give our necessary condition for the boundedness of $T_{a,b}$ when $a > -(1+N)$ we introduce a Békollè -Bonami type class of weights denoted by $(B_p^{a,b,q,Q})$.

**Definition 0.1.7.** *Let $\omega$ be a weight on $\mathbb{B}$. For $Q \le q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if*

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

*where the supremum is taken over the pseudo-balls $B$.*

*For $Q \le q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if*

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

*For $Q > q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if*

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

*where the supremum is taken over the pseudo-balls $B$.*

*For $Q > q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if*

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z)\right) \left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

A necessary condition for the boundedness of $P_{s,t}$ when $-1 > s+t > -(1+N)$ is the following.

**Theorem 0.1.8.** *In the case both $-(N+1) < s+t < -1$ and $s+t+\frac{Q-q}{p} \le -1$ hold, and in the case both $s+t > -1$ and $Q < q$ hold, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

For the remaining cases, we introduce another Békollè -Bonami type class of weights, $(K_p^{s,t,q,Q})$, in order to give our necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$.

**Definition 0.1.9.** *Let $\omega$ be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p}>-1$ and $-1>s+t>-N-1$, we say that $\omega\in(K_p^{s,t,q,Q})$ $(s>-1)$ if*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

*For $Q\geq q$ and $s+t>-1$, we say that $\omega\in(K_p^{s,t,q,Q})$ $(s>-1)$ if*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

A necessary condition for the boundedness of $P_{s,t}$ when $s+t>-(1+N)$ is the following.

**Theorem 0.1.10.** *In the case both $s+t+\frac{Q-q}{p}>-1$ and $-1>s+t>-N-1$ hold, and in the case both $s+t>-1$ and $Q\geq q$ hold, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega\in(K_p^{s,t,q,Q})$.*

We introduce a maximal and a fractional maximal operator that will be used to establish a good lambda inequality in order to give sufficient conditions for the boundedness of $P_{s,t}$.

If $a>-1$ we set

$$m_{a,b}f(z)=\sup_{\zeta\in\mathbb{B},R>1-|\zeta|:z\in B(\zeta,R)}\frac{1}{\mu_a(B(\zeta,R))}\int_{B(\zeta,R)}|f(w)|d\mu_b(w),$$

more generally if $a>-1-N$ we set

$$m'_{a,b}f(z)=\sup_{\zeta\in\mathbb{B},R>1-|\zeta|:z\in B(\zeta,R)}\frac{1}{R^{N+1+a}}\int_{B(\zeta,R)}|f(w)|d\mu_b(w).$$

Before giving our good lambda inequality, we introduce here $(D_p^{s,t,q,Q})$ a Békollè-Bonami type class of weights.

**Definition 0.1.11.** *Let $\omega$ be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p}>-1$ and $-1>s+t>-N-1$, we say that $\omega\in(D_p^{s,t,q,Q})$ $(s>-1)$ if*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

*For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (D_p^{s,t,q,Q})$ ($s > -1$) if*

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$. In each case we denote by $D_p^{s,t,q,Q}(\omega)$ the expression in the left hand side.*

**Remark 0.1.12.** *Constants and standard weights ($\omega(z) = (1-|z|^2)^\eta$) are in $(D_p^{s,t,q,Q})$. We also have $(D_p^{s,t,q,q}) \subseteq (D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. For $Q = q$ we have $(K_p^{s,t,q,Q}) = (D_p^{s,t,q,Q})$.*

Here is our good lambda inequality.

**Theorem 0.1.13.** *Suppose that $1 < p < \infty$. Let $\omega \in (D_p^{s,t,q,Q})$ where both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, or both $s+t > -1$ and $Q \geq q$ hold. There are two positive constants $C$ and $\beta$ such that for all $\gamma$ sufficiently small, $\lambda > 0$ and for all positive locally integrable functions $f$, if $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$, then*

$$\begin{aligned}\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq \\ C D_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (0.3)\end{aligned}$$

To show that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ when $s+t > -1$, we introduce the following maximal and fractional maximal operator. If $s+t > -1$ we set

$$O_{s,t}f(z) = (1-|z|^2)^t m_{s+t,s}f(z);$$

more generally if $s+t > -1-N$ we set

$$O'_{s,t}f(z) = (1-|z|^2)^t m'_{s+t,s}f(z).$$

The following theorem shows together with the good lambda inequality that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$.

**Theorem 0.1.14.** *For $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$, if $\omega d\mu_q \in (D_p^{s,t,q,Q})$, there is a constant $C_{s,t,p,q,Q} > 0$ such that for all $f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z)d\mu_q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}} |f(z)|^p \omega(z)d\mu_q(z).$$

Then for the case $s+t > -1$ and $q = Q$ we have

**Corollary 0.1.15.** *Let $\omega$ be a weight on $\mathbb{B}$. Then for $s+t > -1, s > -1$ the following assertions are equivalent.*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*

2. $T_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$*;*

3. $S_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$*;*

4. $\omega \in (K_p^{s,t,q,Q})$.

Rahm, Tchoundja and Wick settled in [67] the particular case of the operators $P_{s,t}$ for $s+t > -1, s > -1, Q = q = s$. To this aim, they used dyadic methods that have been initiated by Aleman, Pott and Reguera in the unit disk [3].

The outline of Part I is as follows. In the first chapter of part I, we define and give properties of the space $B_q^p$. In the second Chapter of part I, we provide proof of each of our mains theorems stated in this introduction as follows: In Section 2 we briefly give requisite background information. We prove Theorem 0.1.2 in Section 3. From Section 4 on, we look at weighted estimates; there we show Theorem 0.1.5 and Theorem 0.1.6. The proof of Theorem 0.1.8, Theorem 0.1.10, Theorem 0.1.14 are in Section 5. The proof of Theorem 0.1.13 is in Section 6. Corollary 0.1.15 appears in Section 7.

## 0.2 Introduction of Part II

In [40] the authors considered the space $H = L^2(0,1)$ and $\varphi \in H$, viewed as 2-periodic odd function on $\mathbb{R}$ through its Fourier expansion

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin n\pi x, \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty. \tag{0.4}$$

Now, let $\varphi_n \in H$ be the dilated function of $\varphi$ defined by $\varphi_n(x) = \varphi(nx)$. they observe that a natural orthonormal basis of $H$ is formed by the sequence of functions $\sqrt{2}\sin n\pi x$, $n = 1, \cdots$ , which are the dilates of the single function $\psi(x) = \sqrt{2}\sin \pi x$. They wanted to know for which functions $\varphi \in H$ the sequence $(\varphi_n)$ formed an orthonormal basis of $H$. They observed that exact orthonormality is surely too much to be required, because then the only possibility is easily seen to be the function $\psi$! Therefore they weaken a little their ambitions and ask under which conditions the system $(\varphi_n)$ of those dilated functions verifies either of the conditions:

1. The system $(\varphi_n)$ is complete, i.e. the space generated by the $\varphi_n$ is dense in $H$.

2. The system $(\varphi_n)$ is complete and, for some constant $C$, we have:

$$C^{-1}\left(\sum_{n=1}^{\infty} |c_n|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{n=1}^{\infty} c_n\varphi_n\right\|_2 \leq C\left(\sum_{n=1}^{\infty} |c_n|^2\right)^{\frac{1}{2}} \tag{0.5}$$

   where all but a finite number of $c_n$ are non-zero. One then says that $(\varphi_n)$ is a Riesz basis in $H$ (the image of an orthonormal basis under an onto isomorphism).

Surprisingly (cf. Theorem 5.2.1 to come), the (complete) answer to Question 1 will be in terms of the forthcoming spaces of Dirichlet series, which had been suggested by Beurling as early as in 1945.

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{C}$. We call Dirichlet series a series of the form:

$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}, \quad s \in \mathbb{C}.$$

Let $\mathbb{C}_\theta = \{s \in \mathbb{C}; \Re e > \theta\}$ be the right half plane. We will denote by $\mathcal{D}$ the set of series $\sum_{n=1}^{\infty} a_n n^{-s}$ such that the abscissae of convergence $\sigma_c$ is finite($\sigma_c < \infty$), equivalently the set of series $\sum_{n=1}^{\infty} a_n n^{-s}$ whose coefficients have at most polynomial growth. We will denote by $\mathcal{H}^2$ the space of Dirichlet series with square-summable coefficients, that is

$$f \in \mathcal{H}^2 \Leftrightarrow f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{with } \|f\|_2^2 =: \sum_{n=1}^{\infty} |a_n|^2 < \infty. \tag{0.6}$$

We will denote by $H^\infty(\mathbb{C}_0)$ the set of bounded analytic functions on $\mathbb{C}_0$, equipped with the norm $\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)|$. Now, the set $\mathcal{H}^\infty$ of bounded Dirichlet series (later seen to be the set of multipliers of $\mathcal{H}^2$) is by definition:

$$\mathcal{H}^\infty = H^\infty(\mathbb{C}_0) \cap \mathcal{D}. \tag{0.7}$$

In others terms, a function $f$ belongs to $\mathcal{H}^\infty$ if it verifies *two* properties:

1. It is a bounded analytic function in $\mathbb{C}_0$.
2. It can be represented as a convergent Dirichlet series for $\mathbb{R}e\ s$ large enough.

The Dirichlet space $\mathcal{H}^p, 0 < p < \infty$, will be defined as the completion of Dirichlet polynomials for the norm $\|\|_p$ to be defined. This *concretely means* the following: a formal Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is in $\mathcal{H}^p$ if there exists a Cauchy sequence (for the $\|\|_p$-norm) $P_j(s) = \sum_{n=1}^{L} a_n^{(j)} n^{-s}$ of Dirichlet polynomials such that

$$\lim_{j\to\infty} a_n^{(j)} = a_n \text{ for all } n = 1, 2, \ldots$$

Our interest here is to characterize multiplier i.e such a $m : f \longrightarrow mf$ which are bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$ and to characterize composition operators,that is functions $\phi$ so that $f \longrightarrow f\circ\phi$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$. More generally, to study the boundedness of weighted composition operator $uC_\phi : f \longrightarrow u(f\circ\phi)$ from $\mathcal{H}^p$ to $\mathcal{H}^q$. Our result Here, depend of $p, q$ and are stated as follows

**Theorem 0.2.1.** *Let $\mathcal{M}_{p,q}$ be the set of all $m : f \longrightarrow mf$ such that $m$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$. Then*

*(a) $\mathcal{M}_{p,q} \subsetneqq \mathcal{H}^\infty$ if $p < q$.*

*(b)* $\mathcal{H}^\infty \subset \mathcal{M}_{p,q}$ *if* $q < p$.

**Theorem 0.2.2.** *For* $q < p$, *if* $m \in \mathcal{H}^{\frac{pq}{q-p}}$ *then* $m \in \mathcal{M}_{p,q}$.

Let $P(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ be a Dirichlet series, then $\Delta P(z) = \sum \gamma_\alpha z^\alpha$, where

$$z^\alpha = z_1^{\alpha_1} \ldots z_r^{\alpha_r} \ \text{ and } \ \gamma_\alpha = c_n \text{ if } n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}.$$

With these notations, we have the following result.

**Theorem 0.2.3.** $m \in \mathcal{M}_{p,q}$ *if and only if* $d\mu(z) = |\Delta m(z)|^q d\rho(z)$ *is a* $(\mathcal{H}^p,q)$*-Carleson measure.*

**Theorem 0.2.4.** *For* $1 < p \le q < \infty$, $m \in \mathcal{M}_{p,q}$ *if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \left( \frac{1-|a|^2}{|1-\bar{a}w|^2} \right)^{\frac{q}{p}} |m_\sigma \circ \psi_\zeta^{-1}(w)|^q d\sigma(w) < \infty \quad \forall\, 0 < \sigma, \zeta < \infty$$

**Theorem 0.2.5.** *Let* $u$ *be an analytic function on* $\mathbb{D}$. *Let* $\varphi$ *be a Dirichlet series. Suppose* $\phi(s) = c_0 s + \varphi(s)$ *is holomorphic from* $\mathbb{C}_+$ *to* $\mathbb{C}_+$ *for* $c_0 \in \mathbb{N}$. *Let* $1 < p \le q < \infty$. *Suppose* $\varphi_{\eta,\zeta,\sigma} = \psi_\eta \circ \phi_\sigma \circ \psi_\zeta^{-1}$ *(*$\eta = (\sigma + c_0)\zeta$*) satisfies*

$$Sup_{\sigma>0} \lim_{\zeta\to\infty} Sup_{a\in\mathbb{D}} I_{\varphi_{\eta,\zeta,\sigma},-1}(u)(a) < \infty \quad \forall\ 0 < \eta, \zeta < \infty \tag{0.8}$$

*where* $I_{\varphi_{\eta,\zeta,\sigma},-1}(u)(a) = \displaystyle\int_{\partial\mathbb{D}} \left( \frac{1-|a|^2}{|1-\bar{a}\varphi_{\eta,\zeta,\sigma}(w)|^2} \right)^{\frac{q}{p}} |u(w)|^q d\sigma(w)$ *and* $\psi_\beta(s) = \frac{s-\beta}{s+\beta}$ *the Mobius transformation from* $\mathbb{C}_+$ *to* $\mathbb{D}$. *Then* $uC_\phi : f \longrightarrow u(f \circ \phi)$ *is bounded from* $\mathcal{H}^p$ *to* $\mathcal{H}^q$.

The outline of Part II is as follows. In the first chapter of part II, we define and give properties of Dirichlet series. In the second Chapter of part II, we start by the description of Hardy space of Dirichlet series and provide, in the last section of the same chapter, proof of each of our theorems stated in this introduction.

⋆ ⋆ Chapter 1 ⋆ ⋆

# Preliminaries Notions

We give here classical results which will help us in all the continuation of this work. For more details, we invite the reader to consult [33],[34],[37] and [69].

## 1.1 Algebra tools

Definitions and results of this section come from [37].

**Definition 1.1.1.** *(Group)*

*Let $G$ a non empty set endowed with an internal composition law denoted $\bullet$.*
*We say that $(G, \bullet)$ is a group when the following properties are satisfy:*
*a) $\bullet$ is associative, i.e for all $a, b, c \in G$, $a \bullet b \bullet c = (a \bullet b) \bullet c = a \bullet (b \bullet c)$;*
*b) $\bullet$ admits a neutral element $e$, i.e $a \bullet e = e \bullet a = a$ for all $a \in G$;*
*c) $\bullet$ is such that every element admits a symmetric, i.e for $a \in G$ there is an element, $a^{-1} \in G$, such that $a \bullet a^{-1} = e$ ou $a^{-1} \bullet a = e$.*

*A group $(G, \bullet)$ is commutative or Abelian when the law $\bullet$ is commutative i.e for all $a, b \in G$, $(a \bullet b = b \bullet a)$.*

**Definition 1.1.2.** *Let $(G, *)$ and $(H, \bullet)$ two groups and $f$ an application from $G$ to $H$.*
*$f$ is a morphism from $(G, *)$ to $(H, \bullet)$ if:*

$$\forall a, b \in G, f(a * b) = f(a) \bullet f(b).$$

*We say that a map $f$ is an isomorphism if $f$ is a bijective morphism.*

**Definition 1.1.3.** *A vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a set $E$, whose elements are called vectors, endowed with two laws:*
*-an internal law $+$,*
*-an external law on the left $\bullet$,*
*such that*

*1) $(E, +)$ is an Abelian group,*
*2) for $u, v \in E$ and $s, t \in \mathbb{K}$*

*a)* $s \bullet (u + v) = (s \bullet u) + (s \bullet v)$
*b)* $(s + t) \bullet u = (s \bullet u) + (t \bullet u)$
*c)* $(st) \bullet u = s \bullet (t \bullet u)$
*d)* $1 \bullet u = u$ *We denote by* $(E, +, \bullet)$ *a vector space or simply* $E$.

**Definition 1.1.4.** $(A, +, \bullet, *)$ *is an algebra on* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*, if:*
*1)(A,+,•) is a vector space on* $\mathbb{K}$*;*
*2) the law * is well defined from* $A \times A$ *to* $A$*;*
*3) the law * is bilinear.*

## 1.2 Topological tools

Definitions and results of this section come from [33],[34].

**Definition 1.2.1.** *A topological space* $\mathcal{T}$ *is any couple formed of a set* $E$ *and a subset* $\mathcal{O}$ *of* $\mathcal{P}(E)$*, (the elements of* $\mathcal{O}$ *are called open set for* $\mathcal{T}$*), such that:*
*1)* $\emptyset$ *and* $E \in \mathcal{O}$ *,*
*2) every union of open sets is an open set,*
*3) any intersection of a finite number of open sets is an open set.*
*If there is no confusion, we will often call* $E$ *a topological space.*

**Definition 1.2.2.** *A subset* $A$ *of* $E$ *is a closed set if* $E \setminus A$ *is an open set.*

**Definition 1.2.3.** *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *two families of open sets of two topologies* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *on* $E$ *respectively. We say that* $\mathcal{T}_2$ *is thinner than* $\mathcal{T}_1$ *if* $\mathcal{O}_1 \subset \mathcal{O}_2$.

**Definition 1.2.4.** *We call neighborhood of an element* $x$ *of the topological space* $E$*, any part* $V$ *of* $E$ *such that there exists an open set* $O$ *satisfying:*

$$x \in O \subset V.$$

We define in the same way the neighborhood of a set.

**Definition 1.2.5.** *Consider two topological spaces* $E$ *and* $F$*. A map* $f$ *from* $E$ *to* $F$ *is said to be continuous at a point* $a \in E$ *if and only if for every neighborhood* $V$ *of* $f(a)$ *in* $F$*, there exists a neighborhood* $U$ *of* $a$ *in* $E$ *such that* $f(U) \subset V$.
*If* $f$ *is continuous at every point of* $E$*, we say that* $f$ *is continuous on* $E$.

**Theorem 1.2.6.** *Let* $E$ *and* $F$ *two topological spaces and a map* $f$ *from* $E$ *to* $F$*.* $f$ *is continuous on* $E$ *if and only if for all open set* $O \in F$*,* $f^{-1}(O)$ *is an open set of* $E$.

**Definition 1.2.7.** *Consider $E_1, E_2, ..., E_n$, $n$ topological spaces, where $n$ is an integer greater than or equal to 2, and their product $E = E_1 \times E_2 \times ... \times E_n = \prod_{i=1}^{n} E_i$ whose elements are the n-tuples $x = (x_1, x_2, ..., x_n)$. consider the $n$ canonical projections $p_i : E \longrightarrow E_i$ defined by*

$$p_i(x) = x_i.$$

*We call product topology on $E$ the thinner topology making continuous the projections $p_i$.*

The open sets of this topology are the unions of elementary open sets, parts of type $\Omega = w_1 \times w_2 \times ... \times w_n$ where $w_i$ is an open set of $E_i$ We define in the same way the product topology on any product of topological spaces.

**Definition 1.2.8.** *We call topological group a group endowed with a topology compatible with the law of the group, i.e a topology which makes the operation of multiplication and the operation of passing to the inverse both continues.*

**Definition 1.2.9.** *A topological space $E$ is said to be compact if and only if it is separated (two distinct points admit disjoint neighborhoods), and has the following property: from any covering of $E$ by open sets of $E$ we can extract a finite subcovering.*

The following properties holds:

**Proposition 1.2.10.** *1) Any closed part of a compact space is compact.*
*2) In separated $E$, every compact is closed.*
*3) A product space $E = \prod_{i=1}^{n} E_i$ is compact if and only if each $E_i$ is compact*
*4) If the space is separated, the continuous image of a compact is compact.*

**Definition 1.2.11.** *We call a locally compact space any separated topological space $E$ such that each element $x$ of $E$ has at least one compact neighborhood.*

**Proposition 1.2.12.** *1) In a locally compact set $E$, every closed set is locally compact.*
*2) A finite product of locally compact spaces is locally compact.*

## 1.3 Complex integration

**Definition 1.3.1.** *We call a curve in a topological space $X$, a continuous map from a segment [a,b] to $X$.*
*When $\gamma$ is a curve from [a,b] to $X$, such that*

$$\gamma(a) = \gamma(b),$$

*we say that $\gamma$ is a closed curve.*

**Definition 1.3.2.** *We call path any curve in $\mathbb{C}$ of class $C^1$ by pieces, i.e continuous and differentiable except possibly at a finite number of points. If the curve is closed, the path is said to be closed. We note $\gamma^*$, the image of the path $\gamma$.*

**Definition 1.3.3.** *Let $\Omega$ an open set of $\mathbb{C}$, let $f : \Omega \longrightarrow \mathbb{C}$ a function. Let $\gamma$ be a path in $\Omega$ defined in the interval [a,b] . Suppose that f is continuous on $\gamma^*$. We call integral of $f$ along the path $\gamma$, the complex denoted by*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

.

## 1.4 Holomorphic functions

Definitions and results of this section come from [69]. Let $z_0 \in \mathbb{C}$ and $r > 0$. We set by:

- $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ the open disk of $\mathbb{C}$ with center $z_0$ and radius $r$.

- $\overline{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$ is the closed disk of $\mathbb{C}$ with center $z_0$ and radius $r$.

- $\mathcal{C}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ is the circle with center $z_0$ and radius $r$.

Let $\Omega$ an open set of $\mathbb{C}$, let $f : \Omega \longrightarrow \mathbb{C}$ a function.

**Definition 1.4.1.** *We say that $f$ has a limit $l$ at a point $a \in \Omega$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \Omega$, $0 < |z - a| < \delta \Longrightarrow |f(z) - l| < \varepsilon$.*
*We then note $\lim\limits_{z \to a} f(z) = l$.*

**Definition 1.4.2.** *We say that $f$ is continuous at a point $a \in \Omega$, if $f(a)$ exists and $\lim\limits_{z \to a} f(z) = f(a)$. We say that $f$ is continuous on $\Omega$ if $f$ is continuous at every point of $\Omega$.*

**Definition 1.4.3.** *We say that $f$ is derivable at a point $a \in \Omega$, if $\lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a}$ exists.*
*We say that $f$ is derivable on $\Omega$ when $f$ is derivable at each point of $\Omega$. In this case we say that $f$ is holomorphic in $\Omega$.*
***Notation:*** *We denote by $H(\Omega)$ the set of all holomorphic functions in $\Omega$.*

**Proposition 1.4.4.** *Let $f$ and $g$ two functions in $H(\Omega)$.*
*1) We have $f + g, f - g \in H(\Omega)$.*
*2) If $g(z)$ is nonzero for any point $z$ of $\Omega$, then $\frac{f}{g} \in H(\Omega)$.*
*3)Let$\Omega_1$ an open set of $\mathbb{C}$ such that $f(\Omega) \subset \Omega_1$, let $h \in H(\Omega_1)$, then $h \circ f \in \Omega$.*
*Moreover, for all $z \in \Omega$, $(h \circ f)' = f'(z)(h' \circ f)(z)$.*

**Theorem 1.4.5.** *Let $f \in H(\Omega)$. Suppose that $\overline{D(a,r)} \subset \Omega$.*
*Then $f$ admits a series expansion of the form*

$$f(z) = \sum_{n\geq 0} c_n(z-a)^n, \ for\ all\ z \in D(a,r)$$

*. Moreover,*

$$c_n = \frac{f^{(n)}(a)}{n!}, \ for\ all\ n \geq 0$$

.

**Theorem 1.4.6.** *Let $\{f_n\}_{n\geq 1}$ a sequence of elements of $H(\Omega)$, suppose $f_n \to f$ uniformly on all compact subsets of $\Omega$. Then, $f \in H(\Omega)$ and $\{f'_n\}_{n\geq 1}$ converges to $f'$ uniformly on all compact subsets of $\Omega$.*

**Lemma 1.4.7.** *(Schwarz)*
*Let $f$ be a holomorphic function in the unit disk $U$ such that: $\sup\limits_{z\in U} |f(z)| \leq 1$ and $f(0) = 0$.*
*Then:*
*1) For all $z \in U$, $|f(z)| \leq |z|$ and*
*2) $|f'(0)| \leq 1$.*
*If we have equality in 1) for a $z \in U - \{0\}$, or if we have equality in 2), then $f(z) = \lambda z$ with $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.*

**Theorem 1.4.8.** *If a map defined on a complex-valued open set of $\mathbb{C}^n$ is holomorphic with respect to each of the coordinates $z_j$, $j = 1, 2, ..., n$ then it is holomorphic in this open set.*

## 1.5 Complex Analysis Tools

In multi-index notation, we denote by $\gamma = (\gamma_1, ..., \gamma_N) \in \mathbb{N}^N$ an $N$-tuple of nonnegative integers,the length of $\gamma$ is $|\gamma| = \gamma_1 + \cdots + \gamma_N$ and $\gamma!$ is defined by $\gamma! = \gamma_1! \cdots \gamma_N!$, $0^0 = 1$, and $z^\gamma = z_1^{\gamma_1} \cdots z_N^{\gamma_N}$. The number of distinct multi-indices $\gamma$ with $|\gamma| = m$, is $\delta_m = \begin{pmatrix} N-1+m \\ N-1 \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix}$=$\dfrac{\Gamma(a+1)}{\Gamma(b+1)(\Gamma(a-b+1)}$ see [9].

The standard basis vectors of $\mathbb{C}^N$ are $e_j = (0, ..., 0, 1, 0, ..., 0)$ with $1$ in the $j$th position, $j = 1, ..., N$. An overbar $\overline{()}$ indicates complex conjugate for functions and closure for sets.

**Definition 1.5.1.** *Given a vector space $X$, a norm on $X$ is a real-valued function $p : X \to \mathbb{R}$ with the following properties:*
*1) For all $x, y \in X$, $p(x+y) \leq p(x) + p(y)$.*
*2) For all $x \in X$ and all scalars $s$, $p(sx) = |s|p(x)$.*
*3) For all $x \in X$, if $p(x) = 0$ then $x = 0$.*
*A quasi-norm is given by the inequality, $p(f+g) \leq C(p(f) + p(g))$ for some constant $C > 1$, in place of 1).*

We steal use the term norm even when we mean quasi-norm. The inner product of a space of functions $X$ is denoted $[\cdot,\cdot]_X$. The $p$th power summable sequence spaces are denoted $l^p$.

Let $\mathbb{S} = \{z \in \mathbb{C}^N : |z| = 1\}$ be the unit sphere in $\mathbb{C}^N$ and $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ the unit ball. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$ and $\mathbb{S}$ is the unit circle $\mathbb{T}$. We let $\sigma$ be the Lebesque measure on $\mathbb{S}$ normalized so that $\sigma(\mathbb{S}) = 1$. For $0 < p \leq \infty$, we denote the Lebesque classes with respect to $\sigma$ by $L^p(\sigma)$.

Let's also recall the definition of the *Hardy spaces* on $\mathbb{B}$. For $0 < p < \infty$, we say an $f \in H(\mathbb{B})$ belongs to $H^p$ whenever

$$||f||^p_{H^p} = \sup_{0<r<1} \int_{\mathbb{S}} |f(r\zeta)|^p d\sigma(\zeta) < \infty.$$

Since $\sigma$ is finite, clearly $H^\infty \subset H^p$.

The *Pochhammer symbol* $(a)_b$ is given by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}, \tag{1.1}$$

when $a$ and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function $\Gamma$, where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad with \;\; \Re z > 0. \tag{1.2}$$

In particular, $(a)_0 = 1$ and for $k$ a positive integer, we have $(a)_k = a(a+1)\cdots - a+k-1)$. The Stirling formula yields

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\Re c \to \infty) \tag{1.3}$$

where $s \sim y$ means both $x = \mathcal{O}(y)$ and $y = \mathcal{O}(x)$ for all $x, y$ in question. If only $x = \mathcal{O}(y)$, we write $x \lesssim y$.

An $f \in H(\mathbb{B})$ can be written in terms of its *homogeneous expansion* and its Taylor series as

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{|\gamma|=0}^{\infty} f_\gamma z^\gamma$$

in which $f_k$ is a holomorphic homogeneous polynomial in $z_1, ..., z_N$ of degree $k$.

**Definition 1.5.2.** *A pseudo-distance on a non void set $X$ is a non-negative real-valued function $d : X \times X \to \mathbb{R}$ such that*
*1) For any $x, y \in X$, $d(x,y) = 0$ if and only if $x = y$.*
*2) For all $x, y \in X$, $d(x,y) = d(y,x)$.*
*3) There exists a constant $\delta(\delta \geq 1)$ such that for all $x, y, z \in X$,*

$$d(x,z) \leq \delta(d(x,y) + d(y,z)).$$

Throughout this document $d$ is the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z|-|w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle\right| & z, w \in \overline{\mathbb{B}}^* \\ |z| + |w| & z = 0 \, or \, w = 0. \end{cases}$$

Throughout this paper $K$ will be a constant such that

$$d(x,y) \leq K(d(x,z) + d(z,y)) \tag{1.4}$$

for all $x$, $y$ and $z$ in $\mathbb{B}$. We will consider pseudo-balls in $\mathbb{B}$, $B(z,r)$, as points $w$ of $\mathbb{B}$ such that $d(z,w) < r$ and we say that $B(z,r)$ touches the boundary of $\mathbb{B}$ if $r > 1 - |z|$. When $B(z,r)$ is such that $r > k(1-|z|)$ for some absolute value $k < 1$, we say that $B(z,r)$ almost touches the boundary of $\mathbb{B}$.

One can find the following two results in [11, 77].

**Lemma 1.5.3.** *For each $z \in \mathbb{B}$ and $r_0$, $0 < r_0 < 1$, if we set $z^0 = (r_0, 0, \cdots, 0)$, then we have*

1. $|1 - z_1 r_0| \geq \frac{1}{4} d(z, z^0)$;
2. $|z_1 - r_0| \leq d(z, z^0)$;
3. $|z - z^0| \leq d(z, z^0)$;
4. $\sum_{k=2}^{N} |z_k|^2 \leq 2d(z, z^0)$.

**Proposition 1.5.4.** *There is a constant $C_1 > 0$ so that for all $z, w, w_0 \in \mathbb{B}$ with $d(z, w_0) > C_1 d(w, w_0)$ we have*

$$|\langle z, w_0 \rangle - \langle z, w \rangle| \leq \frac{1}{2} |1 - \langle z, w_0 \rangle|.$$

*Then*

$$|1 - \langle z, w \rangle| \geq \frac{1}{2} |1 - \langle z, w_0 \rangle|.$$

**Definition 1.5.5.** *Let $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ be the unit ball of $\mathbb{C}^N = \mathbb{R}^{2N}$. The weighted Lebesgue measure $d\mu_q$ in $\mathbb{B}$ is defined by $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ when $q > -1$ and $\mu$ is the Lebesgue (volume) measure on the unit ball $\mathbb{B}$ and $c_q$ is the normalized constant so that $\mu_q(\mathbb{B}) = 1$, hence*

$$c_q = \frac{\Gamma(N+q+1)}{N!\Gamma(q+1)}.$$

*When $q \leq -1$, we simply write $d\mu_q(z) = (1-|z|^2)^q d\mu(z)$.*

We then obtain the following result that will be heavily used throughout the paper. This extends to all $q$ the result given in [11].

**Lemma 1.5.6.** *For each $w \in \mathbb{B}$, $0 < |w| = r < 1$ and $0 < R < 2$*

$$\mu_q(B(w,R)) \simeq R^{N+1}[\max(R, 1-r)]^q \; if \; q > -1.$$

*Then for $q > -1$, $(\mathbb{B}, d, \mu_q)$ is a homogeneous space in the sense of [20].*

*However, if $B(w,R)$ is away from the boundary ($R < \frac{1-|w|}{2}$), the equivalence remains true if $q \leq -1$, ie*

$$\mu_q(B(w,R)) \simeq R^{N+1}(1-|w|)^q.$$

*Proof.* Assume $q \leq -1$. Since $z \in B(w,R)$, we have $|w| - |z| < R$. Hence $1 - |z| < 1 - |w| + R$ and $1 - |w| < 1 - |z| + R$. Since $R < \frac{1-|w|}{2}$, we have $1 - |w| < 2(1 - |z|)$ so that

$$\frac{1}{2}(1-|w|) \leq 1 - |z| \leq 1 - |w| + R \leq \frac{3}{2}(1-|w|). \tag{1.5}$$

Hence, for all $R \in (0, \frac{1-|w|}{2})$, we have

$$\int_{B(w,R)} (1-|z|)^q d\mu(z) \simeq R^{N+1}(1-|w|)^q.$$

□

We recall the following well-known estimates in [68].

**Proposition 1.5.7.** *Let*

$$I = \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{1+N+c}} d\mu(w)$$

*for $d > -1$ and $c \in \mathbb{R}$. We have, when $|z| \to 1^-$,*

*(i) If $c < d$, then $I \sim 1$;*

*(ii) If $c = d$, then $I \sim \frac{1}{|z|^2} \log \frac{1}{1-|z|^2}$;*

*(iii) If $c > d$, then $I \sim (1-|z|^2)^{-(c-d)}$.*

**Definition 1.5.8.** *$F_s$ is a homogeneous polynomial of degree $s$ on $\mathbb{C}^n$ if $F_s$ is a polynomial and $F_s(\lambda z) = \lambda^s F_s(z)$ for all $\lambda \in \mathbb{C}$ and for all $z \in \mathbb{C}^n$.*

The following result is in [68].

**Theorem 1.5.9.** *Suppose*

*(a) $\Omega$ is an open set in $\mathbb{C}^n$,*

*(b) for $s = 0, 1, 2, \ldots$, $F_s$ is a homogeneous polynomial of degree $s$ and*

*(c)* $Sup_s|F_s(z)| < \infty$ *for every* $z \in \Omega$.

*The series* $\sum_0^\infty F_s(z)$ *converges uniformly on every compact subset of* $\Omega$.

## 1.6 Harmonic Analysis Tools

**Definition 1.6.1.** *Let* $d$ *be a pseudo distance (see Definition 1.5.2) on a non void set* $X$. *A non-negative measure* $\nu$ *is homogeneous or doubling if there is a constant* $\delta_2$ *such that for all* $x \in X$ *and* $r > 0$ *we have*

$$\nu(B(x, 2r)) \leq \delta_2 \nu(B(x, r)) < \infty,$$

*where*

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

*We then say that* $(X, d, \nu)$ *is a homogeneous space.*

The following result can be found in [20] and will be helpful in the proof of Theorem 0.1.2 or Theorem 3.2.1.

**Theorem 1.6.2** ([20]). *Let* $(X, d, \mu)$ *be a homogeneous space and let* $K(x, y)$ *be a function such that* $K(x, .) : y \to K(x, y) \in L^2(X)$. *Suppose the operator T defined by*

$$Tf(x) = \int_X K(x, y) f(y) d\mu(y),$$

*satisfies the following two conditions*

1. *there is a constant* $C_1$ *so that* $\|Tf\|_2 \leq C_1 \|f\|_2$*;*

2. *there are two constants* $C_2$ *and* $C_3$ *so that for all* $y, y_0$ *we have*

$$\int_{d(x,y_0) > C_2 d(y,y_0)} |K(x, y) - K(x, y_0)| d\mu(x) < C_3 \text{ (Hörmander Condition)}.$$

*Then for all p,* $1 \leq p \leq 2$*, there is a constant* $A_p$ *depending only on* $C_i, i = 1, 2, 3$*, so that for all* $f \in L^2 \bigcap L^p$ *we have*

$$\|Tf\|_p \leq A_p \|f\|_p \; if \; p > 1,$$

*and for all* $\lambda > 0$

$$\mu(\{x \in X : |Tf(x)| > \lambda\}) \leq A_1 \frac{\|f\|_1}{\lambda}.$$

One can find the following result in [35].

**Theorem 1.6.3** (Marcinkiewicz Interpolation Theorem). *Let* $p_0, p_1$ *be so that* $1 \leq p_0 < p_1 \leq \infty$. *Let* $T$ *be a sublinear operator defined from* $L^{p_0} + L^{p_1}$ *to the space of measurable functions. Assume that*

*$T$ is simultaneously of weak type $(p_0, p_0)$ with operator norm $A_{p_0,p_0}$ and of weak type $(p_1, p_1)$ with operator norm $A_{p_1,p_1}$. Then for every $0 < t < 1$, $T$ is of (strong) type $(p_t, p_t)$ where*

$$\frac{1}{p_t} = \frac{t}{p_0} + \frac{1-t}{p_1}.$$

*Moreover, if $p_1 < \infty$, then*

$$\|Tf\|_{p_t} \leq A_{p_t,p_t} \|f\|_{p_t}$$

*with*

$$A_{p_t,p_t} = 2\left[p_t\left(\frac{A_{p_0,p_0}^{p_0}}{p_t - p_0} - \frac{A_{p_1,p_1}^{p_1}}{p_1 - p_t}\right)\right]^{\frac{1}{p_t}}.$$

*If $p_1 = \infty$, we can take*

$$A_{p_t,p_t} = 2\left[p_t \frac{A_{p_0,p_0}^{p_0}}{p_t - p_0}\right]^{\frac{1}{p_t}}.$$

Let $(X, d, \nu)$ be a homogeneous space (see Definition 1.6.1).
The fractional maximal function is defined as follows

$$M_\gamma f(z) = \sup_{B: z\in B} \frac{1}{\nu^{1-\gamma}(B)} \int_B |f(w)| d\nu(w), \quad \gamma \in [0, 1),$$

where $B$ are pseudo-balls with respect to $d$. When $\gamma = 0$, $M_0$ is called the Hardy-Littlewood maximal operator. The following result will be used in Section 3.4 in the study of our maximal and fractional maximal function. One can find their proof in [22] or in [73].

**Theorem 1.6.4.** *Let $X$ be a homogeneous space, $0 \leq \gamma < 1$, $1 < p \leq r < \infty$ and a pair of weights $(u, v)$, then the following are equivalent.*

*(i) There exists a constant $C_1 > 0$ so that*

$$\left(\int_X [M_\gamma f(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \leq C_1 \left(\int_X |f(x)|^p u(x) d\nu(x)\right)^{\frac{1}{p}}$$

*for any $f \in L^p(X, ud\nu)$;*

*(ii) There exists a constant $C_2 > 0$ such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x)\right)^{\frac{1}{p}}$$

*for any ball $B \subset X$.*

We will also make use of the following classes, in Section 3.4, in the study of our maximal and fractional maximal function and to establish the good lambda inequality.

**Definition 1.6.5.** *A measure $\omega d\mu_\alpha$ is a Muckenhoupt weight or is in the $(A_p, \alpha)$ $(1 < p < \infty)$ class if there is a constant $C_p(\omega)$ so that for all pseudo-balls $B := B(\zeta, R)$ we have*

$$\left(\frac{1}{\mu_\alpha(B)} \int_B \omega(z) d\mu_\alpha(z)\right) \left(\frac{1}{\mu_\alpha(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_\alpha(z)\right)^{p-1} \leq C_p(\omega).$$

**Definition 1.6.6.** *A measure $\omega d\mu_\alpha$is a Muckenhoupt weight or is in the $(A_\infty, \alpha)$ class if there exist $\delta, \beta$ with $0 < \delta, \beta < 1$, so that for all pseudo-balls $B$ of $\mathbb{B}$ and for all measurable subset $E$ of $B$ we have*

$$\mu_\alpha(E) \leq \delta\mu_\alpha(B) \Rightarrow \omega d\mu_\alpha(E) \leq \beta\omega d\mu_\alpha(B).$$

We give now two properties of Muckenhoupt weights that we will need later (see [36]).

**Lemma 1.6.7.** *If $\sigma \in (A_\infty, \alpha)$ then there are two positive constants $A$ and $\beta_0$ so that for all balls $B$ and a measurable subset $E$ of $B$ we have*

$$\int_E \sigma(x) d\mu_\alpha(x) \leq A \left(\frac{d\mu_\alpha(E)}{d\mu_\alpha(B)}\right)^{\beta_0} \int_B \sigma(x) d\mu_\alpha(x).$$

**Theorem 1.6.8.** *The Hardy-Littlewood maximal operator is bounded on $L^p(\omega d\mu_\alpha), 1 < p < \infty$, if and only if $\omega d\mu_\alpha \in (A_p, \alpha)$.*

The following lemma [75, Lemma 2-Chapter IV] will be used in Section 3.5.

**Lemma 1.6.9.** *Let $(X, \mathcal{A}, \mu)$ a measure space. Let $f$ and $g$ be two positive measurable functions so that for all $t > 0$*

$$\mu(\{x \in X : f(x) > t, g(x) \leq ct\}) \leq a\mu(\{x \in X : f(x) > bt\}),$$

*where $a, b$ and $c$ are positive constants such that $a < b^p$ $(1 < p < \infty)$. Then*

$$\|f\|_p^p \leq \frac{c^{-p}}{1 - ab^{-p}} \|g\|_p^p.$$

We will use the following lemma in Section 3.4 to show Lemma 3.4.1. One can find it in [11, 77].

**Lemma 1.6.10.** *For $a > -1 - N$, there are two constants $C_1, C_2$ $(C_1 > 0)$ so that, for $z, w, w_0 \in \mathbb{B}$ with $|1 - \langle z, w_0\rangle| > C_1 d(w, w_0)$, we have*

$$\left|\frac{1}{(1 - \langle z, w\rangle)^{N+1+a}} - \frac{1}{(1 - \langle z, w_0\rangle)^{N+1+a}}\right| \leq C_2 \frac{d(w, w_0)}{|1 - \langle z, w_0\rangle|^{N+a+2}}.$$

## 1.7 Smoothness of functions

**Definition 1.7.1.** *Let $\Omega$ be a domain or a closure of a domain in $\mathbb{C}^N$. Consider the class $C^m(\Omega)$, $m \in \mathbb{Z}_+$ that is $f \in C^m(\Omega)$ if and only if for any $\alpha(\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$ with $|\alpha| \leq m$, $\partial^\alpha f = \dfrac{\partial^\alpha f}{x_1^{\alpha_1} ... x_N^{\alpha_N}}$*

*exists and belongs to* $C(\Omega) := C^0(\Omega)$ *the set of continuous functions in* $\Omega$.

Let $0 < \epsilon \leq 1$, we define two Lipschitz classes of order $\epsilon$, $Lip_\epsilon(\Omega)$ and $\bigwedge_\epsilon(\Omega)$, as follows:

**Definition 1.7.2.** $f \in C(\Omega)$ *is said to belong to* $Lip_\epsilon(\Omega)$ *or to* $\bigwedge_\epsilon(\Omega)$ *if there exists a constant* $A > 0$ *such that* $|f(z \underline{+} \zeta) - f(z)| \leq A\|\zeta\|^\epsilon$ *or* $|f(z+\zeta) + f(z-\zeta) - 2f(z)| \leq A\|\zeta\|^\epsilon$*, respectively, for all* $z \in \Omega$ *and all* $\zeta \in \mathbb{C}^N$ *such that* $z \underline{+} \zeta \in \Omega$

Note that for every $0 < \epsilon < 1$, $Lip_1(\Omega) \subset \bigwedge_1(\Omega) \subset \bigwedge_\epsilon(\Omega) = Lip_\epsilon(\Omega)$.
We now define $\bigwedge_a(\Omega)$ for every $a > 0$ as follows:

**Definition 1.7.3.** *If* $a = m + \epsilon$ *with* $m \in \mathbb{Z}_+$ *and* $0 < \epsilon \leq 1$*, then* $f \in \bigwedge_a(\Omega)$ *if and only if* $f \in C^m(\Omega)$ *and* $\partial^\alpha f \in \bigwedge_\epsilon(\Omega)$ *for every* $\alpha \in \mathbb{Z}_+^N$ *with* $|\alpha| = m$.

For future reference we recall two classical results due to Hardy-Littlewood and Zygmund see [9].

**Theorem 1.7.4.** *Let* $f \in H(\mathbb{D})$*. Then* $f \in \bigwedge_\epsilon(\overline{\mathbb{D}})$, $0 < \epsilon < 1$ *if and only if* $\sup\{(1-|z|)^{1-\epsilon}|f'(z)| : z \in \mathbb{D}\} < \infty$ *and* $f \in \bigwedge_1(\overline{\mathbb{D}})$ *if and only if* $\sup\{(1-|z|)|f''(z)| : z \in \mathbb{D}\} < \infty$.

## 1.8 Special functions

These definition and basic facts are in [9]

**Definition 1.8.1.** *The familiar beta-function is* $B(r,s) = \dfrac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$ *for* $\Re r > 0, \Re s > 0$ *where* $\Gamma$ *is defined by equation (1.2).*
*For* $a, b, c \in \mathbb{R}$ *and* $z \in \mathbb{B}$*, the hypergeometric function is defined by*

$$F(a,b;c:z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m!(c)_m} z^m, \tag{1.6}$$

*where* $(a)_m$ *is defined in equation (1.1), and hence* $F(a,b;b:z) = (1-z)^a$*. We note the identity*

$$F(a,b;c:z) = (1-z)^{c-a-b} F(c-a, c-b; c:z),$$

*and the integral representation, for* $c > b > 0$,

$$F(a,b;c:z) = \frac{1}{B(b,c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1} dt.$$

*We also consider the related function* $G_{a,b}$*, for* $a, b \in \mathbb{R}$*, defined by*

$$G_{a,b}(z) = \sum_{m=0}^{\infty} \frac{(a)_m}{m!(m+1)^b} z^m, \quad z \in \mathbb{D},$$

*and thus* $G_{a,0}(z) = (1-z)^a$.

We recall that for $r, s \in \mathbb{C}$

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)(\Gamma(r-s+1)}. \tag{1.7}$$

One can see the following result in [9, Corollaire 2.4]

**Theorem 1.8.2.** *For $a, b \in \mathbb{R}$ the function $G_{a,b}$ admits the followings properties:*

*(i) When $a > b$, $G_{a,b}$ is of the form*

$$G_{a,b}(z) = (1-z)^{-(a-b)} F(z) \quad (z \in \mathbb{D})$$

*where $F \in H(\mathbb{D}) \bigcap \bigwedge_{a-b}(\overline{\mathbb{D}})$ (see Definition 1.7.3). If, in addition, $-b = k \in \mathbb{Z}_+$ then $F$ is a polynomial of degree $k$ if $a \neq 1$ or $k = 0$, and of degree $k-1$ otherwise;*

*(ii) When $a = b$ and $-a$ is not in $\mathbb{Z}_+$ then $G_{a,b}$ is of the form*

$$G_{a,b}(z) = \frac{-1}{\Gamma(a)} \frac{1}{z} \log(1-z) + F(z) \quad (z \in \mathbb{D})$$

*where $F \in H(\mathbb{D}) \bigcap \bigwedge_1(\overline{\mathbb{D}})$ (see Definition 1.7.3). If, in addition, $a = 1$ then $F \equiv 0$;*

*(iii) When $a = b = -k$, $k \in \mathbb{Z}_+$ then $G_{a,b}$ is the polynomial*

$$G_{a,b}(z) = \sum_{m=0}^{k} (-1)^m (m+1)^k \binom{k}{m} z^m;$$

*(iv) When $a < b$, $G_{a,b}$ is in $H(\mathbb{D}) \bigcap \bigwedge_{b-a}(\overline{\mathbb{D}})$ (see Definition 1.7.3). If, in addition, $-a = k \in \mathbb{Z}_+$ then $G_{a,b}$ is the polynomial*

$$G_{a,b}(z) = \sum_{m=0}^{k} (-1)^m (m+1)^{-b} \binom{k}{m} z^m.$$

One can see [9] for more details about these special functions.
The following Propositions are in [53, Proposition 2.1,Remarque 2.2]

**Proposition 1.8.3.** *For a multi-index $\alpha$, $0 < p < \infty$, and $s \in \mathbb{C}$ such that $\Re s > -1$, we have*

$$\int_{\partial \mathbb{B}} |\zeta^\alpha|^p d\sigma(\zeta) = \frac{(N-1)! \prod_{j=1}^N \Gamma(1 + \alpha_j p/2)}{\Gamma(N + |\alpha| p/2)}$$

*and*

$$\int_{\mathbb{B}} |z^\alpha|^p (1-|z|^2)^s d\mu(z) = \frac{N! \prod_{j=1}^N \Gamma(1 + \alpha_j p/2)}{\Gamma(N + 1 + s + |\alpha| p/2)}.$$

**Proposition 1.8.4.** $\int_{\mathbb{B}} z^\alpha \overline{z^\beta} d\mu_s = 0$ *if $\alpha \neq \beta$.*

## 1.9 Integration theory

**Definition 1.9.1.** *A tribe on $X$ or $\sigma$-algebra on $X$ is a non-empty set of part of $X$ stable by passage to the complement and by countable union (therefore also by countable intersection).*

**Definition 1.9.2.** *A measurable space is a couple $(X,\mathcal{A})$ where $X$ is a non-empty set and $\mathcal{A}$ is a $\sigma$-algebra on $X$.*

**Definition 1.9.3.** *A map $\mu : \mathcal{A} \longrightarrow [0, +\infty]$ is a measure when the following two propositions are satisfied:*
*1) $\mu(\emptyset) = 0$;*
*2) $\mu$ is $\sigma - additive$ i.e for $E_1, E_2, ... \in \mathcal{A}$ a countable family of parts of $X$ pairwise disjoints,*

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$

### 1.9.1 Haar Measure

**Definition 1.9.4.** *A Haar measure on a locally compact group $G$ is a non-zero quasi-regular Borel measure $\lambda$ invariant under left translation.*
*In other words, for any Borel subset $B$ of $G$, and for any $g$ in $G$, we have:*

$$\lambda(gB) = \lambda(B).$$

For the following theorem see [82].

**Theorem 1.9.5.** *Let $G$ be a topological group. If $G$ is locally compact, then there exists a Haar measure $\lambda$ on $G$ finite on the compact subsets of $G$. Moreover, any complex Borel measure invariant under left translation is written $\alpha\lambda$ where $\alpha$ is a complex number.*

### 1.9.2 $L^p$ spaces

Definitions and results of this subsection come from[69].
Let $(X, \mathcal{A}, \mu)$ be a measure space. For $0 < p < \infty$, $L^p(X, \mu)$ denotes the set of (or class of) function $f$, $\mu$-measurable on $X$ and with values in $\mathbb{C}$ such that $|f|^p$ is $\mu-$integrable.

$L^\infty(X, \mu)$ denotes the set of ( or class of) functions $f$ $\mu$-measurable on $X$ with values in $\mathbb{C}$ such that there is a constant $c > 0$ so that $\mu\left(\{x \in X : |f(x)| > c\}\right) = 0$ or $|f| \leq c$ $\mu$-almost everywhere.

For $1 \leq p < \infty$ and $f \in L^p(X, \mu)$ set

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$

and for $f \in L^\infty(X,\mu)$ set

$$\|f\|_{L^\infty(X,\mu)} = \text{ess sup}\,|f| = \inf\{c > 0 : |f| \leq c\ ,\ \mu - p.p\}.$$

In the following, $L^p(X,\mu)$ will be denoted simply $L^p$ and $\|f\|_{L^p(X,\mu)} = \|f\|_p$ when it will not cause confusion. Unless otherwise stated, when $X = \mathbb{R}^n$ the measure considered will be the Lebesgue measure on $\mathbb{R}^n$.

**Theorem 1.9.6.** *( Hölder inequality)*
*Let $f$ and $g$ two measurable functions on $X$ and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$ then $fg \in L^1$. Moreover,*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Recall that a measure $\nu$ is $\sigma$-finites means there is a sequence $\{X_n\}_n$ pairwise disjoint elements of $\mathcal{A}$ verify $X = \bigcup_n X_n$ et $\nu(X_n) = \infty$.

**Theorem 1.9.7.** *(Minkowski inequality)*

*(1) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ where $\mu$ and $\nu$ are measures $\sigma$-finites. Let $f$ be a measurable function of $(X \times Y, \mathcal{A} \otimes \mathcal{B})$. We then have:*

$$\Big(\int_X (\int_Y f(x,y)\mathrm{d}\nu(y))^p \mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \leq \int_Y \Big(\int_X (f(x,y))^p \mathrm{d}\mu(x)\Big)^{\frac{1}{p}} \mathrm{d}\nu(y),$$

*where $1 < p < \infty$ and $f > 0$.*

*(2) Let $(X, \mathcal{A}, \mu)$ a measure space, $f, g \in L^p$ with $1 \leq p < \infty$ then we have: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

**Theorem 1.9.8.** *Let $\{f_n\}_{n\in\mathbb{N}}$ a sequence of positive $\mu$-integrable functions on $X$. Then*

$$\int_X \liminf f_n \mathrm{d}\mu \leq \liminf \int_X f_n \mathrm{d}\mu.$$

**Theorem 1.9.9.** *Let $\{f_n\}_{n\in\mathbb{N}}$ a increasing sequence of positive $\mu$-integrable functions.*

*If $\sup_{n\in\mathbb{N}} \int_X f_n \mathrm{d}\mu < \infty$ then $\{f_n\}_n$ converges $\mu$-almost everywhere to $f = \sup_{n\in\mathbb{N}} f_n$. Moreover, $f$ is $\mu$-integrable and*

$$\lim_{n\to+\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

**Theorem 1.9.10.** *Let $(X, \mathcal{A}, \mu)$ a measure space and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of $\mu$-integrable functions which converges point-wise $\mu$-almost everywhere and such that there is a $\mu$-integrable function $g$ such that for $n \in \mathbb{N}$, we have: $|f_n| \leq g$ $\mu$-almost everywhere. Then $f = \lim_{n\to+\infty} f_n$ is $\mu$-integrable and :*

$$\lim_{n\to+\infty} \int_X f_n \mathrm{d}\mu = \int_X \lim_{n\to+\infty} f_n \mathrm{d}\mu.$$

**Theorem 1.9.11.** *Let $(X, \mathcal{A}, \mu)$ a measure space and $f : X \times E \longrightarrow \overline{\mathbb{R}}$, a function where $E$ is a metric space. Set*

$$\phi(t) = \int_X f(x,t)\mathrm{d}\mu(x)$$

*and suppose:*

*(1) For all $t \in E$, $f(\cdot, t)$ is $\mu$-integrable on $X$;*

*(2) For almost all $x \in X$, $f(x, \cdot)$ is continuous on $E$;*

*(3) For all $t_0 \in E$, there is $V$, an open set such that $t_0 \in V \subset E$ and there is a $\mu$-integrable function $g$ on $X$ depending only on the variable $x$ satisfying: $\forall t \in V$,*
*$|f(\cdot, t)| \leq g$ $\mu$-almost everywhere.*

*Then the function $\phi$ is continuous on $E$.*

**Theorem 1.9.12.** *Let $(X, \mathcal{A}, \mu)$ a measure space and $f : X \times I \longrightarrow \mathbb{R}$, where $I$ is an open interval of $\mathbb{R}$. Set*

$$\phi(t) = \int_X f(x,t)\mathrm{d}\mu(x)$$

*and suppose:*

*(1) For all $t \in I$, $f(\cdot, t)$ is $\mu$-integrable on $X$;*

*(2) The function $f(x, \cdot)$ is derivable on $I$, for almost every $x \in X$;*

*(3) There is a $\mu$-integrable function $g$ on $X$ such that for all $t \in I$,*

$$\left|\frac{\partial f}{\partial t}(\cdot, t)\right| \leq g \quad \mu\text{-almost everywhere.}$$

*Then the function $\phi$ is derivable on $I$ and*

$$\phi'(t) = \int_X \frac{\partial f}{\partial t}(x,t)\mathrm{d}\mu(x).$$

**Theorem 1.9.13.** *For $1 \leq p \leq \infty$, $(L^p, \|\cdot\|_p)$ is a Banach space.*

**Theorem 1.9.14.** *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $L^p$ which converges in $L^p$ to $f \in L^p$ then we can extract from the sequence $\{f_n\}_n$ a sub-sequence $\{f_{n_k}\}_k$ and we can find a function $g \in L^p$ such that the sequence $\{f_{n_k}\}_k$ converge pointwise to $f$ $\mu$-almost everywhere and for all $k \in \mathbb{N}$, $|f_{n_k}| \leq g$ $\mu$-almost everywhere.*

**Theorem 1.9.15.** *(Fubini theorem)*
*Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ two measured spaces $\sigma - finite$ and $f$ a measurable function on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$.*

*(A) If $f$ is measurable and positive, then:*

*1) the function* $x_1 \longmapsto \int_{X_2} f(x_1,x_2)\mathrm{d}\mu_2(x_2)$ *is measurable on* $(X_1,\mathcal{A}_1)$*;*

*2) the function* $x_2 \longmapsto \int_{X_1} f(x_1,x_2)\mathrm{d}\mu_1(x_1)$ *is measurable on* $(X_2,\mathcal{A}_2)$*;*

*3) we have the following equality:*

$$\int\int_{X_1\times X_2} f\mathrm{d}\mu_1\otimes\mu_2 = \int_{X_1}\left(\int_{X_2} f(x_1,x_2)\mathrm{d}\mu_2(x_2)\right)\mathrm{d}\mu_1(x_1)$$
$$= \int_{X_2}\left(\int_{X_1} f(x_1,x_2)\mathrm{d}\mu_1(x_1)\right)\mathrm{d}\mu_2(x_2).$$

*(B) If $f$ is integrable on* $(X_1\times X_2,\mathcal{A}_1\otimes\mathcal{A}_2,\mu_1\otimes\mu_2)$*, then*

*1) for almost every* $x_1\in X_1$*, the function* $x_2\longmapsto f(x_1,x_2)$ *is integrable on* $X_2$*;*

*2) the function* $x_2 \longmapsto \int_{X_2} f(x_1,x_2)\mathrm{d}\mu_2(x_2)$ *is integrable on* $X_1$*;*

*3) for almost every* $x_2\in X_2$*, the function* $x_1\longmapsto f(x_1,x_2)$ *is integrable on* $X_1$*;*

*4) the function* $x_1 \longmapsto \int_{X_1} f(x_1,x_2)\mathrm{d}\mu_1(x_1)$ *is integrable on* $X_2$*;*

*5) we have the following equality:*

$$\int\int_{X_1\times X_2} f\mathrm{d}\mu_1\otimes\mu_2 = \int_{X_1}\left(\int_{X_2} f(x_1,x_2)\mathrm{d}\mu_2(x_2)\right)\mathrm{d}\mu_1(x_1)$$
$$= \int_{X_1}\left(\int_{X_1} f(x_1,x_2)\mathrm{d}\mu_1(x_1)\right)\mathrm{d}\mu_2(x_2).$$

**Theorem 1.9.16.** *Let* $(X,\mu)$ *be a measure space. For* $1\leq p<\infty$*, the topological dual* $(L^p)'$ *of* $L^p$ *is isometric to the space* $L^q$*, where $q$ is such that* $\frac{1}{p}+\frac{1}{q}=1$*. More precisely, for* $T\in(L^p)'$*, there is a function* $g\in L^q$ *such that for all* $f\in L^p$

$$Tf = \langle f,g\rangle = \int_X f\overline{g}\mathrm{d}\mu.$$

*Moreover, by duality:* $\|T\|_{(L^p)'} = \|g\|_{L^q} = \sup_{\|f\|_{L^p}=1}\left|\int_X f\overline{g}\mathrm{d}\mu\right|$*. Thus we identify* $(L^p)'$ *to* $L^q$ *thanks to the isometric isomorphism:*

$$\begin{aligned} \langle\cdot,\cdot\rangle :\ L^q &\longrightarrow (L^p)' \\ g &\longmapsto \langle\cdot,g\rangle : L^p \longrightarrow \mathbb{C} \\ &\qquad\qquad\qquad f \longmapsto \langle f,g\rangle = \int_X f\overline{g}\mathrm{d}\mu. \end{aligned}$$

## 1.10 Functional analysis tools

Definitions and results of this section come from [33],[34].

**Definition 1.10.1.** *Let $E$ be a vector space on $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We call norm on $E$ any map $N$ from $E$ to $\mathbb{R}$ such that:*
*(1) For all $x \in E,\ N(x) \geq 0$ and $N(x) = 0 \iff x = 0$,*
*(2) For all $x, y \in E,\ N(x+y) \leq N(x) + N(y)$,*
*(3) For all $s \in \mathbb{K}$, for all $x \in E,\ N(sx) = |s|N(x)$.*
*In this case $E$ is then a normed vector space of norm $N$ denoted $(E, N)$.*

**Remark 1.10.2.** *-There can be several norms on $E$*
*-If $E$ is a normed vector space of norm $N$, we equip $E$ with the topology of metric space, it is the one whose open sets are defined as follows:*
*$O$ is an open set for $E$ if for all $x \in O$, there exists $r(x) > 0$ such that*

$$\{y \in E : N(x-y) < r(x)\} \subset O.$$

**Definition 1.10.3.** *Let $E$ be a normed vector space with norm $||.||$ and $l \in E$.*
*We say that a sequence $(u_n)$ of $E$ converges to $l$ if:*

$$\forall \varepsilon > 0,\ \ \exists n_0 \in \mathbb{N}/\ \forall n > n_0,\ \ ||u_n - l|| < \varepsilon.$$

**Definition 1.10.4.** *Let $E$ be a normed vector space of norm $||.||$.*
*We say that $(u_n)$ is a Cauchy sequence if:*

$$\forall \varepsilon > 0,\ \ \exists n_0 \in \mathbb{N}/\ \forall p > n_0,\ \forall q > n_0,\ \ ||u_p - u_q|| < \varepsilon.$$

**Definition 1.10.5.** *Let $H$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We call inner (or scalar) product on $H$ and we note $\langle , \rangle$ any positive definite Hermitian form on $H$. i.e a map $\langle , \rangle : H \times H \longrightarrow \mathbb{K}$ such that for all $x, x', y, y' \in H$ and $s \in \mathbb{K}$ we have:*
*-.$\langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle$;*
*-.$\langle sx, y\rangle = s\langle x, y\rangle$;*
*-.$\langle x, y + y'\rangle = \langle x, y\rangle + \langle x, y'\rangle$;*
*-.$\langle x, sy\rangle = \overline{s}\langle x, y\rangle$;*
*-.$\langle x, y\rangle = \overline{\langle y, x\rangle}$;*
*-.$\langle x, x\rangle \geq 0$;*
*-.$\langle x, x\rangle > 0$ if and only if $x \neq 0$.*

**Theorem 1.10.6.** *If $\langle , \rangle$ is a positive Hermitian form on $H$, then we have:*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle.$$

**Definition 1.10.7.** *- A Pre-hilbertian space is a vector space $H$ endowed with scalar product.*
*- A Hilbertian space (or Hilbert space) is a pre-Hilbertian space in which any Cauchy sequence converges for the associated norm given by*

$$||x|| = \sqrt{\langle x, x\rangle}.$$

**Definition 1.10.8.** *Let $H$ be a pre-Hilbertian space. We say that $(e_n)_{n\in\mathbb{N}}$ is a orthonormal system of $H$ if $\langle e_j, e_k\rangle = \delta_{j,k}$, where $\delta_{j,k} = 1$ if $j = k$ and $0$ otherwise.*

**Theorem 1.10.9.** *Let $H$ be a pre-Hilbertian space. If $(e_n)_{n\in\mathbb{N}}$ is a orthonormal system of $H$, then*

$$\sum |\langle x, e_n\rangle|^2$$

*converges to $\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2$ and for $x \in H$,*

$$\sum_{n=0}^{\infty} |\langle x, e_n\rangle|^2 \leq ||x||^2.$$

**Definition 1.10.10.** *Let $E$ be a normed vector space. A series with general term $u_n$ in $E$ is said to be convergent , with sum U, if and only if the sequence of partial sums $U_n = u_1 + ... + u_n$ converges, with*

$$U = \lim_{n\to+\infty} U_n.$$

*We say that the series converges absolutely if the series of general term $||u_n||$ is convergent.*

**Definition 1.10.11.** *We call Banach algebra a quintuplet $(\mathcal{A}, +, \bullet, *, ||.||)$ such that:*
*- $(\mathcal{A}, +, \bullet, ||.||)$ is a Banach space, a norm vector space in which any Cauchy sequence converges. We also say complete normed vector space*
*- $(\mathcal{A}, +, \bullet, *)$ is an algebra such that*

$$||x * y|| \leq ||x||||y||, \quad \forall x, y \in \mathcal{A}$$

**Theorem 1.10.12.** *The Bertrand series, of positive general term*

$$u_n = \frac{1}{n^{\alpha}(\ln n)^{\beta}}$$

*is convergent for $\alpha > 1$, or for $\alpha = 1$ and $\beta > 1$ and is divergent otherwise.*

**Definition 1.10.13.** *Let $u$ and $v$ be two series of general terms $u_n$ and $v_n$ in $\mathbb{R}$ (or $\mathbb{C}$). The product of $u$ and $v$ is a series $w$ of general terms $w_n = u_0 v_n + u_1 v_{n-1} + ... + u_n v_0$.*

**Definition 1.10.14.** *We call product series in $\mathbb{C}$ an expression of the form*

$$\prod a_n.$$

**Theorem 1.10.15.** *The product series* $\prod a_n$ *converges if and only if the sequence* $(a_n)_n$ *converges to* 1.

**Theorem 1.10.16.** *The product* $w$ *(of sum* $W$*) of an absolutely convergent series* $u$ *(of sum* $U$*) and of a convergent series* $v$ *(of sum* $V$*), is itself convergent, and for the sum we have the equality* $W = UV$.

**Theorem 1.10.17.** *Let a series of general term* $u_n$ *in* $E$*, a complete normed vector space, and* $\varphi$ *a permutation of* $\mathbb{N}$*. If* $u$ *is absolutely convergent, the series* $v$ *of* $v_n = u_{\varphi(n)}$ *is absolutely convergent, and with sum* $V = U$.

**Definition 1.10.18.** *Consider a series of functions* $(u_n)_n$ *, i.e each* $u_n$ *is a function from* $E$ *to* $F$ *where* $E$ *and* $F$ *are two vector spaces with* $F$ *a normed space. We define partial sums as follows:* $U_n(t) = u_0(t) + ... + u_n(t)$ *for all* $t$ *of* $E$.
*1) We will say that the series* $u_n$ *is simply convergent to a function* $U$ *if and only if the sequence of partial sums* $(u_n)_n$ *is simply convergent to* $U$*. i.e*

$$\forall t, \forall \varepsilon > 0,\ \exists n_0,\ \forall n > n_0, ||U_n(t) - U(t)|| < \varepsilon$$

*2) We will say that the series* $u_n$ *converges uniformly if and only if the sequence of partial sums* $(U_n)_n$ *converges uniformly. i.e*

$$\forall \varepsilon > 0,\ \exists n_0,\ \forall n > n_0, ||U_n(t) - U(t)|| < \varepsilon\ \ \forall t \in E$$

*3). If* $F$ *is a Banach space, we will say that the series* $(u_n)$ *is simple absolutely convergent (respectively uniform absolutely) if the series of functions from* $E$ *to* $\mathbb{R}$ *defined by* $t \longrightarrow ||u_n(t)||$ *converges simply (respectively uniformly).*
*4). If* $F$ *is a Banach space, we will say that it converges normally (or in norm, or that there is dominated convergence) if and only if there exists a series of positive term* $\alpha_n$*, convergent, and such that for any integer* $n$ *and for any element* $t \in E$*,*

$$||u_n(t)|| \leq \alpha_n.$$

**Remark 1.10.19.** *For a series of functions* $(u_n)$*, we have:*
*(1) simple absolute convergence implies simple convergence*
*(2) uniform absolute convergence implies uniform convergence*
*(3) normal convergence implies uniform absolute convergence.*

**Theorem 1.10.20.** *Let a series of functions have values in a Banach space* $F$*. The normal convergence is equivalent to the existence of each*

$$||u_n||_\infty = sup\{||u_n(x)||; x \in E\}$$

*and the convergence of the numerical series* $\|u_n\|_\infty$

**Theorem 1.10.21.** *Let* $E$ *be a topological space,* $F$ *a normed vector space. If the series of functions* $(u_n)_n$ *from* $E$ *into* $F$ *converges uniformly, and if the functions* $u_n$ *are continuous, the sum function is continuous.*

**Theorem 1.10.22.** *Let $H$ be a Hilbert space, Let $\varphi$ be a continuous linear form on $H$. Then there is a unique $x \in H$ such that:*

$$\varphi(y) = \langle y, x\rangle, \forall y \in H.$$

*Moreover:*

$$||\varphi|| = ||x||.$$

**Definition 1.10.23.** *Let $X$ be a topological space. We denote by $C^0(X, \mathbb{R})$ the set of continuous functions from $X$ to $\mathbb{R}$.*

**Definition 1.10.24.** *It is said that a part $\mathcal{A}$ of $C^0(X, \mathbb{R})$ separate the points if $\forall (x, y) \in X^2, x \neq y, \exists h \in \mathcal{A}, h(x) \neq h(y)$.*

**Theorem 1.10.25.** *If an algebra $\mathcal{A}$ of $C^0(X, \mathbb{R})$ separate the points and if $\forall x \in X, \exists h \in \mathcal{A}, h(x) \neq 0$, then for any pair $\{x, y\}$ of $X, \forall f \in C^0(X, \mathbb{R}), \forall \varepsilon > 0, \exists g \in \mathcal{A}$ with $|f(x) - g(x)| < \varepsilon$ and $|f(y) - g(y)| < \varepsilon$.*

**Theorem 1.10.26.** *Let $X$ be a compact topological space, and $E = C^0(X, \mathbb{R})$ the vector space normalized by the uniform convergence norm. If $\mathcal{A}$ is an algebra of $E$ which separates the points, and such that $\forall x \in X, \exists f \in \mathcal{A},$ with $f(x) \neq 0,$ then $\overline{\mathcal{A}} = E$.*

**Corollary 1.10.27.** *Let $X$ be a compact topological space, and $E = C^0(X, \mathbb{C})$ the vector space normed by the norm of uniform convergence. If $\mathcal{A}$ is an algebra of $E$ which separates the points, and such that $\forall x \in X, \exists f \in \mathcal{A},$ with $f(x) \neq 0,$ such that $\forall g \in \mathcal{A}, \overline{g} \in \mathcal{A}$, then the algebra $\mathcal{A}$ is everywhere dense in $E$.*

**Theorem 1.10.28.** *(Closed graph theorem)*
*Let $E$ and $F$ be two Banach spaces. Let $u : E \longrightarrow F$ be a linear map. Then if the graph of $u$ is closed in $E \times F$, $u$ is continuous.*

**Theorem 1.10.29.** *Let $H$ be a Hilbert space of functions defined in $X$. If $L_x : H \mapsto \mathbb{C}$ such that $L_x(f) = f(x)$ is continuous, then $H$ is a reproducing kernel Hilbert space.*

**Definition 1.10.30.** *We call reproducing kernel for a Hilbert space $H$ of functions defined in $X$, a function $K$ defined by*

$$K(x, y) = \overline{K_x(y)} \text{ with } K_x \text{ such that } f(x) = \langle f, K_x\rangle_H \; \forall f \in H$$

# ♣ Part I: Weighted estimates for operators associated to the Bergman-Besov kernels ♣

⋆ ⋆ Chapter 2 ⋆ ⋆

# Bergman-Besov spaces $B^p_q$, $p > 0, q \in \mathbb{R}$.

The purpose of this chapter is to provide a complete description of Bergman-Besov spaces $B^p_q(\mathbb{B})$. In particular, we will show that $B^2_q(\mathbb{B})$ are reproducing kernel Hibert spaces. We will obtain inclusion relations among these spaces and the Block-Lipschitz spaces an the space of bounded function. We will exhibit explicit examples in cases that the inclusions are strict and when these inclusions are sharp.

## 2.1 Radial differential operator

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^N$ with respect to the usual hermitian inner product $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_N}$ and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. Let $H(\mathbb{B})$ and $H^\infty$ denote the spaces of all bounded holomorphic functions on $\mathbb{B}$, respectively.

We let $\mu$ be the Lebesgue measure of $\mathbb{B}$ normalized so that $\mu(\mathbb{B}) = 1$. For $q \in \mathbb{R}$, we also define on $\mathbb{B}$ the measures

$$d\mu_q(z) := \left(1 - |z|^2\right)^q d\mu(z),$$

See Definition 1.5.5 for more details.

For $0 < p < \infty$, we denote the Lebesgue classes with respect to $\mu_q$ by $L^p_q(\mathbb{B})$. For $\alpha \in \mathbb{R}$, we also define the weighted classes $L^\infty_\alpha(\mathbb{B})$ by

$$L^\infty_\alpha(\mathbb{B}) := \left\{\varphi \text{ measurable on } \mathbb{B} : \left(1 - |z|^2\right)^\alpha \varphi(z) \in L^\infty(\mathbb{B})\right\}$$

so that $L^\infty_0(\mathbb{B}) = L^\infty(\mathbb{B})$, which are normed by

$$||\varphi||_{L^\infty_\alpha} := \underset{z\in\mathbb{B}}{ess\ sup} \left(1 - |z|^2\right)^\alpha |\varphi(z)|.$$

For $q > -1$ and $0 < p < \infty$, we define the weighted Bergman spaces by $A^p_q(\mathbb{B}) = L^p_q(\mathbb{B}) \cap H(\mathbb{B})$. To extend this family to all real $q$, we need to derivatives. Given $q \in \mathbb{R}$ and $0 < p < \infty$, let $m$ be a nonnegative integer such that $q + pm > -1$. Then the Bergman-Besov space $B^p_q(\mathbb{B})$ consists of all

$f \in H(\mathbb{B})$ for which

$$\left(1-|z|^2\right)^m \frac{\partial^m f}{\partial z_1^{\gamma_1}\cdots\partial z_N^{\gamma_N}} \in L_q^p(\mathbb{B})$$

for every multi-index $\gamma = (\gamma_1, ..., \gamma_N)$ with $\gamma_1 + \cdots + \gamma_N = m$.

The spaces $B_q^2$ are reproducing kernel Hilbert spaces whose kernels occupy large part in our study of all $B_q^p$ spaces. Consequently, even to define the spaces of interest in this work, it is more advantageous to use certain radial differential operators that are compatible with the kernels.

Besov spaces extend weighted Bergman spaces to all $q$. To define them, we first take a radial differential operator $D_s^t$ of order $t$ for any $s, t \in \mathbb{R}$ defined on $H(\mathbb{B})$.

Let us recall here the *Pochhammer symbol* $(a)_b$ is given by equation (1.1)

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when $a$ and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function $\Gamma$ (see equation (1.2)). In particular, $(a)_0 = 1$ and for $k$ a positive integer, we have $(a)_k = a(a+1)\cdots(-a+k-1)$. The Stirling formula yields

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\Re c \to \infty) \tag{2.1}$$

where $s \sim y$ means both $x = \mathcal{O}(y)$ and $y = \mathcal{O}(x)$ for all $x$, $y$ in question. If only $x = \mathcal{O}(y)$, we write $x \lesssim y$.

We recall that $f \in H(\mathbb{B})$ can be written in terms of its *homogeneous expansion* and its Taylor series as

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{|\gamma|=0}^{\infty} f_\gamma z^\gamma$$

in which $f_k$ is a holomorphic homogeneous polynomial in $z_1, ..., z_N$ of degree $k$.

**Definition 2.1.1.** *Let $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which $f_k$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree $k$. We define, $D_s^t$, for $s, t \in \mathbb{R}$*

$$D_s^t f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

*where*

$$d_k(s,t) = \frac{c_k(s+t)}{c_k(s)},$$

*with*

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if} \quad a > -(N+1), \\ \frac{k!}{(1-N-a)_k} & \text{if} \quad a \leq -(N+1). \end{cases}$$

*and $(a)_b$ is define by equation (1.1).*

*We also consider the linear transformation $I_s^t$ defined for $f \in H(\mathbb{B})$ by*

$$I_s^t f(z) = (1-|z|^2)^t D_s^t f(z). \tag{2.2}$$

Note that $d_0(s,t) = 1$, $d_k(s,t) > 0$ for any $k$, and using equation (2.1)

$$d_k(s,t) \sim k^t \qquad (k \to \infty), \tag{2.3}$$

for any $s, t$.

**Theorem 2.1.2.** *Any $D_s^t$ is a continuous operator on $H(\mathbb{B})$.*

*Proof.* Let $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which $f_k$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree $k$. Set $F_k = k^t f_k$ for $k \geq N_0$, where $N_0$ is a large integer. $F_k$ is a homogeneous polynomial of degree $k$ because of $f_k$. Let $z \in \mathbb{B}$, then there exists $1 > r_z > 0$ such that $|z| < r_z$

$$\begin{aligned}
Sup_{k\geq N_0}|F_k(z)| &= Sup_{k\geq N_0}|k^t f_k(z)| \\
&= Sup_{k\geq N_0}|k^t f_k(r_z \tfrac{z}{r_z})| \\
&= Sup_{k\geq N_0}|k^t r_z^k f_k(\tfrac{z}{r_z})| \\
&\leq Sup_{k\geq N_0}|k^t e^{k\ln(r_z)}| Sup_{k\geq N_0}|f_k(\tfrac{z}{r_z})| \\
&\leq C_z Sup_{k\geq N_0}|k^t e^{k\ln(r_z)}| \\
&< \infty
\end{aligned}$$

Where the third equality is because $f_k$ is homogeneous of degree $k$, the second inequality is because $f$ is uniformly convergent on every compact subset of $\mathbb{B}$, and the last inequality is because $k^t e^{k\ln(r_z)} \to 0 \quad (k \to \infty)$. Then by Theorem 1.5.9 one have the uniformly convergence on every compact subset of $\mathbb{B}$ of the series $\sum_0^{\infty} F_k$, so that the sum of the series is holomorphic. Then $D_s^t f \in H(\mathbb{B})$ by Definition 2.1.1 and equation (2.3). $\square$

From (2.3), $D_s^t$ is of order $t$. In particular, $D_s^t z^\gamma = d_{|\gamma|}(s,t) z^\gamma$ for any multi-index $\gamma$, and hence $D_s^t(1) = 1$. More importantly, one verifies easily that

$$D_s^0 = 1, \qquad D_{s+t}^u D_s^t = D_s^{t+u}, \qquad and \qquad (D_s^t)^{-1} = D_{s+t}^{-t} \tag{2.4}$$

for any $s, t, u$. Thus any $D_s^t$ maps $H(\mathbb{B})$ onto itself.

For $a, b \in \mathbb{C}$ and suitable $g$, consider the operator

$$V_b^a g(z) = (1-|z|^2)^a \int_{\mathbb{B}} \frac{1}{(1-\langle z,w\rangle)^{N+1+a+b}} g(w)(1-|w|^2)^b d\mu(w). \tag{2.5}$$

one can found the following result in [53, Theorem 2.4].

**Theorem 2.1.3.** *(a) For $1 \leq p \leq \infty$, $V_b^a$ is bounded on $L^p(\mu_c)$ if and only if*

$$-\Re a < \frac{c+1}{p} < \Re b + 1.$$

*(b) For $0 < p \leq 1$, if*
*$0 < \Re b + 1 - N(\frac{1}{p} - 1)$ and $-\Re a < \frac{c+1}{p} < \Re b + 1 - N(\frac{1}{p} - 1)$, then $V_b^a$ is a continuous map from $L^p(\mu_c) \bigcap H(\mathbb{B})$ to $L^p(\mu_c)$.*

The following result is in [53, Theorem 3.4]

**Theorem 2.1.4.** *Suppose $b > -1$ and $u - t + b > -(N+1)$. Let $f \in H(\mathbb{B})$. Then $(1 - |z|^2)^{u-t} D_r^u(f) = M(D_s^t(f))$ for an operator $M$ of type $V_b^{u-t}$. In particular, $D_r^t(f) = M(D_s^t(f))$ for an operator of type $V_b^0$.*

*Proof.* We show the details only for the case when $s, s+t, r, r+u$ are greater than $-(N+1)$; the other cases are similar. Following [9, p.41], for $b > -1$, we consider

$$h(\lambda) = \sum_{k=0}^{\infty} \frac{(N+1+r+u)_k}{(N+1+r)_k} \frac{(N+1+s)_k}{(N+1+s+t)_k} \frac{(b+1)_{N+k}}{N!k!} \lambda^k,$$

which belongs to $H(\mathbb{D})$. Taking $M$ to be defined as

$$Mf(z) = (1 - |z|^2)^{u-t} \int_{\mathbb{B}} D_s^t f(w) h(\langle z, w \rangle)(1 - |w|^2)^b d\mu(w),$$

it is easy to check that using Proposition 1.8.3 for $f(z) = z^\alpha$ , we have

$$(1 - |z|^2)^{u-t} D_r^u f(z) = M D_s^t f(z).$$

Hence, by virtue of Theorem 2.1.2, we have for $f \in H(\mathbb{B})$

$$(1 - |z|^2)^{u-t} D_r^u f(z) = M D_s^t f(z).$$

By Stirling's formula (see equation (2.1)) we have

$$h(\lambda) \sim \sum_{k=0}^{\infty} \frac{(N+1+b)_k}{k!(k+1)^{t-u}} \lambda^k = G_{N+1+b, t-u}(\lambda),$$

see Definition 1.8.1. Then by the assumption on $b$ and Theorem 1.8.2 (i),

$$h(\lambda) = \frac{h_1(\lambda)}{(1-\lambda)^{N+1+u-t+b}}$$

for a holomorphic $h_1$ in the Lipschitz class $\wedge_{N+1+u-t+b}$ on $\overline{\mathbb{D}}$. This finally implies, $Arg(1-\lambda)$ is bounded, that

$$|h(\lambda)| \leq \frac{C}{|1-\lambda|^{N+1+u-t+b}}$$

as in the proof of [9, Lemma 5.6]. The proof is now completed. □

## 2.2 $B^p_q$: Definition and first properties

In this section, we define the space $B^p_q$ for $p > 0$ and $q \in \mathbb{R}$, and we show that $B^2_q$ are reproducing kernel Hilbert spaces.

**Definition 2.2.1.** *For $q \in \mathbb{R}$ and $0 < p < \infty$,we define the Bergman-Besov space $B^p_q$ to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ (see equation (2.2)) belongs to $L^p_q$ for some $s, t$ satisfying*

$$q + pt > -1 \tag{2.6}$$

*The quantity $||f||_{B^p_q} := ||I^t_s f||_{L^p_q}$ for any such $s, t$ defines a norm on $B^p_q$ for $p \geq 1$ and a quasi-norm (see Definition 1.5.1) for $0 < p < 1$.*

**Definition 2.2.2.** *For $\alpha \in \mathbb{R}$, we define the Bloch-Lipschitz space $B^\infty_\alpha$ to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to $L^\infty_s$ for some $s, t$ satisfying*

$$\alpha + t > 0. \tag{2.7}$$

*The quantity $||f||_{B^\infty_\alpha} := ||I^t_s f||_{L^\infty_q}$ for any such $s, t$ defines a norm on $B^\infty_\alpha$.*

Explicit forms of the norms of $B^p_q$ and $B^\infty_\alpha$ given in Definition 2.2.1 and 2.2.2 are

$$||f||_{B^p_q} = \left( \int_{\mathbb{B}} |D^t_s f(z)|^p (1 - |z|^2)^{q+pt} d\mu(z) \right)^{\frac{1}{p}} \qquad (q + pt > -1), \tag{2.8}$$

$$||f||_{B^\infty_\alpha} = \sup_{z \in \mathbb{B}} |D^t_s f(z)| (1 - |z|^2)^{\alpha+t} \qquad (\alpha + t > 0). \tag{2.9}$$

We first make sure that the $B^p_q$ spaces are well-defined.

**Theorem 2.2.3.** *Suppose $f \in H(\mathbb{B})$, $q \in \mathbb{R}$ and $r, s, t, u \in \mathbb{R}$.*

*(a) Let $0 < p < \infty$. For $q + pt > -1$ the function $I^t_s f$ belongs to $L^p(\mu_q)$ if and only if for some $r$ and $u$ satisfying $q + pu > -1$ the function $I^u_r f$ belongs to $L^p(\mu_q)$, and the $L^p(\mu_q)$ norms of these two functions are equivalent.*

*(b) Let $p = \infty$. For $t > 0$ the function $I^t_s f$ is bounded on $\mathbb{B}$ if and only if for some $r$ and $u > 0$ the function $I^u_r f$ is bounded on $\mathbb{B}$, and the supremums of each of these two functions on $\mathbb{B}$ are equivalent norms for $f$.*

*Proof.* (a) The relation $I^t_s f \in L^p(\mu_q)$ and $I^u_r f \in L^p(\mu_q)$ can be restated in the forms $D^t_s f \in L^p(\mu_{q+pt})$ and $(1 - |z|^2)^{u-t} D^u_r f(z) \in L^p(\mu_{q+pt})$, respectively. So we apply Theorem 2.1.3 with

$c = q + pt$ and $a = u - t$. These values satisfy the first inequalities there since $q + pu > -1$. We take a sufficiently large real $b$ in Theorem 2.1.4 which also satisfies the second inequality in Theorem 2.1.3. Theorem 2.1.3 then says that if $I_s^t \in L^p(\mu_q)$, then $I_r^u \in L^p(\mu_q)$.
In the opposite direction, we interchange the roles of the pairs $s, t$ and $r, u$, and use the condition $q + pt > -1$.

(b) By choosing a large enough $b$ in Theorem 2.1.4 and replacing $b$ by $b - t$, we can write $I_r^u = (M_1 \circ I_s^t)$ with an operator $M_1$ of type $V_{b-t}^u$. Interchanging the pair $s, t$ and $r, u$, we get an operator $M_2$ that is of type $V_{b-u}^t$ such that $I_s^t = (M_2 \circ I_r^u)$. We take first $c = 0$, $a = u$, and $b$ large, and then $c = 0$, $a = t$, and $b$ large in Theorem 2.1.3 (a) to obtain that $V_{b-t}^u$ is bounded on $L^p(\mu)$ and $V_{b-u}^t$ is bounded on $L^p(\mu)$. The inequalities on $a$ and $c$ are satisfied by the hypotheses $u > 0$ and $t > 0$, respectively. Then $I_r^u$ is uniformly bounded on $\mathbb{B}$ if and only if $I_s^t$ is uniformly bounded on $\mathbb{B}$.

□

The following two results are immediate consequence of Theorem 2.2.3 and can be found in [53].

**Corollary 2.2.4.** *The space $B_q^p$ is independent of the particular choice of $s, t$ as long as $q + pt > -1$ holds. The $L^p(\mu_q)$ norms of $I_{s_1}^{t_1} f$ and $I_{s_2}^{t_2} f$ are equivalent as long as $q + pt_1 > -1$ and $q + pt_2 > -1$.*

**Theorem 2.2.5.** *Let $q, p, \alpha, s, t$ be arbitrary. Then the maps $D_s^t : B_q^p \to B_{q+pt}^p$ and $D_s^t : \mathcal{B}_\alpha^\infty \to \mathcal{B}_{\alpha+t}^\infty$ are isomorphisms, and isometries when the parameters of the norms of the spaces are chosen appropriately.*

In order to show that each $B_q^2$ is a reproducing kernel Hilbert space, we recall some facts.
Let $\mathfrak{H}$ a Hilbert space of complex-valued functions on a non-void set $X$ with a reproducing kernel $K$ see 1.10.30. One verifies easily that $K$ is positive-definite, and we write $K \gg 0$, in the sense that for any finite sequence $x_1, ..., x_n \in X$, the matrix $[K(x_i, x_j)]$ is positive-definite.
The following results are in [9].

**Theorem 2.2.6.** *Assume that $K \gg 0$ on $X$. Then there is a unique functional Hilbert space $\mathfrak{H}$ on $X$ with $K$ as its reproducing kernel. Moreover, if $X$ is a complex manifold and $K$ is a sesqui-holomorphic (sesqui-linear and holomorphic) on $X$ then $\mathfrak{H} \subset H(X)$.*
*We will denote $\mathfrak{H} = \mathfrak{H}(K; X)$.*

Thus there is a one-to-one correspondence between positive-definite kernels $K$ on $X$ and functional Hilbert spaces $\mathfrak{H}$.

**Proposition 2.2.7.** *Let $c_j \geq 0$ and let $K_j \gg 0$ be positive-definite kernel on $X$, $j = 1, 2$. Then:*

*(i)* $c_1 K_1 + c_2 K_2 \gg 0$ *on $X$;*

*(ii)* $K_1 . K_2 \gg 0$ *on $X$;*

*(iii)* $\Phi(K_1) \gg 0$ *on* $X$ *whenever* $\Phi$ *is an entire function on* $\mathbb{C}$ *with non-negative Taylor coefficients. In particular* $e^{K_1} \gg 0$ *on* $X$.

For kernels $K_1$ and $K_2$ on $X$, we write $K_1 \gg K_2$ on $X$ if $K_1 - K_2 \gg 0$ on $X$. we have

**Proposition 2.2.8.** *Let* $K_j \gg 0$ *on* $X$ *and let* $\mathfrak{H}_j = \mathfrak{H}_j(K_j : X)$ *be the associated functional Hilbert space on* $X$ *(*$j = 1, 2$*). Let* $c > 0$*. Then the following statements are equivalent:*

*(i)* $c^2 K_1 \gg K_2$ *on* $X$*;*

*(ii)* $\mathfrak{H}_1 \hookrightarrow \mathfrak{H}_2$ *and* $\|f\|_2 \leq \|f\|_1$ *for every* $f \in \mathfrak{H}_2$*;*

*In particular,* $\mathfrak{H}_1 \simeq \mathfrak{H}_2$ *(i.e* $\mathfrak{H}_1 \hookrightarrow \mathfrak{H}_2$ *and* $\mathfrak{H}_2 \hookrightarrow \mathfrak{H}_1$*) if and only if* $(c_1c_2)^2 K_1 \gg c_2^2 K_2 \gg K_1$ *on* $X$ *for some* $c_j > 0$ *(*$j = 1, 2$*).*

Many interesting examples of sesqui-holomorphic kernels on the Ball $\mathbb{B}$ of $\mathbb{C}^N$ are of the form

$$K(z, w) = \sum_{\alpha} c_\alpha z^\alpha \overline{w^\alpha} \quad (z, w \in \mathbb{B})$$

where $c_\alpha := c_\alpha(K) \in \mathbb{C}$, $\alpha \in \mathbb{Z}_+^N$. The family of such kernels is denoted $\mathfrak{H}(\mathbb{B})$. For any $K \in \mathfrak{H}(\mathbb{B})$ we associate the index set

$$\Gamma(K) = \{\alpha \in \mathbb{Z}_+^N : c_\alpha(K) \neq 0\}.$$

By $\mathfrak{H}_+(\mathbb{B})$ we denote the positive cone in $\mathfrak{H}(\mathbb{B})$ which consists of all $K \in \mathfrak{H}(\mathbb{B})$ such that $K \gg 0$ on $\mathbb{B}$. We then have:

**Theorem 2.2.9.** *For* $K \in \mathfrak{H}(\mathbb{B})$*, the following statements are equivalent:*

*(i)* $K \in \mathfrak{H}_+(\mathbb{B})$*;*

*(ii)* $c_\alpha(K) > 0$ *for every* $\alpha \in \Gamma(K)$*.*

*Moreover, in both cases the associate functional Hilbert space* $\mathfrak{H} := \mathfrak{H}(K : \mathbb{B}$ *consists of functions* $f$ *in* $H(\mathbb{B})$ *of the form*

$$f(z) = \sum_{\alpha \in \Gamma(K)} a_\alpha z^\alpha \quad (z \in \mathbb{B})$$

*such that*

$$\|f\|_K^2 = \sum_{\alpha \in \Gamma(K)} c_\alpha^{-1} |a_\alpha|^2 < \infty.$$

*In particular,* $\{\sqrt{c_\alpha} z^\alpha : \alpha \in \Gamma(K)\}$ *forms an orthonormal basis for* $\mathfrak{H}$*.*

**Corollary 2.2.10.** *Let* $K_j \in \mathfrak{H}_+(\mathbb{B})$ *with* $\mathfrak{H}_j := \mathfrak{H}_j(K_j : \mathbb{B})$ *and* $c_\alpha^{(j)} := c_\alpha(K_j)$ *(*$j = 1, 2$*) for* $\alpha \in \mathbb{Z}_+^N$*, and let* $a > 0$*. Then the following are equivalent:*

*(i)* $a^2 K_1 - K_2 \in \mathfrak{H}_+(\mathbb{B})$*;*

*(ii)* $\mathfrak{H}_2 \hookrightarrow \mathfrak{H}_1$ *and* $\|f\|_{K_1} \leq a\|f\|_{K_2}$ *for every* $f \in \mathfrak{H}_2$;

*(iii)* $c_\alpha^{(2)} \leq a^2 c_\alpha^{(1)}$ *for every* $\alpha \in \mathbb{Z}_+^N$*, and thus* $\Gamma(K_2) \subset \Gamma(K_1)$.

*In particular* $\mathfrak{H}_1 \simeq \mathfrak{H}_2$ *if and only if* $c_\alpha^{(1)} \simeq c_\alpha^{(2)}$ *on* $\mathbb{Z}_+^N$ *(and hence* $\Gamma(K_2) = \Gamma(K_1)$*).*

We are now able to show that each $B_q^2$ is a reproducing kernel Hilbert space.

**Definition 2.2.11.** *For* $q \in \mathbb{R}$ *and* $z, w \in \mathbb{B}$*, the Bergman-Besov kernels are*

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(1+N+q)_k}{k!}\langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^{\infty} \frac{k!\langle z,w\rangle^k}{(1-(N-q))_k}, & q \leq -(1+N), \end{cases}$$

*or*

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} & = \displaystyle\sum_{\alpha} \frac{(N+1+q)_{|\alpha|}}{\alpha!} z^\alpha \overline{w}^\alpha, \ \ if\ q > -(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) & = \displaystyle\sum_{\alpha} \frac{(|\alpha|!)^2}{\alpha!(1-N-q)_{|\alpha|}} z^\alpha \overline{w}^\alpha, \ \ if\ q \leq -(N+1) \end{cases}.$$

*where* ${}_2F_1 \in H(\mathbb{D})$ *is the hypergeometric function (see Definition 1.8.1).*

The following growth rate appearing in [52] will be use to obtain that $B_q^2$ are reproducing kernel Hilbert spaces.

**Lemma 2.2.12.** *If* $f \in B_q^p$*, then for any* $s, t$ *satisfying* $q + pt > -(1+N)$*, we have*

$$|D_s^t f(z)| \lesssim \frac{||f||_{B_q^p}}{(1-|z|^2)^{(1+N+q+pt)/p}} \qquad (z \in \mathbb{B})$$

*Proof.* When $f$ belongs to the Bergman-Besov space $B_q^p$, with $q > -1$, then $B_q^p = A_q^p$ and the result is well known. If $f$ belongs to the general Besov space $B_q^p$, by Definition 2.2.1, $D_s^t f \in A_{q+pt}^p$ for any $s, t$ satisfying (2.6) and $||f||_{B_q^p} = ||I_s^t f||_{L_q^p} = ||D_s^t f||_{A_{q+pt}^p}$. The result for $t$ satisfying (2.6) follows by applying the Bergman space case to $D_s^t f$.

What we have so far can be written also in the form $||f||_{\mathcal{B}^\infty_{(1+N+q)/p}} \lesssim ||f||_{B_q^p}$. But, by Remark 2.2.17, the parameter $t$ used in the norm $||f||_{\mathcal{B}^\infty_{(1+N+q)/p}}$ can be as low as to satisfy $q + pt > -(1+N)$. $\square$

**Corollary 2.2.13.** *If* $f \in B_{-(1+N)}^p$*, then* $||f||_{\mathcal{B}^\infty} \lesssim ||f||_{B_{-(1+N)}^p}$.

This corollary appeared also in [38, Proposition 3.3 ], as well as in [?, Corollary 5.5] more generally.

**Theorem 2.2.14.** *Each $B^2_q$ space is a reproducing kernel Hilbert space equipped with the several equivalent inner products*

$$_q\langle f, g\rangle^t_q = \int_{\mathbb{B}} I^t_s f \overline{I^t_s g} d\mu_q$$

*for each $s, t$ such that $q + 2t > -1$, the reproducing kernel of $B^2_q$ is (more precisely is equivalent to) $K_q$.*

*Proof.* By Theorem 1.10.29, we first showed that $L_z : B^2_q \mapsto \mathbb{C}$ such that $L_z(f) = f(z)$ is continuous for any $z \in \mathbb{B}$.
Let $q \in \mathbb{R}$ and $u > 0$ such that $q + 2u > -1$. Take $t$ such that $q + 2(u - t) > -1$. Let $f \in B^2_q$, then by Theorem 2.2.5,

$$D^{-t}_{s+t} f \in B^2_{q-2t},$$

and because $q - 2t < q - 2t + 2u$ we have

$$B^2_{q-2t} \subset B^2_{q-2t+2u}.$$

Set $Q = q - 2t + 2u$. then $Q + 2t = q + 2u > -1$, and equation 2.4, Theorem 2.2.12, Theorem 2.2.5 allow us to have

$$\begin{aligned} |f(z)| = |D^t_s D^{-t}_{s+t} f| &\lesssim \frac{||D^{-t}_{s+t} f||_{B^2_Q}}{(1 - |z|^2)^{\frac{1+N+Q+2t}{2}}} \\ &\lesssim \frac{||f||_{B^2_{Q+2t}}}{(1 - |z|^2)^{\frac{1+N+Q+2t}{2}}} \\ &= \frac{||f||_{B^2_{q+2u}}}{(1 - |z|^2)^{\frac{1+N+q+2u}{2}}} \\ &\leq \frac{||f||_{B^2_q}}{(1 - |z|^2)^{\frac{1+N+q+2u}{2}}}, \end{aligned}$$

where the last inequality is because $B^2_q \subset B^2_{q+2u}$ since $q < q + 2u$. Then $L_z : B^2_q \mapsto \mathbb{C}$ such that $L_z(f) = f(z)$ is continuous for any $z \in \mathbb{B}$.
Now we show that the reproducing kernel of $B^2_q$ is (more precisely is equivalent to) $K_q$.
Let

$$H_q(z, w) = \sum_{\alpha} c_\alpha(H_q) z^\alpha \overline{w^\alpha} = \sum_{l=0}^{\infty} \sum_{|\alpha|=l} c_\alpha(H_q) z^\alpha \overline{w^\alpha} = \overline{H^q_z(w)}$$

be the reproducing kernel of $B^2_q$. By Corollary 2.2.10 it is sufficient to show that $c_\alpha(H_q) \simeq c_\alpha(K_q)$.
First, let us find $c_\alpha(H_q)$. By definition of the reproducing kernel (see Definition 1.10.30), if

$$f(z) = \sum_{\alpha} a_\alpha z^\alpha = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} a_\alpha z^\alpha$$

is in $B^2_q$, then we have for $s, t$ such that $q + 2t > -1$,

$$f(z) =_q \langle f, H^q_z\rangle^t_q.$$

Since

$$D_s^t H_z^q(w) = \sum_{l=0}^{\infty} \sum_{|\alpha|=l} \frac{c_l(s+t)}{c_l(s)} \overline{c_\alpha(H_q)} w^\alpha \overline{z^\alpha}$$

and

$$D_s^t f(w) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{c_k(s+t)}{c_k(s)} a_\alpha w^\alpha$$

then

$$_q\langle f, H_z^q \rangle_s^t = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{|\alpha|=k} \sum_{|\beta|=l} \frac{c_k(s+t)}{c_k(s)} \frac{\overline{c_l(s+t)}}{\overline{c_l(s}} a_\alpha c_\beta(H_q) z^\alpha \int_{\mathbb{B}} w^\alpha \overline{w^\beta} d\mu_q. \tag{2.10}$$

Hence by Proposition 1.8.3 and Proposition 1.8.4 we have

$$_q\langle f, H_z^q \rangle_s^t = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{|c_k(s+t)|^2}{|c_k(s)|^2} c_\alpha(H_q) \frac{N!\Gamma(1+q+2t)\alpha!}{\Gamma(N+1+q+2t+|\alpha|)} a_\alpha z^\alpha$$

so that

$$c_\alpha(H_q) = \frac{|c_k(s)|^2}{|c_k(s+t)|^2} \frac{\Gamma(N+1+q+2t+|\alpha|)}{N!\Gamma(1+q+2t)\alpha!},$$

where $\Gamma$ is the function Gamma see (1.2). By Stirling's formula (See equation (2.1)), we have

$$c_\alpha(H_q) \simeq \frac{|\alpha|^{N+1+q}}{\alpha!}$$

and

$$c_\alpha(K_q) \simeq \frac{|\alpha|^{N+1+q}}{\alpha!}.$$

Then $c_\alpha(H_q) \simeq c_\alpha(K_q)$ and the proof is completed. □

The following lemma is obtain by direct computation and by the virtue of Proposition 1.8.3 and Proposition 1.8.4.

**Lemma 2.2.15.** *For $f(z) = \sum_\alpha a_\alpha z^\alpha \in B_q^2$, we have*

$$||f||_{B_q^2}^2 = \sum_\alpha \frac{N!\alpha!|a_\alpha|^2|d_{|\alpha|}(s,t)|^2}{\Gamma(N+1+q+2t+|\alpha|)},$$

*where $d_{|\alpha|}(s,t)$ is defined in Definition 2.1.1 and $\Gamma$ in* (1.2).

The following result can be found in [54].

**Lemma 2.2.16.** *1. For $q < -(N+1)$, each $|K_q(z,w)|$ is bounded above as $z, w$ vary in $\mathbb{B}$.*

*2. For each $q \in \mathbb{R}$,*

*(a) each $|K_q(z,w)|$ with $q \in \mathbb{R}$ is bounded below by a positive constant as $z, w$ vary in $\mathbb{B}$. In particular, no $K_q(z,w)$ has a zero in $\mathbb{B} \times \mathbb{B}$.*

*(b) there is a $\rho_0 < 1$ such that for $|z| \leq \rho_0$ and all $w \in \mathbb{B}$, we have $\Re K_q(z,w) \geq \frac{1}{2}$.*

*Proof.* 1. We recall that

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} & = \sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!}\langle z,w\rangle^k, \ \text{ if } q > -(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) & = \sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k}\langle z,w\rangle^k, \ \text{ if } q \leq -(N+1) \end{cases},$$

so $K_q(z,w) = \sum_{k=0}^{\infty} c_k(q)\langle z,w\rangle^k = \sum_{k=0}^{\infty} c_k(q) v^k = k_q(v)$ where $v = \langle z,w\rangle$ and $c_k(q)$ is defined in Definition 2.1.1. By Stirling's formula $c_k(q) \sim k^{N+q} \ (k \to \infty)$, so that when $q < -(N+1)$, the power series of $k_q(v)$ converges uniformly for $v \in \overline{\mathbb{D}}$. This shows boundedness. When $q < -(N+1)$, see that $k_q$ is not zero on a set containing $\overline{\mathbb{D}} - \{1\}$. The reason for this is that the first term 1 of the hypergeometric function $k_q$ is positive see [60, Equation 15.13.1]. But also $k_q(1) \neq 0$. Thus $|k_q|$ for $q < -(1+N)$ is bounded below on $\overline{\mathbb{D}}$.
2. If $q = -(1+N)$, then $k_{-(1+N)}(v) = v\log(1-v)^{-1}$. On $\overline{\mathbb{D}} - \{1\}$, $k_{-(1+N)}$ is not zero and $|k_{-(1+N)}(v)|$ blows up as $v \to 1$ within $\overline{\mathbb{D}}$. So $|k_{-(1+N)}|$ is bounded below on $\overline{\mathbb{D}}$.
The claim about $|k_q|$ for $q > -(1+N)$ is obvious and the lower bound can be taken as $2^{-(1+N+q)}$. Then (a) follows.
We have

$$|K_q(z,w)| \leq 1 + C\sum_{k=1}^{\infty} k^{N+q}|\langle z,w\rangle|^k,$$

for some constant $C$ and

$$C\sum_{k=1}^{\infty} k^{N+q}|\langle z,w\rangle|^k \leq C\sum_{k=1}^{\infty} k^{N+q}|z|^k|w|^k \leq C|z|\sum_{k=1}^{\infty} k^{N+q}|z|^{k-1}$$

for all $z,w \in \mathbb{B}$. The last series converges, say, for $|z| = \frac{1}{2}$; call its sum W and set $\rho_0 = \min\{\frac{1}{2}, \frac{1}{2CW}\}$. If $|z| \leq \rho_0$, then

$$|C\sum_{k=1}^{\infty} k^{N+q}\langle z,w\rangle^k| \leq CW|z| \leq \frac{1}{2} \ \ (z \in \mathbb{B}).$$

This is, $|K_q(z,w) - 1| \leq \frac{1}{2}$ for $|z| \leq \rho_0$ and all $z \in \mathbb{B}$. This implies the desired result (b). □

One of the best things about the operator $D^t_s$ is that it allows us to pass easily from one kernel to the other and from one space to the other of $B^2_q$ Hilbert spaces family. For instance

$$D^t_q K_q(z,w) = K_{q+t}(z,w) \tag{2.11}$$

for any $q, t$, where differentiation is performed on the holomorphic variable $z$ (see [53]).

**Remark 2.2.17.** *By now, it is well-known that Definition 2.2.1 and 2.2.2 are independent of $s, t$ under* (2.6) *and* (2.7)*, respectively, and also of the particular type of the derivative. Further, the norms on* *a given space depending on $s,t$ are equivalent to each other under* (2.6) *or* (2.7). *For these, see, for example, [?, Theorem5.12 (i)], [48],[50],[?]. So given $s,t$, $I_s^t$ imbeds $B_q^p$ isometricaly into $L_q^p$ if and only if* (2.6) *holds, and $I_s^t$ imbeds $B_\alpha^\infty$ isometrically into $L_\alpha^\infty$ if and only if* (2.7) *holds.*

*If $q > -1$, we can take $t = 0$ in* (2.6) *and recover the weighted Bergman spaces $B_q^p = A_q^p$. Further, $B_{-1}^2$ is the Hardy spaces while $B_{-(1+N)}^2$ is the Dirichlet space, and $B_{-N}^2$ is the Drury-Arveson space. If $\alpha > 0$, we can take $t = 0$ in* (2.7) *and obtain the weighted Bloch spaces. If $\alpha < 0$, then the corresponding spaces are the holomorphic Lipschitz spaces $\Lambda_{-\alpha} = \mathcal{B}_\alpha^\infty$, see, for example, [?, Section 6.4].*

*The usual Bloch space $B_0^\infty = B^\infty$ corresponds to $\alpha = 0$. Most other authors use $\alpha + 1$ while [?] uses $-\alpha$ where we use $\alpha$. There is no discussion of little Bloch spaces in this thesis.*

**Notation 2.2.18.** *If $X_a$ is a family of spaces indexed by $a \in \mathbb{R}$, the symbol $X_{<a}$ denotes any one of the spaces $X_b$ with $b < a$. For functions, $h_{<a}$ has a similar meaning.*

**Remark 2.2.19.** *Invertibility of $D_s^t$ implies that only the zero function has zero norm in $B_q^p$ or $\mathcal{B}_\alpha^\infty$. The other types of derivatives mentioned in Remark 2.2.17 that can be used in place of the $D_s^t$ are powers of the holomorphic gradient and the radial derivative given by*

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_N}\right) \qquad \text{and} \qquad Rf(z) = \langle \nabla f(z), \overline{z} \rangle.$$

*Integrals of these derivatives define seminorms for the spaces $B_q^p$ or $\mathcal{B}_\alpha^\infty$.*

*The holomorphic automorphism of $\mathbb{B}$ that exchanges $0$ and $z$ $(z \neq 0)$ is the map*

$$\varphi_z(w) = \frac{z - P_z(w) - \sqrt{1-|z|^2}(I - P_2)(w)}{1 - \langle w, z \rangle} \qquad (w \in \mathbb{B}), \tag{2.12}$$

*where $P_2(w) = \langle w, z \rangle z/|z|^2$ is the projection on the complex line passing through $0$ and $z$. It reduces to the well-known function $\varphi_2(w) = (z-w)/(1-\overline{z}w)$ for $w \in \mathbb{D}$ when $N = 1$. The Bergman metric on $\mathbb{B}$ is defined by*

$$d(z,w) = \frac{1}{2} log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} = tanh^{-1}|\varphi_z(w)| \ \ z, w \in \mathbb{B}.$$

*This metric is invariant under composition with the automorphisms of $\mathbb{B}$. We denote the balls centered at a with radius $r$ in the Bergman metric by $B(a,r)$. A sequence $\{a_k\}$ in $\mathbb{B}$ is called separated if there is a $\rho > 0$ such that $B(a_k, a_m) \geq \rho$ for all $k \neq m$, and we call $\rho$ the separation constant.*

## 2.3 Examples of functions in Besov or Bloch spaces

We now develop and collect interesting functions that live in certain Besov or Bloch spaces. These examples can be found in [52]

**Example 2.3.1.** *The functions in the Hilbert spaces $B^2_q$ can be characterized by their Taylor series. Let $q < Q$. Set*

$$h(z) := \sum_k k^{(2(N-1)+q+Q)/4} z_1^k \qquad (z \in \mathbb{B})$$

*Then from Lemma 2.2.15 and by* (2.1)*,*

$$||h||^2_{B^2_q} \sim \sum_k \frac{1}{k^{1-(Q-q)/2}} = \infty \qquad \textit{while} \qquad ||h||^2_{B^2_Q} \sim \sum_k \frac{1}{k^{1+(Q-q)/2}} < \infty$$

**Example 2.3.2.** *An example that is essential for the Bloch-Lipchitz spaces is the family of functions*

$$f_\alpha(z) := \sum_{k=0}^{\infty} c_k(\alpha - (1+N)) z_1^k \qquad (z \in \mathbb{B})$$

*indexed by $\alpha \in \mathbb{R}$. For any branch of the logarithm, by Definition 2.2.11 of Bergman-Besov kernels,*

$$f_\alpha(z) = \frac{1}{(1-z_1)^\alpha} \quad (\alpha > 0) \qquad \textit{and} \qquad f_0(z) = \frac{1}{z_1} log \frac{1}{1-z_1} \qquad (z \in \mathbb{B}) \tag{2.13}$$

*Note that, by* (2.11)*,*

$$D^t_{\alpha-(1+N)} f_\alpha = f_{\alpha+t}. \tag{2.14}$$

*Now Definition 2.2.2 show immediately that $f_1 \in \mathcal{B}^\infty_1$ since*

$$\frac{1-|z|^2}{|1-z_1|} < 2.$$

*The same Definition 2.2.2, shows that also $f_1 \notin \mathcal{B}^\infty_\beta$ if $\beta < 1$. Applying* (2.14) *and Theorem 2.2.5 yields*

$$f_\alpha \in \mathcal{B}^\infty_\alpha \backslash \mathcal{B}^\infty_{<\alpha} \qquad (\alpha \in \mathbb{R}) \tag{2.15}$$

*Assume that*

$$f_{<\frac{1+N}{p}} \in A^p_0 \qquad \textit{but} \qquad f_{\frac{1+N}{p}} \notin A^p_0, \tag{2.16}$$

*then*

$$f_{<\frac{1+N+q}{p}} \in B^p_q \qquad \textit{but} \qquad f_{\frac{1+N+q}{p}} \notin B^p_q. \tag{2.17}$$

*by applying* (2.14) *and Theorem 2.2.5.*

*We give here proof of* (2.16)*. For $\alpha > 0$, by* (2.11)*, for $w = r\zeta$ with $\zeta \in \mathbb{S}$ and $r \geq 0$, we have*

$$|f_\alpha(w)|^p = \frac{1}{|1-w_1|^{\alpha p}} = \frac{1}{|1-\langle r\zeta, e_1\rangle|^{\alpha p}}.$$

*Then by polar coordinates,*

$$\int_{\mathbb{B}} |f_\alpha|^p d\nu \sim \int_0^1 \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|1-\langle re_1, \zeta\rangle|^{\alpha p}} r^{2N-1} dr.$$

*If $\alpha = (1+N)/p$, then $\alpha p = 1+N$, and Proposition 1.5.7, yields*

$$\int_{\mathbb{B}} |f_\alpha|^p d\nu \sim \int_0^1 \frac{r^{2N-1}}{1-r^2} dr$$

*which diverges. If $\alpha < (1+N)/p$, then $\alpha p - N = c < 1$, and Proposition 1.5.7, yields*

$$\int_{\mathbb{B}} |f_\alpha|^p d\nu \lesssim \int_0^1 \frac{r^{2N-1}}{(1-r^2)^c} dr$$

*which converges, where we use $\lesssim$ to incorporate the cases $c \leq 0$ too.*

We say that a sequence $\{n_k\}$ in $\mathbb{N}$ has *Hadamard gaps* if there is a $r > 1$ such that $n_{k+1} \geq \tau n_k$. For such a sequence, if the homogeneous expansion of an $f \in H(\mathbb{B})$ has the form $f = \sum_k f_{n_k}$, we write $f \in HG$. The following theorem is in [52].

**Theorem 2.3.3.** *Let $H = \sum_k H_{n_k} \in HG$.*

*(i) We have $h \in B^p_q$ if and only if $\sum_k n_k^{-(1+q)} \|H_{n_k}\|^p_{L^p(\sigma)} < \infty$.*

*(ii) We have $h \in B^\infty_\alpha$ if and only if $\sup_k n_k^{-\alpha} \|H_{n_k}\|_{L^\infty(\sigma)} < \infty$.*

**Definition 2.3.4.** *Ryll-Wojtaszczyk polynomials are polynomials $(W_m)_m$, each $W_m$ is a homogeneous polynomial of degree $m$ with the properties*

$$\|W_m\|_{L^\infty(\sigma)} = 1 \qquad \textit{and} \qquad \|W_m\|_{L^p(\sigma)} \gtrsim 1 \quad (0 < p < \infty). \tag{2.18}$$

Several of these examples are constructed using the *Ryll-Wojtaszczyk polynomials* $W_m$, $m = 0, 1, 2, \ldots$ see Definition 2.3.4 These polynomials were invented for $p = 2$ in [70, Theorem 1.2]. The general case $p$ is due to [78, Corollary 1]. Clearly also

$$\|W_m\|_{L^p(\sigma)} \leq \|W_m\|_{L^\infty(\sigma)} = 1. \tag{2.19}$$

When $N = 1$, we can simply take $W_m = z^m$. However, taking something like $z_1^m$ for simplicity when $N > 1$ would not be as useful, because it does not satisfy the second property in (2.18).

*Proof.* See [49, Lemmas 9.4 and 9.2], see also [**?**, Proposition 61 and 63]. For unweighted Bergman spaces $A^p_0$, see also [85, Proposition 3]. For the spaces $\mathcal{B}^\infty_\alpha$ with $\alpha > -1$, see also [85, Proposition 2]. □

The example that is indispensable for the Bergman-Besov spaces is the family of functions

**Example 2.3.5.**

**Lemma 2.3.6.** *Let $p, q\ in \mathbb{R}$ and denote by*

$$G_{qp}(z) := 2^{k(1+q)/p} W_{2^k}(z) \qquad (z \in \mathbb{B}).$$

*Then*

*(i)* $G_{qp} \notin B^p_q$

*(ii)* $G_{qp} \in B^p_Q$ *if and only if* $(1+q)/p < (1+Q)/P$.

*(iii)* $G_{qp} \in \mathcal{B}^\infty_\alpha$ *if and only if* $(1+q)/p \leq \alpha$.

*Proof.* (i) By Theorem 2.3.3 and (2.18), it is clear that $G_{qp} \notin B^p_q$. But the real question is to determine those spaces $B^p_Q$ that $G_{qp}$ lies in. The answer is that $G_{qp} \in B^p_Q$ if and only if $(1+q)/p < (1+Q)/P$.
(ii) If $(1+q)/p < (1+Q)/P$, then Theorem 2.3.3 and (2.19),

$$\sum_k 2^{-k(1+Q)} (2^{k(1+q)/p})^P \|W_{2^k}\|^P_{L^p(\sigma)} \leq \sum_k \frac{1}{2^{kP((1+Q)/P - 1(1+q)/p)}} < \infty,$$

and $G_{qp} \in B^p_Q$. On the other hand, if $(1+q)/p \geq (1+Q)/P$, then by Theorem 2.3.3 and (2.18),

$$\sum_k 2^{-k(1+Q)} (2^{k(1+q)/p})^p \|W_{2^k}\|^P_{L^P(\sigma)} \gtrsim \sum_k \frac{1}{2^{kP((1+Q)/P - 1(1+q)/p}} = \infty$$

and $G_{qp} \notin B^P_Q$. By Theorem 2.3.3 and (2.18), it is clear that $G_{qp} \in \mathcal{B}^\infty_\alpha$ if and only if $(1+q)/p \leq \alpha$. □

*Further,* $G_{qp} \in H^\infty$ *by* (2.18) *if* $q < -1$. *On the other hand, if* $G_{1,p} \in H^\infty$ *were true then also* $G_{-1,p} \in H^2$. *Then we would have* $\|G_{-1,p}\|_{H^2} = (\sum_k \|W_{2^k}\|^2_{L^2(\sigma)})^{\frac{1}{2}} < \infty$ *by [?, Proposition 1.4.8]. But this is impossible since* $\sum_k \|W_{2^k}\|_{L^2(\sigma)} = \infty$ *by* (2.18).

The use Ryll-Wojtaszczyk polynomials to show exclusions between pairs of function spaces is advocated in [85] and [62].
We now conctruct functions in every Besov space using the atomic decomposition idea. The atomic decomposition of Besov spaces is developed in several places starting with [19, Theorem 2].

**Example 2.3.7.**

**Proposition 2.3.8.** *Let a sequence* $\{a_k\}$ *in* $\mathbb{B}$ *that is merely separated with separation constant* $2\rho$. *Given* $q$ *and* $p$, *we also take a sequence* $\alpha = \{\lambda_k\}$ *in* $l^p$. *For* $0 < p \leq 1$, *we take an* $s$ *satisfying the inequality* $1 + N + q < p(1 + N + s)$; *for* $1 < p < \infty$, *we take an* $s$ *satisfying* $1 + q < p(1 + s)$. *We set*

$$F_{qp}(z) := \sum_k \lambda_k \left(1 - |a_k|^2\right)^{1+N+s-(1+N+q)/p} K_s(z, a_k) \qquad (z \in \mathbb{B}).$$

*We have* $F_{qp} \in B^p_q$.

*Proof.* We start by showing that the series defining $F_{qp}$ converges uniformly on any compact set $K \subset \mathbb{B}$ and hence $F_{pq} \in H(\mathbb{B})$. Let $z \in K$; then $|K_s(z, a_k)| \sim 1$ for any $k$, and

$$|F_{qp}(z)| \lesssim \sum_k |\lambda_k| \left(1 - |a_k|^2\right)^{1+N+s-(1+N+q)/p}.$$

First, for $0 < p \leq 1$, by the first choice of $s$, the power on $1 - |a_k|^2$ is positive and hence

$$|F_{qp}(z)| \lesssim \sum_k |\lambda_k| \leq \sum_k |\lambda_k|^p < \infty \qquad (z \in K).$$

Second, for $1 < p < \infty$, by the Hölder inequality and that $\lambda \in \ell^p$, we have

$$|F_{qp}(z)| \lesssim \left(\sum_k \left(1 - |a_k|^2\right)^{(p(1+N+s)-(1+N+q))/(p-1)}\right)^{(p-1)/p} \qquad (z \in M)$$

Call the power on $1 - |a_k|^2$ by $\widetilde{p}$. By the second choice of $s$, we see that $\widetilde{p} > N$. Then, since the balls $B(a_k, \rho)$ are disjoint, using Lemma 1.5.6 we obtain

$$\begin{aligned}|F_{qp}(z)|^{p/(p-1)} &\lesssim \sum_k \left(1 - |a_k|^2\right)^{\widetilde{p}} \sim \sum_k \int_{B(a_k,\rho)} \left(1 - |w|^2\right)^{\widetilde{p}-(1+N)} d\nu(w)\\ &= \int_{\cup_k b(a_k,\rho)} (1 - |w|^2)^{\widetilde{p}-(1+N)} d\nu(w) \leq \int_{\mathbb{B}} (1 - |w|^2)^{\widetilde{p}-(1+N)} d\nu(w)\end{aligned}$$

for all $z \in K$. But the last integral is finite. We conclude that $F_{qp} \in H(\mathbb{B})$.

To show that $F_{qp} \in B^p_q$, we consider the linear map $T$ defined by $T : l^p \to H(\mathbb{B})$, such that $T\lambda := F_{qp}$ for $\lambda \in \ell^p$. Let $t > 0$ satisfying (2.6); then $s + t > -1$ for any value of $p$. Using (2.11) we have

$$D^t_s T\lambda(z) = \sum_k \lambda_k \frac{(1 - |a_k|^2)^{1+N+s-(1+N+q)/p}}{(1 - \langle z, a_k\rangle)^{1+N+s+t}} \qquad (z \in \mathbb{B}).$$

First, since $I^t_s F(z) = (1 - |z|^2)^t D^t_s F(z)$, for $0 < p \leq 1$,

$$|I^t_s T\lambda(z)|^p \leq (1 - |z|^2)^{pt} \sum_k |\lambda_k|^p \frac{(1 - |a_k|^2)^{p(1+N+s)-(1+N+q)}}{|1 - \langle z, a_k\rangle|^{p(1+N+s+t)}},$$

then by, Proposition 1.5.7,

$$\begin{aligned}\|T\lambda\|^p_{B^p_q} &\leq \sum_k |\lambda_k|^p (1 - |a_k|^2)^{p(1+N+s)-(1+N+q)} \int_{\mathbb{B}} \frac{(1 - |z|^2)^{q+pt}}{|1 - \langle z, a_k\rangle|^{p(1+N+s+t)}} d\nu(z)\\ &\sim \sum_k |\lambda_k|^p = \|\lambda\|^p_{\ell^p}.\end{aligned}$$

Second, for $1 < p < \infty$, by Theorem 1.5.6,

$$|D^t_s T\lambda(z)| \lesssim \sum_k |\lambda_k| \frac{(1 - |a_k|^2)^{(1+N)-(1+N+q)/p}}{\nu(B(a_k, \rho))} \int_{B(a_k,\rho)} \frac{(1 - |w|^2)^s}{|1 - \langle z, w\rangle|^{1+N+s+t}} d\nu(w)$$

and $|I_s^t T\lambda(z)| \leq V_{s+t}^0 \phi(z)$, where

$$\phi(w) = \sum_k |\lambda_k| \frac{(1-|a_k|^2)^{(1+N)-(1+N+q)/p}}{\nu(B(a_k,\rho))} \chi_\nu(B(a_k,\rho))(w),$$

and $V_{s+t}^0$ is defined in (2.5) , and $\chi_{B(a_k,\rho)}$ denotes the characteristic function. Now since $\{a_k\}$ is separated, from Theorem 1.5.6 we have

$$\begin{aligned}
\|\phi\|_{L_q^p}^p &= \sum_k |\lambda_k|^p \frac{(1-|a_k|^2)^{p(1+N)-(1+N+q)}}{\nu^p(B(a_k,\rho))} \nu_q(B(a_k,\rho)) \\
&\sim \sum_k |\lambda_k|^p \frac{(1-|a_k|^2)^{p(1+N)-(1+N+q)}}{(1-|a_k|^2)^{p(1+N)}} (1-|a_k|^2)^{1+N+q} \\
&= \sum_k |\lambda_k|^p = \|\lambda\|_{\ell^p}^p.
\end{aligned}$$

Applying Theorem 2.1.3 implies that $V_{s+t}^0 : L_q^p \to L_q^p$ is bounded by the conditions imposed above on $s, t$. It follows that $\|T\lambda\|_{B_q^p} \leq \|V_{s+t}^0 \phi\|_{L_q^p} \lesssim \|\lambda\|_{\ell^p}$.

Thus $T : \ell^p \to B_q^p$ is bounded and $F_{qp} = T\lambda \in B_q^p$ for any of $p$.

□

## 2.4 Some inclusions relations on Besov or Bloch spaces

Here we take care of a few better-known and straightforward inclusions that are part of the full picture.

**Theorem 2.4.1.**

*(i) If $q < Q$, then $B_q^p \subset B_Q^p$.*

*(ii) If $\alpha < \beta$, then $\mathcal{B}_\alpha^\infty \subset \mathcal{B}_\beta^\infty$.*

*Proof.* For both (i) and (ii), the inclusion follows directly from Definitions 2.2.1 and 2.2.2 and Remark 2.2.17. □

The inclusion in (i) is strict too, because $G_{qp} \in B_Q^p \backslash B_q^p$ by Example 2.3.5, since $\frac{1+q}{p} < \frac{1+Q}{p}$. A similar example when $N = 1$ appears earlier in [51, Example 4.5].

**Theorem 2.4.2.** $\mathcal{B}_{<0}^\infty \subset H^\infty \subset \mathcal{B}^\infty$

*Proof.* The left hand inclusion on $\mathbb{B}$ is in [**?**, Theorem 6.4.10]. There is a different proof in [50, Theorem 5.4], using Bergman projections. Both proofs show also $\mathcal{B}_{<0}^\infty \subset A(\mathbb{B})$.

Let $N > 1$. Let $f \in H^\infty$, and without loss of generality, assume $f : \mathbb{B} \to \mathbb{D}$ so that $\|f\|_\infty \leq 1$. Let also $w = f(z)$, and set $g = \varphi_w \circ f \circ \varphi_z$, where $\varphi_w$ on $\mathbb{D}$ and $\varphi_z$ on $\mathbb{B}$ are as in (2.12). Then

$g \in H(\mathbb{B})$, $g(0) = 0$, and $|g'(0)\zeta| \leq 1$ for all $\zeta \in \mathbb{B}$, where $g'(0)$ is called the *hyperbolic derivative* of $f$ at $z$; see [46, p.65]. Applying the chain rule yields that $g'(0) = (1 - |w|^2)^{-1}\nabla f(z)\varphi_z'(0)$, where $\varphi_z'(0) = -(1_|z|^2)P_z + \sqrt{1-|z|^2}(I - P_z)$ by [?, Theorem 2.2.2 (ii)]. Since $|w| \leq 1$, using the special value $\zeta = z$, we obtain that $|\nabla f(z)(1 - |z|^2)|z| \leq 1$ for all $z \in \mathbb{B}$. This proves that $f \in \mathcal{B}^\infty$.

By Example 2.3.2, the function $f_0 \in \mathcal{B}^\infty \backslash H^\infty$ shows that the right hand inclusion is strict. If $\alpha < 0$, choose $\beta$ such that $\alpha < \beta < 0$. Then $f_\beta \in H^\infty \backslash \mathcal{B}_\alpha^\infty$ by (2.15), and the left hand inclusion is also strict. □

These are others inclusion relations.

**Theorem 2.4.3.** *Given $B_q^p$, we have the inclusions*

$$\mathcal{B}^\infty_{<\frac{1+q}{p}} \subset B_q^p \subset \mathcal{B}^\infty_{\frac{1+N+q}{p}}$$

**Theorem 2.4.4.** *Let $B_q^p$ be given.*

*(i) If $p \leq P$, then $B_q^p \subset B_Q^P$ if and only if*

$$\frac{1+N+q}{p} \leq \frac{1+N+Q}{P}$$

*(ii) If $P < p$, then $B_q^p \subset B_Q^P$ if and only if*

$$\frac{1+q}{p} < \frac{1+Q}{P}. \tag{2.20}$$

**Theorem 2.4.5.**

*(i) $B_q^p \subset H^\infty$ if and only if $q < -(1+N)$, or $q = -(1+N)$ and $0 < p \leq 1$.*

*(ii) $H^\infty \subset B_q^p$ if and only if $q > -1$, or $q = -1$ and $p \geq 2$.*

## 2.5 Sobolev imbeddings

Our final intention is to compare the inclusions in Theorem 2.4.3 and 2.4.4 with those predicted by the holomorphic versions of the Sobolev imbedding theorems. Following [1, chapter 3], for $1 \leq p < \infty$, we let the *Sobolev space* $W^{m,p}$ be the space of all locally integrable functions on $\mathbb{B}$ all of whose generalized partial derivatives of order up to and including $m = 1, 2, ...$ belong to $L^p$. The subspace of $W^{m,p}$ consisting of holomorphic functions on $\mathbb{B}$ can be regarded as the Besov space $B_{-mp}^p$ in which always $q = -mp \leq -1$.

The Sobolev imbedding theorem we are interested in is [1, Theorem 4.12], and concerns the continuous inclusion of $W^{m,p}$ in Lebesgue or Lipschitz spaces. It is for Sobolev spaces defined on some types of domains in $\mathbb{R}^n$ for all of which $\mathbb{B}$ is a standard example. We read this theorem by setting $n = 2N$ and taking the intersections of all spaces with $H(\mathbb{B})$. We analyze our findings in five regions of the parameters.

**(a)** If $mp < 2N$ (that is, $-2N < q < -1$), the Sobolev imbedding is $W^{m,p} \subset L^{p^*}$, which translates to the holomorphic setting as $B^p_{-mp} \subset B^{p^*}_0$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{m}{2N}$. The sharpest (that is, into smallest space) inclusion that Theorem 2.4.4 (i) gives is $B^p_{-mp} \subset B^{p_2}_0$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{m}{1+N}$. Since $p_2 < p^*$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$. The two results say the same for $N = 1$.

**(b)** If $mp = 2N$ (that is, $q = -2N$), the Sobolev imbedding is $W^{m,p} \subset L^{p^*}$ for any $p^* \geq 1$, which yields $B_{-2N} \subset B^{p^*}_0$. The inclusions that Theorem 2.4.4 (i) gives $B^p_{-2N} \subset B^{p_2}_0$, where $\frac{1}{p_2} \geq \frac{1-N}{1+N}\frac{1}{p}$, which says $0 < p_2 < \infty$ for all $N$. So the two imbeddings are equivalent for $q = -2N$.

**(c)** If $2N < mp < 2N + p$ (that is, $-2N - p < q < -2N$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_\beta$, where $\beta = m - 2N/p$ and $0 < \beta < 1$, which yields $B^p_{-mp} \subset \mathcal{B}^\infty_{-\beta}$. The inclusion that inclusion Theorem 2.4.3 gives is $B^p_{-mp} \subset \mathcal{B}^\infty_\alpha$, where $\alpha = (1 + N)/p - m$. Since this $\alpha < -\beta$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$. The two results say the same for $N = 1$.

**(d)** If $mp = 2N+p$ (that is, $q = -2N-p$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_\beta$ for any $0 < \beta < 1$, yielding $B^p_{-2N-p} \subset \mathcal{B}^\infty_{-\beta}$. If also $p = 1$, then $W^{m,1} \subset \Lambda_1$, that is, $B^1_{2N-1} \subset \mathcal{B}^\infty_{-1}$. The inclusion that Theorem 2.4.3 gives is $B^p_{-2N-p} \subset \mathcal{B}^\infty_\alpha$ for any $1 \leq p < \infty$, where $\alpha = (1 - N)/p - 1$. Since this $\alpha < -\beta$, our inclusion is sharper than that of the Sobolev imbedding theorem for $N > 1$ or $p > 1$. The two results say the same when both $N = 1$ and $p = 1$.

**(e)** If $mp > 2N + p$ (that is, $q < -2N - p$), the Sobolev imbedding is $W^{m,p} \subset \Lambda_1$, which yields $B^p_{-mp} \subset \mathcal{B}^\infty_{-1}$. The inclusion that Theorem 2.4.3 gives is $B^p_{-mp} \subset \mathcal{B}^\infty_\alpha$, where $\alpha = (1+N)/p-m$. Since now $\alpha < -1$, our inclusion is sharper than that of the Sobolev imbedding theorem.

We can compare the number of derivatives lost in the embeddings in (c), (d) and (e), which are all in the form $W^{m,p} \subset \Lambda_\beta$. The number of derivatives needed for $B^p_{-mp}$ is given by $-mp + pt > -1$ and is $t > m - 1/p$, while for $\Lambda_{m-2N/p}$ it is given as $t > m - 2N/p$; hence the number of the lost in the Sobolev imbeddings. This difference has already been noted in [9, PP.39-40].

The fact that our inclusions are stronger and our loss of derivatives is less in general than those predicted by the Sobolev imbedding theorem should come as no surprise, because our spaces consist of holomorphic functions that are very smooth on very nice domain.

Chapter 3

# Weighted estimates for operators associated to the Bergman-Besov kernels

## 3.1 Introduction

As said in the general introduction, weighted inequalities appeared almost simultaneously with the birth of singular integrals that stimulated their development. In particular, a natural question is the characterization of positive functions $\omega$ for which a singular integral maps $L^p(\omega d\mu)$ to itself. A famous example of a singular integral is the Bergman projection, whose boundedness problem, solved elsewhere by Békollè and Bonami, is historically linked to the duality problem for Bergman spaces.

Recall that Besov spaces extend weighted Bergman spaces to all $q$. A function $f \in H(\mathbb{B})$ belongs to the Besov space $B_q^p$ whenever $I_s^t f \in L_q^p$ for some $s, t$ satisfying

$$\begin{cases} q + pt > -1 & \text{if } \ 1 \leq p < \infty \\ t > 0 & \text{if } \ p = \infty. \end{cases}$$

Where for $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which $f_k$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree $k$,

$$I_s^t f(z) = (1 - |z|^2)^t D_s^t f(z),$$

$$D_s^t f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

with

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if } \ a > -(N+1), \\ \frac{k!}{(1-N-a)_k} & \text{if } \ a \leq -(N+1). \end{cases}$$

$(u)_v$ is the Pochhammer symbol defined by $(u)_v = \frac{\Gamma(u+v)}{\Gamma(u)}$ with $\Gamma$ the gamma function see (1.2).

The space $B^2_q$ is a Hilbert space with reproducing kernel $K_q$ defined in Definition 2.2.11, but we recall it here:

$$K_q(z,w)=\begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} &=& \sum_{k=0}^{\infty}\frac{(N+1+q)_k}{k!}\langle z,w\rangle^k, \text{ if } q>-(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) &=& \sum_{k=0}^{\infty}\frac{k!}{(1-N-q)_k}\langle z,w\rangle^k, \text{ if } q\leq -(N+1), \end{cases}$$

where ${}_2F_1\in H(\mathbb{D})$ is the Gauss hypergeometric function see (1.6). For $a,b,s,t\in\mathbb{R}$ the operators that we are interested in are defined by (reproducing) Bergman-Besov kernels. For $f\in L^p(d\mu_q)$ we define

$$\begin{aligned} T^q_{a,b}f(z) &:= T_{a,b}f(z)=\int_{\mathbb{B}}K_a(z,w)f(w)(1-|w|^2)^{b-q}d\mu_q(w),\\ S^q_{a,b}f(z) &:= S_{a,b}f(z)=\int_{\mathbb{B}}|K_a(z,w)||f(w)|(1-|w|^2)^{b-q}d\mu_q(w),\\ P^q_{s,t}f(z) &:= P_{s,t}f(z)=(1-|z|^2)^t\int_{\mathbb{B}}K_{s+t}(z,w)f(w)(1-|w|^2)^{s-q}d\mu_q(w). \end{aligned}$$

Throughout this thesis, $b>-1$ and $s>-1$ because we want our operator to be well defined (see for example Lemma 3.4.1). Our main motivation comes from the operators $P_{s,0}$, and $P^+_{s,N+1+s}$, which are the Bergman projection and Berezin transform respectively, where $P^+_{s,N+1+s}f(z)=(1-|z|^2)^tS_{s+N+1+s,s}f(z)$. The operators $P_{s,t}$, $T_{a,b}$ and $S_{a,b}$ are important in the study of function-theoretic operator theory, see for example [88] when $q=-N-1$. In this chapter we are mainly interested on the weighted estimates for the operators $T_{a,b}$ or $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Here and throughout the paper $\omega$ is a locally integrable positive function called a weight. It follows from (0.1) and (0.2) that it will be enough to study weighted estimates for one of these two operators since the $(L^p(\omega d\mu_q),L^p(\omega d\mu_Q))$ inequality for the family $T_{a,b}$ is equivalent to the $(L^p(\omega d\mu_q),L^p(\omega d\mu_{Q+pt}))$ inequality for the family $P_{s,t}$ and conversely. The results we obtain depend upon the values of $s+t$, $q$ and $Q$. In the case $s+t<-(N+1)$ we have the following two main results. The outline of this chapter is as follows. We prove Theorem 0.1.2 in Section 2. From Section 3 on, we look at weighted estimates; there we show Theorem 0.1.5 and Theorem 0.1.6. The proof of Theorem 0.1.8, Theorem 0.1.10, Theorem 0.1.14 are in Section 4. The proof of Theorem 0.1.13 is in Section 5. Corollary 0.1.15 appears in Section 6.

Recall that, throughout this document $d$ is the pseudo-distance (see Definition 1.5.2) in $\overline{\mathbb{B}}$ defined by

$$d(z,w)=\begin{cases} ||z|-|w||+\left|1-\langle\frac{z}{|z|},\frac{w}{|w|}\rangle\right| & z,w\in\overline{\mathbb{B}}^*\\ |z|+|w| & z=0\,or\,w=0. \end{cases}$$

Throughout this paper $K$ will be a constant such that

$$d(x,y)\leq K(d(x,z)+d(z,y)) \tag{3.1}$$

for all $x$, $y$ and $z$ in $\mathbb{B}$. We will consider pseudo-balls in $\mathbb{B}$, $B(z,r)$, as points $w$ of $\mathbb{B}$ such that

$d(z,w) < r$ and we say that $B(z,r)$ touches the boundary of $\mathbb{B}$ if $r > 1-|z|$. When $B(z,r)$ is such that $r > k(1-|z|)$ for some absolute value $k < 1$, we say that $B(z,r)$ almost touches the boundary of $\mathbb{B}$.

## 3.2 Weak Type $L^1$ Inequality for $P_{s,t}$ and $T_{a,b}$.

In this section, we will prove Theorem 0.1.2 that we first recall here.

**Theorem 3.2.1.** *In the case $q = s$, $s+2t > -1$ and $s+t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from $L_q^1$ to $L_q^{1,\infty}$ and not from $L_q^1$ to $L_q^1$.*

*Proof.* The kernel of $P_{s,t}$ is $H_{s,t}(z,w) = \frac{(1-|z|^2)^t}{(1-\langle z,w\rangle)^{N+1+s+t}}$. We are going to proceed in three steps.

*Step 1:* Show that $H_{s,t}(z,.) \in L_q^2, \forall z \in \mathbb{B}$.

Indeed, we have

$$\int_{\mathbb{B}} |H_{s,t}(z,w)|^2 d\mu_q(w) = \int_{\mathbb{B}} \frac{(1-|z|^2)^{2t}}{|1-\langle z,w\rangle|^{2(N+1+s+t)}} d\mu_q(w)$$
$$\leq \frac{(1-|z|^2)^{2t}}{(1-|z|)^{2(N+1+s+t)}} \int_{\mathbb{B}} (1-|w|^2)^q d\mu(w)$$

where in the second inequality, the member of the right hand side is finite because $q = s > -1$.

*Step 2:* Show that $P_{s,t}$ is bounded from $L_q^2$ to $L_q^2$.

We have to show the boundedness of $T_{s+t,s}$ from $L_q^2$ to $L_{q+2t}^2$. By Kaptanoglu and Ureyen, for $a = s+t$, $b = s$, $p = P = 2$ and $Q = q+2t$ (this is the reason that $q+2t > -1$ is needed), this holds.

*Step 3:* Show that there are two constants $C_1$ and $C_2$ so that $\forall w, w_0 \in \mathbb{B}$ we have

$$\int_{d(z,w_0) > C_1 d(w,w_0)} |H_{s,t}(z,w) - H_{s,t}(z,w_0)| d\mu_q(z) < C_2.$$

This was already done in [11] (see the proof of [11, Proposition 1] choose $a = q+t+1$).

Because of *Step 1*, *Step 2* and *Step 3* we have by using Theorem 1.6.2 that the operators $P_{s,t}$ are bounded from $L_q^1$ to $L_q^{1,\infty}$. Observe that $P_{s,t}$ is bounded from $L_q^1$ to $L_q^1$ if and only if $T_{s+t,s}$ is bounded from $L_q^1$ to $L_{q+t}^1$; and by Theorem 0.1.1, $T_{s+t,s}$ is not bounded from $L_q^1$ to $L_{q+t}^1$ because $q = s$. □

**Remark 3.2.2.** *In the case $a > -(N+1)$, the operators $T_{a,b}^q$ are bounded from $L_q^1$ to $L_q^{1,\infty}$ if we have the following two conditions* *i)* $a \leq b$

*ii)* $-1 < q \leq b$.

*The case $a = b = q > -1$ is due to Békollè in [11]. The remaining cases are obtained by using Theorem 0.1.1.*

**Remark 3.2.3.** *In the special case $b = q$, $T^q_{a,b}$ is self adjoint and bounded from $L^p_q$ to itself for $1 < p < \infty$. Indeed, let $f \in L^p_q$ and $g \in L^{p'}_q$. Then*

$$\begin{aligned}
\langle T^q_{a,b} f, g\rangle_{L^2_q} &= \int_{\mathbb{B}}\int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w)\overline{g(z)}(1-|z|^2)^q d\mu(z)\\
&= \int_{\mathbb{B}} f(w)(1-|w|^2)^{b-q}\overline{\int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z)(1-|w|^2)^q d\mu(w)}\\
&= \int_{\mathbb{B}} f(w)\overline{T^*_{a,b} g(w) d\mu_q(w)}\\
&= \langle f, (T^q_{a,b})^* g\rangle_{L^2_q},
\end{aligned}$$

*where*

$$(T^q_{a,b})^* g(w) = (1-|w|^2)^{b-q}\int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z).$$

*Observe that when $b = q$, $(T^q_{a,b})^* = T^q_{a,b}$ and since $T^q_{a,b}$ is bounded from $L^p_q$ to $L^p_q$ when $1 < p < 2$, then $T^q_{a,b} = \left(T^q_{a,b}\right)^*$ is bounded from $L^{p'}_q$ to $L^{p'}_q$ with $2 < p' < \infty$.*

**Remark 3.2.4.** *By Remark 3.2.2 we have that $T^q_{a,b}$ is of weak type $(1,1)$ and let $A_{1,1}$ be the operator norm. By Theorem 0.1.1 we have that $T^q_{a,b}$ is of strong type $(2,2)$ and let $A_{2,2}$ be the operator norm. Applying Theorem 1.6.3 leads us, with a better estimation of the operator norm*

$$A_{p,p} = 2\left[p\left(\frac{A_{1,1}}{p-1} - \frac{A^2_{2,2}}{2-p}\right)\right]^{\frac{1}{p}},$$

*to a new way to have the boundedness of $T^q_{a,b}$ from $L^p_q$ to $L^p_q$ when $1 < p < 2$. In the special case $b = q$ we have the boundedness from $L^p_q$ to $L^p_q$ when $1 < p < \infty$ because in this case $T^q_{a,b}$ is self adjoint.*

## 3.3 Weighted estimates: Preliminary necessary conditions and the case where $s+t<-N-1$

In this section we will give a proof of our criterion for the weights that provide boundedness of $P_{s,t}$ when $s+t<-N-1$) (respectively $T_{a,b}$ when $a < -N-1$). We start first with some general necessary conditions.

### 3.3.1 Preliminary Necessary Conditions

**Lemma 3.3.1.** *Let $\omega$ be a weight. For $q, Q \in \mathbb{R}$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega$ must be in $L^1(d\mu_{Q+pt})$.*

*Proof.* Let $f(w) = (1-|w|^2)^{-s}\chi_{B(0,R)}(w)$ where $B(0,R)$ is the Euclidean ball. Then

$$\begin{aligned}
P_{s,t}f(w) &= (1-|w|^2)^t \int_{\mathbb{B}} K_{s+t}(w,z)(1-|z|^2)^{-s}\chi_{B(0,R)}(z)(1-|z|^2)^s d\mu(z)\\
&= (1-|w|^2)^t \int_{B(0,R)} K_{s+t}(w,z)d\mu(z)\\
&= (1-|w|^2)^t \overline{\int_{B(0,R)} K_{s+t}(z,w)d\mu(z)}\\
&= (1-|w|^2)^t \overline{K_{s+t}(0,w)}\mu(B(0,R))\\
&= (1-|w|^2)^t \mu(B(0,R)).
\end{aligned}$$

Since $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$,

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z)d\mu_Q(z) < \infty,$$

so that

$$\mu^p(B(0,R)) \int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

then

$$\int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

and $\omega \in L^1(d\mu_{Q+pt})$. $\square$

**Lemma 3.3.2.** *Let $\omega$ be a weight and $q, Q \in \mathbb{R}$. If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(b-q)})$.*

*Proof.* Assume that $T_{a,b}$ is well defined, that is for any $z \in \mathbb{B}$

$$S_{a,b}f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|d\mu_b(w) < \infty,$$

and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. We want to show that $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p(p'-1)(b-q)})$, in other words we want to show $\omega^{-1} \in L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$. Assume that $\omega^{-1}$ is not in $L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$, then by the Riesz representation theorem there exists a positive function $h$ in $L^p(\omega d\mu_{q+p(p'-1)(b-q)})$ so that

$$\langle h, \omega^{-1}\rangle_{\omega,q+p(p'-1)(b-q)} = \infty.$$

This means that

$$\begin{aligned}\infty &= \int_{\mathbb{B}} h(z)d\mu_{q+p(p'-1)(b-q)}(z)\\ &= \int_{\mathbb{B}} h(z)(1-|z|^2)^{p(p'-1)(b-q)}d\mu_q(z)\\ &= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p(p'-1)-1](b-q)}d\mu_b(z)\\ &= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)}d\mu_b(z).\end{aligned}$$

Since $h \in L^p(\omega d\mu_{q+p(p'-1)(b-q)})$, then $g \in L^p(\omega d\mu_q)$, where $g(z) = h(z)(1-|z|^2)^{(p'-1)(b-q)}, \ \forall z \in \mathbb{B}$. So

$$T_{a,b}g(0) = \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)}d\mu_b(z) = \infty,$$

contradicting the fact that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. $\square$

**Proposition 3.3.3.** *In the case $s+t \leq -1$ and $Q \leq q$, or in the case $s+t+\frac{Q-q}{p} \leq -1$, for any pseudo-ball $B$ that touches the boundary, there are no weights $\omega$ so that, both conditions $\omega \in L^1(B, d\mu_{Q+pt})$ and $\omega^{\frac{-1}{p-1}} \in L^1(B, d\mu_{q+p'(s-q)})$ hold at the same time.*

*Proof.* Let $s+t \leq -1$, $Q \leq q$ and $B$ be a pseudo-ball that touches the boundary see Definition 1.5.2. Then

$$\begin{aligned}\infty &= \int_B d\mu_{s+t}(z)\\ &= \int_B (1-|z|^2)^{s+t}d\mu(z)\\ &= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t}d\mu(z)\\ &= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}}\omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}}d\mu(z)\\ &\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt}d\mu(z)\right)^{\frac{1}{p}}\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{\frac{1}{p'}}\\ &\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt}d\mu(z)\right)^{\frac{1}{p}}\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{\frac{1}{p'}}\end{aligned}$$

so that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

Let $s+t+\frac{Q-q}{p}\leq -1$, and $B$ be a pseudo-ball that touches the boundary. Then

$$\begin{aligned}\infty &= \int_B d\mu_{s+t+\frac{Q-q}{p}}(z)\\ &= \int_B (1-|z|^2)^{s+t+\frac{Q-q}{p}}d\mu(z)\\ &= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{Q}{p}+t}d\mu(z)\\ &= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{Q+pt}{p}}\omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}}d\mu(z)\\ &\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt}d\mu(z)\right)^{\frac{1}{p}}\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{\frac{1}{p'}}\end{aligned}$$

so that necessarily $\omega\notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}}\notin L^1(d\mu_{q+p'(s-q)})$.

□

### 3.3.2 The case where $s+t<-N-1$

Here we are going to characterize the boundedness of the operators $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, where $\omega$ is a weight and $s+t<-N-1$.

**Theorem 3.3.4.** *In the case $s+t<-(N+1)$, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q\leq q$.*

*Proof.* This is due to Proposition 3.3.3, Lemma 3.3.1 and Lemma 3.3.2. □

**Theorem 3.3.5.** *In the case $s+t<-(N+1)$, if $Q>q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty.$$

*Moreover,*

$$\|P_{s,t}\|^p\simeq\left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

*Proof.* Assume that

$$\left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty.$$

We have

$$\begin{aligned}\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) &\lesssim \int_{\mathbb{B}}(1-|z|^2)^{pt}\left(\int_{\mathbb{B}}|f(v)|d\mu_s(v)\right)^p\omega(z)d\mu_Q(z)\\ &= \int_{\mathbb{B}}\left(\int_{\mathbb{B}}|f(v)|d\mu_s(v)\right)^p\omega(z)d\mu_{Q+pt}(z)\\ &= \left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}|f(z)|d\mu_s(z)\right)^p,\end{aligned}$$

where the first inequality is due to the fact that $K_{s+t}$ is bounded when $s+t<-(N+1)$. We also have

$$\begin{aligned}\left(\int_{\mathbb{B}}|f(z)|d\mu_s(z)\right)^p &= \left(\int_{\mathbb{B}}|f(z)|(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{s-q}d\mu_q(z)\right)^p\\ &\leq \left(\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z)\right)\left(\int_{\mathbb{B}}((\omega(z))^{\frac{-1}{p}})^{p'}|(1-|z|^2)^{p'(s-q)}d\mu_q(z)\right)^{\frac{p}{p'}}.\end{aligned}$$

Then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when

$$\left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty.$$

Now assume that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Let $\rho_0$ be as in Lemma 2.2.16, then for positive functions $f$ we have

$$\begin{aligned}\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) &= \int_{\mathbb{B}}\left|\int_{\mathbb{B}}K_{s+t}(z,w)f(w)d\mu_s(w)\right|^p\omega(z)d\mu_{Q+pt}(z)\\ &\geq \int_{|z|\leq\rho_0}|\int_{\mathbb{B}}\Re K_{s+t}(z,w)f(w)d\mu_s(w)|^p\omega(z)d\mu_{Q+pt}(z)\\ &= \frac{1}{2^p}\left(\int_{\mathbb{B}}f(w)d\mu_s(w)\right)^p\int_{|z|\leq\rho_0}\omega(z)d\mu_{Q+pt}(z).\end{aligned}$$

By continuity of $P_{s,t}$ there exists a constant $C_{s,t,p,q,Q}>0$ such that

$$\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z)\leq C_{s,t,p,q,Q}\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z),$$

for all $f\in L^p(\omega d\mu_q)$. Hence,

$$\frac{1}{2^p}\left(\int_{|z|\leq\rho_0}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}|f(z)|d\mu_s(z)\right)^p\leq C_{s,t,p,q,Q}\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z)$$

for all positive $f\in L^p(\omega d\mu_q)$. Let $f(z)=(\omega(z))^{\frac{-1}{p-1}}(1-|z|^2)^{(p'-1)(s-q)},\ \forall z\in\mathbb{B}$. Then $f\in L^p(\omega d\mu_q)$ by Lemma 3.3.2. Replacing this choice of $f$ in the last inequality we obtain

$$\left(\int_{|z|\leq\rho_0}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z)^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}\leq 2^pC_{s,t,p,q,Q}<\infty.$$

Then

$$\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty.$$

Using Lemma 3.3.1, we get

$$\left(\int_{\mathbb{B}}\omega(z)d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty.$$

□

## 3.4 Weighted estimates: The case where $a > -N-1$ and $s+t > -N-1$

Here we are going to start the study of the boundedness of the operators $T_{a,b}$ and $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ in the case $a > -(N+1)$ and $s+t > -(N+1)$ respectively.

### 3.4.1 Necessary Conditions

To obtain our necessary conditions, we are going to use the following lemma. This is an extension to the analogue lemma in [11].

**Lemma 3.4.1.** *Let $B_1 = B(w_0, R)$ be a pseudo-ball of sufficiently small radius $R$ touching the boundary of $\mathbb{B}$. There is a pseudo-ball $B_2$ with the same radius touching the boundary of $\mathbb{B}$, sufficiently far from $B_1$, but such that $d(B_1, B_2) \simeq R$, for which for all non negative functions $f$ with support in $B_i$ we have if $a > -1-N$*

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{R^{N+1+a}}\int_{B_i} f(w)d\mu_b(w),$$

*for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. In particular, if $a > -1$ then*

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{\mu_a(B_i)}\int_{B_i} f(w)d\mu_b(w)$$

*for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. The constant $C_{a,b}$ does not depend of $B_i$, $B_j$ or $f$.*

*Proof.* Let $w_0$ be the center of $B_i$. If $R$ is sufficiently small, we take $B_j$ such that for all $z$ in $B_j$ and for all $w$ in $B_i$, we have $d(z, w_0) \geq C_1 d(w, w_0)$ where $C_1$ is as in Lemma 1.6.10. Let $f$ be a non negative function with support in $B_i$ and let $z \in B_j$, we have

$$\begin{aligned} T_{a,b}f(z) = & \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\int_{B_i} f(w)d\mu_b(w) \\ & + \int_{B_i}\left[\frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\right] f(w)d\mu_b(w). \end{aligned}$$

Then

$$\begin{aligned} |T_{a,b}f(z)| \geq & \frac{1}{|1-\langle z, w_0\rangle|^{N+1+a}}\int_{B_i} f(w)d\mu_b(w) \\ & - \int_{B_i}\left|\frac{1}{(1-\langle z, w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z, w_0\rangle)^{N+1+a}}\right| f(w)d\mu_b(w) \end{aligned}$$

By Lemma 1.6.10 and Proposition 1.5.4 we have

$$\left|\frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right| \le \frac{1}{2}\frac{1}{|1-\langle z,w_0\rangle|^{N+a+1}},$$

so that

$$|T_{a,b}f(z)| \ge \frac{1}{2}\frac{1}{|1-\langle z,w_0\rangle|^{N+1+a}}\int_{B_i} f(w)d\mu_b(w).$$

Since our pseudo-balls touch the boundary and since $d(B_i,B_j) \simeq R$, we have

$$|1-\langle z,w_0\rangle| \lesssim R.$$

Hence

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2}\frac{1}{R^{N+1+a}}\int_{B_i} f(w)d\mu_b(w).$$

By Lemma 1.5.6 and [77, Lemma 2.8], if $a > -1$ and because $B_i$ touches the boundary we have $\mu_a(B_i) \simeq R^{N+1+a}$, so that

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2}\frac{1}{\mu_a(B_i)}\int_{B_i} f(w)d\mu_b(w).$$

□

We are now ready to prove Theorem 0.1.9.

**Theorem 3.4.2.** *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \le q$ then if $a > -1$ we have*

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty.$$

*More generally if $a > -(N+1)$, then*

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty,$$

*where $R_B$ is the radius of $B$.*

*Proof.* Assume that $a > -(N+1)$ and $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \le q$. Then there exists a constant $C_{a,b,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p\omega(z)d\mu_Q(z) \le C_{a,b,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z).$$

Let $f$ be a positive function with support in $B_i$ (we take $B_i, B_j$ as in Lemma 3.4.1). By Lemma 3.4.1, we then have in the case $a > -1$

$$C_{a,b}^p\int_{B_j}\frac{1}{\mu_a^p(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\omega(z)d\mu_Q(z) \le C_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z),$$

hence

$$\frac{1}{\mu_a^p(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_Q(z)\right) \leq C'_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z) \leq C'_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_Q(z). \tag{3.2}$$

Choosing $f = \frac{\mu_a(B_i)}{\mu_b(B_i)}\chi_{B_i}$ in the last inequality we get

$$\omega(B_j) \leq C'_{a,b,p,q,Q}\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)}\omega(B_i),$$

where $\omega(B_k) = \displaystyle\int_{B_k}\omega(z)d\mu_Q(z)$, $k = i, j$.
As $B_i$ and $B_j$ touch the boundary of $\mathbb{B}$ and have the same radius, by Lemma 1.5.6 we have

$$\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \simeq \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}.$$

We then have

$$\omega(B_j) \leq C''_{a,b,p,q,Q}\frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}\omega(B_i).$$

Interchanging $B_i$ and $B_j$ (See Lemma 3.4.1) we get

$$\omega(B_i) \leq C''_{a,b,p,q,Q}\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)}\omega(B_j),$$

so that

$$\frac{\mu_b^p(B_i)}{\mu_a^p(B_i)}\omega(B_i) \leq C''_{a,b,p,q,Q}\omega(B_j)$$

which together with (3.2) leads to

$$\frac{\mu_b^p(B_i)}{\mu_a^{2p}(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_Q(z)\right) \leq C'''_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(b-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 3.3.2) in that last inequality we obtain

$$\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1} \leq C'''_{a,b,p,q,Q}.$$

When $-1-N < a < -1$ it is sufficient to replace $\mu_a(B_i)$ by $R^{N+1+a}$ in the proof. □

**Theorem 3.4.3.** *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q > q$ then if $a > -1$ we have*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1} < \infty.$$

*where the supremum is taken over the pseudo-balls* $B$.
*More generally if* $a > -(N+1)$, *then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls* $B$ *with radius* $R_B$.

*Proof.* The proof is similar to the proof of Theorem 3.4.2, except that we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}}\chi_{B_i}(z)$. □

**Theorem 3.4.4.** *In the case* $-(N+1) < s+t < -1$ *and* $Q \leq q$, *or in the case* $s+t+\frac{Q-q}{p} \leq -1$, *there are no weights* $\omega$ *such that* $P_{s,t}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_Q)$.

*Proof.* This is due to Proposition 3.3.3, Lemma 3.3.1 and Lemma 3.3.2. □

**Theorem 3.4.5.** *In the case* $-(N+1) < s+t < -1$, *with* $s+t+\frac{Q-q}{p} > -1$ *if* $P_{s,t}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_Q)$ *then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls* $B$ *with radius* $R_B$.

*Proof.* Assume that $-1 > s+t > -(N+1)$, with $s+t+\frac{Q-q}{p} > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Then there exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z).$$

Let $f$ be a positive function with support in $B_i$ (we take $B_i$, $B_j$ of radius $R$ as in Lemma 3.4.1). By Lemma 3.4.1, we then have

$$C_{s,t}^p\int_{B_j}\frac{1}{R^{(N+1+s+t)p}}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\omega(z)d\mu_{Q+pt}(z) \leq C_{s,t,p,q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z),$$

hence

$$\frac{1}{R^{(N+1+s+t)p}}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z). \tag{3.3}$$

Choosing $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t}\chi_{B_i}(z)$ in the last inequality we get, because $N+s+t > -1$,

$$R^{Q-q}\int_{B_j}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_i}\omega(z)d\mu_{Q+pt}(z).$$

Interchanging $B_i$ and $B_j$ (See Lemma 3.4.1) we get

$$R^{Q-q}\int_{B_i}\omega(z)d\mu_{Q+pt}(z) \lesssim \int_{B_j}\omega(z)d\mu_{Q+pt}(z),$$

which together with (3.3) lead to

$$\frac{R^{Q-q}}{R^{p(N+1+s+t)}}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(s-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 3.3.2) in that last inequality, we obtain

$$\left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}}\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}}\int_{B_i}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} \leq C'''_{s,t,p,q,Q}.$$

□

In the same way we have

**Theorem 3.4.6.** *If $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \geq q$, then if $s+t > -1$ we have*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty$$

*where $R_B$ is the radius of $B$.*

*Proof.* The proof is similar to the proof of Theorem 3.4.5, we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t}\chi_{B_i}$. □

**Theorem 3.4.7.** *In the case $s+t > -1$ there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$.*

*Proof.* We are going to proceed in two steps.

*Step 1:* Show that in the case $s+t > -1$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, for $Q < q$, then we have

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Assume that $s+t > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$. There exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z), \quad f \in L^p(\omega d\mu_q).$$

Let $f$ be a positive function with support in $B_i$ (we take $B_i, B_j$ as in Lemma 3.4.1). By Lemma 3.4.1, we then have

$$\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z) \quad (3.4)$$
$$\leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_Q(z).$$

Choosing $f(z) = (1-|z|^2)^t\chi_{B_i}(z)$ in the last inequality we get

$$\int_{B_j}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_i}\omega(z)d\mu_{Q+pt}(z).$$

Interchanging $B_i$ and $B_j$ (see Lemma 3.4.1) we get

$$\int_{B_i}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_j}\omega(z)d\mu_{Q+pt}(z),$$

which together with (3.4) lead to

$$\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z) = (\omega^{\frac{-1}{p-1}}(z))(1-|z|^2)^{(p'-1)(s-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 3.3.2) in that last inequality we obtain

$$\left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} \leq C'''_{s,t,p,q,Q}.$$

*Step 2:* Show that

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} = \infty.$$

Let

$$II = \left(\frac{1}{\mu_{s+t}(B)}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

Let $B$ be a pseudo-ball that touches the boundary and $R_B$ its radius. By Hölder's inequality we have

$$\begin{aligned}\mu_{s+t}(B) &= \int_B d\mu_{s+t}(z)\\ &= \int_B (1-|z|^2)^{s+t} d\mu(z)\\ &= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t} d\mu(z)\\ &= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{q}} d\mu(z)\\ &\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{\frac{1}{p'}}.\end{aligned}$$

Note that for $z \in B$ we have $1-|z| < 2R_B$. Then

$$\begin{aligned}\mu^p_{s+t}(B) &\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{p-1}\\ &= \left(\int_B \omega(z)(1-|z|^2)^{q-Q+Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{p-1}\\ &\leq (2R_B)^{q-Q} \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{p-1}.\end{aligned}$$

Hence

$$\begin{aligned}(2R_B)^{Q-q} &\leq \frac{1}{\mu^p_{s+t}(B)} \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right) \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{p-1}\\ &= II.\end{aligned}$$

Taking the $\sup$ over smaller radii, we get $\sup II = \infty$. □

### 3.4.2 The Associated Maximal and Fractional Maximal Functions

We introduce for $b > -1$ and $s > -1$ the following maximal functions. If $a > -1$ we set

$$m_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w), \tag{3.5}$$

more generally if $a > -1-N$ we set

$$m'_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w). \tag{3.6}$$

If $a > -1$ we set

$$M_{a,b}f(z) = \sup_{B: z\in B} \frac{1}{\mu_a(B)} \int_B |f(w)| d\mu_b(w), \tag{3.7}$$

and more generally if $a > -1-N$ we set

$$M'_{a,b}f(z) = \sup_{B: z\in B} \frac{1}{R^{N+1+a}} \int_B |f(w)| d\mu_b(w). \tag{3.8}$$

If $s+t > -1$ we set

$$O_{s,t}f(z) = (1-|z|^2)^t \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_{s+t}(B(\zeta,R))}\int_{B(\zeta,R)}|f(w)|d\mu_s(w), \tag{3.9}$$

more generally if $s+t > -1-N$ we set

$$O'_{s,t}f(z) = (1-|z|^2)^t \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+s+t}}\int_{B(\zeta,R)}|f(w)|d\mu_s(w). \tag{3.10}$$

Let finally define the following fractional maximal function

$$M_\gamma f(z) = \sup_{B: z\in B} \frac{1}{\mu_b^{1-\gamma}(B)}\int_B |f(w)|d\mu_b(w), \quad \gamma\in(0,1). \tag{3.11}$$

Note that for $a<b$ we get by Lemma 1.5.6 $M_{a,b}\sim M_\gamma$ with $\gamma = 1-\frac{N+1+a}{N+1+b}$.

For all $k\in(0,1)$, we define the operator of regularisation $R_k^b$

$$R_k^b f(z) = \frac{1}{\mu_b(B_k(z))}\int_{B_k(z)} f(\zeta)d\mu_b(\zeta), \tag{3.12}$$

where $B_k(z) = \{w\in\mathbb{B} : d(z,w) < k(1-|z|)\}$.

We will need the following lemmas to show Theorem 0.1.14 (see [11]).

**Lemma 3.4.8.** *Let $k\in(0,\frac{1}{2})$. If $z'\in B_k(z)$, then $z\in B_{k'}(z')$, where $k' = \frac{k}{1-k}$.*

*Proof.* We have

$$||z|-|z'|| \le d(z,z') < k(1-|z|).$$

Therefore $1-|z| \le \frac{1}{1-k}(1-|z'|)$, so that $d(z,z') < k(1-|z|) \le \frac{k}{1-k}(1-|z'|)$. □

**Lemma 3.4.9.** *If $B := B(x,R)$ touches the boundary then if we take $B' = B(x, K(1+2k_1)R)$, then for all $w\in B$, $B_{k_1}(w)\subset B'$ (See (1.4) for the definition of $K$).*

*Proof.* Let $w\in B$ and $z\in B_{k_1}(w)$. We have

$$\begin{aligned} d(x,z) &\le K(d(x,w)+d(w,z)) \\ &\le K\left(R + k_1(1-|w|)\right) \\ &\le K\left(R+k_1(2R)\right) = K(1+2k_1)R. \end{aligned}$$

□

The next three lemmas are generalizations of their analogues in [11] to the maximal functions $m'_{a,b}$ and $O'_{s,t}$. For the sake of completeness and for the reader convenience, we include their proofs.

**Lemma 3.4.10.** *For all $k \in (0,1)$, there is a constant $C_k$ such that for all positive locally integrable function $f$ we have if $a > -1$*

$$m_{a,b} f \leq C_k m_{a,b}(R_k^b f),$$

*and more generally if $a > -1-N$*

$$m'_{a,b} f \leq C_k m'_{a,b}(R_k^b f).$$

*Proof.* We have to show that for all $z$ and all pseudo-balls $B$ containing $z$ which touch the boundary of $\mathbb{B}$, there is a pseudo-ball $B'$ containing $z$, which touches the boundary of $\mathbb{B}$ so that

$$\frac{1}{\mu_a(B)} \int_B f(w) d\mu_b(w) \leq C_k \frac{1}{\mu_a(B')} \int_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w).$$

By Lemma 3.4.8, $\chi_{B_k(w)}(\zeta) \geq \chi_{B_{k_1}(\zeta)}(w)$, where $k_1 = \frac{k}{k+1}$. If $B = B(x,R)$ $(R > 1-|x|)$, by Lemma 3.4.9, for $B' = B(x, K(1+2k_1)R)$, we have $\forall w \in B, B_{k_1}(w) \subset B'$. Note that

$$\mu_b(B_{k_1}(\zeta)) \simeq \mu_b(B_k(w)) \text{ when } \mathrm{w} \in \mathrm{B}_{\mathrm{k}_1}(\zeta). \tag{3.13}$$

Hence

$$\begin{aligned}
\int_{B'} R_k^b f(w) d\mu_b(w) &= \int_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int_B f(\zeta) \chi_{B_k(w)}(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&\geq \int_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int_B f(\zeta) \chi_{B_{k_1}(\zeta)}(w) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int_B \left[ \int_{B'} \frac{1}{\mu_b(B_k(w))} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Using (3.13), we get

$$\begin{aligned}
\int_{B'} R_k^b f(w) d\mu_b(w) &\gtrsim \int_B \frac{1}{\mu_b(B_{k_1}(\zeta))} \left[ \int_{B'} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta) \\
&= \int_B f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Since $\mu_a$ is a homogeneous measure we have

$$\frac{1}{\mu_a(B)} \int_B f(w) d\mu_b(w) \lesssim \frac{1}{\mu_a(B')} \int_{B'} R_k^b f(w) d\mu_b(w).$$

For $m'_{a,b}$ it is sufficient to observe that $B$ and $B'$ have equivalent radii. □

The following lemma appears as a corollary of the preceding one by observing that

$$O_{s,t} f(z) = (1-|z|^2)^t m_{s+t,s} f(z)$$

and

$$O'_{s,t}f(z) = (1-|z|^2)^t m'_{s+t,s}f(z).$$

**Lemma 3.4.11.** *For all $k \in (0,1)$, there is a constant $C_k$ such that for all positive locally integrable functions $f$ we have if $s+t > -1$*

$$O_{s,t}f \leq C_k O_{s,t}(R_k^s f),$$

*and more generally if $s+t > -1-N$*

$$O'_{s,t}f \leq C_k O'_{s,t}(R_k^s f).$$

One can find the following lemma in [11] but for $b = Q$.

**Lemma 3.4.12.** *For all $k \in (0, \frac{1}{2})$, there are two constants $C$ and $k' < 1$ depending only on $k, b, Q, N$ such that for all $f, g \in L^1(d\mu_b)$, $f \geq 0, g \geq 0$*

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C\int_{\mathbb{B}} g(z)[R_{k'}^{b,Q}f(z)]d\mu_b(z),$$

*where*

$$R_{k'}^{b,Q}f(z) = \frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_Q(\zeta). \tag{3.14}$$

*Proof.* By Lemma 3.4.8, $\chi_{B_k(z)}(w) \leq \chi_{B_{k'}(w)}(z)$, where $k' = \frac{k}{1-k}$. Because of (3.13) there is a constant $C$ such that

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z).$$

We want to form the quantity $f(z)[R_k^b g(z)]$ on the left while controlling it on the right in order to use Fubini's theorem to bring out the quantity $g(z)[R_{k'}^{b,Q}f(z)]$. Then, for $w \in B_k(z)$

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w).$$

We form $R_k^b g(z)$ on the left

$$\int_{B_k(z)} \frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w)d\mu_b(w) \leq \int_{B_k(z)} \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w),$$

by a multiplication by $f(z)$ we have

$$f(z)R_k^b g(z) \leq Cf(z)\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w).$$

After integration, we obtain

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C\int_{\mathbb{B}}\left[\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)f(z)d\mu_b(w)\right]d\mu_Q(z).$$

Recall that $(z \in \mathbb{B}$ and $w \in B_k(z)) \implies (z \in B_{k'}(w)$ and $w \in \mathbb{B})$, hence using Fubini's theorem

$$\int_{\mathbb{B}} f(z) R_k^b g(z) d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[ \int_{B_{k'}(w)} \frac{1}{\mu_b(B_{k'}(w))} \chi_{B_{k'}(w)}(z) f(z) d\mu_Q(z) \right] d\mu_b(w),$$

hence

$$\int_{\mathbb{B}} f(z) R_k^b g(z) d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[ \frac{1}{\mu_b(B_{k'}(w))} \int_{B_{k'}(w)} f(z) d\mu_Q(z) \right] d\mu_b(w),$$

then

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)] d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)] d\mu_b(z).$$

□

The following result will be used in the proof of Theorem 0.1.14. This extends Lemma 9 in [11]. We give a proof for the reader convenience.

**Lemma 3.4.13.** *Let $k \in (0,1)$. There are two constants $c, C$ depending only on $a, b, N, k$ such that for all positive locally integrable functions g if $a > -1$*

$$c\, m_{a,b} g \leq R_k^b(m_{a,b} g) \leq C m_{a,b} g,$$

*and more generally if $a > -1-N$*

$$c\, m'_{a,b} g \leq R_k^b(m'_{a,b} g) \leq C m'_{a,b} g.$$

*Proof.* It is sufficient to show that there are two constants $0 < c < C$ such that $\forall w \in B_k(z)$

$$c\, m_{a,b} g(z) \leq m_{a,b} g(w) \leq C m_{a,b} g(z).$$

We are going to show the two inequalities. More precisely we are going to show that there are two constants $0 < c < C$ such that, for each pseudo-ball $B$ containing $z$ and touching the boundary, there is a pseudo-ball $B'$ containing $w$ and touching the boundary so that

$$\frac{c}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{1}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta)$$

and show for each pseudo-ball $B$ containing $z$ touching the boundary, there is a pseudo-ball $B'$ containing $w$ touching the boundary so that

$$\frac{1}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{C}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta).$$

In each case, by Lemma 3.4.9, it is sufficient if $B = B(x, R)$ to take $B' = B(x, K(1+2kK)R)$. For the result with $m'_{a,b}$ it is sufficient to notice that $B$ and $B'$ have equivalent radii. □

In the same way as Lemma 3.4.13, since $O_{s,t} f(z) := (1-|z|^2)^t m_{s+t,s} f(z)$ and $O'_{s,t} f(z) := (1-|z|^2)^t m'_{s+t,s} f(z)$, we obtain the following result.

**Lemma 3.4.14.** *Let $k \in (0,1)$. There are two constants $c, C$ depending only on $s, t, N, k$ such that for all locally integrable function g if $s+t > -1$*

$$c\, O_{s,t}g \leq R_k^b(O_{s,t}g) \leq C O_{s,t}g,$$

*and more generally if $s+t > -1-N$*

$$c\, O'_{s,t}g \leq R_k^b(O'_{s,t}g) \leq C O'_{s,t}g.$$

Now we give a useful characterization of elements in $(B_p^{a,b,q,Q})$, see Definition 0.1.7.

**Lemma 3.4.15.** *For $a > -1$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z).$$

*More generally if $a > -1-N$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}}\int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ of radius R that touch the boundary.*

*Proof.* Assume $\omega \in (B_p^{a,b,q,Q})$. Let $0 \leq f \in L^p(\omega d\mu_q)$ and $B$ be a pseudo-ball of radius $R$ such that $R > 1-|z|$. Then

$$\begin{aligned}
\left(\int_B f(z)d\mu_b(z)\right)^p &= \left(\int_{\mathbb{B}} f(z)(1-|z|^2)^{b-q}d\mu_q(z)\right)^p\\
&= \left(\int_B f(z)(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{b-q}d\mu_q(z)\right)^p\\
&\leq \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B ((\omega(z))^{\frac{-1}{p}})^{p'}(1-|z|^2)^{p'(b-q)}d\mu_q(z)\right)^{\frac{p}{p'}}\\
&= \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq\\
&\quad\left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left[\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\omega(B)\right]\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}
\end{aligned}$$

and because $\omega \in B_p^{a,b,q,Q}$, there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z).$$

For the general case it is sufficient to replace $\mu_a(B)$ with $R^{N+1+a}$.

If we assume that there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}} \int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ of radius R such that $R > 1-|z|$ it is sufficient to take $f(z) = (1-|z|^2)^{(p'-1)(b-q)}\omega^{\frac{-1}{p-1}}(z)\chi_B(z)$ to get $\omega \in (B_p^{a,b,q,Q})$. □

**Remark 3.4.16.** *The result remains true even if $B$ almost touches the boundary.*

In the same way, for $(D_p^{s,t,q,Q})$ (see Definition 0.1.11), we have the following lemma.

**Lemma 3.4.17.** *For $Q \geq q$ and $s+t > -1$, $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ that touch the boundary.*

*For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -1-N$, $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \leq C_{s,t,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ that touch the boundary.*

**Remark 3.4.18.** *The result remains true even if $B$ almost touches the boundary.*

**Corollary 3.4.19.** *For $C_1 > 1$, if $\omega \in D_p^{s,t,q,Q}$ then there exists a constant $C_2 > 0$ such that for any pseudo-ball $B := B(y,r)$ which touches or almost touches the boundary, we have*

$$\int_{B(y,C_1r)} \omega(\zeta)d\mu_{Q+pt}(\zeta) \leq C_2 \int_{B(y,r)} \omega(\zeta)d\mu_{Q+pt}(\zeta).$$

**Proposition 3.4.20.** *Let $X$ be an homogeneous space. Let $w$ be a weight in $X$. For $-N-1 < a \leq b$, $Q > q$ and $k \in (0, \frac{1}{2})$, assume that there exists a constant $C_2 > 0$ such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x)d\nu(x)\right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x)d\nu(x)\right)^{\frac{1}{p}} \tag{3.15}$$

*for any pseudo-ball $B \subset X$, where $v(z) = R^{b,Q}_{k'}\omega(z)$, $u(z) = R^{b,Q}_{k'}\omega(z)(1-|z|^2)^{2p(b-a)+Q-q}$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$. If $\omega \in (B^{a,b,q,Q}_p)$, there is a constant $C_{a,b,p,q,Q} > 0$ such that for all $f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (m_{a,b}f(z))^p\,\omega(z)d\mu_Q(z) \leq \int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z).$$

*Proof.* Let set

$$III = \int_{\mathbb{B}} (m_{a,b}f(z))^p\,\omega(z)d\mu_Q(z).$$

Using in this order Lemma 3.4.10, Lemma 3.4.13, Hölder's inequality and Lemma 3.4.12 we have,

$$\begin{aligned} III &\leq C^p_k \int_{\mathbb{B}} (m_{a,b}R^b_k f(z))^p\omega(z)d\mu_Q(z)\\ &\leq C^p_k A^p \int_{\mathbb{B}} \left[R^b_k(m_{a,b}R^b_k f(z))\right]^p\omega(z)d\mu_Q(z)\\ &\leq C^p_k A^p \int_{\mathbb{B}} R^b_k\left[(m_{a,b}R^b_k f(z))^p\right]\omega(z)d\mu_Q(z)\\ &\leq C^p_k A^p C \int_{\mathbb{B}} (m_{a,b}R^b_k f(z))^p R^{b,Q}_{k'}\omega(z)d\mu_b(z). \end{aligned}$$

For $a \leq b$, assume that there exists a constant $C_2 > 0$ such that

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x)d\nu(x)\right)^{\frac{1}{r}} \leq C_2\left(\int_B u^{1-p'}(x)d\nu(x)\right)^{\frac{1}{p}}$$

for any ball $B \subset X$, where $v(z) = R^{b,Q}_{k'}\omega(z)$, $u(z) = R^{b,Q}_{k'}\omega(z)(1-|z|^2)^c$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$, with $c$ to be determined. Then we have

$$\begin{aligned} \int_{\mathbb{B}} (m_{a,b}f(z))^p\omega(z)d\mu_Q(z) &\leq C^p_k A^p C \int_{\mathbb{B}} (m_{a,b}R^b_k f(z))^p R^{b,Q}_{k'}\omega(z)d\mu_b(z)\\ &\leq C^p_k A^p C \int_{\mathbb{B}} (M_{a,b}R^b_k f(z))^p R^{b,Q}_{k'}\omega(z)d\mu_b(z)\\ &\lesssim C^p_k A^p C \int_{\mathbb{B}} (M_\gamma R^b_k f(z))^p R^{b,Q}_{k'}\omega(z)d\mu_b(z)\\ &\leq C^p_k A^p C' \int_{\mathbb{B}} (R^b_k f(z))^p(1-|z|^2)^c R^{b,Q}_{k'}\omega(z)d\mu_b(z), \end{aligned}$$

where for the last inequality we used Theorem 1.6.4. Now let us control $(R^b_k f(z))^p R^{b,Q}_{k'}\omega(z)$. We have

$$\begin{aligned} (R^b_k f(z))^p R^{b,Q}_{k'}\omega(z) &= \left(\frac{1}{\mu_b(B_k(z))}\int_{B_k(z)} f(\zeta)d\mu_b(\zeta)\right)^p\left(\frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right)\\ &\leq \left(\frac{1}{\mu_b(B_k(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p\left(\frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right)\\ &\lesssim \left(\frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p\left(\frac{1}{\mu_b(B_{k'}(z))}\int_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right), \end{aligned}$$

where the second inequality is because $k' > k$, the third one is because $\mu_b(B_{k'}(z)) \backsimeq \mu_b(B_k(z))$. Then

$$(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) \lesssim \frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_a^2(B_{k'}(z))}\int_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p\left(\int_{B_{k'}(z)}\omega(\zeta)d\mu_Q(\zeta)\right),$$

so that

$$(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) \lesssim C_{a,b,p,q,Q}\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))}\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),$$

because of Lemma 3.4.15 since it is possible to dilate the pseudo-balls $B_k$ so that they touch the boundary and the fact that the measures $d\mu_q$ and $d\mu_{q+p'(b-a)}$ are homogeneous. By Lemma 1.5.6, since $B_{k'}(z) = B(z, k'(1-|z|))$ with $0 < k' < 1$, we have

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))} \backsimeq (1-|z|^2)^{2pa-2pb-b-(Q-q)-(N+1)}.$$

Recall that we already have

$$\int_{\mathbb{B}}(m_{a,b}f(z))^p\omega(z)d\mu_Q(z) \le C_k^pA^pC'\int_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Let us set

$$IV = \int_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Hence, using the previous control of $(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)$ and Fubini's theorem, we have

$$\begin{aligned}
IV &\lesssim \int_{\mathbb{B}}\left(\int_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{c+(b-a)+2pa-2pb-b-Q+q-1-N}d\mu_a(z)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k'}(z)}(\zeta)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c-2p(b-a)-Q+q-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int_{\mathbb{B}}(1-|\zeta|^2)^{c-2p(b-a)-(Q-q)}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).
\end{aligned}$$

The proof is complete if we take

$$c = 2p(b-a) + (Q-q).$$

□

**Remark 3.4.21.** *The result in Proposition 3.4.20 says that if we assume that the Sawyer type condition (3.15) holds, then the necessary condition $w \in (B_p^{a,b,q,Q})$ for the boundedness of $T_{a,b}$ from $L^p(wd\mu_q)$ to $L^p(wd\mu_Q)$ is also sufficient by the good lambda inequality in Theorem 0.1.13. Since we do not know if $w \in (B_p^{a,b,q,Q})$ implies the Sawyer type condition, we will provide in the sequel a testable sufficient* *condition for the boundedness of $T_{a,b}$ in this situation.*

**Lemma 3.4.22.** *Let $k \in (0, \frac{1}{4})$. For $s+t > -1$, or for $-N-1 < s+t < -1$ with $s+t+\frac{Q-q}{p} > -1$, if $\omega \in D_p^{s,t,q,Q}$, we have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$.*

*Proof.* We have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$ if

$$V := \left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B \sigma(z)d\mu_{s+t+\frac{Q-q}{p}}(z)\right)\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B \sigma^{\frac{-1}{p-1}}(z)d\mu_{s+t+\frac{Q-q}{p}}(z)\right)^{p-1} \leq C_p(\omega).$$

Note that $\sigma(z) \simeq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(z)$ since $d\mu_b(B_{k'}(z)) \simeq (1-|z|^2)^{n+1+b}$. We consider two cases.

*First case:* $B := B(y,r)$ with $r \ll 1-|y|$.

In this case, since $R_{k'}^{a,b}f(x)$ is defined as in (3.14), Corollary 3.4.19 shows that there are two constants $0 < c < C$ such that

$$cR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y) \leq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(x) \leq CR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y)$$

for all $x \in B$. We then have

$$V \simeq \frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)}\left(\frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)}\right)^{p-1} = 1.$$

*Second case:* $B := B(y,r)$ touches the boundary.

Recall that our measures are homogeneous, and recall that if $z \in B$ and $x \in B_{k'}(z)$, then $z \in B_{k''}(x)$ and $x \in B' := B(y, 2k'Kr + Kr)$, where $k'' = \frac{k'}{1-k'}$. Let

$$VI = \int_B \left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)} \omega(x)d\mu_{Q+pt}(x)\right)(1-|z|^2)^{-t-\frac{Q-q}{p}}d\mu_{s+t+\frac{Q-q}{p}}(z).$$

Then by Fubini's theorem we have,

$$\begin{aligned} VI &= \int_B \left( \frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right) d\mu_s(z) \\ &= \int_B \int_{B_{k'}(z)} \frac{1}{\mu_s(B_{k'}(z))} \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\ &= \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k'}(z))} \chi_B(z) \chi_{B_{k'}(z)}(x) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\ &\lesssim \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{1}{\mu_s(B_{k''}(x))} \chi_{B'}(x) \chi_{B_{k''}(x)}(z) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\ &\lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x). \end{aligned}$$

So we have

$$VI = \int_B R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(z) \lesssim \int_{B'} \omega(x) d\mu_{Q+pt}(x). \tag{3.16}$$

Let us now control $(R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'}$. We have

$$\begin{aligned} (R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'} &= \left( \frac{1}{\mu_s(B_{k'}(z))} (1-|z|^2)^{-t-\frac{Q-q}{p}} \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{1-p'} \\ &\simeq \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1}. \end{aligned}$$

Setting

$$VII = \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1},$$

we have, by Hölder's inequality

$$\begin{aligned} VII &= \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} \omega^{\frac{1}{p}}(x) \omega^{\frac{-1}{p}}(x)(1-|x|^2)^{s-\frac{q}{p}+\frac{Q+pt}{p}} d\mu(x) \right) \right]^{p'-1} \\ &\leq \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{\frac{1}{p}-1} \left( \int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p'}} \right]^{p'-1} \\ &\leq \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \right]^{\frac{1}{p}}. \end{aligned}$$

Let us set

$$VIII = \int_B \left( R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \right)^{\frac{-1}{p-1}} d\mu_{s+t+\frac{Q-q}{p}}(z),$$

then we have

$$VIII \leq \int_B \left[\left(\int_{B_{k'}(z)} \omega(x)d\mu_{Q+pt}(x)\right)^{-1}\left(\int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x)d\mu_{q+p'(s-q)}(x)\right)\omega(z)(1-|z|^2)^{Q+pt}\right]^{\frac{1}{p}} \omega^{\frac{-1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}}d\mu(z)$$

so that by Hölder's inequality and Fubini's theorem, we have

$$VIII \lesssim \left(\int_B \omega^{\frac{-p'}{p}}(z)d\mu_{q+p'(s-q)}(z)\right)^{\frac{1}{p'}} \left(\int_{B'}\left[\left(\int_{B_{k''}(x)} \omega(\zeta)d\mu_{Q+pt}(\zeta)\right)^{-1}\left(\int_{B_{k''}(x)} \omega(z)d\mu_{Q+pt}(z)\right)\right]\omega^{\frac{-p'}{p}}(x)d\mu_{q+p'(s-q)}(x)\right)^{\frac{1}{p}}.$$

Finally, we obtain

$$VIII \lesssim \int_{B'} \omega^{\frac{-p'}{p}}(z)d\mu_{q+p'(s-q)}(z). \tag{3.17}$$

Where on the last but one inequality we used Fubini's theorem (as in the control of VI) and the fact that for $x \in B_{k'}(z)$ we have $z \in B_{k''}(x)$ and

$$\int_{B_{k''}(x)} \omega(\zeta)d\mu_{Q+pt}(\zeta) \lesssim \int_{B_{k'}(z)} \omega(\zeta)d\mu_{Q+pt}(\zeta).$$

This is a variant of Corollary 3.4.19 or simply the application of Lemma 3.4.17 for $f(\zeta) = (1-|\zeta|^2)^{t+\frac{Q-q}{p}}1_{B_{k'}(z)}(\zeta)$ with $B := B(z, 2Kk'(1+k'')(1-|z|)) \supseteqq B_{k''}(x)$ (Lemma 3.4.9) and the fact that our measures are homogeneous. Since $\omega \in D_p^{s,t,q,Q}$, we use (3.16) and (3.17) to conclude that $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$. $\square$

**Theorem 3.4.23.** *In the case both $Q \geq q$ and $s+t > -1$ hold, and in the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, if $\omega \in D_p^{s,t,q,Q}$, there is a constant $C_{s,t,p,q,Q} > 0$ such that $\forall f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (O_{s,t}f(z))^p\omega(z)d\mu_Q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z).$$

*Proof.* Using in this order Lemma 3.4.11, Lemma 3.4.14, Hölder's inequality and Lemma 3.4.12 we have

$$\begin{aligned}
\int_{\mathbb{B}} (O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &= \int_{\mathbb{B}} (m_{s+t,s}f(z))^p\omega(z)d\mu_{Q+pt}(z)\\
&\leq C_k^p \int_{\mathbb{B}} (m_{s+t,s}R_k^s f(z))^p\omega(z)d\mu_{Q+pt}(z)\\
&\leq C_k^p A^p \int_{\mathbb{B}} [R_k^s(m_{s+t,s}R_k^s f(z))]^p\omega(z)d\mu_{Q+pt}(z)\\
&\leq C_k^p A^p \int_{\mathbb{B}} R_k^s[(m_{s+t,s}R_k^s f(z))^p]\omega(z)d\mu_{Q+pt}(z)\\
&\leq C_k^p A^p C\int_{\mathbb{B}} (m_{s+t,s}R_k^s f(z))^p R_{k'}^{s,Q+pt}\omega(z)d\mu_s(z).
\end{aligned}$$

By Lemma 3.4.22, $R^{s,Q+pt}_{k'}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in A_{p,s+t+\frac{Q-q}{p}}$. Using natural domination between our maximal operator defined by equation 3.5 and equation 3.7, we have

$$\begin{aligned}\int_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\leq C_k^pA^pC\int_{\mathbb{B}}(m_{s+t,s}R_k^sf(z))^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z)\\ &\leq C_k^pA^pC\int_{\mathbb{B}}(M_{s+t,s}R_k^sf(z))^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z)\\ &= C_k^pA^pC\int_{\mathbb{B}}(M_{s+t,s+t}[(1-|z|^2)^{-t}R_k^sf(z)])^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z).\end{aligned}$$

Because in each case we have $\frac{Q-q}{p} \geq 0$, using Theorem 1.6.8, we have that

$$\begin{aligned}\int_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\lesssim \int_{\mathbb{B}}(M_{s+t+\frac{Q-q}{p},s+t}[(1-|z|^2)^{-t}R_k^sf(z)])^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z)\\ &\lesssim \int_{\mathbb{B}}(M_{s+t+\frac{Q-q}{p},s+t+\frac{Q-q}{p}}[(1-|z|^2)^{-t-\frac{Q-q}{p}}R_k^sf(z)])^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z)\\ &\lesssim \int_{\mathbb{B}}(1-|z|^2)^{-pt-(Q-q)}(R_k^sf(z))^pR^{s,Q+pt}_{k'}\omega(z)d\mu_s(z). \qquad (3.18)\end{aligned}$$

Now let us control $IX = (R_k^sf(z))^pR^{s,Q+pt}_{k'}\omega(z)$. We have

$$\begin{aligned}IX &= \left(\frac{1}{\mu_s(B_k(z))}\int_{B_k(z)}f(\zeta)d\mu_s(\zeta)\right)^p\left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\ &\lesssim \left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)}f(\zeta)d\mu_s(\zeta)\right)^p\left(\frac{1}{\mu_s(B_{k'}(z))}\int_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\ &\lesssim \frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}\int_{B_{k'}(z)}f(\zeta)d\mu_s(\zeta)\right)^p\left(\int_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\ &\lesssim C_{s,t,p,q,Q}\frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\int_{B_{k'}(z)}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\ &\lesssim C_{s,t,p,q}(1-|z|^2)^{pt+(Q-q)-s-N-1}\int_{B_{k'}(z)}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),\end{aligned}$$

where for the last but one inequality we used Lemma 3.4.17. Hence using (3.18) and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\lesssim \int_{\mathbb{B}}\left(\int_{B_{k'}(z)}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{-1-N}d\mu(z)\\ &\lesssim \int_{\mathbb{B}}\left(\int_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\ &\lesssim \int_{\mathbb{B}}f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).\end{aligned}$$

Here we used Lemma 3.4.8 with $k'' := (k')'$. The proof is complete. □

## 3.5 Good lambda inequality and sufficient conditions

In this section we will establish the good lambda inequality that allow us to provide sufficient conditions for the boundedness of our operators. We first need some preliminary results. The first result extends to the maximal function $m'_{s+t,s}$ the analogue result in [11]. For the sake of completeness, we include the proof.

**Proposition 3.5.1.** *Let $\beta > 0$ and $s+t > -1-N$ there is a constant $A > 0$ such that for all $\xi \in \mathbb{B}$ $R > 1-|z_0|$ and a positive locally integrable function $f$ and for all $z \in B(z_0, R)$ if $s+t > -1$, then*

$$R^\beta \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi) \leq A m_{s+t,s} f(z).$$

*More generally if $-1-N < s+t$ then*

$$R^\beta \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi) \leq A m'_{s+t,s} f(z).$$

*Proof.* Recall that if $s+t > -1$, by Lemma 1.5.6, there is a constant $a > 0$ such that for all $k \in \mathbb{N}$, we have $\mu_{s+t}(B(z_0, 2^{k+1}R)) \leq a(2^{k+1}R)^{N+1+s+t}$, so that setting

$$X = R^\beta \int_{d(z_0,\xi)\geq R} \frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}} \mathrm{d}\mu_s(\xi)$$

we have

$$\begin{aligned} X &\leq \sum_{k=0}^{+\infty} \frac{1}{2^{k(N+1+s+t+\beta)} R^{N+1+s+t}} \int_{d(z_0,\xi)<2^{k+1}R} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a 2^{N+1+s+t} \sum_{k=0}^{+\infty} 2^{-k\beta} \frac{1}{\mu_{s+t}(B(z_0, 2^{k+1}R))} \int_{B(z_0,2^{n+1}R)} f(\xi)\mathrm{d}\mu_s(\xi) \\ &\leq a 2^{N+1+s+t} m_{s+t,s} f(z) \sum_{n=0}^{+\infty} 2^{-k\beta} = \frac{a 2^{N+1+s+t}}{1-2^{-\beta}} m_{s+t,s} f(z). \end{aligned}$$

We can take $A = \dfrac{a2^{N+1+s+t}}{1-2^{-\beta}}$ to conclude. $\square$

**Proposition 3.5.2.** *Let $\omega \in (D_p^{s,t,q,Q})$, we set again $\sigma(z) = R_k^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$ with $k \in (0, 1/2)$. Set $B = B(z', r)$ with $1-|z'| < cr$ and $L = \left\{z \in B : 1-|z| < C_0'\gamma^{\frac{1}{N+1+s+t}} r\right\}$ where $C_0' > 0$, $0 < \gamma < 1$, $r > 0$ and $c > 0$ are constants. Then if we set $L' = \left\{z \in \bar{B} : 1-|z| < 2C_0'\gamma^{\frac{1}{N+1+s+t}} r\right\}$ and $\bar{B} = B(z', ar)$ with $a = K(C_0'+1)$, there are two constants $C_1$ and $C_2 > 0$ independent of $\gamma$ such that if $s+t+\frac{Q-q}{p} > -1$ then*

$$\omega d\mu_{Q+pt}(L) \leq C_1 \sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \quad \textit{and} \quad \mu_{s+t+\frac{Q-q}{p}}(L') \leq C_2 \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}} \mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{3.19}$$

*Proof.* Let $k_1 = \frac{k}{1+k}$, then for $z \in L$ and $\xi \in B_{k_1}(z)$ we have $z \in B = B(z', r)$, $1-|z| <$

$C_0'\gamma^{\frac{1}{N+1+s+t}}r$ and $z \in B_k(\xi)$ because $k_1' = k$. Then

$$d(z', \xi) \leq K(d(z', z) + d(z, \xi)) < K[r + k_1(1 - |z|)] < K[r + k_1 C_0'\gamma^{\frac{1}{N+1+s+t}}r] < (C_0' + 1)rK$$

because $0 < k_1, \gamma < 1$. Then $\xi \in \overline{B} = B(z', ar)$ with $a = K(C_0' + 1)$. Moreover,

$$1 - |\xi| < 1 - |z| + d(z, \xi) < (k_1 + 1)(1 - |z|) < 2C_0'\gamma^{\frac{1}{N+1+s+t}}r$$

because $0 < k_1 < 1$, so that $\xi \in L' = \left\{z \in \overline{B} : 1 - |z| < 2C_0'\gamma^{\frac{1}{N+1+s+t}}r\right\}$. Then we have

$$\chi_L(z)\chi_{B_{k_1}(z)}(\xi) \leq \chi_{L'}(\xi)\chi_{B_k(\xi)}(z).$$

Remember that

$$\mu_{s+t+\frac{Q-q}{p}}(B_k(\xi)) \simeq \mu_{s+t+\frac{Q-q}{p}}(B_{k_1}(z)).$$

Hence,

$$\begin{aligned}
\omega d\mu_{Q+pt}(L) &= \int_L \omega(z) d\mu_{Q+pt}(z) \\
&= \int_L \left(\frac{\omega(z)}{\mu_s(B_{k_1}(z))}\int_{B_{k_1}(z)} d\mu_s(\xi)\right) d\mu_{Q+pt}(z) \\
&= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_L(z)\chi_{B_{k_1}(z)}(\xi)\omega(z)}{\mu_s(B_{k_1}(z))} d\mu_{Q+pt}(z)\right) d\mu_s(\xi) \\
&\lesssim \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{L'}(\xi)\chi_{B_k(\xi)}(z)\omega(z)}{\mu_s(B_k(\xi))} d\mu_{Q+pt}(z)\right) d\mu_s(\xi) \\
&= \int_{\mathbb{B}} \left(\frac{\chi_{L'}(\xi)}{\mu_s(B_k(\xi))}\int_{B_k(\xi)} \omega(z) d\mu_{Q+pt}(z)\right) (1 - |\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \int_{L'} \left(R_k^{s,Q+pt} w(\xi)\right) (1 - |\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \sigma d\mu_{s+t+\frac{Q-q}{p}}(L').
\end{aligned}$$

Let us show the second inequality. Let $z \in \overline{B}$, then, $d(z', z) = ||z'| - |z|| + \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then $||z'| - |z|| < ar$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Moreover, as $1 - |z'| < cr$ then $1 - |z| = 1 - |z'| + |z'| - |z| < cr + ar$. Then for $z \in L'$, setting $\beta_1 = 2C_0'\gamma^{\frac{1}{N+1+s+t}}r$ and $\beta_2 = (c + a)r$, we get $1 - |z| < \beta_1$, $1 - |z| < \beta_2$ and $\left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar$. Then, $L' \subset \{z \in \mathbb{B} : 1 - |z| < \min(\beta_1, \beta_2),\ \left|1 - \frac{\langle z', z\rangle}{|z'||z|}\right| < ar\}$. In spherical coordinates we have, for $s + t + \frac{Q-q}{p} > -1$

$$\begin{aligned}
\mu_{s+t+\frac{Q-q}{p}}(L') &\lesssim (s + t + \frac{Q-q}{p} + 1) \int\limits_{1-\rho<\min(\beta_1,\beta_2)} (1 - \rho^2)^{s+t+\frac{Q-q}{p}} \rho \mathrm{d}\rho \int\limits_{|1-\frac{\langle z', z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\leq (s + t + \frac{Q-q}{p} + 1) \int\limits_{1-\min(\beta_1,\beta_2)<\rho<1} (1 - \rho)^{s+t+\frac{Q-q}{p}} \mathrm{d}\rho \int\limits_{|1-\frac{\langle z', z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\lesssim r^N[-(1 - \rho)^{s+t+\frac{Q-q}{p}+1}]_{1-\min(\beta_1,\beta_2)}^1 = r^N(\min(\beta_1, \beta_2))^{s+t+\frac{Q-q}{p}+1}.
\end{aligned}$$

Then,

$$\mu_{s+t+\frac{Q-q}{p}}(L') \lesssim r^N \beta_1^{s+t+\frac{Q-q}{p}+1} \simeq r^{N+s+t+\frac{Q-q}{p}+1}\gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}}. \tag{3.20}$$

On the other hand, as $1-|z'|<cr$, we have by Lemma 1.5.6

$$\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \simeq (ar)^{N+1}(\max(1-|z'|, ar))^{s+t+\frac{Q-q}{p}} \lesssim r^{N+1+s+t+\frac{Q-q}{p}}.$$

□

We are now ready to prove our good lambda inequality (Theorem 0.1.13) that we recall here for the reader convenience. It is used to show that $S_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$ when $m'_{s+t,s}f \in L^p(\omega d\mu_{Q+pt})$.

**Theorem 3.5.3** (Good lambda inequality)**.** *Let $\omega \in (D_p^{s,t,q,Q})$ $(1<p<+\infty)$ in the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, or both $s+t>-1$ and $Q \geq q$ hold. There are two positive constants $C$ and $\beta$ such that for all $\gamma$ sufficiently small, $\lambda > 0$ and for all positive locally integrable functions $f$, we have*

$$\begin{aligned}\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq& \\ CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}).&\end{aligned} \tag{3.21}$$

*Proof.* Let $\lambda > 0$, $0<\gamma<1$ and $f$ a positive locally integrable function. Let $E_\lambda = \{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}$. By the Whitney decomposition Lemma (see [36]), there are a positive integer $J$, $\delta > 1$ and a sequence of pseudo-balls $\{B_j\}_{j=1}^\infty$, with $B_j = B(z_j, r_j)$, such that

- $E_\lambda = \bigcup_{j=1}^{\infty} B_j$;
- every point of $E_\lambda$ is at most in $J$ balls $B_j$;
- the balls $B'_j = B(z_j, \delta r_j)$ touch the complement of $E_\lambda$ in $\mathbb{B}$.

To obtain (3.21), it is then sufficient to show that

$$\omega\mu_{Q+pt}(\{z \in B : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega\mu_{Q+pt}(B), \tag{3.22}$$

where $B = B(z', r)$ is a ball in the Whitney decomposition of $E_\lambda$. From the third property of the Whitney decomposition, there is $z_0 \in B' = B(z', \delta r)$ such that $S_{s+t,s}f(z_0) \leq \lambda$. Without loss of generality, assume that there is $\xi_0 \in B$ such that $m'_{s+t,s}f(\xi_0) \leq \gamma\lambda$. Let $\widetilde{B} = B(z_0, R)$ with $R = \max(1-|z_0|, C_0 r)$ where we choose $C_0 \geq \max(c_1 K(1+\delta), \delta)$ where $c_1$ is the constant $C_1$ in Lemma 1.6.10.

We set $f_1 = 1_{\widetilde{B}} f$ and $f_2 = 1_{\mathbb{B}\setminus\widetilde{B}} f$, then $f = f_1 + f_2$ and we have

$$S_{s+t,s} f_2(z) \leq \int\limits_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{f(\xi)}{|1-\langle z_0, \xi\rangle|^{N+1+s+t}} \right| d\mu_s(\xi) + \int\limits_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{1}{|1-\langle z, \xi\rangle|^{N+1+s+t}} - \frac{1}{|1-\langle z_0, \xi\rangle|^{N+1+s+t}} \right| f(\xi) d\mu_s(\xi).$$

So that, by Lemma 1.6.10 and Proposition 3.5.1, we finally have, for $z \in B$

$$S_{s+t,s} f_2(z) \leq S_{s+t,s} f(z_0) + A' m'_{s+t,s} f(\xi_0) \leq \lambda + A'\gamma\lambda.$$

So, since

$$S_{s+t,s} f(z) \leq S_{s+t,s} f_1(z) + S_{s+t,s} f_2(z) \leq S_{s+t,s} f_1(z) + \lambda + A'\gamma\lambda,$$

we have that $S_{s+t,s} f(z) > 2\lambda$ implies that $S_{s+t,s} f_1(z) > (1 - A'\gamma)\lambda$. Therefore, to prove (3.22), it will be enough to show that

$$\omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s} f_1(z) > b\lambda\}) \leq C D_p^{s,t,q,Q}(\omega) \gamma^{\beta} \omega d\mu_{Q+pt}(B), \tag{3.23}$$

where $b = 1 - A'\gamma$ with an appropriate choice of $\gamma$. We are going to discuss according to the values of the radius $R = \max(1 - |z_0|, C_0 r)$ of $\widetilde{B} = B(z_0, R)$. Let $E'_\lambda = \{S_{s+t,s} f_1 \geq b\lambda\} \cap B$.

*First case:* $C_0 r \leq 1 - |z_0|$.

Then, $\widetilde{B} = B(z_0, 1-|z_0|)$. Therefore for all $z \in B$ and $\xi \in \widetilde{B}$, $|1-\langle z, \xi\rangle| \geq 1-|z| > C'(1-|z_0|)$, so that for all $z \in B$,

$$S_{s+t,s} f_1(z) = \int\limits_{\widetilde{B}} \frac{f(\xi) d\mu_s(\xi)}{|1-\langle z, \xi\rangle|^{N+1+s+t}} \leq \frac{1}{(C'(1-|z_0|))^{N+1+s+t}} \int\limits_{\widetilde{B}} f(\xi) d\mu_s(\xi).$$

Then

$$S_{s+t,s} f_1(z) < C'' m'_{s+t,s} f(\xi_0) \leq C''\gamma\lambda.$$

Hence, if we take $0 < \gamma < \gamma_0 = \min(\frac{1}{A'}, \frac{b}{C''})$ then it remains only to prove the following case.

*Second case:* $1 - |z_0| < C_0 r$.

Then $\widetilde{B} = B(z_0, C_0 r)$ and $E'_\lambda \subseteq L$ for $L$ defined in Proposition 3.5.2. In fact, if $z \in E'_\lambda$, then

$z \in B$ and

$$\begin{aligned} b\lambda \leq S_{s+t,s}f_1(z) &= \int_{\widetilde{B}} \frac{f(\xi)d\mu_s(\xi)}{|1-\langle z,\xi\rangle|^{N+1+s+t}} \leq \frac{1}{(1-|z|)^{N+1+s+t}} \int_{\widetilde{B}} f(\xi)d\mu_s(\xi) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}} m'_{s+t,s}f(\xi_0) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}} \gamma\lambda. \end{aligned}$$

For $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$, with $k' \in (0,\frac{1}{2})$, by Lemma 3.4.22 we have $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$ so that $\sigma \in (A_{\infty,s+t+\frac{Q-q}{p}})$ because $(A_{p,s+t+\frac{Q-q}{p}}) \subseteq (A_{\infty,s+t+\frac{Q-q}{p}})$.

Given the fact that $L'$ is a measurable subset of $\bar{B} = B(z', ar)$, we have by Proposition 3.5.2 and Lemma 1.6.7

$$\begin{aligned} \omega d\mu_{Q+pt}(L) &\leq C\sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \\ &\leq C\left(\frac{\mu_{s+t+\frac{Q-q}{p}}(L')}{\mu_{s+t+\frac{Q-q}{p}}(\bar{B})}\right)^{\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \\ &\leq C\gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \end{aligned}$$

As $E'_\lambda$ is a subset of $L = \{z \in \bar{B} : 1-|z| < C'_0\gamma^{\frac{1}{N+1+s+t}} r\}$, it follows that for $\beta = \frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0$ we have

$$\omega d\mu_{Q+pt}(E'_\lambda) \leq C\gamma^\beta \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{3.24}$$

One shows by Fubini's theorem that $\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \leq C\omega d\mu_{Q+pt}(\widetilde{\bar{B}})$ with $\widetilde{\bar{B}} = B(z', (2k+1)arK)$. And by Corollary 3.4.19 we get $\omega d\mu_{Q+pt}(\widetilde{\bar{B}}) \leq CD_p^{s,t,q,Q}(\omega)\omega d\mu_{Q+pt}(B)$. Then

$$\omega d\mu_{Q+pt}(E'_\lambda) = \omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s}f_1(z) > b\lambda\}) \leq CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(B).$$

This ends the proof. □

The following results appear as consequences of Theorem 3.5.3 and Lemma 1.6.9.

**Theorem 3.5.4.** *Let $p > 1$. For $Q \geq q$ and $s+t > -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s}f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m_{s+t,s}f(z))^p d\mu_{Q+pt}(z).$$

*Proof.* It is enough to apply Lemma 1.6.9 to $S_{s+t,s}f$ and $m_{s+t,s}f$ with $t = 2\lambda$, $c = \frac{\gamma}{2}$, $b = \frac{1}{2}$ and $a = CD_p^{s,t,q,Q}(\omega)\gamma^\beta$. We just need to take $\gamma$ small enough so that $a < b^p$. □

**Theorem 3.5.5.** *Let $p > 1$. For $s+t+\frac{Q-q}{p} > -1$ and $-N-1 < s+t < -1$, if $\omega \in D_p^{s,t,q,Q}$ there* *is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s}f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m'_{s+t,s}f(z))^p d\mu_{Q+pt}(z).$$

## 3.6 Final remark

This part is simply a direct application of the two preceding sections and Remark 0.1.12. Therefore for $1 < p < +\infty$ we have the following two corollaries.

**Corollary 3.6.1.** *Let $\omega$ be a weight on $\mathbb{B}$. Then for $s + t > -1$ and $q = Q$, the following assertions are equivalent.*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*
2. *$T_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
3. *$S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
4. *$\omega \in (K_p^{s,t,q,q})$.*

*Proof.* By Remark 0.1.12, we have that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that by Theorem 3.5.4, since $(1 - |z|^2)^t S_{s+t,s}f(z) = P_{s,t}f(z)$ and $O_{s,t}f(z) = (1 - |z|^2)^t m_{s+t,s}f(z)$ and Theorem 3.4.23, we have (4) implies (1). □

**Corollary 3.6.2.** *Let $\omega$ be a weight on $\mathbb{B}$. In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, and in the case both $s + t > -1$ and $Q \geq q$ hold, if $\omega \in (D_p^{s,t,q,Q})$ then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, so that $S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{Q+pt})$.*

In Theorem 3.4.5 and in Theorem 3.4.6 we show that being in $(K_p^{s,t,q,Q})$ is a necessary condition for the continuity of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, while in Corollary 3.6.2, we have that being in $D_p^{s,t,q,Q}$ is a sufficient one. When $Q = q$, we find out that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that we have a necessary and sufficient condition. But when $Q > q$ we have $(D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. It is an open question to find in this case a necessary and sufficient condition for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.

# ♣ Part II: Hardy Dirichlet spaces ♣

⋆ ⋆ **Chapter 4** ⋆ ⋆

# Dirichlet Series

In this chapter we define and give some properties of Dirichlet series and we establish the isometric isomorphism between $\mathcal{H}(\mathbb{B})$ (the set of Dirichlet series to be defined) and the Hardy space $H^2(\mathbb{D}^\infty)$ on the polydisc of infinite dimension.

In the following lines, for a complex number $s$, $\Re(s)$ will designate the real part of $s$. For more details, we we invite the reader to consult [64],[40] and [65].

## 4.1 Definitions and first results

Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbb{C}$. We call Dirichlet series a series of the form:

$$f(s) = \sum_{n\geq 1} \frac{a_n}{n^s}, \quad s \in \mathbb{C}.$$

**Lemma 4.1.1.** *Let $0 < \alpha < \beta$. For all $z$ in $\mathbb{C}$, such that $x = \Re(z) > 0$, we have*

$$|e^{-\alpha z} - e^{-\beta z}| \leq \frac{|z|}{x}(e^{-\alpha x} - e^{-\beta x}).$$

*Proof.* We have

$$e^{-\alpha z} - e^{-\beta z} = z\int_\alpha^\beta e^{-tz} dt$$

To obtain the result, we pass to the module, noting that $|e^{-tz}| = e^{-tx}$. □

**Theorem 4.1.2.** *If the series $\sum_{n\geq 1} \frac{a_n}{n^s}$, $s \in \mathbb{C}$ converges for a $s_0 \in \mathbb{C}$. Then it converges in the half-plane $\Re(s) > \Re(s_0)$. Moreover, it converges uniformly on any sector of the type*

$$\Delta_{s_0,\alpha} = \{s \in \mathbb{C}; \Re(s) > \Re(s_0), |Arg(s - s_0)| \leqslant \alpha\},\ 0 < \alpha < \frac{\pi}{2}.$$

*Proof.* Let $s \in \mathbb{C}$, such that $\Re(s) > \Re(s_0)$. To Show that the series of general term $a_n n^{-s}$ converges, set $a_n n^{-s} = a_n n^{-s-s_0+s_0}$.

Let $u_n = a_n n^{-s_0}$ and $v_n = n^{s_0 - s}$, then $a_n n^{-s} = u_n v_n$. Let

$$A_{p,q} = \sum_{n=p}^{q} u_n v_n = \sum_{n=p}^{q} a_n n^{-s}.$$

For $n \geq p$, let $U_{p,n} = \sum_{i=p}^{n} u_i$. Thus,

$$U_{p,p} = u_p \; and \; for \; n > p, u_n = U_{p,n} - U_{p,n-1} \quad (*)$$

As $\sum_{n \geq 1} \frac{a_n}{n^{s_0}}$ converges, one have

$$\forall \varepsilon > 0, \exists N(\varepsilon) \; such \; that \; for \; p,q \in \mathbb{N} \; we \; have \; q \geqslant p > N(\varepsilon) \Rightarrow |U_{p,q}| < \varepsilon. \tag{4.1}$$

Let $\varepsilon > 0$.

- For $q > p$, using (*)

$$\begin{aligned}
A_{p,q} &= u_p v_p + \sum_{n=p+1}^{q} u_n v_n \\
&= U_{p,p} v_p + \sum_{n=p+1}^{q} u_n v_n \\
&= U_{p,p} v_p + \sum_{n=p+1}^{q} (U_{p,n} - U_{p,n-1}) v_n \\
&= U_{p,p} v_p + \sum_{n=p+1}^{q} U_{p,n} v_n - \sum_{n=p+1}^{q} U_{p,n-1} v_n \\
&= U_{p,p} v_p + \sum_{n=p+1}^{q} U_{p,n} v_n - \sum_{n=p}^{q-1} U_{p,n} v_{n+1} \\
&= U_{p,p} v_p + U_{p,q} v_q + \sum_{n=p+1}^{q-1} U_{p,n} v_n - U_{p,p} v_{p+1} - \sum_{n=p+1}^{q-1} U_{p,n} v_{n+1} \\
&= U_{p,q} v_q + U_{p,p}(v_p - v_{p+1}) + \sum_{n=p+1}^{q-1} U_{p,n}(v_n - v_{n+1}) \\
&= U_{p,q} v_q + \sum_{n=p}^{q-1} U_{p,n}(v_n - v_{n+1}). \quad (**)
\end{aligned}$$

By (4.1), there is $N(\epsilon)$ such that for $q > p > N(\epsilon)$, one have, using (**)

$$\begin{aligned}
|A_{p,q}| &\leqslant \varepsilon |v_q| + \varepsilon \sum_{n=p}^{q-1} |v_n - v_{n+1}| \\
&\leqslant \varepsilon \Big(1 + \sum_{n=p}^{q-1} |v_n - v_{n+1}|\Big)
\end{aligned}$$

where we used the fact that $|v_n| = \frac{1}{n^{\Re(s) - \Re(s_0)}} < 1$ since $\Re(s) > \Re(s_0)$.

On the other hand, we have

$$\begin{aligned}|v_n - v_{n+1}| &= \left|\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right| \\ &= \left|e^{-(s-s_0)lnn} - e^{-(s-s_0)ln(n+1)}\right|.\end{aligned}$$

By Lemma 4.1.1 with $z = s - s_0$, $\alpha = \ln n$ and $\beta = \ln(n+1)$
one gets

$$|v_n - v_{(n+1)}| \leqslant \frac{|s - s_0|}{\Re(s) - \Re(s_0)} \left(\frac{1}{n^{\Re(s)-\Re(s_0)}} - \frac{1}{(n+1)^{\Re(s)-\Re(s_0)}}\right).$$

Hence,

$$\begin{aligned}|A_{p,q}| &\leqslant \varepsilon \left[1 + \frac{|s-s_0|}{\Re(s)-\Re(s_0)} \sum_{n=p}^{q-1} \left(\frac{1}{n^{\Re(s)-\Re(s_0)}} - \frac{1}{(n+1)^{\Re(s)-\Re(s_0)}}\right)\right] \\ &= \varepsilon \left[1 + \frac{|s-s_0|}{\Re(s)-\Re(s_0)} \left(\frac{1}{p^{\Re(s)-\Re(s_0)}} - \frac{1}{q^{\Re(s)-\Re(s_0)}}\right)\right] \quad (***) \\ &\leq \varepsilon(1 + \frac{|s-s_0|}{\Re(s)-\Re(s_0)}).\end{aligned}$$

- By (4.1), for $q = p \geq N(\varepsilon)$ one have

$$|A_{p,p}| = |u_p||v_p| < |u_p| < \varepsilon.$$

Therefore we have just shown that the series of general term $a_n n^{-s}$ satisfies the Cauchy criterion, so it converges. Now show the second part
Let $s \in \Delta_{s_0,\alpha}$ . Set $z = s - s_0$ and $\theta = arg(z)$. Then $cos(\theta) = \frac{\Re(s)-\Re(s_0)}{|s-s_0|}$ and $|\theta| \leqslant \alpha$ so that

$$\frac{|s - s_0|}{\Re(s) - \Re(s_0)} = \frac{1}{cos(\theta)} \leqslant \frac{1}{cos(\alpha)}.$$

This implies that for $s \in \Delta_{s_0,\alpha}$, from (***) we will have $|A_{p,q}| \leq \varepsilon(1 + \frac{1}{cos(\alpha)})$. Here the estimate does not depend on $s$, therefore on this set the series of general term $a_n n^{-s}$ satisfies the Cauchy criterion uniformly, hence the uniform convergence, which is what we wanted. □

Here is a result similar to that of the existence of the radius of convergence for the power series.

**Corollary 4.1.3.** *Let $(a_n)_{n\geq 1}$ a sequence in $\mathbb{C}$. Set*

$$E_c = \{\sigma \in \mathbb{R} : \sum_{n=1}^{\infty} a_n n^{-\sigma} \ \ converges \ \}$$

*and*

$$\sigma_c := \begin{cases} inf E_c, & E_c \neq \emptyset, \\ \infty, & E_c = \emptyset., \end{cases}$$

*Then*

- *If* $\Re(s) > \sigma_c$, *the series* $\sum_{n\geq 1} a_n n^{-s}$ *converges;*

- *If* $\Re(s) < \sigma_c$, *the series* $\sum_{n\geq 1} a_n n^{-s}$ *diverges.*

$\sigma_c$ is the abscissa of convergence of the Dirichlet series and the half-plane $\Re(s) > \sigma_c$ is the half-plane of convergence.

*Proof.* Suppose $E_c = \emptyset$. We have to show that for all $s \in \mathbb{C}$ the series $\sum_n a_n n^{-s}$ diverges. Suppose there is $s_0 \in \mathbb{C}$ such that $\sum_n a_n n^{-s_0}$ converges. Then from Theorem 4.1.2, for all $s \in \mathbb{C}$, such that $\Re(s) > \Re(s_0)$ the series $\sum_n a_n n^{-s}$ converges. Thus, if we take $\sigma = \Re(s)$ for $s$ such that $\Re(s) > \Re(s_0)$, then $\sigma \in E_c$, thus $E_c \neq \emptyset$ which is absurd. Thus for all $s \in \mathbb{C}$ the series $\sum_n a_n n^{-s}$ diverges.
Suppose $E_c \neq \emptyset$. If $E_c = \mathbb{R}$, then $\sigma_c = -\infty$ and for $s \in \mathbb{C}$ as $\sum_n a_n n^{-\Re s}$ converges, then $\sum_n a_n n^{-s}$ converges.
Suppose $E_c \neq \emptyset$ and $E_c \neq \mathbb{R}$. In this case $\sigma_c$ is finite. Let $s \in \mathbb{C}$, such that $\Re(s) > \sigma_c$. Then there exists $\sigma_0 \in E_c$ such that $\Re(s) > \sigma_0 > \sigma_c$. By Theorem 4.1.2 we deduce that the series $\sum_n a_n n^{-s}$ converges.
It remains to show that if for all $s$ such that $\Re(s) < \sigma_c$, the series $\sum_n a_n n^{-s}$ diverges. Indeed if there exists a $s \in \mathbb{C}$, such that $\Re(s) < \sigma_c$ and the series $\sum_n a_n n^{-s}$ converges, then for $\Re(s) < \sigma_1 < \sigma_c$ we deduce from Theorem 4.1.2 that the series $\sum_n a_n n^{-\sigma_1}$ converges, which contradicts the fact that $\sigma_c$ is the lower bound of $E_c$. □

**Corollary 4.1.4.** *Let* $(a_n)_{n\geq 1}$ *a sequence of* $\mathbb{C}$. *Set*

$$E_a = \{\sigma \in \mathbb{R} : \sum_{n\geqslant 1} a_n n^{-\sigma} \ converges \ \ absolutely \ \}$$

*and*

$$\sigma_a := \begin{cases} inf E_a, & E_a \neq \emptyset, \\ \infty, & E_a = \emptyset. \end{cases}$$

*Then,*

- *If* $\Re(s) > \sigma_a$, *the series* $\sum_{n\geq 1} a_n n^{-s}$ *converges absolutely;*

- *If* $\Re(s) < \sigma_a$, *the series* $\sum_{n\geqslant 1} a_n n^{-s}$ *does not converge absolutely.*

$\sigma_a$ is the abscissa of absolute convergence of the Dirichlet series and the half-plane $\Re(s) > \sigma_a$ is the half-plane of absolute convergence.

*Proof.* Observe that

$$E_a = \{\sigma \in \mathbb{R} : \sum_{n\geqslant 1} |a_n| n^{-\sigma} \ \ converge\}.$$

It suffices then to use Corollary 4.1.3 to conclude. □

Here is a result which links the abscissa of convergence to the abscissa of absolute convergence of a Dirichlet series.

**Proposition 4.1.5.** *Let $(a_n)_{n\geq 1}$ a sequence in $\mathbb{C}$. Suppose $E_c \neq \emptyset$ then*

$$\sigma_a - 1 \leq \sigma_c \leq \sigma_a.$$

*Proof.* If the series $\sum_n a_n n^{-s}$ converges absolutely , then it converges, hence $\sigma_c \leq \sigma_a$.
Let us now show that $\sigma_a - 1 \leq \sigma_c$ . If we show that for all $\sigma > \sigma_c + 1$, the series $\sum_n a_n n^{-\sigma}$ converges absolutely, it will be done. Let $\sigma > \sigma_c+1$, then $\sigma-1 > \sigma_c$, so there exists $\sigma_0$ such that $\sigma-1 > \sigma_0 > \sigma_c$. As the series $\sum_n a_n n^{-\sigma_0}$ converges (because $\sigma_0 > \sigma_c$), the sequence $(a_n n^{-\sigma_0})_n$ is bounded. Let M>0 be such that $|a_n n^{-\sigma_0}| < M$, we therefore have

$$|a_n n^{-\sigma}| = \frac{|a_n|}{n^{\sigma_0}} \frac{1}{n^{\sigma-\sigma_0}} \leq \frac{M}{n^{\sigma-\sigma_0}}.$$

The series $\sum_n n^{-(\sigma-\sigma_0)}$ converges since $\sigma - \sigma_0 > 1$ (Riemann series), then the series $\sum_n a_n n^{-\sigma}$ is absolutely convergent. Hence $\sigma_a \leq \sigma_c + 1$. □

The following result tells us that the sum of a Dirichlet series is a holomorphic function in the half-plane of convergence.

**Theorem 4.1.6.** *Let $(a_n)_{n\geq 1}$ a sequence in $\mathbb{C}$ such that $\sigma_c < \infty$. The function*

$$F : s \to F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

*is holomorphic on the half-plane $\Re(s) > \sigma_c$ and, for all $k \geq 0$, one have*

$$F^{(k)}(s) = \sum_{n=1}^{\infty} \frac{a_n(-\ln n)^k}{n^s}, \Re(s) > \sigma_c.$$

*Proof.* Let K be a compact set contained in the half-plane $\Re(s) > \sigma_c$. There are $s_0 \in \mathbb{C}$ and $0 \leq \alpha < \frac{\pi}{2}$ such that $\Re(s_0) > \sigma_c$ and $K \subset \Delta_{s_0,\alpha}$ where $\Delta_{s_0,\alpha} = \{s \in \mathbb{C}; \Re(s) > \Re(s_0), |Arg(s-s_0)| \leq \alpha\}$. According to Theorem 4.1.2, the series $\sum_n a_n n^{-s}$ converges uniformly to $F$ on $\Delta_{s_0,\alpha}$, hence on $K$ . Theorem 1.4.6 then allows us to deduce that $F$ is holomorphic on $\Re(s) > \sigma_c$ and that

$$F^{(k)}(s) = \sum_{n=1}^{\infty} a_n(-\ln n)^k n^{-s}.$$

□

The next result shows uniqueness for the coefficients of a Dirichlet series.

**Proposition 4.1.7.** *Let $(a_n)_{n\geq 1}$ a sequence in $\mathbb{C}$ such that $E_c \neq \emptyset$. Let*

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \Re(s) > \sigma_c.$$

*If for all $N \geq 0$, one has*

$$\lim_{\sigma\to\infty} N^{\sigma} F(\sigma) = 0$$

*then $a_n = 0$, for all $n \geq 1$.*

*Proof.* We proceed by strong induction on $n$. Let's show that $a_1 = 0$.
Let $\sigma_1 > \sigma_c$. By Theorem 4.1.2, the series converges uniformly on $[\sigma_1, \infty[$. Hence for N=1, we have

$$\lim_{\sigma\to+\infty} F(\sigma) = \sum_{n=1}^{\infty} \lim_{\sigma\to+\infty} (a_n n^{-\sigma}).$$

Since

$$\lim_{\sigma\to+\infty} (a_n n^{-\sigma}) = \{ {}^{a_1,\ if\ n=1}_{0,\ else.},$$

as $\lim\limits_{\sigma\to+\infty} F(\sigma) = 0$, we get that $a_1 = 0$.
Suppose $a_1 = a_2 = ... = a_{n-1} = 0$. Let us show that $a_n = 0$. For $N = n$,

$$N^{\sigma} F(\sigma) = \sum_{k=n}^{\infty} a_k N^{\sigma} k^{-\sigma} = \sum_{l=1}^{\infty} a_{n+l-1} N^{\sigma} (n+l-1)^{-\sigma}.$$

By stretching $\sigma \to +\infty$ , we obtain as in the previous case that $a_n = 0$ □

Here is a direct consequence of Proposition 4.1.7.

**Corollary 4.1.8.** *Let $F(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ and $G(s) = \sum\limits_{n=1}^{\infty} b_n n^{-s}$ two Dirichlet series. Suppose that there is a real $\sigma_0$ such that $F(\sigma) = G(\sigma)$, for all $\sigma > \sigma_0$. Then $a_n = b_n$ , for all $n \geqslant 1$.*

*Proof.* Let $H(\sigma) = F(\sigma) - G(\sigma) = \sum\limits_{n=1}^{\infty} (a_n - b_n) n^{-\sigma} = 0$ for $\sigma > \sigma_0$. So, for $N \geqslant 1$

$$\lim_{\sigma\to\infty} N^{\sigma} H(\sigma) = 0.$$

So for all $n \geq 1$, $a_n - b_n = 0$ by Proposition 4.1.7, hence the result. □

## 4.2 Euler product

We give here result due to Euler see [64]

**Definition 4.2.1.** *We say that a map $f$ is multiplicative arithmetic if $f$ is defined from $\mathbb{N}^*$ to $\mathbb{C}$ such that the image by $f$ of $1$ is equal to $1$, and, if $n$ and $m$ are two strictly positive integers, prime to each other, then*

$$f(nm) = f(n)f(m).$$

*If moreover for any prime number $p$ we have*

$$f(p^n) = (f(p))^n,$$

*$f$ is said to be completely multiplicative.*

In the following lines $\mathbb{P}$ denotes the set of all prime numbers.

**Theorem 4.2.2.** *Let $f$ be a multiplicative arithmetic function and $\sigma_a$ the abscissa of absolute convergence of the Dirichlet series $\sum_n f(n)n^{-s}$. Then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\in\mathbb{P}} \left( \sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} \right), \quad \Re(s) > \sigma_a.$$

*Moreover, the infinite product converges uniformly on any half-plane $\Re(s) \geq \Re(s_0) > \sigma_a$.*

*Proof.* Let $T \geq 0$. Set $\aleph(T)$ the set of natural numbers whose prime factors are less than or equal to $T$. Let $p_1, ..., p_k$ be the list of prime numbers $\leq T$. We have, by multiplicity of $f$

$$\begin{aligned} \prod_{2\leq p\leq T p\in\mathbb{P}} \left( \sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} \right) &= \sum_{n_1,\dots,n_k\geq 0} \frac{f(p_1^{n_1})...f(p_k^{n_k})}{(p_1^{n_1}...p_k^{n_k})^s} \\ &= \sum_{n\in\aleph(T)} \frac{f(n)}{n^s}. \end{aligned}$$

Hence

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{2\leq p\leq T, p\in\mathbb{P}} \left( \sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} \right)| = |\sum_{n\notin\aleph(T)} \frac{f(n)}{n^s}| \leq \sum_{n\notin\aleph(T)} \frac{|f(n)|}{n^{\Re(s)}}.$$

Note that, $\{n \in \mathbb{N} : n \notin \aleph(T)\} \subset \{n \in \mathbb{N} : n > T\}$ hence

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{2\leq p\leq T, p\in\mathbb{P}} \left( \sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} \right)| \leq \sum_{n>T} \frac{|f(n)|}{n^{\Re(s)}}$$

For $\Re(s) \geq \Re(s_0) > \sigma_a$, we have

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{2\leq p\leq T, p\in\mathbb{P}} \left( \sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}} \right)| \leq \sum_{n>T} \frac{|f(n)|}{n^{\Re(s_0)}}$$

and since the series $\sum_n f(n)n^{-s}$ converges absolutely on $\Re(s) > \sigma_a$, then

$$\lim_{T\to\infty} \sum_{n>T} \frac{|f(n)|}{n^{\Re(s_0)}} = 0.$$

Thus,

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{2\le p\le T, p\in\mathbb{P}} \left(\sum_{n=0}^{\infty} \frac{f(p^n)}{p^{ns}}\right)| \to 0,\ when\ T \to \infty.$$

this yields the result. □

**Corollary 4.2.3.** *Let $f$ be a completely multiplicative arithmetic function and $\sigma_a$ the abscissa of absolute convergence of the Dirichlet series $\sum_n f(n)n^{-s}$. Then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{f(p)}{p^s}\right)^{-1},\ \Re(s) > \sigma_a.$$

*Proof.* Since $f$ is completely multiplicative, we have $f(p^n) = (f(p))^p$. By Theorem 4.2.2 we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p\in\mathbb{P}} \left(\sum_{n=0}^{\infty} (\frac{f(p)}{p^s})^n\right).$$

Since the left hand side is finite, so is the right hand side. Hence for any prime $p$, the series

$$\sum_n (\frac{f(p)}{p^s})^n$$

converges. So we have $|f(p)| < |p^s|$ and we get that:

$$\sum_{n=0}^{\infty} \left(\frac{f(p)}{p^s}\right)^n = \left(1 - \frac{f(p)}{p^s}\right)^{-1},$$

and the result follows. □

Consider two sums of the type

$$\sum f(n)n^{-s}\ \ and\ \ \sum g(n)n^{-s}, \tag{4.2}$$

where

$f, g : \mathbb{N}^* \to \mathbb{C}$ are two arithmetic functions that are zero except at a finite number of points. Thus the two sums of (4.2) are finite and

$$(\sum_{n=1}^{\infty} \frac{f(n)}{n^s})(\sum_{n=1}^{\infty} \frac{g(n)}{n^s}) = \sum_{n,m\ge 1} \frac{f(n)g(m)}{n^s m^s}.$$

By grouping in the double sum the terms with the same denominator, we have

$$(\sum_{n=1}^{\infty}\frac{f(n)}{n^s})(\sum_{n=1}^{\infty}\frac{g(n)}{n^s}) = \sum_{n=1}^{\infty}(\sum_{d,d':dd'=n} f(d)g(d'))n^{-s} = \sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s},$$

where

$$(f*g)(n) = \sum_{d,d'\in\mathbb{N}^*:dd'=n} f(d)g(d'). \tag{4.3}$$

In this section we have worked with $f$ and $g$ null except at a finite number of points, which avoided convergence problems. Here is a more general result see [64].

**Theorem 4.2.4.** *Let*

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \;\; and \;\; G(s) = \sum_{n=1}^{\infty} g(n)n^{-s},$$

*two Dirichlet series of abscissa of absolute convergence $\sigma_a^F$ and $\sigma_a^G$ respectively. Let us denote by $\sigma_a$ the abscissa of absolute convergence of the Dirichlet series $\sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s}$. Then, the following hold:*

*(i) We have $\sigma_a \leqslant max(\sigma_a^F,\sigma_a^G)$.*

*(ii) For all $s\in\mathbb{C}$ such that $\Re(s) > max(\sigma_a^F,\sigma_a^G)$, we have*

$$\sum_{n=1}^{+\infty}(f*g)(n)n^{-s} = F(s)G(s).$$

*Proof.* To show that $\sigma_a \leqslant max(\sigma_a^F,\sigma_a^G)$, it suffices to show that for all $s\in\mathbb{C}$ such that $\Re(s) > max(\sigma_a^F,\sigma_a^G)$, the series $\sum_{n=1}^{\infty}(f*g)(n)n^{-s}$ converges absolutely. Let $s\in\mathbb{C}$ such that $\Re(s) > max(\sigma_a^F,\sigma_a^G)$, then for fixed $N$, we have

$$\begin{aligned}
\sum_{n\le N}|(f*g)(n)n^{-s}| &= \sum_{n\le N}\frac{|(f*g)(n)|}{n^{\Re(s)}}\\
&= \sum_{n\le N}|\sum_{dd'=n} f(d)g(d')|n^{-\Re(s)}\\
&\le \sum_{n\le N}\sum_{dd'=n}|f(d)||g(d')|n^{-\Re(s)}\\
&\le \left(\sum_{d\le N}\frac{|f(d)|}{d^{\Re(s)}}\right)\left(\sum_{d'\le N}\frac{|g(d')|}{d'^{\Re(s)}}\right).
\end{aligned}$$

By letting $N\to\infty$ we have that $\sum_{n=1}^{\infty}(f*g)(n)n^{-s}$ converges absolutely,this proves (i). To show (ii), since the series $\sum f(n)n^{-s}$ converges absolutely and $\sum g(n)n^{-s}$ converges because it converges absolutely on the half-plane $\Re(s) > max(\sigma_a^F, \sigma_a^G)$ then from Theorem 1.10.16, the series

$$\sum_{n,m\geqslant 1} \frac{f(n)g(m)}{(mn)^s}$$

converges to $F(s)G(s)$. In fact this double series converges absolutely, according to Theorem 1.10.17 we can therefore group the terms arbitrarily and we obtain

$$F(s)G(s) = \sum_{n=1}^{\infty}\left(\sum_{dd'=n;d,d'\in\mathbb{N}} f(d)g(d')\right)n^{-s} = \sum_{n=1}^{\infty}(f*g)(n)n^{-s}.$$

□

*Example: Riemann zeta function*

**Definition 4.2.5.** *The Riemann zeta function, is defined by*

$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}.$$

The Riemann zeta function converges absolutely on the open set $\Re(s) > 1$ and by Theorem 4.1.6 this series is holomorphic on this open set.

**Theorem 4.2.6.** *For $\Re(s) > 1$, one have*

$$\zeta(s) = \prod_{p\in\mathbb{P}}\left(1 - \frac{1}{p^s}\right)^{-1}.$$

*Proof.* It suffices to notice that the function defined on $\mathbb{N}^*$ by $f(n) = 1$ is completely multiplicative and we conclude using Corollary 4.2.3. □

We are now interested in a particular class of Dirichlet series; that is the series

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

such that

$$\sum_{n=1}^{\infty}|a_n|^2 < \infty. \tag{4.4}$$

## 4.3 Hilbert space of Dirichlet series

Let $f$ be a Dirichlet series satisfying (4.4), we have

$$|f(s)| = \sum_{n=1}^{\infty} |\frac{a_n}{n^s}| \leqslant (\sum_{n=1}^{\infty} |a_n|^2)(\sum_{n=1}^{\infty} n^{-2\Re(s)}).$$

We deduce that there is absolute convergence of f for $\Re(s) > \frac{1}{2}$, therefore $\sigma_a \leq \frac{1}{2}$. We can have $\sigma_a = \frac{1}{2}$. For this, take the Dirichlet series $D$ with $a_n = \frac{1}{n^{\frac{1}{2}}\ln(n+1)}$, we have $\sum |a_n|^2 < \infty$ (see Theorem 1.10.12).
Since

$$D(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1/2}\ln(n+1)},$$

by Theorem 1.10.12, $D(s)$ does not converge absolutely if $\Re(s) \leq 1/2$.

**Definition 4.3.1.** *We denote by*

$$\mathcal{H} = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : ||f||^2_{\mathcal{H}} = \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

*the Hilbert space of Dirichlet series.*

**Remark 4.3.2.** *Observe that elements of $\mathcal{H}$ are defined in $\Re(s) > 1/2$.*

**Theorem 4.3.3.** *We define the product*

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} a_n \overline{b_n},$$

*where $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$. Then $\langle , \rangle_{\mathcal{H}}$ is a scalar product and $\mathcal{H}$ is a Hilbert space.*

*Proof.* The product $\langle , \rangle_{\mathcal{H}}$ satisfies Definition 1.10.5 so that $\langle , \rangle_{\mathcal{H}}$ is a scalar product. It remains to show that $\mathcal{H}$ is complete. Let $(f^{(p)})_p$ be a Cauchy sequence of elements of $\mathcal{H}$, show that it converges in $\mathcal{H}$. We denote $f^{(p)}(s) = \sum u_n^{(p)} n^{-s}$. The sequence $(f^{(p)})_p$ being Cauchy, we have

$$\forall \varepsilon > 0, \exists p_0, \forall p > p_0, \forall q > p_0, ||f^{(p)} - f^{(q)}||^2_{\mathcal{H}} = \sum_{n=1}^{\infty} |u_n^{(p)} - u_n^{(q)}|^2 < \varepsilon. \tag{4.5}$$

For each $n$ fixed in $\mathbb{N}^*$, we have $|u_n^{(p)} - u_n^{(q)}|^2 < \varepsilon$ as soon as $p > p_0, q > p_0$, $p_0$ being associated with $\varepsilon$. Thus, the sequence $(u_n^{(p)})_p$ with fixed $n$ is of Cauchy in $\mathbb{C}$ which is a complete space, so it converges. Let $u_n = \lim_{p\to\infty} u_n^{(p)}$. Relation (4.5) also implies that, for $N$ fixed in $\mathbb{N}^*$, we have

$$\forall \varepsilon > 0, \exists p_0, \forall p > p_0, \forall q > p_0, \forall N \in \mathbb{N}^* \sum_{n=1}^{N} |u_n^{(p)} - u_n^{(q)}|^2 < \varepsilon. \tag{4.6}$$

In (4.6) doing $q \to \infty$ we have

$$\forall \varepsilon > 0, \exists p_0, \forall p > p_0, \forall N \in \mathbb{N}^* \sum_{n=1}^{N} |u_n^{(p)} - u_n|^2 < \varepsilon, \tag{4.7}$$

which implies the convergence of the series $(|u_n^{(p)} - u_n|^2)_n$ since the sequence of partial sums is bounded by $\varepsilon$. So we deduce from (4.7) that

$$\forall \varepsilon > 0, \exists p_0, \forall p > p_0, \sum_{n=1}^{+\infty} |u_n^{(p)} - u_n|^2 < \varepsilon. \tag{4.8}$$

Fixing $p > p_0$, denoting $u$ the sequence of $u_n$ and $u^{(p)}$ the sequence of $u_n^{(p)}$, we have $u^{(p)} - u \in l^2(\mathbb{C})$. Note that $u^{(p)} \in l^2(\mathbb{C})$ because $f^{(p)} \in \mathcal{H}$ thus $u \in l^2(\mathbb{C})$. Let $f(s) = \sum u_n n^{-s}$, then $f \in \mathcal{H}$. Therefore (4.8) reads

$$\forall \varepsilon > 0, \exists p_0, \forall p > p_0, ||f^{(p)} - f||_{\mathcal{H}}^2 < \varepsilon.$$

This translates the convergence of the sequence $(f^{(p)})_p$ to $f$ in $\mathcal{H}$. □

**Notation 4.3.4.** *Let $\mathbb{C}_\theta = \{s \in \mathbb{C}; \Re(s) > \theta\}$ where $\theta$ is a real.*

**Theorem 4.3.5.** *The space $\mathcal{H}$ is a reproducing kernel Hilbert space with reproducing kernel $K$ given by*

$$K(s,a) = \zeta(s + \overline{a}),$$

*where $\zeta$ is defined in Definition 4.2.5.*
*Moreover we have*

$$||K_a||_{\mathcal{H}}^2 = K_a(a) = \varsigma(2\Re(a)).$$

*Proof.* By Theorem 1.10.29, it suffices to show that for all $a \in \mathbb{C}_{1/2}$ the linear map $L_a : f \mapsto f(a)$ is continuous. Let $a \in \mathbb{C}_{1/2}$. We have, using Theorem 1.10.6

$$\begin{aligned} |L_a(f)| &\leq |f(a)| \\ &= |\sum_{n=1}^{\infty} a_n n^{-a}| \\ &\leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^{-2a}\right)^{\frac{1}{2}} \\ &= C_a ||f||_{\mathcal{H}}, \end{aligned}$$

which proves the continuity of $L_a$. By Theorem 1.10.22, for any $a$ in $\mathbb{C}_{1/2}$, there exists a unique element $K_a$ of $\mathcal{H}$ with the property that

$$f(a) = \langle f, K_a \rangle \; for \; all \; f \in \mathcal{H}.$$

Let $K_a(s) = \sum_{n=1}^{\infty} b_{n,a} n^{-s}$. So, for $a \in \mathbb{C}_{1/2}$ and $f(s) = n^{-s}$, we have

$$n^{-a} = \overline{b_{n,a}}$$

so

$$K_a(s) = \sum_{n=1}^{\infty} \overline{n^{-a}} n^{-s} = \sum_{n=1}^{\infty} n^{-s-\overline{a}} = \zeta(s+\overline{a}),$$

where $\zeta$ is defined in Definition 4.2.5.
Furthermore

$$||K_a||^2_{\mathcal{H}} = K_a(a) = \varsigma(2\Re(a)).$$

□

## 4.4 The space $H^2(\mathbb{D}^\infty)$: the Hardy space $H^2$ on the polydisk of infinite dimension

A fundamental observation made by Bohr in [1] is the following; Let us set

$$z_1 = 2^{-s}, z_2 = 3^{-s}, ..., z_m = p_m^{-s}, ...,$$

where $p_m$ designates the $m$-th prime number. Given the decomposition of any integer into a product of prime factors, $f \in \mathcal{H}$ such that $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ can be considered as a power series over an infinite number of variables, where the variables are the $z_m = p_m^{-s}$.
Indeed if we set $z = (z_m)_m = (z_1, z_2, z_3, ...)$ and we write $n = p_{k_1}^{v_1} p_{k_2}^{v_2} ... p_{k_r}^{v_r}$, the decomposition of $n$ into prime factors, we thus have formally $f$ which is on the form

$$\mathfrak{Q}f(z) = \sum_{n=1}^{+\infty} a_n z_{k_1}^{v_1} z_{k_2}^{v_2} ... z_{k_r}^{v_r}. \tag{4.9}$$

From now on, for $f$ in $\mathcal{H}$, we call $\mathfrak{Q}f$ the power series given by (4.9) associated with $f$, without taking into account of the relation that links $z$ to $s$.

**Definition 4.4.1.** *A quasi-character on* $\mathbb{N}^*$ *is a mapping* $\phi : \mathbb{N}^* \longrightarrow \mathbb{C}$*, such that*

$$\phi(mn) = \phi(m)\phi(n) \text{ and } \phi(1) = 1 \text{ with } \phi(n) \in \mathbb{D} \text{ for } n > 1.$$

Another way to think of the $\mathfrak{Q}f$ extension of $f$ is to write

$$\mathfrak{Q}f(\phi) = \sum_{n=1}^{\infty} a_n \phi(n), \tag{4.10}$$

where $\phi$ is a quasi-character and $\mathbb{D}$ is the unit disk. From the decomposition of any integer into a product of prime factors, we deduce that $\phi$ is entirely determined by the data of its values on the prime numbers $(p_n)$. Thus for $m-th$ prime number $p_m$ by setting $\phi(p_m) = z_m$. and by identifying $\phi$ with $z = (z_m)_m$, we see that it is equivalent to consider either expression (4.9) or (4.10 ) of $\mathfrak{Q}f$.

Now, taking $f(n) = |\phi(n)|^2 n^s$ in Corollary 4.2.3

$$\begin{aligned}
|\mathfrak{Q}f(z)|^2 &= |\mathfrak{Q}f(\phi)|^2 \\
&\leqslant (\sum_{n=1}^{+\infty} |a_n|^2)(\sum_{n=1}^{\infty} |\phi(n)|^2) \\
&= ||f||^2_{\mathcal{H}} \prod_{p\in\mathbb{P}} (1-|\phi(p)|^2)^{-1} = ||f||^2_{\mathcal{H}} \prod_{m=1}^{\infty} (1-|\phi(p_m)|^2)^{-1} \\
&= ||f||^2_{\mathcal{H}} \prod_{m=1}^{\infty} (1-|z_m|^2)^{-1}.
\end{aligned}$$

Then $\mathfrak{Q}f(z)$ is well defined if

$$|z_m| < 1 \textit{ and } \sum_{m=1}^{\infty} |z_m|^2 < \infty.$$

Then

$$|\mathfrak{Q}f(z)| \leqslant C(z)||f||_{\mathcal{H}}.$$

**Definition 4.4.2.** *We call polydisk of infinite dimension, the set of $z = (z_1, z_2, ...)$ with $|z_m| < 1$ for all integer $m$. We denote it $\mathbb{D}^\infty$.*

**Definition 4.4.3.** *We call $l^2(\mathbb{N})$ the set of sequences $(a_n)_n \subset \mathbb{C}$ such that*

$$\sum_{m=1}^{\infty} |a_m|^2 < +\infty.$$

**Definition 4.4.4.** *We call $H^2(\mathbb{D}^\infty)$ the set of series of the form (4.9) with $z \in \mathbb{D}^\infty \cap l^2(\mathbb{N})$.*

**Proposition 4.4.5.** *We have that $H^2(\mathbb{D}^\infty)$ is a Hilbert space endowed with the scalar product*

$$\langle \mathfrak{Q}f, \mathfrak{Q}g \rangle_{H^2(\mathbb{D}^\infty)} = \sum_{n=1}^{+\infty} a_n \overline{b_n},$$

*where*

$$\mathfrak{Q}f(z) = \sum_{n=1}^{\infty} a_n z_{k_1}^{v_1} z_{k_2}^{v_2} ... z_{k_r}^{v_r} \textit{ and } \mathfrak{Q}g(z) = \sum_{n=1}^{\infty} b_n z_{k_1}^{v_1} z_{k_2}^{v_2} ... z_{k_r}^{v_r}.$$

*Proof.* We proceed as in the proof of Theorem 4.3.3. □

It should be observed that $H^2(\mathbb{D}^\infty)$ differs from $\mathcal{H}$ only by the set where their elements are taken, one on $\mathbb{D}^\infty \cap l^2(\mathbb{N})$ the other on $\mathbb{C}_{1/2}$.

### 4.4.1 Definition of the space $H^\infty(\mathbb{D}^\infty)$

Hilbert showed that it is possible to go beyond $\mathbb{D}^\infty \cap l^2(\mathbb{N})$ under certain conditions see [40]. Let $(\beta_m)_m$ be a sequence of positive numbers such that $0 < \beta_m \leq 1$.

Consider the following set

$$\Omega = \{z = (z_m)_m \in \mathbb{D}^\infty : |z_m| < \beta_m \; for \; all \; m\}$$

Let us set

$$z^{(m)} = (z_1, z_2, ..., z_m, 0, 0, ...).$$

for $z = (z_1, z_2, ...) \in \Omega$.

Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series, we only ask it to have a finite convergence abscissa, we no longer ask to satisfy the condition $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. We want to define $\mathfrak{Q}f$ on such a $f$.

**Definition 4.4.6.** *We denote by $\mathfrak{Q}f(z^{(m)})$ the series obtained by replacing in (4.9) $z$ by $z^{(m)}$*

**Definition 4.4.7.** *For $z \in \Omega$, let $\mathfrak{Q}f(z) = \lim_{m\to\infty} \mathfrak{Q}f(z^{(m)})$.*

**Definition 4.4.8.** *We say that $\mathfrak{Q}f$ is finitely bounded on $\Omega$ if the series $\mathfrak{Q}f(z^{(m)})$ ($m$-th abscissa of $\mathfrak{Q}f$) absolutely converges on $\Omega$ for all $m$, and if there exists a constant $C = C(f)$ such that*

$$|\mathfrak{Q}f(z^{(m)})| \leqslant C, \quad z \in \Omega,$$

*$C$ does not depend on $m$.*

**Definition 4.4.9.** *The set of $\mathfrak{Q}f$ which are finitely bounded on $\mathbb{D}^\infty$ is denoted $H^\infty(\mathbb{D}^\infty)$. We define the following norm on $H^\infty(\mathbb{D}^\infty)$*

$$||\mathfrak{Q}f||_{H^\infty(\mathbb{D}^\infty)} = \sup_{m\geq 1, z\in\mathbb{D}^\infty} |\mathfrak{Q}f(z^{(m)})|.$$

**Definition 4.4.10.** *We denote by $c_0(\mathbb{N})$ the space of sequences $(z_m)_m \subset \mathbb{C}$ which converges to zero.*

**Theorem 4.4.11.** *Given $f(s) = \sum_n a_n n^{-s}$, we have $\mathfrak{Q}f \in H^\infty(\mathbb{D}^\infty)$ is holomorphic and bounded on $\mathbb{D}^\infty \bigcap c_0(\mathbb{N})$.*

*Proof.* Let $\mathfrak{Q}f \in H^\infty(\mathbb{D}^\infty)$. Let us show that $\mathfrak{Q}f(z^{(m)})$ is holomorphic. For that let us show that $\mathfrak{Q}f(z^{(m)})$ is holomorphic with respect to each variable.

Fix $z_1, ..., z_{j-1}, z_{j+1}, ..., z_m$ and set

$$g_j(z_j) = \mathfrak{Q}f(z^{(m)}) = \sum_{v_1,...,v_m\geq 0} a_{v_1...v_m} z_1^{v_1} ... z_j^{v_j} ... z_m^{v_m}, \quad z_j \in \mathbb{D}$$

and

$$h_j(z_j) = \sum_{v_1,...,v_m\geq 0} v_j a_{v_1...v_m} z_1^{v_1} ... z_j^{v_j-1} ... z_m^{v_m} \quad z_j \in \mathbb{D}.$$

It suffices to show that $\lim_{h\to 0} \dfrac{g_j(z_j + h) - g_j(z_j)}{h}$ exists.

Let $\varepsilon > 0$ and $r_j$ such that $0 < |z_j| < r_j < 1$. Consider $h \in \mathbb{C}$ with $0 < |h| \leq r_j - |z_j|$. Then

$|z_j+h| \leq r_j$. Note that

$$\begin{aligned}
\frac{g_j(z_j+h)-g_j(z_j)}{h} &= \sum_{v_1,\dots,v_m\geq 0} a_{v_1\dots v_m} z_1^{v_1}\dots[\frac{(z_j+h)^{v_j}-z_j^{v_j}}{h}]\dots z_m^{v_m}\\
&= \sum_{v_1,\dots,v_m\geq 0} a_{v_1\dots v_m} z_1^{v_1}\dots[\frac{h}{h}((z_j+h)^{v_j-1}+(z_j+h)^{v_j-2}z_j+(z_j+h)z_j^{v_j-2}+z_j^{v_j-1})]\dots z_m^{v_m}\\
&= \sum_{v_1,\dots,v_m\geq 0} a_{v_1\dots v_m} z_1^{v_1}\dots[(z_j+h)^{v_j-1}+(z_j+h)^{v_j-2}z_j+(z_j+h)z_j^{v_j-2}+z_j^{v_j-1}]\dots z_m^{v_m}
\end{aligned}$$

$\frac{g_j(z_j+h)-g_j(z_j)}{h} - h_j(z_j) = \sum_{v_1,\dots,v_m\geq 0} a_{v_1\dots v_m} z_1^{v_1}\dots[(z_j+h)^{v_j-1}+(z_j+h)^{v_j-2}z_j+(z_j+h)z_j^{v_j-2}+z_j^{v_j-1}-v_jz_j^{v_j-1}]\dots z_m^{v_m}$.

Set $u(z_j,h)=(z_j+h)^{v_j-1}+(z_j+h)^{v_j-2}z_j+\dots+(z_j+h)z_j^{v_j-2}+z_j^{v_j-1}-v_jz_j^{v_j-1}$ then

$$|u(z_j,h)| \leq 2v_j r_j^{v_j-1}.$$

On $\mathbb{D}$, $g_j$ converges absolutely i.e $\sum_{v_1,\dots,v_m\geq 0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots|r_j^{v_j}|\dots|z_m^{v_m}|$ converges. Let $r$ such that $r_j<r<1$. One have

$$\begin{aligned}
v_j a_{v_1\dots v_m} z_1^{v_1}\dots r_j^{v_j-1}\dots z_m^{v_m} &= \frac{v_j}{r_j} a_{v_1\dots v_m} z_1^{v_1}\dots(\frac{r_j}{r})^{v_j} r_j^{v_j}\dots z_m^{v_m}\\
&= \frac{v_j}{r_j}(\frac{r_j}{r})^{v_j} a_{v_1\dots v_m} z_1^{v_1}\dots r^{v_j}\dots z_m^{v_m}
\end{aligned}$$

Let

$$u_{v_j} = \frac{v_j}{r_j}(\frac{r_j}{r})^{v_j},$$

then

$$\frac{u_{v_j+1}}{u_{v_j}} = \frac{v_j+1}{v_j}\frac{r_j}{r}, \lim_{v_j\to+\infty}\frac{u_{v_j+1}}{u_{v_j}} = \frac{r_j}{r} \ \ and \ \ |\frac{r_j}{r}|<1$$

according to the Cauchy criterion, the series of $u_{v_j}$ converges, therefore it is bounded. So there exists $M>0$ such that $|\frac{v_j}{r_j}(\frac{r_j}{r})^{v_j}|<M$.

Then

$$|v_j a_{v_1\dots v_m} z_1^{v_1}\dots r_j^{v_j-1}\dots z_m^{v_m}| < M|a_{v_1\dots v_m}||z_1^{v_1}|\dots|r^{v_j}|\dots|z_m^{v_m}|$$

Thus $\sum_{v_1,\dots,v_m\geq 0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots[2v_jr_j^{v_j-1}]\dots|z_m^{v_m}|$ converges as soon as the m-1 other components are fixed. Then

$$\exists n_0, \sum_{v_1,\dots,v_m> n_0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots[2v_jr_j^{v_j-1}]\dots|z_m^{v_m} < \frac{\varepsilon}{2},$$

one then have, for all $h$ with $0<|h|\leq r_j-|z_j|$,

$$|\frac{g_j(z_j+h)-g_j(z_j)}{h} - h_j(z_j)| \leq \frac{\varepsilon}{2} + \sum_{v_1,\dots,v_m\leq n_0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots|u(z_j,h)|\dots|z_m^{v_m}|$$

and $\lim_{h\to 0}\sum_{v_1,\dots,v_m\leq n_0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots|u(z_j,h)|\dots|z_m^{v_m}|=0$ as a finite sum of limits each worth 0. Thus to $\varepsilon$ we associate $\alpha>0$ such that $0<|h|<\alpha$ implies

$$\sum_{v_1,\dots,v_m\leq n_0} |a_{v_1\dots v_m}||z_1^{v_1}|\dots[2v_jr_j^{v_j-1}]\dots|z_m^{v_m}| < \frac{\varepsilon}{2}$$

and finally we have

$$0 < |h| < inf(\alpha, r_j - |z_j|) \Rightarrow |\frac{g_j(z_j + h) - g_j(z_j)}{h} - h_j(z_j)| \leq \varepsilon.$$

Hence $\mathfrak{Q}f(z^{(m)})$ is holomorphic with respect to each variable and by Theorem 1.4.8 $\mathfrak{Q}f(z^{(m)})$ is holomorphic. To show that $\mathfrak{Q}f$ is holomorphic it suffices then to show that $\mathfrak{Q}f$ is the uniform limit of $\mathfrak{Q}f(z^{(m)})$ and Theorem 1.4.6 will allow us to conclude that $\mathfrak{Q}f$ is holomorphic. Let $z \in \mathbb{D}^\infty \bigcap c_0(\mathbb{N})$ and let $n, m$ such that $n < m$, then

$$|\mathfrak{Q}f(z^{(m)}) - \mathfrak{Q}f(z^{(n)})| = |\sum_{v_1,\dots,v_m \geq 0, / \exists i, n < i \leq m \;\; and \;\; v_i \neq 0} a_{v_1 \dots v_m} z_1^{v_1} \dots z_n^{v_n} \dots z_m^{v_m}|.$$

Let, $n < j \leq m$. Set

$$L_j(z_j) = \sum_{v_1,\dots,v_m \geq 0, / \exists i, n < i \leq m \;\; and \;\; v_i \neq 0} a_{v_1 \dots v_m} z_1^{v_1} \dots z_n^{v_n} \dots z_m^{v_m}$$

and

$$P_j(z_j) = \frac{L_j(z_j)}{2||\mathfrak{Q}f||_{H^\infty(\mathbb{D}^\infty)}}, \; on \; \mathbb{D}$$

$L_j$ and $P_j$ are holomorphic on $\mathbb{D}$ as the sum of holomorphic functions on $\mathbb{D}$,

$$|P_j(z_j)| < 1 \; sur \; \mathbb{D}$$

By Lemma 1.4.7, one has

$$|P_j(z_j)| < |z_j| \; i.e \; |L_j(z_j)| < 2||\mathfrak{Q}f||_{H^\infty(\mathbb{D}^\infty)}|z_j| \; for \; all \; n < j \leq m.$$

Let $\epsilon > 0$, as the $z_j$ tend to zero, there is a $j_0$ from which $|z_j| < \epsilon$ for $j > j_0$.

For $n > j_0$ one have

$$\begin{aligned} |L_j(z_j)| &= |\sum_{v_1,\dots,v_m \geq 0, / \exists i, n < i \leq m \;\; and \;\; v_i \neq 0} a_{v_1 \dots v_m} z_1^{v_1} \dots z_n^{v_n} \dots z_m^{v_m}| \\ &< 2||\mathfrak{Q}f||_{H^\infty(\mathbb{D}^\infty)} max\{|z_j| : n < j \leq m\} \\ &< 2\epsilon||\mathfrak{Q}f||_{H^\infty(\mathbb{D}^\infty)}, \end{aligned}$$

thus the $\mathfrak{Q}f(z^{(m)})$ are uniformly Cauchy and therefore converges uniformly. Since by definition $\mathfrak{Q}f(z) = \lim_{m \to \infty} \mathfrak{Q}f(z^{(m)})$ the $\mathfrak{Q}f(z^{(m)})$ converges uniformly to $\mathfrak{Q}f$.
Let us show that $\mathfrak{Q}f$ is bounded. Each $\mathfrak{Q}f(z^{(m)})$ is bounded because $\mathfrak{Q}f \in H^\infty(\mathbb{D}^\infty)$ and since $\mathfrak{Q}f(z^{(m)})$ converges uniformly to $\mathfrak{Q}f$, then $\mathfrak{Q}f$ is bounded. □

### 4.4.2 The space $H^2(\mathbb{T}^\infty)$

We denote by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathbb{T}$ is a subgroup of the multiplicative group of the group $(\mathbb{C}^*, .)$. We consider the space $H^2(\mathbb{D}^\infty)$ as the Hardy space $H^2$ on the infinite polydisk. It can be identified with a space of functions on $\mathbb{T}^\infty$. Definitions and results of this section come from [82].

**Definition 4.4.12.** *A character over a group $G$ is a morphism from $G$ to the multiplicative group $(\mathbb{T}, .)$. The set of characters on $G$ which are continuous is called the dual of $G$ and is denoted by $\widehat{G}$.*

Let $\chi$ be a character on the multiplicative group $(\mathbb{N}^*, .)$, in other words $\chi : \mathbb{N}^* \longrightarrow \mathbb{C}$ is such that

(i) $\chi(mn) = \chi(m)\chi(n), m, n = 1, 2, 3, ...,$

(ii) $|\chi(n)| = 1.$

We observe that $\chi$ is entirely determined by the data of its values on the prime number $(p_m)_m$. It is possible to extend $\chi$ on $\mathbb{Q}^+$. It suffices to set $\chi(n^{-1}) = \chi(n)^{-1} = \overline{\chi(n)}$ and $\chi(nm^{-1}) = \chi(n)\chi(m)^{-1}$. The set of $\chi$ defined on $\mathbb{Q}_+$ is denoted $\mathcal{E}$

**Theorem 4.4.13.** *The set $\mathcal{E}$ is in bijection with the infinite dimensional polycircle $\mathbb{T}^\infty$.*

*Proof.* If we take $z = (z_1, z_2, ...) \in \mathbb{T}^\infty$, we then define $\chi$ as

$$\chi(2) = z_1,\ \chi(3) = z_2, ...,\ \chi(p_m) = z_m, ...,\ where\ p_m\ is\ the\ m-th\ prime\ number$$

and we extend the definition multiplicatively over $\mathbb{N}^*$ using the fact that any integer can be broken down into a product of prime factors, then over $\mathbb{Q}_+$. We therefore have a bijection between $\mathcal{E}$ and $\mathbb{T}^\infty$. $\square$

We equip $\mathbb{T}^\infty$ with the product topology, $\mathbb{Q}_+$ with the discrete topology so that each $\chi \in \mathcal{E}$ is continuous. We equip $\mathcal{E} = \widehat{\mathbb{Q}_+}$ with the law $*$ defined as follows:

$$(\chi * \psi)(x) = \chi(x)\psi(x) \quad x \in \mathbb{Q}_+.$$

**Proposition 4.4.14.** *Let $(G_j)_{j\in J}$ be a family of compact topological groups (without assumption on the cardinality of $J$), the product group $G = \prod_{j\in J} G_j$ is a compact topological group.*

*Proof.* We endow $G$ with the product topology, i.e with the least fine topology making the projections $\pi_j : G \to G_j$ continuous. Let us show that $G$ is a topological group; we must show that the product on $G$ and the inversion are continuous. Let us denote by $p : G \times G \to G$ the point-by-point product defined on $G$ and $p_j$ the product on $G_j$. We know that $p$ is continuous if and only if $\pi_j \circ p$ is continuous. Let $g, g' \in G$ then we have

$$\pi_j \circ p(g, g') = g_j g'_j = p_j(\pi_j(g), \pi_j(g')).$$

$\pi_j \circ p$ is then continuous as composition of continuous maps. We also show that the inversion on $G$ is continuous. We then have that $G$ is compact as a product of compact topological groups. $\square$

The next result is in [82]

**Theorem 4.4.15.** *Let $G$ be a discrete Abelian group. Let $\widehat{G}$ be the dual of $G$. $\widehat{G}$ is a closed subgroup of $\mathbb{T}^G$ and therefore a compact Abelian topological group.*

Since $\mathbb{Q}_+$ is discrete, a direct application of Theorem 4.4.15 shows that the topological group $(\mathcal{E}, *)$(see Definition 4.4.12) is compact. Thus $\mathbb{T}^\infty$ and $\mathcal{E}$ are compact.
Since $\mathcal{E}$ is compact, by Theorem 1.9.5 there exists a unique Haar measure on $\mathcal{E}$ that we identify with the product measure $\rho$ defined on $\mathbb{T}^\infty$. More precisely if we denote $\lambda$ the measure of length on $\mathbb{T}$ normalized such that $\lambda(\mathbb{T}) = 1$ and if for the Borel $E_1, E_2, ..., E_N \subset \mathbb{T}$ we note $E = E_1 \times E_2 \times ... \times E_N \times \mathbb{T} \times \mathbb{T} \times ... \subset \mathbb{T}^\infty$ then we have

$$\rho(E) = \lambda(E_1)\lambda(E_2)...\lambda(E_N);$$

We admit that this one has total mass 1. In the following lines we assume $\mathbb{T}^\infty$ equipped with the measure $\rho$.

We can now use the bijection between $\mathbb{T}^\infty$ and $\mathcal{E}$ to define an integration on $\mathcal{E}$ in the following manner:
If $n = p_{k_1}^{v_1} p_{k_2}^{v_2} ... p_{k_a}^{v_a}$ and $m = p_{l_1}^{u_1} p_{l_2}^{u_2} ... p_{l_e}^{u_e}$, as $\chi$ is multiplicative then

$$\chi(n) = z_{k_1}^{v_1} z_{k_2}^{v_2} ... z_{k_a}^{v_a} \quad (* * **)$$

$$and$$

$$\chi(m) = z_{l_1}^{u_1} z_{l_2}^{u_2} ... z_{l_e}^{u_e}$$

so we set

$$\int_{\mathcal{E}} \overline{\chi(n)} \chi(m) d\rho(\chi) = \int_{\mathbb{T}^\infty} \overline{z_{k_1}^{v_1} z_{k_2}^{v_2} ... z_{k_a}^{v_a}} z_{l_1}^{u_1} z_{l_2}^{u_2} ... z_{l_e}^{u_e} d\rho(z).$$

We arrange (****) to have the same decomposition with certain exponents equal to zero. So

$$\int_{\mathcal{E}} \overline{\chi(n)} \chi(m) d\rho(\chi) = \prod_{i=1}^{r} \int_{\mathbb{T}} \overline{z_{k_i}^{v_i}} z_{k_i}^{u_i} |dz_{k_i}| = \{ {}^{0 \ if \ n \neq m}_{1 \ else}$$

Now let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$ where $\mathcal{H}$ is defined in Definition 4.3.1.

**Proposition 4.4.16.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$. The sequence of functions $(f_n)_{n \in \mathbb{N}^*}$ where for all $n \in \mathbb{N}^*$,*

$$\begin{aligned} f_n : \ & \mathcal{E} \longrightarrow \mathbb{C} \\ & \chi \longrightarrow f_n(\chi) = \sum_{i=1}^{n} a_i \chi(i) \end{aligned}$$

*is a Cauchy sequence in $L^2(\mathcal{E},\rho)$. This sequence therefore converges in $L^2(\mathcal{E},\rho)$ to a function denoted $\mathfrak{B}f(\chi)$.*

*Proof.* Let m and n such that $m < n$, then

$$\begin{aligned}
||f_n(\chi)-f_m(\chi)||^2_{L^2(\mathcal{E},\rho)} &= ||\sum_{i=1}^{n} a_i\chi(i) - \sum_{i=1}^{m} a_i\chi(i)||^2_{L^2(\mathcal{E},\rho)}\\
&= ||\sum_{i=m+1}^{n} a_i\chi(i)||^2_{L^2(\mathcal{E},\rho)}\\
&= \int_{\mathcal{E}}(\overline{\sum_{j=m+1}^{n} a_j\chi(j))(\sum_{i=m+1}^{n}} a_i\chi(i))d\rho(\chi)\\
&= \sum_{j,i=m+1}^{n} \overline{a_j}a_i \int_{\mathcal{E}} \overline{\chi(j)}\chi(i)d\rho(\chi)\\
&= \sum_{i=m+1}^{n} |a_i|^2.
\end{aligned}$$

We have $\sum_{i=m+1}^{n} |a_i|^2 \to 0$ when $m,n\to\infty$ since $\sum_{n=1}^{\infty}|a_n|^2 < \infty$. □

**Proposition 4.4.17.** *The function $\mathfrak{B}f(\chi)$ is completely characterised by the fact that*

$$\int_{\mathcal{E}}(\overline{\chi}(q)\mathfrak{B}f(\chi)d\rho(\chi) = a_q;\quad q\in\mathbb{Q}_+,$$

*with $a_q = 0,\ for\ q\in\mathbb{Q}_+\backslash\mathbb{N}$.*

*Proof.* From Proposition 4.4.16, for $f_n(\chi) = \sum_{i=1}^{n} a_i\chi(i)$, the sequence $(f_n)_{n\in\mathbb{N}^*}$converges to $\mathfrak{B}f(\chi)$ in $L^2(\mathcal{E},\rho)$ we can extract a subsequence always denoted $(f_n)$ which simply converges for almost everything $\chi$ to $\mathfrak{B}f(\chi)$.

- $For$ $q\in\mathbb{N}$ and $n > q$ we have

$$\int_{\mathcal{E}}\overline{\chi(q)}f_n(\chi)d\rho(\chi) = a_q$$

$\overline{\chi(q)}f_n(\chi)$ is then integrable. $\overline{\chi(q)}f_n(\chi)$ simply converges for almost everything $\chi$ because $(f_n)$ simply converges for almost everything $\chi$. We deduce that $\overline{\chi(q)}f_n(\chi)$ is bounded for almost all $\chi$ and the fact that $\mathcal{E}$ has total mass $1$ ensures that the constants are integrable we can therefore apply the dominated convergence theorem to conclude that

$$\int_{\mathcal{E}}\overline{\chi(q)}\mathfrak{B}f(\chi)d\rho(\chi) = a_q$$

-Suppose $q\in\mathbb{Q}_+\backslash\mathbb{N}$. without compromising generality suppose $q = m^{-1}$ with $m\in\mathbb{N}^*$ then

$$\chi(q) = \chi(m^{-1}) = \chi(m)^{-1} = \overline{\chi(m)}.$$

Thus

$$\int_{\mathcal{E}}\overline{\chi(q)}f_n(\chi)d\rho(\chi) = \int_{\mathcal{E}}\chi(m)f_n(\chi)d\rho(\chi)$$

$\int_{\mathcal{E}} \chi(m)\chi(k)d\rho(\chi) = 0$ for all $k \in \mathbb{N}$ so $\int_{\mathcal{E}} \chi(m)f_n(\chi)d\rho(\chi) = 0$ and by an argument similar to the previous case we have

$$\int_{\mathcal{E}} \overline{\chi(q)}\mathfrak{B}f(\chi)d\rho(\chi) = 0$$

hence the result. □

**Lemma 4.4.18.** *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$ *then*

$$\int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho = \sum_{n=1}^{\infty} |a_n|^2 = ||f||^2_{\mathcal{H}}$$

*Proof.* By Proposition 4.4.16 the sequence of function $(f_n)_{n\in\mathbb{N}^*}$ where for all $n \in \mathbb{N}^*$,

$$\begin{aligned} f_n : \ & \mathcal{E} \longrightarrow \mathbb{C} \\ & \chi \longrightarrow f_n(\chi) = \sum_{i=1}^{n} a_i\chi(i) \end{aligned}$$

converges to $\mathfrak{B}f(\chi)$ in $L^2(\mathcal{E}, \rho)$. Set $||f_n(\chi) - \mathfrak{B}f(\chi)||^2_{L^2(\mathcal{E})} = u_n$ then

$$\begin{aligned} u_n &= ||\sum_{i=1}^{n} a_i\chi(i) - \mathfrak{B}f(\chi)||^2_{L^2(\mathcal{E},\rho)} \\ &= \int_{\mathcal{E}} (\sum_{i=1}^{n} a_i\chi(i) - \mathfrak{B}f(\chi))\overline{(\sum_{i=1}^{n} a_i\chi(i) - \mathfrak{B}f(\chi))}d\rho(\chi) \\ &= \int_{\mathcal{E}} (\sum_{i,j=1}^{n} a_i\overline{a_j}\overline{\chi(j)}\chi(i) - \mathfrak{B}f(\chi)\sum_{i=1}^{n} \overline{a_i}\overline{\chi(i)} - \overline{\mathfrak{B}f(\chi)}\sum_{i=1}^{n} a_i\chi(i) + |\mathfrak{B}f(\chi)|^2)d\rho(\chi) \\ &= \sum_{i=1}^{n} |a_i|^2 - \sum_{i=1}^{n} \overline{a_i}\int_{\mathcal{E}} \overline{\chi(i)}\mathfrak{B}f(\chi)d\rho(\chi) - \sum_{i=1}^{n} a_i \overline{\int_{\mathcal{E}} \overline{\chi(i)}\mathfrak{B}f(\chi)d\rho(\chi)} + \int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho \\ &= \sum_{i=1}^{n} |a_i|^2 - \sum_{i=1}^{n} \overline{a_i}a_i - \sum_{i=1}^{n} a_i\overline{a_i} + \int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho \\ &= -\sum_{i=1}^{n} |a_i|^2 + \int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho. \end{aligned}$$

Since $\lim_{n\to\infty} ||f_n(\chi) - \mathfrak{B}f(\chi)||^2_{L^2(\mathcal{E})} = 0$ then

$$\int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho = \sum_{n=1}^{\infty} |a_n|^2 = ||f||^2_{\mathcal{H}}$$

hence the result. □

From the relations between $\mathfrak{B}f(\chi)$ and the coefficients $\{a_n\}_n$ in Proposition 4.4.17, and Lemma 4.4.18, we will denote $\mathfrak{B}f(\chi)$ by $\sum_n a_n\chi(n)$ for all $\chi \in \mathcal{E}$. Note that $\mathfrak{B}f(\chi) = \sum_n a_n\chi(n)$ when $\sum_n a_n\chi(n)$ makes sense in $\mathbb{C}$, for all $\chi \in \mathcal{E}$; this is the case when only a finite number of coefficients $a_n$ is not zero. Hence $\mathfrak{B}f$ is seen as an extension on the characters of $\mathfrak{Q}f$ defined in (4.10) on the quasi-characters $\phi$.

**Definition 4.4.19.** *We call $H^2(\mathcal{E})$ (or even $H^2(\mathbb{T}^\infty)$) the closed subspace of $L^2(\mathcal{E},\rho)$ formed of functions which can be written as a series of the form*

$$\sum_{n=1}^{\infty} a_n\chi(n) \text{ with } \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

**Definition 4.4.20.** *We call $H^\infty(\mathcal{E})$ the intersection $L^\infty(\mathcal{E},\rho)\bigcap H^2(\mathcal{E})$ which is a subspace of $L^\infty(\mathcal{E},\rho)$.*

Since we seek to identify the Hardy space on $\mathbb{T}^\infty$ with the Hardy space on $\mathbb{D}^\infty$, we need of an operator $\mathcal{F}: H^2(\mathbb{T}^\infty) \longrightarrow H^2(\mathbb{D}^\infty)$ which for a function $g(\chi) \sim \sum_{n=1}^{\infty} b_n\chi(n)$ of $H^2(\mathbb{T}^\infty)$ associates

$$\mathcal{F}g(\phi) = \sum_{n=1}^{\infty} b_n\phi(n)$$

where $\phi$ is a quasi-character such as $\phi \in \mathbb{D}^\infty \bigcap l^2(\mathbb{N})$. The following lemma tells us that $\mathcal{F}$ is an isometric isomorphism and that

$$\mathfrak{Q} = \mathcal{F}\mathfrak{B}.$$

The following Lemmas are in [40].

**Lemma 4.4.21.** *The operator $\mathfrak{B}: \mathcal{H} \longrightarrow H^2(\mathbb{T}^\infty)$ and the operator $\mathcal{F}: H^2(\mathbb{T}^\infty) \longrightarrow H^2(\mathbb{D}^\infty)$ are isometric isomorphisms and we have $\mathfrak{Q} = \mathcal{F}\mathfrak{B}$.*

*Proof.* By construction of $\mathfrak{F}$ we have $\mathfrak{Q} = \mathcal{F}\mathfrak{B}$. To complete the proof it suffices to show that these operators preserve the norm. It has already been demonstrated that

$$\int_{\mathcal{E}} |\mathfrak{B}f|^2 d\rho = ||f||^2_{\mathcal{H}}.$$

Let $g(\chi) \sim \sum_{n=1}^{\infty} b_n\chi(n)$ in $H^2(\mathbb{T}^\infty)$. By definition of $\mathcal{F}$ and by the norm on $H^2(\mathbb{D}^\infty)$ we have $\sum_{n=1}^{\infty} |b_n|^2 = ||\mathcal{F}g||^2_{H^2(\mathbb{D}^\infty)}$. Note that $g(\chi) = \mathfrak{B}f(\chi)$ with $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, hence

$$\int_{\mathcal{E}} |g|^2 d\rho = ||\mathcal{F}g||^2_{H^2(\mathbb{D}^\infty)}.$$

□

**Lemma 4.4.22.** *The restriction of $\mathcal{F}$ to $H^\infty(\mathcal{E})$ is also an isometric isomorphism from $H^\infty(\mathbb{T}^\infty) \longrightarrow H^\infty(\mathbb{D}^\infty)$.*

⋆ ⋆ Chapter 5 ⋆ ⋆

# Multipliers and composition operators on Hardy spaces of Dirichlet Series

In this chapter we define the Hardy spaces of Dirichlet Series and we study their multipliers, and weighted composition operators between two different Hardy spaces of Dirichlet Series.

## 5.1 The Banach space $\mathcal{H}^\infty$

### 5.1.1 Definition and first properties of $\mathcal{H}^\infty$

Given $\theta \in \mathbb{R}$, recall that $\mathbb{C}_\theta = \{s \in \mathbb{C}; \Re s > \theta\}$. We will denote by $\mathcal{D}$ the set of series $\sum_{n=1}^{\infty} a_n n^{-s}$ such that $\sigma_c < \infty$ (the set of convergent Dirichlet series see Definition 4.1.3), equivalently of series $\sum_{n=1}^{\infty} a_n n^{-s}$ whose coefficients have at most polynomial growth. We will denote by $\mathcal{H}^2$ the space of Dirichlet series with square-summable coefficients:

$$f \in \mathcal{H}^2 \Leftrightarrow f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{with} \ \sum_{n=1}^{\infty} |a_n|^2 =: \|f\|_2^2 < \infty. \tag{5.1}$$

We will denote by $H^\infty(\mathbb{C}_0)$ the set of bounded analytic functions on $\mathbb{C}_0$, equipped with the norm

$$\|f\|_\infty = \sup_{s \in \mathbb{C}_0} |f(s)|. \tag{5.2}$$

**Definition 5.1.1.** *The set $\mathcal{H}^\infty$ of bounded Dirichlet series (later seen to be the set of multipliers of $\mathcal{H}^2$) is by definition:*

$$\mathcal{H}^\infty = H^\infty(\mathbb{C}_0) \cap \mathcal{D}. \tag{5.3}$$

In others terms, a function $f$ belongs to $\mathcal{H}^\infty$ if it verifies *two* properties:

1. It is a bounded analytic function in $\mathbb{C}_0$.
2. Moreover, it can be represented as a convergent Dirichlet series for $\Re s$ large enough.

Let us examine some examples:

1. $f(s) = \frac{s-1}{s+1} \notin \mathcal{H}^\infty$. Indeed, $f$ is analytic and bounded by $1$ in $\mathbb{C}_0$, but $f \notin \mathcal{D}$.
2. $f(s) = e^{-s} \notin \mathcal{H}^\infty$, for the same reason.
3. $f(s) = (1 - 2^{-s})\zeta(s+1) = \sum_{n=1}^{\infty}(-1^{n-1}n^{-s-1}) \notin \mathcal{H}^\infty$. We have well the membership of $f$ in $\mathcal{D}$, but $f$ is not bounded in $\mathbb{C}_0$, for example because of Theorem 5.1.7 and the fact that $\sum_p \frac{1}{p} = \infty$.
4. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^\infty(\mathbb{B})$, the space of functions which are analytic an bounded in the open unit disk $\mathbb{D}$, equipped with the norm $\|g\|_\infty = \sup_{z\in\mathbb{D}} |g(z)|$. Then the mapping

$$g \mapsto \Delta(g) = f, \; f(s) = g(2^{-s}), \;\; \text{is an isometry : } H^\infty \xrightarrow{\text{into}} \mathcal{H}^\infty.$$

   This is obvious since $g(s) = \sum_{n=0}^{\infty} b_n (2^n)^{-s} \in \mathcal{H}^\infty$.

To estimate the coefficients of a function in $\mathcal{H}^\infty$, we have the following starting point, of basic utility appearing in [65].

**Theorem 5.1.2.** *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$, then for all $n \in \mathbb{N}^*$*

$$|a_n| \leq \|f\|_\infty, \text{ and } \sigma_a(f) \leq 1. \tag{5.4}$$

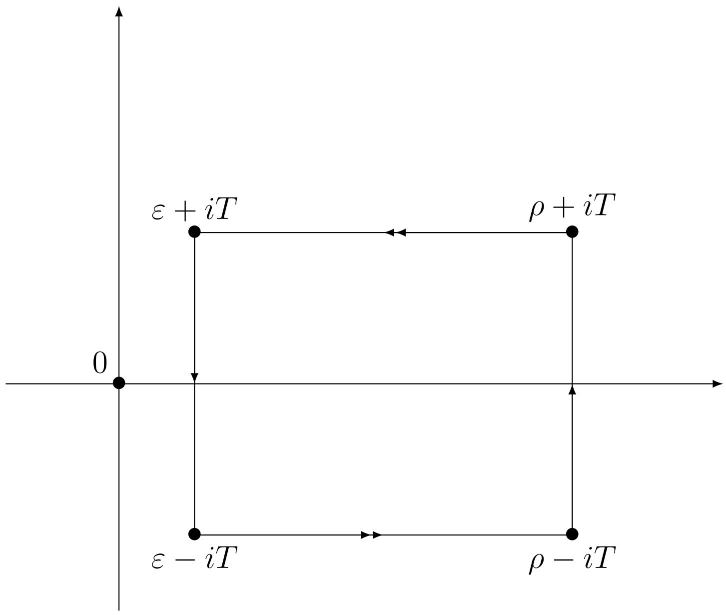
*Proof.* Let $\rho > 0$ such that $\sum_{n=1}^{\infty} |a_n| n^{-\rho} < \infty$. Such a $\rho$ exists since $\sigma_a(f) \leq \sigma_c(f) + 1$. We can then write the Fourier-Bohr formulas see [15]:

$$a_n n^{-\rho} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\rho + it) n^{it} \mathrm{d}t,$$

equivalently

$$a_n = \lim_{T\to\infty} \frac{1}{2iT} \int_{\rho - iT}^{\rho + iT} f(s) n^s \mathrm{d}s.$$

By the Cauchy integral theorem for a rectangle, we have as well




$$a_n = \lim_{T\to\infty} \frac{1}{2iT}\left(\int_{\varepsilon+iT}^{\rho+iT} f(s)n^s\mathrm{ds} + \int_{\varepsilon-\mathrm{iT}}^{\varepsilon+\mathrm{iT}} \mathrm{f(s)n^sds} - \int_{\varepsilon-\mathrm{iT}}^{\rho-\mathrm{iT}} \mathrm{f(s)n^sds}\right).$$

The first and third terms are dominated by $\rho n^\rho \|f\|_\infty/2T$ and tend to $0$ as $T\to\infty$. The second term is majorized by $n^\varepsilon\|f\|_\infty$, so that $|a_n| \leq \|f\|_\infty$, by letting $\varepsilon$ tend to $0$. This implies that $\sigma_a(f) \leq 1$, since $\sum\limits_{n\geq 1}|\frac{a_n}{n^\rho}| \leq \sum\limits_{n\geq 1}|\frac{\|f\|_\infty}{n^\rho}| < \infty$ for $\rho > 1$. □

The following result gives some information on the structure of $\mathcal{H}^\infty$ see [65, Theorem 6.2.1].

**Theorem 5.1.3.** *The space $\mathcal{H}^\infty$, equipped with the norm $\|\ \|_\infty$, is a unital and non-separable Banach algebra. Its inversible elements $f$ are characterized by the relation $|f(s)| \geq \delta$ for some $\delta > 0$.*

The partial sums of a function in $\mathcal{H}^\infty$ are well-controlled as shown by the following theorem see [65, Theorem 6.2.2].

**Theorem 5.1.4.** *(Balasubramanian-Calado-Queffélec)*
*Let us consider a function $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}^\infty$ and $S_N(s) = \sum_{n=1}^N a_n n^{-s}$, its partial sum of index $N \geq 2$. Then*

$$\|S_N\|_\infty \leq C\log N\|f\|_\infty \tag{5.5}$$

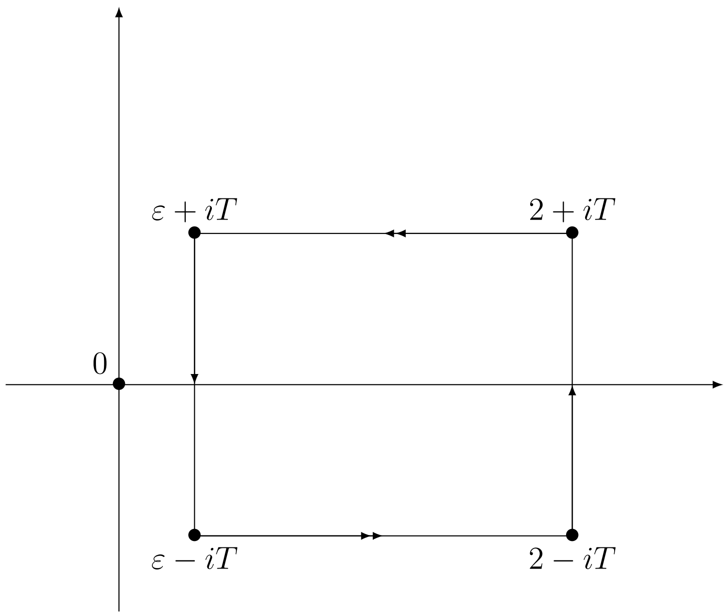
*where $C$ is a numerical constant.*

*Proof.* From Theorem 5.1.2, we know that $|a_n| \leq \|f\|_\infty$, so that $\sigma_a(f) \leq 1$. We can apply the Perron-Landau formula [65, Theorem 2.3 of Chapter 4] with $\rho = 2$ and $x = N + \frac{1}{2}$ a half-integer, namely

$$A(x) =:= \sum_{n\leq x} a_n = \frac{1}{2i\pi}\int_{2-iT}^{2+iT} f(s)\frac{x^s}{s}\mathrm{ds} + O\left(\frac{x^2}{T}\sum_{n\geq 1}\frac{|a_n|}{n^2|\log\frac{x}{n}|}\right).$$

We easily see that $|\log(x/n)|$ is $\geq 1/4(N+\frac{1}{2})$ if $n > x$ and is $\geq 1/4N$ if $n < x$, so that the error term is dominated by $Cx^3(\|f\|_\infty/T) = C\|f\|_\infty$ if we adjust $T = x^3$, the constant $C$ being absolute. Now, we shift the integral term using the Cauchy integral formula as follows (with $0 < \varepsilon < 2$):

$$\int_{2-iT}^{2+iT} f(s)\frac{x^s}{s}\mathrm{d}s = \int_{\varepsilon-iT}^{\varepsilon+iT} f(s)\frac{x^s}{s}\mathrm{d}s + \int_{\varepsilon}^{2} f(u+iT)\frac{x^{u+iT}}{u+iT}\mathrm{d}u - \int_{\varepsilon}^{2} f(u-iT)\frac{x^{u-iT}}{u-iT}\mathrm{d}u.$$




The last two integrals are uniformly dominated by $(x^2/T)\|f\|_\infty = \|f\|_\infty/x$ since $T = x^3$. For the first integral, we have

$$\begin{aligned}
\left|\int_{\varepsilon-T}^{\varepsilon+T} f(s)\frac{x^s}{s}\mathrm{d}s\right| &= \left|\int_{-T}^{T} f(\varepsilon+it)\frac{x^{\varepsilon+it}}{\varepsilon+it}i\mathrm{d}t\right| \\
&\leq \int_{-T}^{T} \|f\|_\infty \frac{x^\varepsilon}{\sqrt{\varepsilon^2+t^2}}\mathrm{d}t \\
&= 2x^\varepsilon\|f\|_\infty \int_0^{T/\varepsilon} \frac{\mathrm{d}u}{\sqrt{u^2+1}} \\
&\leq 2x^\varepsilon\|f\|_\infty \left(1 + \int_1^{T/\varepsilon} \frac{\mathrm{d}u}{u}\right) \\
&\leq 4x^\varepsilon\|f\|_\infty \log\left(\tfrac{T}{\varepsilon}\right).
\end{aligned}$$

Now, we take $\epsilon = 1/\log x$ so that $x^\varepsilon = e$, and we finally obtain (changing $C$ if necessary):

$$|A(x)| \leq C\|f\|_\infty \log(x^3\log x) \leq C\|f\|_\infty \log x \leq C'\|f\|_\infty \log N \quad (**)$$

which ends the proof of Theorem 5.1.4. Indeed, if $g(s) = f(s+s_0)$ where $s_0 \in \mathbb{C}_0$, we have from (**) that

$$\left|\sum_{n=1}^{N} a_n n^{-s_0}\right| \leq C\log N\|g\|_\infty \leq C\log N\|f\|_\infty.$$

This implies that $\|S_N\|_\infty \leq C\log N\|f\|_\infty$. $\square$

### 5.1.2 Some applications of the control of partial sums

As a first consequence of Theorem 5.1.4, we have a simple proof of the following Theorem of Bohr in [**?**], see [65, Theorem 6.2.3].

**Theorem 5.1.5.** (Bohr's theorem)
*Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$. Then, the associate Dirichlet series converges uniformly in each half-plane $\mathbb{C}_\varepsilon$, $\varepsilon > 0$. In other terms, we have $\sigma_u(f) \leq 0$.*

*Proof.* We must show that $\sum_{n=1}^{\infty} a_n n^{-s-\varepsilon}$ converges uniformly in $\mathbb{C}_0$ for each $\varepsilon > 0$. To that effect, we set $S_n(s) = \sum_{j=1}^{n} a_j j^{-s}$, $S_0(s) = 0$ and perform and Abel's transformation:

$$\sum_{n=1}^{N} a_n n^{-s-\varepsilon} = \sum_{n=1}^{N} [S_n(s) - S_{n-1}(s)] n^{-\varepsilon} = \sum_{n=1}^{N-1} S_n(s)[n^{-\varepsilon} - (n+1)^{-\varepsilon}] + S_N(s) N^{-\varepsilon}.$$

Now, the series in the RHS is normally convergent in $\mathbb{C}_0$, since its general term is dominated by $C_\varepsilon (\log n)/(n^{\varepsilon+1}) \|f\|_\infty$ by Theorem 5.1.4, while the remaining term $S_N(s)N^{-\varepsilon}$ is dominated in $\mathbb{C}_0$ by $C \log N / N^\varepsilon \|f\|_\infty$, which ends the proof. $\square$

An interesting aspect of that theorem is that, in the end, the Dirichlet series attached to a function $f \in \mathcal{H}^\infty$ converges to that function where it is defined, namely in $\mathbb{C}_0$!

**Remark 5.1.6.** Bohr [**?**] actually proved his theorem on uniform convergence for more general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \text{ with } \lambda_{n+1} - \lambda_n \geq \delta e^{-A\lambda_n}$$

where $\delta$ and $A$ are positive constants. This condition holds with $A = 1$ for $\lambda_n = \log n$ (ordinary Dirichlet series) but fails to hold if for example $\lambda_n = (\log n)^\alpha$ with $0 < \alpha < 1$. This general version can be proved essentially in the same way as Theorem 5.1.4, using the elementary Perron-Landau formulas of Chapter 4 with $x = (\lambda_N + \lambda_{N+1})/2$ and $y = e^{x-\lambda_n}$, to get control of $A^*(x) = \sum_{\lambda_n \leq x} a_n$.

As a second consequence of Theorem 5.1.4, we can now extend Bohr's inequality from Dirichlet polynomials to an arbitrary element of $\mathcal{H}^\infty$ see [65, Theorem 6.2.4]

**Theorem 5.1.7.** (Bohr's inequality)
*Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$. Then we have*

$$\sum_p |a_p| \leq \|f\|_\infty, \tag{5.6}$$

*where the some is on prime number.*

*Proof.* For $g \in \mathcal{H}^\infty$ and $\varepsilon > 0$, set $(T_\varepsilon g)(s) = g(s + \varepsilon)$. We have $T_\varepsilon g \in \mathcal{H}_\infty$ and $\|T_\varepsilon g\|_\infty \leq \|g\|_\infty$. Now, by Theorem 5.1.5, there is an integer $N_0 = N_0(\varepsilon)$ such that

$$\|T_\varepsilon S_N\|_\infty \leq \|T_\varepsilon f\|_\infty + \varepsilon \leq \|f\|_\infty + \varepsilon, \ \forall N \geq N_0.$$

The Bohr inequality for the Dirichlet polynomial $T_\varepsilon S_N$ therefore gives

$$\sum_{p\leq N}|a_p|p^\varepsilon \leq \|T_\varepsilon S_N\|_\infty \leq \|f\|_\infty + \varepsilon.$$

Let $N\to\infty$ to get $\sum_p |a_p|p^\varepsilon \leq \|f\|_\infty + \varepsilon$. Let now $\varepsilon \to 0$ and use Theorem 1.9.8 to get (5.6). □

The following theorem is [65, Theorem 6.2.5].

**Theorem 5.1.8.** *(Carlson's identity)*
*Let $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}^\infty$ and $\varepsilon > 0$. Then, we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |f(\varepsilon+it)|^2 \mathrm{d}t = \sum_{n=1}^\infty |a_n|^2 n^{-2\varepsilon} \leq \|f\|_\infty^2. \tag{5.7}$$

*In particular, we have the contractive inclusion $\mathcal{H}^\infty \subset \mathcal{H}_2$, namely*

$$f\in\mathcal{H}^\infty \Rightarrow f\in\mathcal{H}^2 \textit{ and } \|f\|_2 \leq \|f\|_\infty.$$

*Proof.* It suffices to notice that the series $\sum_{n=1}^\infty a_n n^{-\varepsilon} n^{-it}$ converges uniformly on $\mathbb{R}$ by Theorem 5.1.5, and that for a series of the type $f(t) = \sum_{n=1}^\infty b_n e^{-i\lambda_n t}$, uniformly convergent on $\mathbb{R}$, we always have (here with $\lambda_n = \log n$)

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |f(t)|^2 \mathrm{d}t = \sum_{n=1}^\infty |b_n|^2.$$

This can be checked by hand thanks to uniform convergence, or can be viewed as the Parseval formula for $f$ in the Bohr compactification $\overline{\mathbb{R}}$ of $\mathbb{R}$. In any case, we have (5.7) and we get the second assertion by letting $\varepsilon$ tend to $0$. □

## 5.2 The Hilbert space $\mathcal{H}^2$

In this section we recall the definition of the Hilbert space $\mathcal{H}^2$ and we give some basic properties.

### 5.2.1 Definition

In the introduction, we saw that a companion space to $\mathcal{H}^\infty$ is the space $\mathcal{H}^2$ of Dirichlet series with square-summable coefficients:

$$f\in\mathcal{H}^2 \Leftrightarrow f(s) = \sum_{n=1}^\infty a_n n^{-s} \text{ with } \|f\|_2^2 = \sum_{n=1}^\infty |a_n|^2.$$

It is easy to see that $\mathcal{H}^2$ is a Hilbert space with the scalar product

$$\langle f, g\rangle = \sum_{n=1}^\infty a_n \overline{b_n}$$

and that moreover it is a Hilbert space of analytic functions on the half-plane $\mathbb{C}_{1/2}$ (whereas $\mathcal{H}^\infty$ is a space of analytic functions on the half-plane $\mathbb{C}_0$!), with the natural orthonormal basis $e_n$, $e_n(s) = n^{-s}, n \geq 1$, so that its reproducing kernel $K_a, a \in \mathbb{C}_{1/2}$ is given by

$$K_a(s) = \sum_{n=1}^{\infty} e_n(s)\overline{e_n(a)} = \zeta(s+\bar{a}) \text{ and } f(a) = \langle f, K_a\rangle \ \textit{for all } f \in \mathcal{H}^2.$$

We shall see later that $\mathcal{H}^\infty$ is exactly the space of multipliers of $\mathcal{H}^2$. Now, the answer to the initial question of this chapter is the following criterion [**?**], which combines the use of $\mathcal{H}^2$ and $\mathcal{H}^\infty$ see [65, Theorem 6.4.1] or [40].

**Theorem 5.2.1.** *Let $(\varphi_n)$ be the system of dilated functions given by (**??**) and $S_\varphi \in \mathcal{H}^2$ be the associated generating function, namely: $S_\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then the following are equivalent:*

1. *The dilated system $(\varphi_n)$ is a Riesz basis of $H$.*
2. *Both $S_\varphi$ and $1/S_\varphi$ belong to $\mathcal{H}^\infty$.*

An immediate, but useful, property is:

**Proposition 5.2.2.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2$. Then, $\sigma_a(f) \leq 1/2$ and the result is optimal in general.*

*Proof.* Just apply the Cauchy-Schwarz inequality: If $\Re e\ s = \sigma > 1/2$, then

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{-2\sigma}\right)^{1/2} < \infty.$$

The function $f(s) = \sum_{n=2}^{\infty} n^{-s}/\sqrt{n}\log n$ is in $\mathcal{H}^2$ and $\sigma_a(f) = 1/2$. □

Here is a more refined result, due to Hedenmalm and Saksman [41]:

**Theorem 5.2.3.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2$. Then, the series*

$$\sum_{n=1}^{\infty} a_n n^{-1/2+it}$$

*converges for almost every real $t$.*

*Proof.* We indicate (see [55]) a simpler proof than the original one, based on Carleson's convergence theorem for integrals: let $f : [1, \infty[\to \mathbb{C}$ be defined by

$$f(x) = a_n \text{ if } n \leq x < n+1,\ n = 1, 2, \ldots.$$

Clearly, $\int_1^\infty |f(x)|^2 \mathrm{d}x = \sum_{n=1}^\infty |a_n|^2$. Now, the change of variable $x = e^y$ shows that $g : \mathbb{R}^+ \to \mathbb{C}$ defined by $g(y) = f(e^y)e^{y/2}$ is in $L^2(\mathbb{R}^+)$ with $\int_0^\infty |g(y)|^2 \mathrm{d}y = \int_1^\infty |f(x)|^2 \mathrm{d}x$. Using Carleson's theorem for integrals ([12]), we get that $\lim_{A\to\infty} \int_0^A g(y)e^{ity}\mathrm{d}y = \int_0^\infty g(y)e^{ity}\mathrm{d}y$ exists for almost all $t$, equivalently that

$$\lim_{A\to\infty} \int_1^A \frac{f(x)}{\sqrt{x}} e^{it\log x}\mathrm{d}x \text{ exists for almost all } t. \tag{5.8}$$

Now,

$$\begin{aligned} \int_1^{N+1} \frac{f(x)}{\sqrt{x}} e^{it\log x}\mathrm{d}x &= \sum_{n=1}^N \int_n^{n+1} a_n \frac{e^{it\log x}}{\sqrt{x}}\mathrm{d}x \\ = \sum_{n=1}^N \int_n^{n+1} a_n &\left( \frac{e^{it\log x}}{\sqrt{x}} - \frac{e^{it\log x}}{\sqrt{n}} \right) \mathrm{d}x + \sum_{n=1}^N a_n n^{-\frac{1}{2}+it} \\ &= \sum_{n=1}^N b_n(t) + \sum_{n=1}^N a_n n^{-\frac{1}{2}+it} \end{aligned}$$

with $b_n(t) = O(n^{-3/2})$, since $a_n \to 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left( \frac{e^{it\log x}}{\sqrt{x}} \right) = O(x^{-3/2})$. The integral $\int_1^\infty \frac{f(x)}{\sqrt{x}} e^{it\log x}\mathrm{d}x$ and the series $\sum_{n=1}^\infty a_n n^{-\frac{1}{2}+it}$ are thus simultaneously convergent and the result follows from (5.8). □

## 5.3 The embedding theorem

### 5.3.1 Relation between $\mathcal{H}^2$ and $H^2(\mathbb{C}_{\frac{1}{2}})$

The usual Hardy-Hilbert space of the half-plane $\mathbb{C}_{1/2}$, $H^2(\mathbb{C}_{1/2})$, is defined as the space of functions $f$ analytic in the half-plane $\mathbb{C}_{1/2}$ such that

$$\|f\|^2_{H^2(\mathbb{C}_{1/2})} = \sup_{\sigma>1/2} \int_{-\infty}^{\infty} |f(\sigma+it)|^2 \mathrm{d}t < \infty.$$

Those functions have almost everywhere radial limits $f(1/2+it)$ and the norm can be calculated by computing the $L^2$-norm. This space is more flexible than $\mathcal{H}^2$, and familiar to analysts, therefore the comparison of the two spaces, when available, sheds some light on $\mathcal{H}^2$. In particular, the following theorem [59] turns out to be especially useful:

**Theorem 5.3.1.** *(Olsen-Seip)*
*Let $F \in \mathcal{H}^2$. Then, the function $f$ defined by $f(s) = \frac{F(s)}{s}$ is in $\mathcal{H}^2(\mathbb{C}_{1/2})$, and moreover*

$$\|f\|^2_{H^2(\mathbb{C}_{1/2})} \lesssim \|F\|_{\mathcal{H}^2}.$$

*Proof.* Without loss of generality, we can assume that $F(s) = \sum a_n n^{-s}$ is a Dirichlet polynomial. Then, using in a crucial way the [65, embedding theorem of Chapter 1] with $a_n/\sqrt{n}$ instead of $a_n$, we get:

$$\begin{array}{rcl} \|f\|^2_{H^2(\mathbb{C}_{1/2})} & = & \int_{-\infty}^{\infty} \left|F\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{1}{4}+t^2\right)^{-1} \mathrm{d}t \\ & = & \sum_{k\in\mathbb{Z}} \int_k^{k+1} \left|F\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{1}{4}+t^2\right)^{-1} \mathrm{d}t \\ & \lesssim & \sum_{k\in\mathbb{Z}} \int_k^{k+1} \left|F\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{1}{4}+k^2\right)^{-1} \mathrm{d}t \\ & \lesssim & \sum_{k\in\mathbb{Z}} (1+k^2)^{-1} \sum_n |a_n|^2 \ll \sum_n |a_n|^2. \end{array}$$

So

$$\|f\|^2_{H^2(\mathbb{C}_{1/2})} \lesssim \|F\|_{\mathcal{H}^2}.$$

□

**Remark** **5.3.2.** *The embedding theorem and some variants play an important role [32] in the characterization of bounded composition operators on $\mathcal{H}^2$ see [65, Theorem 6.4.5].*
*Let $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ be an analytic function so that the map $f \mapsto f \circ \varphi = C_\varphi(f)$ (a composition operator) sends $\mathcal{H}^2$ to $\mathcal{H}^2$. Then:*

**Theorem** **5.3.3.** *(Gordon-Hedenmalm)*
*The function $\varphi$ determines a bounded composition operator $C_\varphi$ on $\mathcal{H}^2$ if and only if*

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} = c_0 s + \psi(s)$$

*where $c_0$ is a non-negative integer and $\psi$ is a Dirichlet series that converges in $\mathbb{C}_\theta$ for some $\theta > 0$ and has the following mapping properties:*

1. *If $c_0 \geq 1$, then either $\psi = 0$ or $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$.*
2. *If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.*

Further work on composition operators on spaces of Dirichlet series can be found in [28, 27, 7, 66].

### 5.3.2 Multipliers of $\mathcal{H}^2$

The Banach space $\mathcal{H}^2$ is not an algebra (i.e. is not stable under products), but nearly! Indeed, we have the following property, in which $d(n)$ denotes the number of divisors of the integer $n$:

**Theorem** **5.3.4.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}^2$ and $h(s) = \sum_{n=1}^{\infty} c_n n^{-s}$, where $h = fg$ and $c = a * b$ (see (4.3)). Then, we have*

$$\sum_{n=1}^{\infty} \frac{|c_n|^2}{d(n)} \leq \|f\|^2_{\mathcal{H}^2} \|g\|^2_{\mathcal{H}^2}. \tag{5.9}$$

*In particular, $\sigma_a(fg) \leq 1/2$.*

*Proof.* Since $c_n = \sum_{ij=n} a_i b_j$. Cauchy-Schawrz gives

$$|c_n|^2 \leq \left(\sum_{ij=n} 1\right)\left(\sum_{ij=n} |a_i|^2|b_j|^2\right) = d(n)\left(\sum_{ij=n} |a_i|^2|b_j|^2\right).$$

Therefore

$$\sum_{n=1}^{\infty} \frac{|c_n|^2}{d(n)} \leq \sum_{n=1}^{\infty}\left(\sum_{ij=n} |a_i|^2|b_j|^2\right) = \left(\sum_{i=1} |a_i|^2\right)\left(\sum_{i=1} |b_i|^2\right) = \|f\|_{\mathcal{H}^2}^2 \|g\|_{\mathcal{H}^2}^2.$$

Let $\sigma > 1/2$, and $\varepsilon > 0$ such that $\varepsilon < 2\sigma - 1$. As it is well-known [76, page 84]: $d(n) \leq C_\epsilon n^\epsilon$. Cauchy-Schawrz once more gives

$$\sum_{n=1}^{\infty} |c_n| n^{-\sigma} \leq \left(\sum_{n=1}^{\infty} \frac{|c_n|^2}{d(n)}\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{-2\sigma} d(n)\right)^{1/2} < \infty.$$

□

To see how far $\mathcal{H}^2$ is from being an algebra, we have the following basic theorem, which gives a link between both spaces $\mathcal{H}^2$ and $\mathcal{H}^\infty$ see [**?**] or [65, theorem 6.4.7].

**Theorem 5.3.5.** *(Hedenmalm-Lindqvist-Seip)*
*The space of multipliers of $\mathcal{H}^2$ is exactly the space $\mathcal{H}^\infty$. In other words, a function $f$ defined on $\mathcal{H}^2$ verifies $fg \in \mathcal{H}^2$ for all $g \in \mathcal{H}^2$ if and only if $f \in \mathcal{H}^\infty$. Moreover*

$$\|f\|_\infty = \sup_{g \in \mathcal{H}^2, \|g\|_2 \leq 1} \|fg\|_2. \tag{5.10}$$

## 5.4 The Banach spaces $\mathcal{H}^p$

We extend the basics results given in sections before to the general Banach space of Dirichlet series

### 5.4.1 A basic identity

Let $p \geq 1$. We have already defined $\mathcal{H}^2$ and $\mathcal{H}^\infty$. We wish to define more generally a space $\mathbb{H}^p$ on the unit disk. To that effect, we recall the Birkhoff-Oxtoby theorem, see [65, Theorem 6.5.1], in which $\mu$ denotes the Haar measure of $\mathbb{T}^\infty$ and $e$ the vector $(1, 1, \ldots, 1, \ldots)$.

**Definition 5.4.1.** *The Kronecker flow $(K_t)_{\geq 0}$ on $\mathbb{T}^\infty$ associated with a sequence $(x_1, \ldots, x_d, \ldots)$ of independent real numbers is the family of the mappings $K_t : \mathbb{T}^\infty \to \mathbb{T}^\infty$ defined for each $z = (z_j)_{j\geq 1} \in \mathbb{T}^\infty$ by*

$$K_t(z) = (e^{ix_1 t} z_1, e^{ix_2 t} z_2, ..., e^{ix_d t} z_d, ...).$$

**Theorem 5.4.2.** *(Birkhoff-Oxtoby Theorem for the Kronecker flow)*
*Let us denote by $(K_t)$ the Kronecker flow on $\mathbb{T}^\infty$ associated with a sequence $(x_1, \dots, x_d, \dots)$ of independent real numbers, and let $f : \mathbb{T}^\infty \to \mathbb{C}$ be a continuous function. Then we have, as $T \to \infty$:*

$$\frac{1}{2T}\int_{-T}^{T} f(K_t e)\mathrm{d}t \to \int_{\mathbb{T}^\infty} f(z)\mathrm{d}\mu(z); \tag{5.11}$$

*and*

$$\frac{1}{T}\int_{0}^{T} f(K_t e)\mathrm{d}t \to \int_{\mathbb{T}^\infty} f(z)\mathrm{d}\mu(z). \tag{5.12}$$

*Now, denote by $\mathcal{P}$ the set of Dirichlet polynomials $P(s) = \sum_{n=1}^{M} c_n n^{-s}$ (here, $n^{-s}$ appears essentially as a free variable). If $P \in \mathcal{P}$, let $\Delta P(z) = \sum \gamma_\alpha z^\alpha$ the associated algebraic poynomial, following the Bohr point of view, where we recall that*

$$z^\alpha = z_1^{\alpha_1} \dots z_r^{\alpha_r} \text{ and } \gamma_\alpha = c_n \text{ if } n = p_1^{\alpha_1} \dots p_r^{\alpha_r}.$$

*With these notations, we have the following result.*

**Theorem 5.4.3.** *For each $P \in \mathcal{P}$ and each $1 \leq p < \infty$, one has*

$$\|P\|_p = \lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} |P(it)|^p \mathrm{d}t\right]^{\frac{1}{p}} = \|\Delta P\|_{L^p(\mathbb{T}^\infty)}.$$

*In particular, $P \mapsto \|P\|_p$ is a norm on $\mathcal{P}$.*

*Proof.* Let $x_j = \log p_j$ where $p_j$ is the $j$-th prime number, and let $(k_T)$ be the Kronecker flow associated with this independent sequence. Let also $f(z) = |\Delta P(z)|^p$, which is continuous and clearly verifies $f(K_t e) = |P(it)|^p$. Applying the Birkhoff-Oxtoby Theorem above to $f$, we, at once, obtain Theorem 5.4.3, since moreover

$$\|P\|_p = 0 \Rightarrow \|\Delta P\|_{L^p(\mathbb{T}^\infty)} = 0 \Rightarrow \Delta P = 0 \Rightarrow P = 0.$$

□

### 5.4.2 Definition of Hardy-Dirichlet spaces

**Definition 5.4.4.** *For $0 < p < \infty$,the Dirichlet space $\mathcal{H}^p, 0 < p < \infty$, will be defined as the completion of Dirichlet polynomials for the norm $\|\|_p$ above. This concretely means the following: a formal Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is in $\mathcal{H}^p$ if there exists a Cauchy sequence (for the $\|\|_p$-norm) $P_j(s) = \sum_{n=1}^{M} a_n^{(j)} n^{-s}$ of Dirichlet polynomials such that*

$$\lim_{j\to\infty} a_n^{(j)} = a_n \text{ for all } n = 1, 2, \dots$$

With this definition, the following first two items are easy, since

$$\|\Delta P\|_{L^p(\mathbb{T}^\infty)} \leq \|\Delta P\|_{L^q(\mathbb{T}^\infty)} \text{ for } p \leq q.$$

**Proposition 5.4.5.** *Let $1 \leq p \leq q \leq \infty$. Then:*

1. *The space $\mathcal{H}^p$ is a Banach space, isometric to $H^p(\mathbb{T}^\infty)$.*
2. *For all $f \in \mathcal{H}^q$ we have $\|f\|_p \leq \|f\|_q$, then $\mathcal{H}^q \subset \mathcal{H}^p$.*
3. *The definition of $\mathcal{H}^p$ coincides with the previous one for $p = 2$.*

*Proof.* 1. Recall that, by definition,

$$H^p(\mathbb{T}^\infty) = \{f \in L^p(\mathbb{T}^\infty); \hat{f}(\alpha) = 0 \text{ if } \alpha \notin \mathbb{N}^{(\infty)}\}.$$

The set of all elements $\alpha \in \mathbb{Z}^{(\infty)} = \widehat{\mathbb{T}^\infty}$ with all coordinates non-negative, i.e. the set $\mathbb{N}^{(\infty)}$, is called *the narrow cone* by Helson. The rest is an easy consequence of Theorem 5.4.3 and of a density argument.

2. This is clear from the identity of Theorem 5.4.3.

3. This is clear, since the "new" space $\mathcal{H}^2$ is just the completion of the prehilbertian space of finitely supported sequences of complex numbers.

□

A nice property of the Banach space $\mathcal{H}^p$ was recently established by Aleman, Olsen, and Saksman [2] is the following

**Theorem 5.4.6.** *(Aleman-Olsen-Saksman)*
*Let $1 < p < \infty$. Then, the sequence $(n^{-s})$ is a Schauder basis of $\mathcal{H}^p$. In other terms, if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^p$, the partial sums of that function converges to it in $\mathcal{H}^p$.*

That theorem, and the previous proposition, say nothing of a half-plane where the Dirichlet series of some element $\sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^p$ would converge pointsize. Since the biggest space is $\mathcal{H}^1$, a nice theorem, due to Helson [42], will give a complete answer, and is a significant extension of Theorem 6.4.6. But the proof requires preliminaries of harmonic analysis which we begin by describing.

### 5.4.3 A detour through harmonic analysis

We begin with a version of Hardy's inequality, which expresses that the canonical injection of the Hardy space $H^1$ into the Bergman space $B^2$. The usual inequality in Hardy space is as follows

$$\text{If } f(z) = \sum_{n=0}^{\infty} c_n z^n \in H^1, \text{ then } \left(\sum_{n=0}^{\infty} \frac{|c_n|}{n+1}\right) \leq \pi\|f\|_1.$$

The next inequality, independently due to Helson ([42]) and Vukotic ([81]), is in some respects worse, since we get rid of the constant $\pi > 1$. It will *lend itself to iterations*, which would not be the case of Hardy's inequality.

**Lemma 5.4.7.** *Let $f(s) = \sum_{n=0}^{\infty} c_n z^n \in H^1$. Then, we have the contractive inequality*

$$\left(\sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}\right)^{\frac{1}{2}} \leq \|f\|_1. \tag{5.13}$$

*Proof.* Let $f(s) = \sum_{n=0}^{\infty} c_n z^n$. We can assume that $\|f\|_1 = 1$. We use the classical and usual fact according to which $f$ can be written as

$$f = gh \text{ with } \|g\|_2 = \|h\|_2 = 1, \text{ with } c_n = \sum_{k=0}^{n} a_k b_{n-k} \text{ where}$$

$f(s) = \sum_{n=0}^{\infty} a_n z^n$ and $h(s) = \sum_{n=0}^{\infty} b_n z^n$ and

$$\sum_{k=0}^{\infty} |a_k|^2 = \sum_{k=0}^{\infty} |b_k|^2 = 1.$$

We next use Cauchy-Schwarz inequality *without separating* the $a_k$ and the $b_{n-k}$. We get

$$\frac{|c_n|^2}{n+1} \leq \sum_{k=0}^{n} |a_k|^2 |b_{n-k}|^2.$$

Summing with respect to $n$ and using a Cauchy product in the right-hand-side, we get (5.13). Since $r \leq 1/\sqrt{2} \Rightarrow r^{2n} \leq 1/(n+1)\ \forall n \geq 0$, (5.13) proves in passing the contractivity of the Poisson kernel $P_r : H^1 \to H^2$ for $r \leq 1/\sqrt{2}$, a result first proved in ([83]) in the general case of $P_r : H^p \to H^q$ (contractivity holds iff $r \leq \sqrt{p/q}$). This notion of "hypercontractivity" had been introduced by A. Bonami in her thesis see [16], for other semi-groups. $\square$

As we already mentioned, this lemma can be iterated using a clever induction and a reverse Hölder inequality to give [42] the following Lemma, see [65, Lemma 6.5.6]

**Lemma 5.4.8.** *(Helson)*
*Let $f(s) = \sum_{\alpha \in \mathbb{N}^r}^{\infty} c_\alpha z^\alpha \in H^1(\mathbb{T}^r), r \geq 1$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, set $w(\alpha) = (1+\alpha_1)\cdots(1+\alpha_r)$. Then, we have*

$$\left(\sum_{\alpha \in \mathbb{N}^r} \frac{|c_\alpha|^2}{w(\alpha)}\right)^{\frac{1}{2}} \leq \|f\|_1. \tag{5.14}$$

Two nice applications of Lemma 5.4.8 are a "dimension-free" inequality and the Lemma of Helson previously mentioned:

**Proposition 5.4.9.** *Let $\mathcal{P}(r,d)$ the set of homogeneous polynomials of degree $d$ in $r$ variables, namely the set of*

$$P(z)=\sum_{\alpha} c_\alpha z^\alpha \text{ with } \alpha=(\alpha_1,\dots,\alpha_r) \text{ and } |\alpha|=\alpha_1+\cdots+\alpha_r=d.$$

*If $P\in\mathcal{P}(r,d)$, it verifies*

$$\|P\|_2 \le 2^{d/2}\|P\|_1. \tag{5.15}$$

*Proof.* For $\alpha=(\alpha_1,\dots,\alpha_r)$ and $|\alpha|=d$, note that

$$w(\alpha)=(1+\alpha_1)\cdots(1+\alpha_r)\le 2^{\alpha_1}\cdots 2^{\alpha_r}=2^d.$$

We then have from Lemma 5.4.8:

$$\|P\|_2=\left(\sum|c_\alpha|^2\right)^{1/2}\le\left(2^d\sum\frac{|c_\alpha|^2}{w(\alpha)}\right)^{1/2}\le 2^{d/2}\|P\|_1.$$

□

**Theorem 5.4.10.** *(Helson)*
*Let $f(s)=\sum_{n=1}^{\infty}c_n n^{-s}\in\mathcal{H}^1$. Then, we have*

$$\left(\sum_{n=1}^{\infty}\frac{|c_n|^2}{d(n)}\right)^{1/2}\le\|f\|_1$$

*where $d(n)$ denotes the number of divisors of $n$. In particular, $\sigma_a(f)\le 1/2$.*

*Proof.* Let $f(s)=\sum_{n=1}^{M}c_n n^{-s}$ be a Dirichlet polynomial, and let $\Delta f(z)=\sum_\alpha\gamma_\alpha z^\alpha$ be the associated algebraic polynomial (Bohr's point of view). If $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, we have $d(n)=(1+\alpha_1)\cdots(1+\alpha_r)=w(\alpha)$.Therefore, using Lemma 5.4.8 and Theorem 5.4.3, we get

$$\left(\sum_n\frac{|c_n|^2}{d(n)}\right)^{1/2}=\left(\sum\frac{|\gamma_\alpha|^2}{w(\alpha)}\right)^{1/2}\le\|\Delta f\|_1=\|f\|_1.$$

and by density, of Dirichlet polynomials in $H^2$, the inequality holds for any $f\in\mathcal{H}^1$. It implies that $\sigma_a(f)\le 1/2$ since $d(n)\le C_\epsilon n^\epsilon$ for each $\varepsilon>0$ and it remain to follows the same way as in the proof of Theorem 5.3.4. □

## 5.5 Composition operator and multipliers

In this section we establish our results stated at the introduction of part II.
Let recall these two theorems due to Cuckovic-Zhao [23].

**Theorem 5.5.1.** *Let $u$ be an analytic function on $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Let $0 < p \leq q < \infty$ then the weighted composition operator $uC_\varphi : f \longrightarrow u(f \circ \varphi)$ is bounded from $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$ if and only if $\underset{a\in\mathbb{D}}{Sup} I_{\varphi,-1}(u)(a) < \infty$ where $I_{\varphi,-1}(u)(a) = \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{\frac{q}{p}} |u(w)|^q d\sigma(w)$.*

**Theorem 5.5.2.** *Let $u$ be an analytic function on $\mathbb{D}$ and $\varphi$ be an analytic self-map of $\mathbb{D}$. Let $1 \leq q < p < \infty$ then the weighted composition operator $uC_\varphi : f \longrightarrow u(f \circ \varphi)$ is bounded from $H^p(\mathbb{D})$ into $H^q(\mathbb{D})$ if and only if $\int_0^{2\pi} \left(\int_{G(\theta)} \frac{d\mu_u(w)}{1-|w|^2}\right)^{\frac{p}{p-q}} d\theta < \infty$, where $\mu_u = \nu_u \circ \varphi^{-1}$ and $d\nu_u(z) = |u(z)|^q d\sigma(z)$ with $d\sigma(z)$ the normalized measure of $\partial\mathbb{D}$, and $G(\theta)$ is the Stolz angle at $\theta$, which is defined for real $\theta$ as the convex hull of the set $\{e^{i\theta}\} \bigcup \{z : |z| < \frac{\sqrt{2}}{2}\}$.*

This our first result.

**Theorem 5.5.3.** *Let $u$ be an analytic function on $\mathbb{D}$. Let $\varphi$ be a Dirichlet series. Suppose $\phi(s) = c_0 s + \varphi(s)$ is holomorphic from $\mathbb{C}_+$ to $\mathbb{C}_+$ for $c_0 \in \mathbb{N}^*$. Let $1 < p \leq q < \infty$, Suppose $\varphi_{\eta,\zeta,\sigma} := \psi_\eta \circ \phi_\sigma \circ \psi_\zeta^{-1}$ ($\eta = (\sigma + c_0)\zeta$) satisfies*

$$\underset{\sigma>0}{Sup} \lim_{\zeta\to\infty} \underset{a\in\mathbb{D}}{Sup} I_{\varphi_{\eta,\zeta,\sigma},-1}(u)(a) < \infty \quad for\ all\ \ \sigma, \eta, \zeta \in \mathbb{R}^{+*}\ and\ \eta = (\sigma + c_0)\zeta, \tag{5.16}$$

*where*

$$I_{\varphi_{\eta,\zeta,\sigma},-1}(u)(a) = \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi_{\eta,\zeta,\sigma}(w)|^2}\right)^{\frac{q}{p}} |u(w)|^q d\sigma(w)$$

*and*

$$\psi_\beta(s) = \frac{s-\beta}{s+\beta}$$

*the Mobius transformation from $\mathbb{C}_+$ to $\mathbb{D}$. Then $uC_\phi : f \longrightarrow u(f \circ \phi)$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$.*

*Proof.* Let $0 < \zeta < \infty$ large enough. Let $H_i^p(_+, \zeta)$ the image of the Hardy space $H^p(\mathbb{D})$ by the transformation $\psi_\zeta$ which preserves the norm. This space is independent of $\zeta$. more precisely

$$\|f\|^p_{H_i^p(_+,\zeta)} = \|f \circ \psi_\zeta^{-1}\|^p_{H^p(\mathbb{D})} = \int_{-\pi}^{\pi} |f \circ \psi_\zeta^{-1}(e^{it})|^p dt = \int_{\mathbb{R}} |f(it)|^p d\lambda_\zeta(t), \tag{5.17}$$

where $d\lambda_\zeta(t) = \dfrac{\zeta}{\pi(t^2+\zeta^2)} dt$ and for the last equality we made the change of variable $e^{it} = \psi_\zeta(ix)$. Let $c_0 \in \mathbb{N}^*$. Then $\phi$ is an holomorphic self-map of $\mathbb{C}_+$ so that $\varphi_{\eta,\zeta} = \psi_\eta \circ \phi \circ \psi_\zeta^{-1}$ is an holomorphic self-map of $\mathbb{D}$. Assume $1 < p \leq q < \infty$. Set $C_{\eta,\zeta} = \underset{a\in\mathbb{D}}{Sup} I_{\varphi_{\eta,\zeta},-1}(u)(a) < \infty \quad for\ all\ \ 0 < \eta, \zeta < \infty$. Then by Theorem 5.5.1

$$\|uF \circ \varphi_{\eta,\zeta}\|_{H^q} \leq C_{\eta,\zeta} \|F\|_{H^p} \quad for\ all F \in H^p(\mathbb{D}).$$

Hence for

$$F = f \circ \varphi_\eta^{-1}\ with\ f \in H_i^p(_+, \eta),$$

we have

$$\|uf\circ\phi\|_{H_i^q(+,\zeta)} \le C_{\eta,\zeta}\|f\|_{H_i^p(+,\zeta)} \quad \forall f\in H_i^p(+,\eta),$$

hence using (5.17), we have

$$\|uf\circ\phi\|_{H_i^q(+,\zeta)} \le C_{\eta,\zeta}\|f\|_{H_i^p(+,\eta)} \quad for\ all f\in H_i^p(+,\eta). \tag{5.18}$$

Let $f(s)=\sum_{n=1}^{N} a_n n^{-s}$ be a Dirichlet polynomial, for $N\in\mathbb{N}$. We have that $f$ is bounded on $\mathbb{C}_+$, therefore $f\in H_i^p(+,\zeta)(0<\zeta<\infty)$. One knows that $n^{-\phi}$, $f\circ\phi$ are Dirichlet series see [64]. Since $\phi$ is by assumption a self-map of $\mathbb{C}_+$, then $f\circ\phi$ is a bounded holomorphic function on $\mathbb{C}_+$ so that $f\circ\phi\in\mathcal{H}^\infty$ and so $f\circ\phi\in\mathcal{H}^q$. Let $g\in\mathcal{H}^\infty$, for $0<\sigma<\infty$ by Theorem 5.4.3, we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(\sigma+it)|^q dt=\|g_\sigma\|_{\mathcal{H}^q}\ \ where\ \ g_\sigma(s)=g(s+\sigma), \tag{5.19}$$

and

$$Sup_{\sigma>0}\|g_\sigma\|_{\mathcal{H}^q}=\|g\|_{\mathcal{H}^q}\ \ see\ \ [65].$$

By equation (5.19) one has

$$\|g_\sigma\|^q_{H_i^q(+,\zeta)}=\int_{\mathbb{R}}|g(\sigma+it)|^q d\lambda_\zeta(t)\to\|g_\sigma\|^q_{\mathcal{H}^q},\ \ \zeta\to\infty. \tag{5.20}$$

Replacing $\phi$ by $\phi_\sigma$ in equation (5.18) one have

$$\|uf\circ\phi_\sigma\|_{H_i^q(+,\zeta)}\le C_{\eta,\zeta,\sigma}\|f\|_{H_i^p(+,\eta)}, \tag{5.21}$$

where $C_{\eta,\zeta,\sigma}=Sup_{a\in\mathbb{D}}I_{\varphi_{\eta,\zeta,\sigma},-1}(u)(a)$ and $\varphi_{\eta,\zeta,\sigma}=\psi_\eta\circ\phi_\sigma\circ\psi_\zeta^{-1}$. Since $\eta=(\sigma+c_0)\zeta$ and $\zeta\to\infty$, then by equation (5.20) and equation (5.21) we have

$$\|uf\circ\phi_\sigma\|_{\mathcal{H}^q}\le\lim_{\zeta\to\infty}C_{\eta,\zeta,\sigma}\|f\|_{\mathcal{H}^p}, \tag{5.22}$$

and then

$$\|uf\circ\phi\|_{\mathcal{H}^q}\le Sup_{\sigma>0}\lim_{\zeta\to\infty}C_{\eta,\zeta,\sigma}\|f\|_{\mathcal{H}^p}.$$

The result then holds since

$$\underset{\sigma>0}{Sup}\lim_{\zeta\to\infty}C_{\eta,\zeta,\sigma}<\infty.$$

□

It is knew that $m:f\longrightarrow mf$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^p$ if and only if $m\in\mathcal{H}^\infty$ This is due to Bailleul in his thesis. Such et $m$ is call a multiplier.

**Theorem 5.5.4.** *Let $\mathcal{M}_{p,q}$ be the set of all $m:f\longrightarrow mf$ such that $m$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$. Then* *(a)* $\mathcal{M}_{p,q} \subsetneqq \mathcal{H}^\infty$ *if* $p < q$.

*(b)* $\mathcal{H}^\infty \subset \mathcal{M}_{p,q}$ *if* $q < p$.

*Proof.* (a) Let $m \in \mathcal{M}_{p,q}$. As $1 \in \mathcal{H}^p$, One has $m \in \mathcal{H}^q$. Because $p < q$, $\mathcal{H}^q \subsetneqq \mathcal{H}^p$ see Proposition 5.4.5, so that $mf \in \mathcal{H}^p$ and $m$ is a multiplier on $\mathcal{H}^p$, then $m \in \mathcal{H}^\infty$. Note that be cause $\mathcal{H}^q \subsetneqq \mathcal{H}^p$, $1 \notin \mathcal{M}_{p,q}$. But $1 \in \mathcal{H}^\infty$, and then $\mathcal{M}_{p,q} \subsetneqq \mathcal{H}^\infty$ if $p < q$.
(b) For $q < p$, $\mathcal{H}^p \subsetneqq \mathcal{H}^q$. Let $m \in \mathcal{H}^\infty$. Then $m$ is a multiplier on $\mathcal{H}^q$. So that, for all $f \in \mathcal{H}^p$, because $\mathcal{H}^p \subset \mathcal{H}^q$, $mf \in \mathcal{H}^q$ and $m \in \mathcal{M}_{p,q}$. □

**Theorem 5.5.5.** *For* $q < p$, *if* $m \in \mathcal{H}^{\frac{pq}{q-p}}$ *then* $m \in \mathcal{M}_{p,q}$.

*Proof.* Let $r = \frac{p}{q}$. Choose $t$ such that
$$\frac{q}{p} + \frac{1}{t} = 1.$$
By Holder's inequality, one has
$$\int_{\mathbb{T}^\infty} |\Delta f(z)|^q |\Delta m(z)|^q d\rho(z) \leq \left(\int_{\mathbb{T}^\infty} |\Delta f(z)|^p d\rho(z)\right)^{\frac{q}{p}} \left(\int_{\mathbb{T}^\infty} |\Delta m(z)|^{qt} d\rho(z)\right)^{\frac{1}{t}}.$$
Then we will have
$$\int_{\mathbb{T}^\infty} |\Delta f(z)|^q |\Delta m(z)|^q d\rho(z) \leq \|f\|^q_{\mathcal{H}^p}$$
if
$$\left(\int_{\mathbb{T}^\infty} |\Delta m(z)|^{qt} d\rho(z)\right)^{\frac{1}{t}} < \infty.$$
Note that $qt = \frac{pq}{p-q}$, then if $m \in \mathcal{H}^{\frac{pq}{q-p}}$ then $m \in \mathcal{M}_{p,q}$.

□

Now let's give some criterion. Observe that $m \in \mathcal{M}_{p,q}$ if and only if there is a constant $C$ such that $\|mf\|_{\mathcal{H}^q} \leq C\|f\|_{\mathcal{H}^p}$ for all $f \in \mathcal{H}^p$, this is
$$\left(\int_{\mathbb{T}^\infty} |\Delta f(z)|^q |\Delta m(z)|^q d\rho(z)\right)^{\frac{1}{q}} \leq C\|f\|_{\mathcal{H}^p}, \quad for\ all f \in \mathcal{H}^p$$
this means that $d\mu(z) = |\Delta m(z)|^q d\rho(z)$ is a ($\mathcal{H}^p$,q)-Carleson measure. So the following holds

**Theorem 5.5.6.** $m \in \mathcal{M}_{p,q}$ *if and only if* $d\mu(z) = |\Delta m(z)|^q d\rho(z)$ *is a (*$\mathcal{H}^p$*,q)-Carleson measure.*

We also obtain the following criterion for the space $\mathcal{M}_{p,q}$ for $1 < p \leq q < \infty$.

**Theorem 5.5.7.** *For* $1 < p \leq q < \infty$, *we have that* $m \in \mathcal{M}_{p,q}$ *if and only if*
$$\int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}w|^2}\right)^{\frac{q}{p}} |m_\sigma \circ \psi_\zeta^{-1}(w)|^q d\sigma(w) < \infty \quad for\ all\ 0 < \sigma, \zeta < \infty$$

*Proof.* Since $\|mf\|_{\mathcal{H}^q} \leq C\|f\|_{\mathcal{H}^p}$ means that

$$\underset{\sigma>0}{Sup}\|(mf)_\sigma\|_{\mathcal{H}^q} \leq C\underset{\sigma>0}{Sup}\|f_\sigma\|_{\mathcal{H}^p},$$

this is equivalent to

$$\lim_{\zeta\to\infty}\|(mf)_\sigma\|_{\mathcal{H}^q} \leq C\lim_{\zeta\to\infty}\|f_\sigma\|_{\mathcal{H}^p} \quad for\ all\ 0<\sigma<\infty.$$

we then use Theorem 5.5.1 to have

$$\int_{\partial\mathbb{D}}\left(\frac{1-|a|^2}{|1-\overline{a}w|^2}\right)^{\frac{q}{p}}|m_\sigma\circ\psi_\zeta^{-1}(w)|^q d\sigma(w)<\infty \quad for\ all\ 0<\sigma,\zeta<\infty.$$

□

# ♣ Conclusion ♣

The aim of this thesis was twofold, namely, in partI, characterize the weights for which we have the boundedness of standard weighted integral operators induced by the Bergman-Besov kernels (see Definition 2.2.11) acting between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$ in terms of Békollè - Bonami type condition on the weights, employing the proof strategy originated by Békollè [11], and, in part II, to study the boundedness of weighted composition operator $uC_\phi : f \longrightarrow u(f \circ \phi)$ from $\mathcal{H}^p$ to $\mathcal{H}^q$, where $\mathcal{H}^r$ designed the Hardy-Dirichlet space (see Section 5.4.2).

In part I, for $a, b, s, t \in \mathbb{R}$ the operators that we were interested in were defined by (reproducing) Bergman-Besov kernels (see Definition 2.2.11). For $f \in L^p(d\mu_q)$ we define

$$\begin{aligned}
T^q_{a,b} f(z) &:= T_{a,b} f(z) = \int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w), \\
S^q_{a,b} f(z) &:= S_{a,b} f(z) = \int_{\mathbb{B}} |K_a(z,w)||f(w)|(1-|w|^2)^{b-q} d\mu_q(w), \\
P^q_{s,t} f(z) &:= P_{s,t} f(z) = (1-|z|^2)^t \int_{\mathbb{B}} K_{s+t}(z,w) f(w)(1-|w|^2)^{s-q} d\mu_q(w).
\end{aligned}$$

We first made the following observation

$$P_{s,t} f(z) = (1-|z|^2)^t T_{s+t,s} f(z) \tag{5.23}$$

and

$$T_{a,b} f(z) = (1-|z|^2)^{b-a} P_{b,a-b} f(z), \tag{5.24}$$

Which allows us to work with the desired one an get the result for the others. The results we obtained depend upon the values of the parameters $a, bs, t, q, Q$: see Theorem 0.1.2, Theorem 0.1.5 and Theorem 0.1.6, Theorem 0.1.8, Theorem 0.1.10, Theorem 0.1.14, Theorem 0.1.13 and Corollary 0.1.15. In Theorem 3.4.5 and in Theorem 3.4.6 we show that being in $(K^{s,t,q,Q}_p)$ is a necessary condition for the continuity of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, while in Corollary 3.6.2, we have that being in $D^{s,t,q,Q}_p$ is a sufficient one. When $Q = q$, we find out that $(K^{s,t,q,q}_p) = (D^{s,t,q,q}_p)$, so that we have a necessary and sufficient condition. But when $Q > q$ we have $(D^{s,t,q,Q}_p) \subseteq (K^{s,t,q,Q}_p)$. It is an open question to find in this case a necessary and sufficient condition for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.

In part II, our interest here was to characterize multiplier i.e such a $m : f \longrightarrow mf$ which are bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$ and to characterize composition operators,that is functions $\phi$ so that $f \longrightarrow f \circ \phi$ is bounded from $\mathcal{H}^p$ to $\mathcal{H}^q$. More generally, to study the boundedness of weighted composition operator $uC_\phi : f \longrightarrow u(f \circ \phi)$ from $\mathcal{H}^p$ to $\mathcal{H}^q$. Our result Here, depend of the parameters $p, q$: see Theorem 0.2.1, Theorem 0.2.2, Theorem 0.2.3, Theorem 0.2.4, Theorem 0.2.5. In Theorem 0.2.1 and in Theorem 0.2.5, it is an open question to find if they give a complete characterisation.

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# ♣ Index ♣

# Index

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# Weighted estimates for operators associated to the Bergman-Besov kernels

## Estimations pondérées pour les opérateurs associés aux noyaux de Bergman-Besov

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**ABSTRACT.** We characterize the weights for which we have the boundedness of standard weighted integral operators induced by the Bergman-Besov kernels acting between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$ in terms of Békollè - Bonami type condition on the weights. To accomplish this we employ the proof strategy originated by Békollè.
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**KEYWORDS.** Bergman-Besov space, weighted inequalities, Bergman-Besov projection

### 1. Introduction

Weighted inequalities appeared almost simultaneously with the birth of singular integrals that stimulated their development. In particular, a natural question is the characterization of positive functions $\omega$ for which a singular integral maps $L^p(\omega d\mu)$ to itself. A famous example of a singular integral is the Bergman projection, whose boundedness problem, solved elsewhere by Békollè and Bonami, is historically linked to the duality problem for Bergman spaces.

The inner product and the norm in $\mathbb{C}^N$ are $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_N\overline{w_N}$ and $|z| = \sqrt{\langle z, z\rangle}$. We let $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ where $q > -1$ and $\mu$ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$, and $c_q$ is the normalized constant, that is $\mu_q(\mathbb{B}) = 1$. Take $(c_q = \frac{\Gamma(n+q+1)}{N!\Gamma(q+1)})$. When $N = 1$, $\mathbb{B}$ is the unit disc $\mathbb{D}$. For $a > -1$, it is a well-known result of Békollè and Bonami that the Bergman projection $T_a$ defined by

$$T_a f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1-\langle z, x\rangle)^{N+1+a}} d\mu_a(x)$$

is bounded on $L^p(\omega d\mu_a)$ if and only if the weight $\omega$ belongs to the so-called Békollè - Bonami class [3]. The Bergman projection can be extended to all $a$ less than or equal to $-1$. Therefore a natural question is whether the Békollè - Bonami result can be generalized. In this paper we work with more general operators than the extended Bergman projection, and more generally we characterize weights for which we have the boundedness between two weighted Lebesgue classes on the unit ball of $\mathbb{C}^N$.

We set $L^p_q := L^p(d\mu_q)$, the Lebesgue space on $\mathbb{B}$ relative to $\mu_q$ with $1 \leq p \leq +\infty$. Let $H(\mathbb{B})$ denote the space of holomorphic functions in the unit ball $\mathbb{B}$. For $q > -1$, a function $f \in H(\mathbb{B})$ belongs to the weighted Bergman space $A^p_q$ whenever $f \in L^p(d\mu_q)$. The norm $\|f\|_{A^p_q}$ is simply the $L^p_q$ norm of $f$.

 

Besov spaces extend weighted Bergman spaces to all $q$. To define them, we first take a radial differential operator $D_s^t$ of order $t$ for any $s, t \in \mathbb{R}$ defined on $H(\mathbb{B})$. Let $f \in H(\mathbb{B})$ be given by its convergent homogeneous expansion $f = \sum_{k=0}^{\infty} f_k$ in which $f_k$ is a homogeneous polynomial in $z_1, \ldots, z_N$ of degree $k$. We define, for $s, t \in \mathbb{R}$

$$D_s^t f := \sum_{k=0}^{\infty} d_k(s,t) f_k = \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k$$

where

$$c_k(a) = \begin{cases} \frac{(N+1+a)_k}{k!} & \text{if} \quad a > -(N+1), \\ \frac{k!}{(1-N-a)_k} & \text{if} \quad a \leq -(N+1). \end{cases}$$

Consider the linear transformation $I_s^t$ defined for $f \in H(\mathbb{B})$ by

$$I_s^t f(z) = (1-|z|^2)^t D_s^t f(z).$$

We say that a function $f \in H(\mathbb{B})$ belongs to the Besov space $B_q^p$ whenever $I_s^t f \in L_q^p$ for some $s, t$ satisfying

$$\begin{cases} q + pt > -1 & \text{if } 1 \leq p < \infty \\ \quad t > 0 & \text{if } p = \infty. \end{cases}$$

It is well known [9] that the $L_q^p$-norm, $\|I_s^t f\|_{L_q^p}$, of any one of the functions $I_s^t f$ is an equivalent norm for $\|f\|_{B_q^p}$, the norm of $f$ in $B_q^p$. When $q > -1$ we have $A_q^p = B_q^p$. The space $B_q^2$ is a Hilbert space with reproducing kernel $K_q$ (see [9] or [2, Theorem 1.9] or [16]) defined by

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} & = & \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!} \langle z,w\rangle^k, & \text{if } q > -(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) & = & \displaystyle\sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k} \langle z,w\rangle^k, & \text{if } q \leq -(N+1), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(u)_v$ is the Pochhammer symbol defined by $(u)_v = \frac{\Gamma(u+v)}{\Gamma(u)}$ with $\Gamma$ the gamma function. Namely, for a number $s$ satisfying $q+1 < p(s+1)$, if $t$ satisfies $q + pt > -1$ then for $f \in B_q^2$ (see [9, Theorem 1.2])

$$(P_s \circ I_s^t) f = \frac{N!}{(1+s+t)_N} f,$$

where

$$P_s f(z) = \int_{\mathbb{B}} K_s(z,w) f(w) (1-|w|^2)^s d\mu(w),$$

is the extended Bergman projection ($s$ may be smaller than or equal to $-1$).

For $a, b, s, t \in \mathbb{R}$ the operators that we are interested in are defined by (reproducing) Bergman-Besov kernels. For $f \in L^p(d\mu_q)$ we define

$$T_{a,b}^q f(z) \ := \ T_{a,b} f(z) = \int_{\mathbb{B}} K_a(z,w) f(w) (1-|w|^2)^{b-q} d\mu_q(w),$$

 

$$S^q_{a,b}f(z) \ := \ S_{a,b}f(z) = \int\limits_{\mathbb{B}} |K_a(z,w)||f(w)|(1-|w|^2)^{b-q}d\mu_q(w),$$

$$P^q_{s,t}f(z) \ := \ P_{s,t}f(z) = (1-|z|^2)^t \int\limits_{\mathbb{B}} K_{s+t}(z,w)f(w)(1-|w|^2)^{s-q}d\mu_q(w).$$

Throughout the paper $b > -1$ and $s > -1$ because we want our operator to be well defined (see for example Lemma 5.1). Note that

$$P_{s,t}f(z) = (1-|z|^2)^t T_{s+t,s}f(z) \tag{1.1}$$

and

$$T_{a,b}f(z) = (1-|z|^2)^{b-a}P_{b,a-b}f(z). \tag{1.2}$$

Our main motivation comes from the operators $P_{s,0}$, and $P^+_{s,N+1+s}$, which are the Bergman projection and Berezin transform respectively, where $P^+_{s,N+1+s}f(z) = (1-|z|^2)^t S_{s+N+1+s,s}f(z)$. The operators $P_{s,t}$, $T_{a,b}$ and $S_{a,b}$ are important in the study of function-theoretic operator theory, see for example [17] when $q = -N-1$.

The boundedness of the operators $T^q_{a,b}$ was already studied by Kaptanoglu and Ureyen [10] in the cases where the operators $T^q_{a,b}$ act from $L^p_q$ to $L^P_Q$, with $q \in \mathbb{R}, 1 \le p, P \le \infty, Q > -1$.

**Theorem 1.1.** *[10, Theorem 1.2] Let $a, b, q, Q \in \mathbb{R}$, $1 \le p \le P \le \infty$, and assume $Q > -1$ when $P < \infty$. Then, the following three conditions are equivalent.*

1. $T_{a,b} : L^p_q \to L^P_Q$;
2. $S_{a,b} : L^p_q \to L^P_Q$;
3. (a) $\frac{1+q}{p} < 1 + b$ *and* $a \le b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \le P < \infty$;

   (b) $\frac{1+q}{p} \le 1 + b$ *and* $a \le b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 = p \le P \le \infty$, *but at least one inequality must be strict;*

   (c) $\frac{1+q}{p} < 1 + b$ *and* $a < b + \frac{1+N+Q}{P} - \frac{1+N+q}{p}$ *for* $1 < p \le P = \infty$.

This result is useful for our work, especially for the case $p = P$ and $q = Q$, to investigate the case where these operators $P_{s,t}$ and $T_{a,b}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$. Here the weak Lebesgue space, $L^{1,\infty}_q$, is the space of measurable functions $f$ for which there exists $A > 0$ such that for all $\lambda > 0$, $\lambda\mu_q\{z \in \mathbb{B} : |f(z)| > \lambda\} \le A$. Our main result in this direction is the following.

**Theorem 1.2.** *In the case $q = s$, $s + 2t > -1$ and $s + t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$ and not from $L^1_q$ to $L^1_q$.*

In this paper we are mainly interested on the weighted estimates for the operators $T_{a,b}$ or $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Here and throughout the paper $\omega$ is a locally integrable positive function called a weight. It follows from (1.1) and (1.2) that it will be enough to study weighted estimates for one of these two operators since the $(L^p(\omega d\mu_q), L^p(\omega d\mu_Q))$ inequality for the family $T_{a,b}$ is equivalent to the $(L^p(\omega d\mu_q), L^p(\omega d\mu_{Q+pt}))$ inequality for the family $P_{s,t}$ and conversely. In the special case of the Bergman projection $T_{a,0}$, Békollè [3] obtained the characterisation of the weights $\omega$ in terms of the Békollè -Bonami condition.

Let $d$ be the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z| - |w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle\right| & z, w \in \overline{\mathbb{B}}^* \\ |z| + |w| & z = 0 \, or \, w = 0. \end{cases}$$

**Definition 1.3** (Békollè - Bonami class)**.** Let $a > -1$. Let $\omega$ be a weight on $\mathbb{B}$. We say that $\omega d\mu_a$ belongs to $(B_p^a)$, $1 < p < \infty$, if there is a constant $B_p^a(\omega)$ such that for every ball B (with respect to the pseudo-distance $d$) of $\mathbb{B}$ that intersects the boundary of $\mathbb{B}$, we have

$$\left( \frac{1}{\mu_a(B)} \int_B \omega(z) d\mu_a(z) \right) \left( \frac{1}{\mu_a(B)} \int_B \omega^{\frac{-1}{p-1}}(z) d\mu_a(z) \right)^{p-1} \leq B_p^a(\omega).$$

For $a > -1$, let

$$T_a f(z) := T_{a,0} f(z) := \int_{\mathbb{B}} \frac{f(x)}{(1 - \langle z, x \rangle)^{N+1+a}} d\mu_a(x)$$

be the Bergman projection. Békollè showed in [3] that

**Theorem 1.4.** *Let $\omega$ be a weight on $\mathbb{B}$. The operator $T_a$, $a > -1$, is well defined and continuous on $L^p(\omega d\mu_a)$, $1 < p < \infty$, if and only if $\omega d\mu_a \in (B_p^a)$.*

The results we obtain depend upon the values of $s + t$, $q$ and $Q$. In the case $s + t < -(N + 1)$ we have the following two main results.

**Theorem 1.5.** *In the case $s + t < -(N + 1)$, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

**Theorem 1.6.** *Let $\omega$ be a weight on $\mathbb{B}$. In the case $s + t < -(N + 1)$, if $Q > q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left( \int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

*Moreover*

$$\|P_{s,t}\|^p \simeq \left( \int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1}.$$

In order to give our necessary condition for the boundedness of $T_{a,b}$ when $a > -(1 + N)$ we introduce a Békollè -Bonami type class of weights denoted by $(B_p^{a,b,q,Q})$.

 

**Definition 1.7.** Let $\omega$ be a weight on $\mathbb{B}$. For $Q \leq q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{\mu_b(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z) \right) \left( \frac{\mu_b(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$.

For $Q \leq q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z) \right) \left( \frac{\mu_b(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.

For $Q > q$ and $a > -1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B \omega(z) d\mu_Q(z) \right) \left( \frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$.

For $Q > q$ and $a > -N-1$, we say that $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B \omega(z) d\mu_Q(z) \right) \left( \frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(b-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.

A necessary condition for the boundedness of $P_{s,t}$ when $-1 > s+t > -(1+N)$ is the following.

**Theorem 1.8.** *In the case both $-(N+1) < s+t < -1$ and $s+t+\frac{Q-q}{p} \leq -1$ hold, and in the case both $s+t > -1$ and $Q < q$ hold, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

For the remaining cases, we introduce another Békollè -Bonami type class of weights, $(K_p^{s,t,q,Q})$, in order to give our necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$.

**Definition 1.9.** Let $\omega$ be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (K_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.

For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (K_p^{s,t,q,S})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.

A necessary condition for the boundedness of $P_{s,t}$ when $s+t > -(1+N)$ is the following.

**Theorem 1.10.** *In the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, and in the case both $s+t > -1$ and $Q \geq q$ hold, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega \in (K_p^{s,t,q,Q})$.*

We introduce a maximal and a fractional maximal operator that will be used to establish a good lambda inequality in order to give sufficient conditions for the boundedness of $P_{s,t}$.

If $a > -1$ we set

$$m_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w),$$

more generally if $a > -1-N$ we set

$$m'_{a,b}f(z) = \sup_{\zeta\in\mathbb{B}, R>1-|\zeta|: z\in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w).$$

Before giving our good lambda inequality, we introduce here $(D_p^{s,t,q,Q})$ a Békollè-Bonami type class of weights.

**Definition 1.11.** Let $\omega$ be a weight on $\mathbb{B}$. For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$, we say that $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{1}{R_B^{N+1+s+t+\frac{Q-q}{p}}} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.

For $Q \geq q$ and $s+t > -1$, we say that $\omega \in (D_p^{s,t,q,Q})$ $(s > -1)$ if

$$\sup_{B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

where the supremum is taken over the pseudo-balls $B$ with radius $R_B$. In each case we denote by $D_p^{s,t,q,Q}(\omega)$ the expression in the left hand side.

**Remark 1.12.** Constants and standard weights ($\omega(z) = (1-|z|^2)^\eta$) are in $(D_p^{s,t,q,Q})$. We also have $(D_p^{s,t,q,q}) \subseteq (D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. For $Q = q$ we have $(K_p^{s,t,q,Q}) = (D_p^{s,t,q,Q})$.

 

Here is our good lambda inequality.

**Theorem 1.13.** *Suppose that* $1 < p < \infty$. *Let* $\omega \in (D_p^{s,t,q,Q})$ *where both* $s+t+\frac{Q-q}{p} > -1$ *and* $-1 > s+t > -N-1$ *hold, or both* $s+t > -1$ *and* $Q \geq q$ *hold. There are two positive constants* $C$ *and* $\beta$ *such that for all* $\gamma$ *sufficiently small,* $\lambda > 0$ *and for all positive locally integrable functions* $f$, *if* $-N-1 < s+t$ *and* $s+t+\frac{Q-q}{p} > -1$, *then*

$$\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \leq \gamma\lambda\}) \leq \\ CD_p^{s,t,q,Q}(\omega)\gamma^{\beta}\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (1.3)$$

To show that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ when $s+t > -1$, we introduce the following maximal and fractional maximal operator. If $s+t > -1$ we set

$$O_{s,t}f(z) = (1-|z|^2)^t m_{s+t,s}f(z);$$

more generally if $s+t > -1-N$ we set

$$O'_{s,t}f(z) = (1-|z|^2)^t m'_{s+t,s}f(z).$$

The following theorem shows together with the good lambda inequality that $(D_p^{s,t,q,Q})$ is sufficient for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when $-N-1 < s+t$ and $s+t+\frac{Q-q}{p} > -1$.

**Theorem 1.14.** *For* $-N-1 < s+t$ *and* $s+t+\frac{Q-q}{p} > -1$, *if* $\omega d\mu_q \in (D_p^{s,t,q,Q})$, *there is a constant* $C_{s,t,p,q,Q} > 0$ *such that for all* $f \in L^p(\omega d\mu_q)$,

$$\int_{\mathbb{B}} (O_{s,t}f(z))^p \omega(z) d\mu_q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Then for the case $s+t > -1$ and $q = Q$ we have

**Corollary 1.15.** *Let* $\omega$ *be a weight on* $\mathbb{B}$. *Then for* $s+t > -1, s > -1$ *the following assertions are equivalent.*

1. $P_{s,t}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_q)$;
2. $T_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$;
3. $S_{s+t,s}$ *is well defined and continuous from* $L^p(\omega d\mu_q)$ *to* $L^p(\omega d\mu_{q+pt})$;
4. $\omega \in (K_p^{s,t,q,Q})$.

Rahm, the third and the fourth authors settled in [11] the particular case of the operators $P_{s,t}$ for $s+t > -1, s > -1, Q = q = s$. To this aim, they used dyadic methods that have been initiated by Aleman, Pott and Reguera in the unit disk [1]. It might be interesting to see if the dyadic methods used in those papers extend to the situation studied in this paper.

The outline of the paper is as follows. In Section 2 we briefly give requisite background information. We prove Theorem 1.2 in Section 3. From Section 4 on, we look at weighted estimates; there we show Theorem 1.5 and Theorem 1.6. The proof of Theorem 1.8, Theorem 1.10, Theorem 1.14 are in Section 5. The proof of Theorem 1.13 is in Section 6. Corollary 1.15 appears in Section 7.

 

## 2. Main tools

### 2.1. *Complex Analysis Tools*

Throughout this paper $d$ is the pseudo-distance in $\overline{\mathbb{B}}$ defined by

$$d(z,w) = \begin{cases} ||z|-|w|| + \left|1 - \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle\right| & z,w \in \overline{\mathbb{B}}^* \\ |z|+|w| & z = 0 \, or \, w = 0. \end{cases}$$

Throughout this paper $K$ will be a constant such that

$$d(x,y) \leq K(d(x,z) + d(z,y)) \tag{2.1}$$

for all $x$, $y$ and $z$ in $\mathbb{B}$. We will consider pseudo-balls in $\mathbb{B}$, $B(z,r)$, as points $w$ of $\mathbb{B}$ such that $d(z,w) < r$ and we say that $B(z,r)$ touches the boundary of $\mathbb{B}$ if $r > 1-|z|$. When $B(z,r)$ is such that $r > k(1-|z|)$ for some absolute value $k < 1$, we say that $B(z,r)$ almost touches the boundary of $\mathbb{B}$.

One can find the following two results in [3, 15].

**Lemma 2.1.** *For each $z \in \mathbb{B}$ and $r_0$, $0 < r_0 < 1$, if we set $z^0 = (r_0, 0, \cdots, 0)$, then we have*

1. $|1 - z_1 r_0| \geq \frac{1}{4} d(z, z^0)$;
2. $|z_1 - r_0| \leq d(z, z^0)$;
3. $|z - z^0| \leq d(z, z^0)$;
4. $\displaystyle\sum_{k=2}^{N} |z_k|^2 \leq 2d(z, z^0)$.

**Proposition 2.2.** *There is a constant $C_1 > 0$ so that for all $z, w, w_0 \in \mathbb{B}$ with $d(z,w_0) > C_1 d(w,w_0)$ we have*

$$|\langle z, w_0 \rangle - \langle z, w \rangle| \leq \frac{1}{2}|1 - \langle z, w_0 \rangle|.$$

*Then*

$$|1 - \langle z, w \rangle| \geq \frac{1}{2}|1 - \langle z, w_0 \rangle|.$$

We recall that $d\mu_q(z) = c_q(1-|z|^2)^q d\mu(z)$ where $q > -1$ and $\mu$ be the Lebesgue (volume) measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N : |z| < 1\}$ of $\mathbb{C}^N = \mathbb{R}^{2N}$, and $c_q$ is the normalized constant, that is $\mu_q(\mathbb{B}) = 1$. When $q \leq -1$, we simply write $d\mu_q(z) = (1-|z|^2)^q d\mu(z)$. We then obtain the following result that will be heavily used throughout the paper. This extends to all $q$ the result given in [3].

**Lemma 2.3.** *For each $w \in \mathbb{B}$, $0 < |w| = r < 1$ and $0 < R < 2$*

$$\mu_q(B(w,R)) \simeq R^{N+1}[\max(R, 1-r)]^q \; if \; q > -1.$$

*Then for $q > -1$, $(\mathbb{B}, d, \mu_q)$ is a homogeneous space in the sense of [5].*

*However, if $B(w,R)$ is away from the boundary ($R < \frac{1-|w|}{2}$), the equivalence remains true if $q \leq -1$, ie*

$$\mu_q(B(w,R)) \simeq R^{N+1}(1-|w|)^q.$$

 

*Proof.* Assume $q \leq -1$. Since $z \in B(w,R)$, we have $|w|-|z|<R$. Hence $1-|z|<1-|w|+R$ and $1-|w|<1-|z|+R$. Since $R<\frac{1-|w|}{2}$, we have $1-|w|<2(1-|z|)$ so that

$$\frac{1}{2}(1-|w|) \leq 1-|z| \leq 1-|w|+R \leq \frac{3}{2}(1-|w|). \tag{2.2}$$

Hence, for all $R \in (0, \frac{1-|w|}{2})$, we have

$$\int_{B(w,R)} (1-|z|)^q d\mu(z) \simeq R^{N+1}(1-|w|)^q.$$

□

We recall the following well-known estimates in [12].

**Proposition 2.4.** *Let*

$$I = \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{1+N+c}} d\mu(w)$$

*for $d>-1$ and $c \in \mathbb{R}$. We have, when $|z| \to 1^-$,*

*(i)* $I \sim 1$ *if* $c<d$;

*(ii)* $I \sim \frac{1}{|z|^2}\log\frac{1}{1-|z|^2}$ *if* $c=d$;

*(iii)* $I \sim (1-|z|^2)^{-(c-d)}$ *if* $c>d$.

The following results can be found in [9] and [10] respectively.

**Theorem 2.5.** *Let $q \in \mathbb{R}$. Let $t,s \in \mathbb{R}$ such that $q+2t>-1$. Equipped with the following equivalent scalar product*

$${}_q\langle f,g\rangle_s^t = \int_{\mathbb{B}} I_s^t f(z)\overline{I_s^t g(z)} d\mu_q(z),$$

*$B_q^2$ is a Hilbert space with reproducing kernel given by*

$$K_q(z,w) = \begin{cases} \frac{1}{(1-\langle z,w\rangle)^{N+1+q}} &= \displaystyle\sum_{k=0}^{\infty} \frac{(N+1+q)_k}{k!}\langle z,w\rangle^k, \ \text{ if } \ q>-(N+1) \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) &= \displaystyle\sum_{k=0}^{\infty} \frac{k!}{(1-N-q)_k}\langle z,w\rangle^k, \ \text{ if } \ q \leq -(N+1). \end{cases}$$

**Lemma 2.6.**

1. *For $q<-(N+1)$, each $|K_q(z,w)|$ is bounded above as $z,w$ vary in $\mathbb{B}$.*
2. *For each $q \in \mathbb{R}$,*
   (a) *$|K_q(z,w)|$ is bounded below by a positive constant as $z,w$ vary in $\mathbb{B}$. In particular, $K_q(z,w)$ is zero free in $\mathbb{B}\times\mathbb{B}$.*
   (b) *there is a $\rho_0<1$ such that for $|z| \leq \rho_0$ and all $w \in \mathbb{B}$, we have $\Re K_q(z,w) \geq \frac{1}{2}$.*

## 2.2. Harmonic Analysis Tools

The following result can be found in [5] and will be helpful in the proof of Theorem 3.1.

**Theorem 2.7** ([5]). *Let $(X, d, \mu)$ be a homogeneous space and let $K(x, y)$ be a function such that $K(x, .) : y \to K(x, y) \in L^2(X)$. Suppose the operator T defined by*

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y),$$

*satisfies the following two conditions*

1. *there is a constant $C_1$ so that $\|Tf\|_2 \leq C_1\|f\|_2$;*
2. *there are two constants $C_2$ and $C_3$ so that for all $y, y_0$ we have*

$$\int_{d(x,y_0)>C_2 d(y,y_0)} |K(x, y) - K(x, y_0)|d\mu(x) < C_3 \ \text{(Hörmander Condition)}.$$

*Then for all p, $1 \leq p \leq 2$, there is a constant $A_p$ depending only on $C_i, i = 1, 2, 3$, so that for all $f \in L^2 \bigcap L^p$ we have $\|Tf\|_p \leq A_p\|f\|_p$ if $p > 1$, and $\forall\lambda > 0$*

$$\mu(\{x \in X : |Tf(x)| > \lambda\}) \leq A_1 \frac{\|f\|_1}{\lambda}.$$

One can find the following result in [7].

**Theorem 2.8** (Marcinkiewicz Interpolation Theorem). *Let $p_0, p_1$ be so that $1 \leq p_0 < p_1 \leq \infty$. Let $T$ be a sublinear operator defined from $L^{p_0} + L^{p_1}$ to the space of measurable functions. Assume that $T$ is simultaneously of weak type $(p_0, p_0)$ with operator norm $A_{p_0,p_0}$ and of weak type $(p_1, p_1)$ with operator norm $A_{p_1,p_1}$. Then for every $0 < t < 1$, $T$ is of (strong) type $(p_t, p_t)$ where*

$$\frac{1}{p_t} = \frac{t}{p_0} + \frac{1-t}{p_1}.$$

*Moreover, if $p_1 < \infty$, then $\|Tf\|_{p_t} \leq A_{p_t,p_t}\|f\|_{p_t}$ with*

$$A_{p_t,p_t} = 2\left[p_t(\frac{A^{p_0}_{p_0,p_0}}{p_t - p_0} - \frac{A^{p_1}_{p_1,p_1}}{p_1 - p_t})\right]^{\frac{1}{p_t}}.$$

*If $p_1 = \infty$, we can take*

$$A_{p_t,p_t} = 2\left[p_t\frac{A^{p_0}_{p_0,p_0}}{p_t - p_0}\right]^{\frac{1}{p_t}}.$$

The fractional maximal function is defined as follows

$$M_\gamma f(z) = \sup_{B:z\in B} \frac{1}{\nu^{1-\gamma}(B)} \int_B |f(w)|d\nu(w), \ \ \gamma \in [0, 1).$$

When $\gamma = 0$ it is the Hardy-Littlewood maximal operator. The following result will be used in Section 5 in the study of our maximal and fractional maximal function. One can find their proof in [6] or in [13].

**Theorem 2.9.** *Let $X$ be a homogeneous space, $0 \leq \gamma < 1$, $1 < p \leq r < \infty$ and a pair of weights $(u, v)$, then the following are equivalent.*

*(i) There exists a constant $C_1 > 0$ so that*

$$\left( \int_X [M_\gamma f(x)]^r v(x) d\nu(x) \right)^{\frac{1}{r}} \leq C_1 \left( \int_X |f(x)|^p u(x) d\nu(x) \right)^{\frac{1}{p}}$$

*for any $f \in L^p(X, ud\nu)$;*

*(ii) There exists a constant $C_2 > 0$ such that*

$$\left( \int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x) \right)^{\frac{1}{r}} \leq C_2 \left( \int_B u^{1-p'}(x) d\nu(x) \right)^{\frac{1}{p}}$$

*for any ball $B \subset X$.*

We will also make use of the following class, in Section 5, in the study of our maximal and fractional maximal function and to establish the good lambda inequality.

**Definition 2.10.** A measure $\omega d\mu_\alpha$ is in the $(A_p, \alpha)$ $(1 < p < \infty)$ class if there is a constant $C_p(\omega)$ so that for all pseudo-balls $B := B(\zeta, R)$ we have

$$\left( \frac{1}{\mu_\alpha(B)} \int_B \omega(z) d\mu_\alpha(z) \right) \left( \frac{1}{\mu_\alpha(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_\alpha(z) \right)^{p-1} \leq C_p(\omega).$$

**Definition 2.11.** A measure $\omega d\mu_\alpha$is a Muckenhoupt weight or is in the $(A_\infty, \alpha)$ class if there exist $\delta, \beta$ with $0 < \delta, \beta < 1$, so that for all pseudo-balls $B$ of $\mathbb{B}$ and for all measurable subset $E$ of $B$ we have

$$\mu_\alpha(E) \leq \delta \mu_\alpha(B) \Rightarrow \omega d\mu_\alpha(E) \leq \beta \omega d\mu_\alpha(B).$$

We give now two properties of Muckenhoupt weight that we will need later (see [8]).

**Lemma 2.12.** *If $\sigma \in (A_\infty, \alpha)$ then there are two positive constants $A$ and $\beta_0$ so that for all balls $B$ and a measurable subset $E$ of $B$ we have*

$$\int_E \sigma(x) d\mu_\alpha(x) \leq A \left( \frac{d\mu_\alpha(E)}{d\mu_\alpha(B)} \right)^{\beta_0} \int_B \sigma(x) d\mu_\alpha(x).$$

**Theorem 2.13.** *The Hardy-Littlewood maximal operator is bounded on $L^p(\omega d\mu_\alpha)$, $1 < p < \infty$, if and only if $\omega d\mu_\alpha \in (A_p, \alpha)$.*

The following known lemma [14, Lemma 2-Chapter IV] will be used in Section 6.

**Lemma 2.14.** *Let $(X, \mathcal{A}, \mu)$ a measure space. Let $f$ and $g$ be two positive measurable functions so that for all $t > 0$*

$$\mu(\{x \in X : f(x) > t, g(x) \leq ct\}) \leq a\mu(\{x \in X : f(x) > bt\}),$$

*where $a, b$ and $c$ are positive constants such that $a < b^p$ $(1 < p < \infty)$. Then*

$$\|f\|_p^p \leq \frac{c^{-p}}{1 - ab^{-p}} \|g\|_p^p.$$

 

We will use the following lemma in Section 5 to show Lemma 5.1. One can find it in [3, 15].

**Lemma 2.15.** *For $a > -1 - N$, there are two constants $C_1, C_2$ $(C_1 > 0)$ so that, for $z, w, w_0 \in \mathbb{B}$ with $|1 - \langle z, w_0\rangle| > C_1 d(w, w_0)$, we have*

$$\left|\frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right| \leq C_2 \frac{d(w,w_0)}{|1-\langle z,w_0\rangle|^{N+a+2}}.$$

## 3. Weak Type $L^1$ Inequality for $P_{s,t}$ and $T_{a,b}$.

In this section, we will prove Theorem 1.2 that we first recall here.

**Theorem 3.1.** *In the case $q = s$, $s + 2t > -1$ and $s + t > -1$ with $s > -1$ the operators $P_{s,t}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$ and not from $L^1_q$ to $L^1_q$.*

*Proof.* The kernel of $P_{s,t}$ is $H_{s,t}(z,w) = \frac{(1-|z|^2)^t}{(1-\langle z,w\rangle)^{N+1+s+t}}$. We are going to proceed in three steps.

*Step 1:* Show that $H_{s,t}(z,.) \in L^2_q, \forall z \in \mathbb{B}$.

Indeed, we have

$$\int_{\mathbb{B}} |H_{s,t}(z,w)|^2 d\mu_q(w) = \int_{\mathbb{B}} \frac{(1-|z|^2)^{2t}}{|1-\langle z,w\rangle|^{2(N+1+s+t)}} d\mu_q(w)$$
$$\leq \frac{(1-|z|^2)^{2t}}{(1-|z|)^{2(N+1+s+t)}} \int_{\mathbb{B}} (1-|w|^2)^q d\mu(w)$$

where in the second inequality, the member of the right hand side is finite because $q = s > -1$.

*Step 2:* Show that $P_{s,t}$ is bounded from $L^2_q$ to $L^2_q$.

We have to show the boundedness of $T_{s+t,s}$ from $L^2_q$ to $L^2_{q+2t}$. By Kaptanoglu and Ureyen, for $a = s+t$, $b = s$, $p = P = 2$ and $Q = q + 2t$ (this is the reason $q + 2t > -1$ is needed), this holds.

*Step 3:* Show that there are two constants $C_1$ and $C_2$ so that $\forall w, w_0 \in \mathbb{B}$ we have

$$\int_{d(z,w_0)>C_1 d(w,w_0)} |H_{s,t}(z,w) - H_{s,t}(z,w_0)| d\mu_q(z) < C_2.$$

This was already done in [3] (see the proof of [3, Proposition 1] choose $a = q + t + 1$).

Because of *Step 1*, *Step 2* and *Step 3* we have by using Theorem 2.7 that the operators $P_{s,t}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$. Observe that $P_{s,t}$ is bounded from $L^1_q$ to $L^1_q$ if and only if $T_{s+t,s}$ is bounded from $L^1_q$ to $L^1_{q+t}$; and by Theorem 1.1, $T_{s+t,s}$ is not bounded from $L^1_q$ to $L^1_{q+t}$ because $q = s$. □

**Remark 3.2.** In the case $a > -(N+1)$, the operators $T^q_{a,b}$ are bounded from $L^1_q$ to $L^{1,\infty}_q$ if we have the following two conditions

i) $a \leq b$

ii) $-1 < q \leq b$.

 

The case $a = b = q > -1$ is due to Békollè in [3]. The remaining cases are obtained by using Theorem 1.1.

**Remark 3.3.** In the special case $b = q$, $T^q_{a,b}$ is self adjoint and bounded from $L^p_q$ to itself for $1 < p < \infty$. Indeed, let $f \in L^p_q$ and $g \in L^{p'}_q$. Then

$$\begin{aligned}
\langle T^q_{a,b} f, g\rangle_{L^2_q} &= \int_{\mathbb{B}}\int_{\mathbb{B}} K_a(z,w) f(w)(1-|w|^2)^{b-q} d\mu_q(w)\overline{g(z)}(1-|z|^2)^q d\mu(z)\\
&= \int_{\mathbb{B}} f(w)(1-|w|^2)^{b-q} \overline{\int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z)}(1-|w|^2)^q d\mu(w)\\
&= \int_{\mathbb{B}} f(w)\overline{T^*_{a,b} g(w) d\mu_q(w)}\\
&= \langle f, (T^q_{a,b})^* g\rangle_{L^2_q},
\end{aligned}$$

where

$$(T^q_{a,b})^* g(w) = (1-|w|^2)^{b-q}\int_{\mathbb{B}} K_a(w,z) g(z)(1-|z|^2)^q d\mu(z).$$

Observe that when $b = q$, $(T^q_{a,b})^* = T^q_{a,b}$ and since $T^q_{a,b}$ is bounded from $L^p_q$ to $L^p_q$ when $1 < p < 2$, then $T^q_{a,b} = \left(T^q_{a,b}\right)^*$ is bounded from $L^{p'}_q$ to $L^{p'}_q$ with $2 < p' < \infty$.

**Remark 3.4.** By Remark 3.2 we have that $T^q_{a,b}$ is of weak type $(1,1)$ and let $A_{1,1}$ be the operator norm. By Theorem 1.1 we have that $T^q_{a,b}$ is of strong type $(2,2)$ and let $A_{2,2}$ be the operator norm. Applying Theorem 2.8 leads us, with a better estimation of the operator norm

$$A_{p,p} = 2\left[p\left(\frac{A_{1,1}}{p-1} - \frac{A^2_{2,2}}{2-p}\right)\right]^{\frac{1}{p}},$$

to a new way to have the boundedness of $T^q_{a,b}$ from $L^p_q$ to $L^p_q$ when $1 < p < 2$. In the special case $b = q$ we have the boundedness from $L^p_q$ to $L^p_q$ when $1 < p < \infty$ because in this case $T^q_{a,b}$ is self adjoint.

## 4. Weighted estimates: Preliminary necessary conditions and the case where $s + t < -N - 1$

In this section we will give a proof of our criterion for the weights that provide boundedness of $P_{s,t}$ when $s + t < -N - 1$) (respectively $T_{a,b}$ when $a < -N - 1$). We start first with some general necessary conditions.

### 4.1. *Preliminary Necessary Conditions*

**Lemma 4.1.** *Let $\omega$ be a weight. For $q, Q \in \mathbb{R}$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega$ must be in $L^1(d\mu_{Q+pt})$.*

*Proof.* Let $f(w) = (1 - |w|^2)^{-s}\chi_{B(0,R)}(w)$ where $B(0, R)$ is the Euclidean ball. Then

$$\begin{aligned}
P_{s,t}f(w) &= (1 - |w|^2)^t \int_{\mathbb{B}} K_{s+t}(w, z)(1 - |z|^2)^{-s}\chi_{B(0,R)}(z)(1 - |z|^2)^s d\mu(z) \\
&= (1 - |w|^2)^t \int_{B(0,R)} K_{s+t}(w, z)d\mu(z) \\
&= (1 - |w|^2)^t \overline{\int_{B(0,R)} K_{s+t}(z, w)d\mu(z)} \\
&= (1 - |w|^2)^t \overline{K_{s+t}(0, w)}\mu(B(0, R)) \\
&= (1 - |w|^2)^t \mu(B(0, R)).
\end{aligned}$$

Since $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$,

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z)d\mu_Q(z) < \infty,$$

so that

$$\mu^p(B(0, R)) \int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

then

$$\int_{\mathbb{B}} \omega(z)d\mu_{Q+pt}(z) < \infty,$$

and $\omega \in L^1(d\mu_{Q+pt})$. □

**Lemma 4.2.** *Let $\omega$ be a weight and $q, Q \in \mathbb{R}$. If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, then $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p'(b-q)})$.*

*Proof.* Assume that $T_{a,b}$ is well defined, that is for any $z \in \mathbb{B}$

$$S_{a,b}f(z) = \int_{\mathbb{B}} |K_a(z, w)||f(w)|d\mu_b(w) < \infty,$$

and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. We want to show that $\omega^{\frac{-1}{p-1}} \in L^1(d\mu_{q+p(p'-1)(b-q)})$, in other words we want to show $\omega^{-1} \in L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$. Assume that $\omega^{-1}$ is not in $L^{p'}(\omega d\mu_{q+p(p'-1)(b-q)})$, then by the Riesz representation theorem there exists a positive function $h$ in $L^p(\omega d\mu_{q+p(p'-1)(b-q)})$ so that

$$\langle h, \omega^{-1}\rangle_{\omega,q+p(p'-1)(b-q)} = \infty.$$

This means that

$$\infty = \int_{\mathbb{B}} h(z)d\mu_{q+p(p'-1)(b-q)}(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{p(p'-1)(b-q)} d\mu_q(z) \\
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p(p'-1)-1](b-q)} d\mu_b(z) \\
&= \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z).
\end{aligned}
$$

Since $h \in L^p(\omega d\mu_{q+p(p'-1)(b-q)})$, then $g \in L^p(\omega d\mu_q)$, where $g(z) = h(z)(1-|z|^2)^{(p'-1)(b-q)}, \ \forall z \in \mathbb{B}$. So

$$
T_{a,b}g(0) = \int_{\mathbb{B}} h(z)(1-|z|^2)^{[p'-1](b-q)} d\mu_b(z) = \infty,
$$

contradicting the fact that $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. $\square$

**Proposition 4.3.** *In the case $s+t \leq -1$ and $Q \leq q$, or in the case $s+t+\frac{Q-q}{p} \leq -1$, for any pseudo-ball $B$ that touches the boundary, there are no weights $\omega$ so that, both conditions $\omega \in L^1(B, d\mu_{Q+pt})$ and $\omega^{\frac{-1}{p-1}} \in L^1(B, d\mu_{q+p'(s-q)})$ hold at the same time.*

*Proof.* Let $s+t \leq -1$, $Q \leq q$ and $B$ be a pseudo-ball that touches the boundary. Then

$$
\begin{aligned}
\infty &= \int_B d\mu_{s+t}(z) \\
&= \int_B (1-|z|^2)^{s+t} d\mu(z) \\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t} d\mu(z) \\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z) \\
&\leq \left( \int_B \omega(z)(1-|z|^2)^{q+pt} d\mu(z) \right)^{\frac{1}{p}} \left( \int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{\frac{1}{p'}} \\
&\leq \left( \int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z) \right)^{\frac{1}{p}} \left( \int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{\frac{1}{p'}}
\end{aligned}
$$

so that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

Let $s+t+\frac{Q-q}{p} \leq -1$, and $B$ be a pseudo-ball that touches the boundary. Then

$$
\begin{aligned}
\infty &= \int_B d\mu_{s+t+\frac{Q-q}{p}}(z) \\
&= \int_B (1-|z|^2)^{s+t+\frac{Q-q}{p}} d\mu(z)
\end{aligned}
$$

$$= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{Q}{p}+t} d\mu(z)$$
$$= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{Q+pt}{p}} \omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z)$$
$$\leq \left(\int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z)\right)^{\frac{1}{p}} \left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z)\right)^{\frac{1}{p'}}$$

so that necessarily $\omega \notin L^1(d\mu_{Q+pt})$ or $\omega^{\frac{-1}{p-1}} \notin L^1(d\mu_{q+p'(s-q)})$.

□

## 4.2. *The case where* $s+t<-N-1$

Here we are going to characterize the boundedness of the operators $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, where $\omega$ is a weight and $s+t<-N-1$.

**Theorem 4.4.** *In the case $s+t<-(N+1)$, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$.*

*Proof.* This is due to Proposition 4.3, Lemma 4.1 and Lemma 4.2. □

**Theorem 4.5.** *In the case $s+t<-(N+1)$, if $Q>q$, then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ if and only if*

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

*Moreover,*

$$\|P_{s,t}\|^p \simeq \left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

*Proof.* Assume that

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

We have

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p \omega(z) d\mu_Q(z) \lesssim \int_{\mathbb{B}} (1-|z|^2)^{pt} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v)\right)^p \omega(z) d\mu_Q(z)$$
$$= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f(v)| d\mu_s(v)\right)^p \omega(z) d\mu_{Q+pt}(z)$$

$$= \left( \int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p,$$

where the first inequality is due to the fact that $K_{s+t}$ is bounded when $s+t < -(N+1)$. We also have

$$\begin{aligned} \left( \int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p &= \left( \int_{\mathbb{B}} |f(z)| (\omega(z))^{\frac{-1}{p}} (\omega(z))^{\frac{1}{p}} (1-|z|^2)^{s-q} d\mu_q(z) \right)^p \\ &\leq \left( \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z) \right) \left( \int_{\mathbb{B}} ((\omega(z))^{\frac{-1}{p}})^{p'} |(1-|z|^2)^{p'(s-q)} d\mu_q(z) \right)^{\frac{p}{p'}}. \end{aligned}$$

Then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ when

$$\left( \int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty.$$

Now assume that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Let $\rho_0$ be as in Lemma 2.6, then for positive functions $f$ we have

$$\begin{aligned} \int_{\mathbb{B}} |P_{s,t} f(z)|^p \omega(z) d\mu_Q(z) &= \int_{\mathbb{B}} \left| \int_{\mathbb{B}} K_{s+t}(z,w) f(w) d\mu_s(w) \right|^p \omega(z) d\mu_{Q+pt}(z) \\ &\geq \int_{|z|\leq\rho_0} | \int_{\mathbb{B}} \Re K_{s+t}(z,w) f(w) d\mu_s(w)|^p \omega(z) d\mu_{Q+pt}(z) \\ &= \frac{1}{2^p} \left( \int_{\mathbb{B}} f(w) d\mu_s(w) \right)^p \int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z). \end{aligned}$$

By continuity of $P_{s,t}$ there exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t} f(z)|^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z),$$

for all $f \in L^p(\omega d\mu_q)$. Hence,

$$\frac{1}{2^p} \left( \int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} |f(z)| d\mu_s(z) \right)^p \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z)$$

for all positive $f \in L^p(\omega d\mu_q)$. Let $f(z) = (\omega(z))^{\frac{-1}{p-1}} (1-|z|^2)^{(p'-1)(s-q)}, \ \forall z \in \mathbb{B}$. Then $f \in L^p(\omega d\mu_q)$ by Lemma 4.2. Replacing this choice of $f$ in the last inequality we obtain

$$\left( \int_{|z|\leq\rho_0} \omega(z) d\mu_{Q+pt}(z) \right) \left( \int_{\mathbb{B}} (\omega(z)^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} \leq 2^p C_{s,t,p,q,Q} < \infty.$$

Then

$$\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Using Lemma 4.1, we get

$$\left(\int_{\mathbb{B}} \omega(z) d\mu_{Q+pt}(z)\right)\left(\int_{\mathbb{B}} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

□

## 5. Weighted estimates: The case where $a > -N-1$ and $s+t > -N-1$

Here we are going to start the study of the boundedness of the operators $T_{a,b}$ and $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ in the case $a > -(N+1)$ and $s+t > -(N+1)$ respectively.

### 5.1. *Necessary Conditions*

To obtain our necessary conditions, we are going to use the following lemma. This is an extension to the analogue lemma in [3].

**Lemma 5.1.** *Let $B_1 = B(w_0, R)$ be a pseudo-ball of sufficiently small radius $R$ touching the boundary of $\mathbb{B}$. There is a pseudo-ball $B_2$ with the same radius touching the boundary of $\mathbb{B}$, sufficiently far from $B_1$, but such that $d(B_1, B_2) \simeq R$, for which for all non negative functions $f$ with support in $B_i$ we have if $a > -1-N$*

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{R^{N+1+a}}\int_{B_i} f(w)d\mu_b(w),$$

*for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. In particular, if $a > -1$ then*

$$|T_{a,b}f(z)| \geq C_{a,b}\frac{1}{\mu_a(B_i)}\int_{B_i} f(w)d\mu_b(w)$$

*for all $z \in B_j$, $i \neq j$, $i,j = 1,2$. The constant $C_{a,b}$ does not depend of $B_i$, $B_j$ or $f$.*

*Proof.* Let $w_0$ be the center of $B_i$. If $R$ is sufficiently small, we take $B_j$ such that for all $z$ in $B_j$ and for all $w$ in $B_i$, we have $d(z,w_0) \geq C_1 d(w,w_0)$ where $C_1$ is as in Lemma 2.15. Let $f$ be a non negative function with support in $B_i$ and let $z \in B_j$, we have

$$T_{a,b}f(z) = \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\int_{B_i} f(w)d\mu_b(w) + \int_{B_i}\left[\frac{1}{(1-\langle z,w\rangle)^{N+1+a}} - \frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right] f(w)d\mu_b(w).$$

 

Then

$$|T_{a,b}f(z)| \geq \frac{1}{|1-\langle z,w_0\rangle|^{N+1+a}}\int_{B_i} f(w)d\mu_b(w)$$
$$-\int_{B_i}\left|\frac{1}{(1-\langle z,w\rangle)^{N+1+a}}-\frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right| f(w)d\mu_b(w)$$

By Lemma 2.15 and Proposition 2.2 we have

$$\left|\frac{1}{(1-\langle z,w\rangle)^{N+1+a}}-\frac{1}{(1-\langle z,w_0\rangle)^{N+1+a}}\right| \leq \frac{1}{2}\frac{1}{|1-\langle z,w_0\rangle|^{N+a+1}},$$

so that

$$|T_{a,b}f(z)| \geq \frac{1}{2}\frac{1}{|1-\langle z,w_0\rangle|^{N+1+a}}\int_{B_i} f(w)d\mu_b(w).$$

Since our pseudo-balls touch the boundary and since $d(B_i,B_j)\simeq R$, we have

$$|1-\langle z,w_0\rangle| \lesssim R.$$

Hence

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2}\frac{1}{R^{N+1+a}}\int_{B_i} f(w)d\mu_b(w).$$

By Lemma 2.3 and [15, Lemma 2.8], if $a>-1$ and because $B_i$ touches the boundary we have $\mu_a(B_i)\simeq R^{N+1+a}$, so that

$$|T_{a,b}f(z)| \gtrsim \frac{1}{2}\frac{1}{\mu_a(B_i)}\int_{B_i} f(w)d\mu_b(w).$$

□

We are now ready to prove Theorem 1.9.

**Theorem 5.2.** *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q\leq q$ then if $a>-1$ we have*

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{\mu_a^2(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty.$$

*More generally if $a>-(N+1)$, then*

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B \omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B)}{R_B^{2(N+1+a)}}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty,$$

*where $R_B$ is the radius of $B$.*

 

*Proof.* Assume that $a > -(N+1)$ and $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \leq q$. Then there exists a constant $C_{a,b,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |T_{a,b}f(z)|^p \omega(z) d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

Let $f$ be a positive function with support in $B_i$ (we take $B_i, B_j$ as in Lemma 5.1). By Lemma 5.1, we then have in the case $a > -1$

$$C_{a,b}^p \int_{B_j} \frac{1}{\mu_a^p(B_i)} \left( \int_{B_i} f(w) d\mu_b(w) \right)^p \omega(z) d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z),$$

hence

$$\begin{aligned} \frac{1}{\mu_a^p(B_i)} \left( \int_{B_i} f(w) d\mu_b(w) \right)^p \left( \int_{B_j} \omega(z) d\mu_Q(z) \right) &\leq C'_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z) \\ &\leq C'_{a,b,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_Q(z). \end{aligned} \tag{5.1}$$

Choosing $f = \frac{\mu_a(B_i)}{\mu_b(B_i)} \chi_{B_i}$ in the last inequality we get

$$\omega(B_j) \leq C'_{a,b,p,q,Q} \frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \omega(B_i),$$

where $\omega(B_k) = \int_{B_k} \omega(z) d\mu_Q(z)$, $k = i, j$. As $B_i$ and $B_j$ touch the boundary of $\mathbb{B}$ and have the same radius, by Lemma 2.3 we have

$$\frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \simeq \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)}.$$

We then have

$$\omega(B_j) \leq C''_{a,b,p,q,Q} \frac{\mu_a^p(B_j)}{\mu_b^p(B_j)} \omega(B_i).$$

Interchanging $B_i$ and $B_j$ (See Lemma 5.1) we get

$$\omega(B_i) \leq C''_{a,b,p,q,Q} \frac{\mu_a^p(B_i)}{\mu_b^p(B_i)} \omega(B_j),$$

so that

$$\frac{\mu_b^p(B_i)}{\mu_a^p(B_i)} \omega(B_i) \leq C''_{a,b,p,q,Q} \omega(B_j)$$

which together with (5.1) leads to

$$\frac{\mu_b^p(B_i)}{\mu_a^{2p}(B_i)}\left(\int_{B_i} f(w)d\mu_b(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_Q(z)\right)\leq C'''_{a,b,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z)=\omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(b-q)}\chi_{B_i}(z)$ ($f\in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality we obtain

$$\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_b(B_i)}{\mu_a^2(B_i)}\int_{B_i}(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}\leq C'''_{a,b,p,q,Q}.$$

When $-1-N<a<-1$ it is sufficient to replace $\mu_a(B_i)$ by $R^{N+1+a}$ in the proof. □

**Theorem 5.3.** *If $T_{a,b}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q>q$ then if $a>-1$ we have*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty.$$

*where the supremum is taken over the pseudo-balls $B$.*

*More generally if $a>-(N+1)$, then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B\omega(z)d\mu_Q(z)\right)\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R_B^{2(N+1+a)}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(b-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

*Proof.* The proof is similar to the proof of Theorem 5.2, except that we choose the first testing function to be $f(z)=(1-|z|^2)^{\frac{Q-q}{p}}\chi_{B_i}(z)$. □

**Theorem 5.4.** *In the case $-(N+1)<s+t<-1$ and $Q\leq q$, or in the case $s+t+\frac{Q-q}{p}\leq -1$, there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.*

*Proof.* This is due to Proposition 4.3, Lemma 4.1 and Lemma 4.2. □

**Theorem 5.5.** *In the case $-(N+1)<s+t<-1$, with $s+t+\frac{Q-q}{p}>-1$ if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ then*

$$\sup_{B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B\omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{R_B^{\frac{Q-q}{p}}}{R_B^{N+1+s+t}}\int_B(\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}<\infty$$

*where the supremum is taken over the pseudo-balls $B$ with radius $R_B$.*

*Proof.* Assume that $-1>s+t>-(N+1)$, with $s+t+\frac{Q-q}{p}>-1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$. Then there exists a constant $C_{s,t,p,q,Q}>0$ such that

$$\int_{\mathbb{B}}|P_{s,t}f(z)|^p\omega(z)d\mu_Q(z)\leq C_{s,t,p,q,Q}\int_{\mathbb{B}}|f(z)|^p\omega(z)d\mu_q(z).$$

 

Let $f$ be a positive function with support in $B_i$ (we take $B_i, B_j$ of radius $R$ as in Lemma 5.1). By Lemma 5.1, we then have

$$C_{s,t}^p \int_{B_j} \frac{1}{R^{(N+1+s+t)p}} \left( \int_{B_i} f(w) d\mu_s(w) \right)^p \omega(z) d\mu_{Q+pt}(z) \leq C_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z),$$

hence

$$\frac{1}{R^{(N+1+s+t)p}} \left( \int_{B_i} f(w) d\mu_s(w) \right)^p \left( \int_{B_j} \omega(z) d\mu_{Q+pt}(z) \right) \leq C'_{s,t,p,q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z). \quad (5.2)$$

Choosing $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t} \chi_{B_i}(z)$ in the last inequality we get, because $N+s+t > -1$,

$$R^{Q-q} \int_{B_j} \omega(z) d\mu_{Q+pt}(z) \leq \int_{B_i} \omega(z) d\mu_{Q+pt}(z).$$

Interchanging $B_i$ and $B_j$ (See Lemma 5.1) we get

$$R^{Q-q} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \lesssim \int_{B_j} \omega(z) d\mu_{Q+pt}(z),$$

which together with (5.2) lead to

$$\frac{R^{Q-q}}{R^{p(N+1+s+t)}} \left( \int_{B_i} f(w) d\mu_s(w) \right)^p \left( \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \leq C'_{s,t,p,q,Q} \int_{B_i} |f(z)|^p \omega(z) d\mu_q(z).$$

Then choosing $f(z) = \omega^{\frac{-1}{p-1}}(z)(1-|z|^2)^{(p'-1)(s-q)} \chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality, we obtain

$$\left( \frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{R^{\frac{Q-q}{p}}}{R^{N+1+s+t}} \int_{B_i} (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} \leq C'''_{s,t,p,q,Q}.$$

□

In the same way we have

**Theorem 5.6.** *If $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q \geq q$, then if $s+t > -1$ we have*

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left( \frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B \omega(z) d\mu_{Q+pt}(z) \right) \left( \frac{R_B^{\frac{Q-q}{p}}}{\mu_{s+t}(B)} \int_B (\omega(z))^{\frac{-1}{p-1}} d\mu_{q+p'(s-q)}(z) \right)^{p-1} < \infty$$

*where $R_B$ is the radius of $B$.*

 

*Proof.* The proof is similar to the proof of Theorem 5.5, we choose the first testing function to be $f(z) = (1-|z|^2)^{\frac{Q-q}{p}+t}\chi_{B_i}$. □

**Theorem 5.7.** *In the case $s+t > -1$ there are no weights $\omega$ such that $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$.*

*Proof.* We are going to proceed in two steps.

*Step 1:* Show that in the case $s + t > -1$, if $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, for $Q < q$, then we have

$$\sup_{pseudo-balls\ B: B\cap\partial\mathbb{B}\neq\varnothing} \left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} < \infty.$$

Assume that $s + t > -1$ and $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$ for $Q < q$. There exists a constant $C_{s,t,p,q,Q} > 0$ such that

$$\int_{\mathbb{B}} |P_{s,t}f(z)|^p\omega(z)d\mu_Q(z) \leq C_{s,t,p,q,Q}\int_{\mathbb{B}} |f(z)|^p\omega(z)d\mu_q(z), \quad f \in L^p(\omega d\mu_q).$$

Let $f$ be a positive function with support in $B_i$ (we take $B_i, B_j$ as in Lemma 5.1). By Lemma 5.1, we then have

$$\begin{aligned}\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_j}\omega(z)d\mu_{Q+pt}(z)\right) &\leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z) \\ &\leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_Q(z).\end{aligned} \tag{5.3}$$

Choosing $f(z) = (1-|z|^2)^t\chi_{B_i}(z)$ in the last inequality we get

$$\int_{B_j}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_i}\omega(z)d\mu_{Q+pt}(z).$$

Interchanging $B_i$ and $B_j$ (see Lemma 5.1) we get

$$\int_{B_i}\omega(z)d\mu_{Q+pt}(z) \leq \int_{B_j}\omega(z)d\mu_{Q+pt}(z),$$

which together with (5.3) lead to

$$\frac{1}{\mu^p_{s+t}(B_i)}\left(\int_{B_i} f(w)d\mu_s(w)\right)^p\left(\int_{B_i}\omega(z)d\mu_{Q+pt}(z)\right) \leq C'_{s,t,p,q,Q}\int_{B_i}|f(z)|^p\omega(z)d\mu_q(z).$$

Then choosing $f(z) = (\omega^{\frac{-1}{p-1}}(z))(1-|z|^2)^{(p'-1)(s-q)}\chi_{B_i}(z)$ ($f \in L^p(\omega d\mu_q)$ by Lemma 4.2) in that last inequality we obtain

$$\left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} \leq C'''_{s,t,p,q,Q}.$$

*Step 2:* Show that

$$\sup_{pseudo-balls\ B:B\cap\partial\mathbb{B}\neq\varnothing}\left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1} = \infty.$$

Let

$$II = \left(\frac{1}{\mu_{s+t}(B)}\int_B \omega(z)d\mu_{Q+pt}(z)\right)\left(\frac{1}{\mu_{s+t}(B)}\int_B (\omega(z))^{\frac{-1}{p-1}}d\mu_{q+p'(s-q)}(z)\right)^{p-1}.$$

Let $B$ be a pseudo-ball that touches the boundary and $R_B$ its radius. By Hölder's inequality we have

$$\begin{aligned}
\mu_{s+t}(B) &= \int_B d\mu_{s+t}(z)\\
&= \int_B (1-|z|^2)^{s+t}d\mu(z)\\
&= \int_B (1-|z|^2)^{s-\frac{q}{p}+\frac{q}{p}+t}d\mu(z)\\
&= \int_B \omega^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{q+pt}{p}}\omega^{-\frac{1}{p}}(z)(1-|z|^2)^{s-\frac{q}{q}}d\mu(z)\\
&\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt}d\mu(z)\right)^{\frac{1}{p}}\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{\frac{1}{p'}}.
\end{aligned}$$

Note that for $z \in B$ we have $1-|z| < 2R_B$. Then

$$\begin{aligned}
\mu^p_{s+t}(B) &\leq \left(\int_B \omega(z)(1-|z|^2)^{q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1}\\
&= \left(\int_B \omega(z)(1-|z|^2)^{q-Q+Q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1}\\
&\leq (2R_B)^{q-Q}\left(\int_B \omega(z)(1-|z|^2)^{Q+pt}d\mu(z)\right)\left(\int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)}d\mu(z)\right)^{p-1}.
\end{aligned}$$

Hence

$$(2R_B)^{Q-q} \leq \frac{1}{\mu^p_{s+t}(B)} \left( \int_B \omega(z)(1-|z|^2)^{Q+pt} d\mu(z) \right) \left( \int_B \omega^{-\frac{1}{p-1}}(z)(1-|z|^2)^{q+p'(s-q)} d\mu(z) \right)^{p-1}$$
$$= II.$$

Taking the sup over smaller radii, we get $\sup II = \infty$. □

## 5.2. *The Associated Maximal and Fractional Maximal Functions*

We introduce for $b > -1$ and $s > -1$ the following maximal functions. If $a > -1$ we set

$$m_{a,b} f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{\mu_a(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_b(w), \tag{5.4}$$

more generally if $a > -1-N$ we set

$$m'_{a,b} f(z) = \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{R^{N+1+a}} \int_{B(\zeta,R)} |f(w)| d\mu_b(w). \tag{5.5}$$

If $a > -1$ we set

$$M_{a,b} f(z) = \sup_{B : z \in B} \frac{1}{\mu_a(B)} \int_B |f(w)| d\mu_b(w), \tag{5.6}$$

and more generally if $a > -1-N$ we set

$$M'_{a,b} f(z) = \sup_{B : z \in B} \frac{1}{R^{N+1+a}} \int_B |f(w)| d\mu_b(w). \tag{5.7}$$

If $s + t > -1$ we set

$$O_{s,t} f(z) = (1-|z|^2)^t \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{\mu_{s+t}(B(\zeta,R))} \int_{B(\zeta,R)} |f(w)| d\mu_s(w), \tag{5.8}$$

more generally if $s + t > -1-N$ we set

$$O'_{s,t} f(z) = (1-|z|^2)^t \sup_{\zeta \in \mathbb{B}, R > 1-|\zeta| : z \in B(\zeta,R)} \frac{1}{R^{N+1+s+t}} \int_{B(\zeta,R)} |f(w)| d\mu_s(w). \tag{5.9}$$

Let finally define the following fractional maximal function

$$M_\gamma f(z) = \sup_{B : z \in B} \frac{1}{\mu_b^{1-\gamma}(B)} \int_B |f(w)| d\mu_b(w), \quad \gamma \in (0,1). \tag{5.10}$$

Note that for $a < b$ we get by Lemma 2.3 $M_{a,b} \sim M_\gamma$ with $\gamma = 1 - \frac{N+1+a}{N+1+b}$.

For all $k \in (0,1)$, we define the operator of regularisation $R_k^b$

$$R_k^b f(z) = \frac{1}{\mu_b(B_k(z))} \int\limits_{B_k(z)} f(\zeta) d\mu_b(\zeta), \tag{5.11}$$

where $B_k(z) = \{w \in \mathbb{B} : d(z,w) < k(1-|z|)\}$.

We will need the following lemmas to show Theorem 1.14 (see [3]).

**Lemma 5.8.** *Let* $k \in (0, \frac{1}{2})$*. If* $z' \in B_k(z)$*, then* $z \in B_{k'}(z')$*, where* $k' = \frac{k}{1-k}$*.*

*Proof.* We have

$$||z| - |z'|| \leq d(z, z') < k(1-|z|).$$

Therefore $1 - |z| \leq \frac{1}{1-k}(1-|z'|)$, so that $d(z,z') < k(1-|z|) \leq \frac{k}{1-k}(1-|z'|)$. □

**Lemma 5.9.** *If* $B := B(x,R)$ *touches the boundary then if we take* $B' = B(x, K(1+2k_1)R)$*, then for all* $w \in B, B_{k_1}(w) \subset B'$ *(See* (2.1) *for the definition of* $K$*).*

*Proof.* Let $w \in B$ and $z \in B_{k_1}(w)$. We have

$$\begin{aligned} d(x,z) &\leq K(d(x,w) + d(w,z)) \\ &\leq K\left(R + k_1(1-|w|)\right) \\ &\leq K\left(R + k_1(2R)\right) = K(1+2k_1)R. \end{aligned}$$

□

The next three lemmas are generalizations of their analogues in [3] to the maximal functions $m'_{a,b}$ and $O'_{s,t}$. For the sake of completeness and for the reader convenience, we include their proofs.

**Lemma 5.10.** *For all* $k \in (0,1)$*, there is a constant* $C_k$ *such that for all positive locally integrable function* $f$ *we have if* $a > -1$

$$m_{a,b} f \leq C_k m_{a,b}(R_k^b f),$$

*and more generally if* $a > -1 - N$

$$m'_{a,b} f \leq C_k m'_{a,b}(R_k^b f).$$

*Proof.* We have to show that for all $z$ and all pseudo-balls $B$ containing $z$ which touch the boundary of $\mathbb{B}$, there is a pseudo-ball $B'$ containing $z$, which touches the boundary of $\mathbb{B}$ so that

$$\frac{1}{\mu_a(B)} \int\limits_B f(w) d\mu_b(w) \leq C_k \frac{1}{\mu_a(B')} \int\limits_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int\limits_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w).$$

By Lemma 5.8, $\chi_{B_k(w)}(\zeta) \geq \chi_{B_{k_1}(\zeta)}(w)$, where $k_1 = \frac{k}{k+1}$. If $B = B(x,R)$ $(R > 1-|x|)$, by Lemma 5.9, for $B' = B(x, K(1+2k_1)R)$, we have $\forall w \in B, B_{k_1}(w) \subset B'$. Note that

$$\mu_b(B_{k_1}(\zeta)) \simeq \mu_b(B_k(w)) \ \text{ when } \ \mathrm{w \in B_{k_1}(\zeta)}. \tag{5.12}$$

Hence

$$\begin{aligned}
\int\limits_{B'} R_k^b f(w) d\mu_b(w) &= \int\limits_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int\limits_{B_k(w)} f(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int\limits_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int\limits_{B} f(\zeta) \chi_{B_k(w)}(\zeta) d\mu_b(\zeta) \right] d\mu_b(w) \\
&\geq \int\limits_{B'} \left[ \frac{1}{\mu_b(B_k(w))} \int\limits_{B} f(\zeta) \chi_{B_{k_1}(\zeta)}(w) d\mu_b(\zeta) \right] d\mu_b(w) \\
&= \int\limits_{B} \left[ \int\limits_{B'} \frac{1}{\mu_b(B_k(w))} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Using (5.12), we get

$$\begin{aligned}
\int\limits_{B'} R_k^b f(w) d\mu_b(w) &\gtrsim \int\limits_{B} \frac{1}{\mu_b(B_{k_1}(\zeta))} \left[ \int\limits_{B'} \chi_{B_{k_1}(\zeta)}(w) d\mu_b(w) \right] f(\zeta) d\mu_b(\zeta) \\
&= \int\limits_{B} f(\zeta) d\mu_b(\zeta).
\end{aligned}$$

Since $\mu_a$ is a homogeneous measure we have

$$\frac{1}{\mu_a(B)} \int\limits_{B} f(w) d\mu_b(w) \lesssim \frac{1}{\mu_a(B')} \int\limits_{B'} R_k^b f(w) d\mu_b(w).$$

For $m'_{a,b}$ it is sufficient to observe that $B$ and $B'$ have equivalent radii. □

The following lemma appears as a corollary of the preceding one by observing that

$$O_{s,t} f(z) = (1 - |z|^2)^t m_{s+t,s} f(z)$$

and

$$O'_{s,t} f(z) = (1 - |z|^2)^t m'_{s+t,s} f(z).$$

**Lemma 5.11.** *For all $k \in (0,1)$, there is a constant $C_k$ such that for all positive locally integrable functions $f$ we have if $s + t > -1$*

$$O_{s,t} f \leq C_k O_{s,t}(R_k^s f),$$

*and more generally if $s + t > -1 - N$*

$$O'_{s,t} f \leq C_k O'_{s,t}(R_k^s f).$$

One can find the following lemma in [3] but for $b = Q$.

**Lemma 5.12.** *For all $k \in (0, \frac{1}{2})$, there are two constants $C$ and $k' < 1$ depending only on $k, b, Q, N$ such that for all $f, g \in L^1(d\mu_b)$, $f \geq 0, g \geq 0$*

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)]d\mu_b(z),$$

*where*

$$R_{k'}^{b,Q} f(z) = \frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} f(\zeta)d\mu_Q(\zeta). \tag{5.13}$$

*Proof.* By Lemma 5.8, $\chi_{B_k(z)}(w) \leq \chi_{B_{k'}(w)}(z)$, where $k' = \frac{k}{1-k}$. Because of (5.12) there is a constant $C$ such that

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z).$$

We want to form the quantity $f(z)[R_k^b g(z)]$ on the left while controlling it on the right in order to use Fubini's theorem to bring out the quantity $g(z)[R_{k'}^{b,Q} f(z)]$. Then, for $w \in B_k(z)$

$$\frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w) \leq \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w).$$

We form $R_k^b g(z)$ on the left

$$\int_{B_k(z)} \frac{1}{\mu_b(B_k(z))}\chi_{B_k(z)}(w)g(w)d\mu_b(w) \leq \int_{B_k(z)} \frac{C}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w),$$

by a multiplication by $f(z)$ we have

$$f(z)R_k^b g(z) \leq Cf(z) \int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)d\mu_b(w).$$

After integration, we obtain

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} \left[\int_{B_k(z)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)g(w)f(z)d\mu_b(w)\right] d\mu_Q(z).$$

Recall that $(z \in \mathbb{B}$ and $w \in B_k(z)) \Longrightarrow (z \in B_{k'}(w)$ and $w \in \mathbb{B})$, hence using Fubini's theorem

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[\int_{B_{k'}(w)} \frac{1}{\mu_b(B_{k'}(w))}\chi_{B_{k'}(w)}(z)f(z)d\mu_Q(z)\right] d\mu_b(w),$$

hence

$$\int_{\mathbb{B}} f(z)R_k^b g(z)d\mu_Q(z) \leq C \int_{\mathbb{B}} g(w) \left[\frac{1}{\mu_b(B_{k'}(w))}\int_{B_{k'}(w)} f(z)d\mu_Q(z)\right] d\mu_b(w),$$

then

$$\int_{\mathbb{B}} f(z)[R_k^b g(z)]d\mu_Q(z) \leq C \int_{\mathbb{B}} g(z)[R_{k'}^{b,Q} f(z)]d\mu_b(z).$$

□

The following result will be used in the proof of Theorem 1.14. This extends Lemma 9 in [3]. We give a proof for the reader convenience.

**Lemma 5.13.** *Let $k \in (0,1)$. There are two constants $c, C$ depending only on $a, b, N, k$ such that for all positive locally integrable functions g if $a > -1$*

$$c\, m_{a,b} g \leq R_k^b(m_{a,b} g) \leq C m_{a,b} g,$$

*and more generally if $a > -1 - N$*

$$c\, m'_{a,b} g \leq R_k^b(m'_{a,b} g) \leq C m'_{a,b} g.$$

*Proof.* It is sufficient to show that there are two constants $0 < c < C$ such that $\forall w \in B_k(z)$

$$c\, m_{a,b} g(z) \leq m_{a,b} g(w) \leq C m_{a,b} g(z).$$

We are going to show the two inequalities. More precisely we are going to show that there are two constants $0 < c < C$ such that, for each pseudo-ball $B$ containing $z$ and touching the boundary, there is a pseudo-ball $B'$ containing $w$ and touching the boundary so that

$$\frac{c}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{1}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta)$$

and show for each pseudo-ball $B$ containing $z$ touching the boundary, there is a pseudo-ball $B'$ containing $w$ touching the boundary so that

$$\frac{1}{\mu_a(B)} \int_B |g(\zeta)| d\mu_b(\zeta) \leq \frac{C}{\mu_a(B')} \int_{B'} |g(\zeta)| d\mu_b(\zeta).$$

In each case, by Lemma 5.9, it is sufficient if $B = B(x, R)$ to take $B' = B(x, K(1 + 2kK)R)$. For the result with $m'_{a,b}$ it is sufficient to notice that $B$ and $B'$ have equivalent radii. □

In the same way as Lemma 5.13, since $O_{s,t} f(z) := (1 - |z|^2)^t m_{s+t,s} f(z)$ and $O'_{s,t} f(z) := (1 - |z|^2)^t m'_{s+t,s} f(z)$, we obtain the following result.

**Lemma 5.14.** *Let $k \in (0,1)$. There are two constants $c, C$ depending only on $s, t, N, k$ such that for all locally integrable function g if $s + t > -1$*

$$c\, O_{s,t} g \leq R_k^b(O_{s,t} g) \leq C O_{s,t} g,$$

*and more generally if $s + t > -1 - N$*

$$c\, O'_{s,t} g \leq R_k^b(O'_{s,t} g) \leq C O'_{s,t} g.$$

Now we give a useful characterization of elements in $(B_p^{a,b,q,Q})$, see Definition 1.7.

 

**Lemma 5.15.** *For $a > -1$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z).$$

*More generally if $a > -1 - N$ and $Q > q$, $\omega \in (B_p^{a,b,q,Q})$ $(b > -1)$ if and only if there is a constant $C_{a,b,p,q,Q} > 0$ such that*

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}} \int_B f(z)d\mu_b(z)\right)^p \int_B \omega(z)d\mu_Q(z) \leq C_{a,b,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ of radius R that touch the boundary.*

*Proof.* Assume $\omega \in (B_p^{a,b,q,Q})$. Let $0 \leq f \in L^p(\omega d\mu_q)$ and $B$ be a pseudo-ball of radius $R$ such that $R > 1 - |z|$. Then

$$\begin{aligned}\left(\int_B f(z)d\mu_b(z)\right)^p &= \left(\int_{\mathbb{B}} f(z)(1-|z|^2)^{b-q}d\mu_q(z)\right)^p \\ &= \left(\int_B f(z)(\omega(z))^{\frac{-1}{p}}(\omega(z))^{\frac{1}{p}}(1-|z|^2)^{b-q}d\mu_q(z)\right)^p \\ &\leq \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B ((\omega(z))^{\frac{-1}{p}})^{p'}(1-|z|^2)^{p'(b-q)}d\mu_q(z)\right)^{\frac{p}{p'}} \\ &= \left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left(\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}.\end{aligned}$$

Hence

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq$$
$$\left(\int_B f^p(z)\omega(z)d\mu_q(z)\right)\left[\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\omega(B)\right]\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)}\int_B \omega^{\frac{-1}{p-1}}(z)d\mu_{q+p'(b-q)}(z)\right)^{p-1}$$

and because $\omega \in B_p^{a,b,q,Q}$, there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{\mu_a^2(B)} \int_B f(z)d\mu_b(z)\right)^p \omega(B) \leq C_{a,b,p,q,Q} \int_B f^p(z)\omega(z)d\mu_q(z).$$

For the general case it is sufficient to replace $\mu_a(B)$ with $R^{N+1+a}$.

If we assume that there is a constant $C_{a,b,p,q,Q} > 0$ such that

$$\left(\frac{\mu_{b+\frac{Q-q}{p}}(B)}{R^{2(N+1+a)}}\int_B f(z)d\mu_b(z)\right)^p \omega(B) \le C_{a,b,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ of radius R such that $R > 1-|z|$ it is sufficient to take $f(z) = (1-|z|^2)^{(p'-1)(b-q)}\omega^{\frac{-1}{p-1}}(z)\chi_B(z)$ to get $\omega \in (B_p^{a,b,q,Q})$. □

**Remark 5.16.** The result remains true even if $B$ almost touches the boundary.

In the same way, for $(D_p^{s,t,q,Q})$ (see Definition 1.11), we have the following lemma.

**Lemma 5.17.** *For $Q \ge q$ and $s+t > -1$, $\omega \in (D_p^{s,t,q,Q})$ ($s > -1$) if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \le C_{s,t,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ that touch the boundary.*

*For $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -1-N$, $\omega \in (D_p^{s,t,q,Q})$ ($s > -1$) if and only if there is a constant $C_{s,t,p,q,Q} > 0$ such that*

$$\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)}\int_B f(z)d\mu_s(z)\right)^p \int_B \omega(z)d\mu_{Q+pt}(z) \le C_{s,t,p,q,Q}\int_B f^p(z)\omega(z)d\mu_q(z),$$

*for all positive $f \in L^p(\omega d\mu_q)$ and all pseudo-balls $B$ that touch the boundary.*

**Remark 5.18.** The result remains true even if $B$ almost touches the boundary.

**Corollary 5.19.** *For $C_1 > 1$, if $\omega \in D_p^{s,t,q,Q}$ then there exists a constant $C_2 > 0$ such that for any pseudo-ball $B := B(y,r)$ which touches or almost touches the boundary, we have*

$$\int_{B(y,C_1r)} \omega(\zeta)d\mu_{Q+pt}(\zeta) \le C_2 \int_{B(y,r)} \omega(\zeta)d\mu_{Q+pt}(\zeta).$$

**Proposition 5.20.** *Let $X$ be an homogeneous space. Let $w$ be a weight in $X$. For $-N-1 < a \le b$, $Q > q$ and $k \in (0,\frac{1}{2})$, assume that there exists a constant $C_2 > 0$ such that*

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x)d\nu(x)\right)^{\frac{1}{r}} \le C_2\left(\int_B u^{1-p'}(x)d\nu(x)\right)^{\frac{1}{p}} \tag{5.14}$$

*for any pseudo-ball $B \subset X$, where $v(z) = R^{b,Q}_{k'}\omega(z)$, $u(z) = R^{b,Q}_{k'}\omega(z)(1-|z|^2)^{2p(b-a)+Q-q}$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$. If $\omega \in (B^{a,b,q,Q}_p)$, there is a constant $C_{a,b,p,q,Q} > 0$ such that for all $f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (m_{a,b}f(z))^p \, \omega(z) d\mu_Q(z) \leq \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

*Proof.* Let set

$$III = \int_{\mathbb{B}} (m_{a,b}f(z))^p \, \omega(z) d\mu_Q(z).$$

Using in this order Lemma 5.10, Lemma 5.13, Hölder's inequality and Lemma 5.12 we have,

$$\begin{aligned}
III &\leq C_k^p \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} \left[R_k^b(m_{a,b}R_k^b f(z))\right]^p \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p \int_{\mathbb{B}} R_k^b \left[(m_{a,b}R_k^b f(z))^p\right] \omega(z) d\mu_Q(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p R^{b,Q}_{k'}\omega(z) d\mu_b(z).
\end{aligned}$$

For $a \leq b$, assume that there exists a constant $C_2 > 0$ such that

$$\left(\int_B [M_\gamma(\chi_B u^{1-p'})(x)]^r v(x) d\nu(x)\right)^{\frac{1}{r}} \leq C_2 \left(\int_B u^{1-p'}(x) d\nu(x)\right)^{\frac{1}{p}}$$

for any ball $B \subset X$, where $v(z) = R^{b,Q}_{k'}\omega(z)$, $u(z) = R^{b,Q}_{k'}\omega(z)(1-|z|^2)^c$, $d\nu = d\mu_b$ $p = r$ and $\gamma = 1 - \frac{N+1+a}{N+1+b}$, with $c$ to be determined. Then we have

$$\begin{aligned}
\int_{\mathbb{B}} (m_{a,b}f(z))^p \omega(z) d\mu_Q(z) &\leq C_k^p A^p C \int_{\mathbb{B}} (m_{a,b}R_k^b f(z))^p R^{b,Q}_{k'}\omega(z) d\mu_b(z) \\
&\leq C_k^p A^p C \int_{\mathbb{B}} (M_{a,b}R_k^b f(z))^p R^{b,Q}_{k'}\omega(z) d\mu_b(z) \\
&\lesssim C_k^p A^p C \int_{\mathbb{B}} (M_\gamma R_k^b f(z))^p R^{b,Q}_{k'}\omega(z) d\mu_b(z) \\
&\leq C_k^p A^p C' \int_{\mathbb{B}} (R_k^b f(z))^p (1-|z|^2)^c R^{b,Q}_{k'}\omega(z) d\mu_b(z),
\end{aligned}$$

where for the last inequality we used Theorem 2.9. Now let us control $(R_k^b f(z))^p R^{b,Q}_{k'}\omega(z)$. We have

$$(R_k^b f(z))^p R^{b,Q}_{k'}\omega(z) = \left(\frac{1}{\mu_b(B_k(z))} \int_{B_k(z)} f(\zeta) d\mu_b(\zeta)\right)^p \left(\frac{1}{\mu_b(B_{k'}(z))} \int_{B_{k'}(z)} \omega(\zeta) d\mu_Q(\zeta)\right)$$

 

$$\leq \left(\frac{1}{\mu_b(B_k(z))}\int\limits_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p \left(\frac{1}{\mu_b(B_{k'}(z))}\int\limits_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right)$$
$$\lesssim \left(\frac{1}{\mu_b(B_{k'}(z))}\int\limits_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p \left(\frac{1}{\mu_b(B_{k'}(z))}\int\limits_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right),$$

where the second inequality is because $k' > k$, the third one is because $\mu_b(B_{k'}(z)) \backsimeq \mu_b(B_k(z))$. Then

$$(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) \lesssim$$
$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))}\left(\frac{\mu_{b+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_a^2(B_{k'}(z))}\int\limits_{B_{k'}(z)} f(\zeta)d\mu_b(\zeta)\right)^p \left(\int\limits_{B_{k'}(z)} \omega(\zeta)d\mu_Q(\zeta)\right),$$

so that

$$(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z) \lesssim C_{a,b,p,q,Q}\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))}\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),$$

because of Lemma 5.15 since it is possible to dilate the pseudo-balls $B_k$ so that they touch the boundary and the fact that the measures $d\mu_q$ and $d\mu_{q+p'(b-a)}$ are homogeneous. By Lemma 2.3, since $B_{k'}(z) = B(z, k'(1-|z|))$ with $0 < k' < 1$, we have

$$\frac{\mu_a^{2p}(B_{k'}(z))}{\mu_b^{p+1}(B_{k'}(z))\mu^p_{b+\frac{Q-q}{p}}(B_{k'}(z))} \backsimeq (1-|z|^2)^{2pa-2pb-b-(Q-q)-(N+1)}.$$

Recall that we already have

$$\int\limits_{\mathbb{B}}(m_{a,b}f(z))^p\omega(z)d\mu_Q(z) \leq C_k^p A^p C' \int\limits_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Let us set

$$IV = \int\limits_{\mathbb{B}}(1-|z|^2)^{c+(b-a)}(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)d\mu_a(z).$$

Hence, using the previous control of $(R_k^b f(z))^p R_{k'}^{b,Q}\omega(z)$ and Fubini's theorem, we have

$$IV \lesssim \int\limits_{\mathbb{B}}\left(\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{c+(b-a)+2pa-2pb-b-Q+q-1-N}d\mu_a(z)$$
$$\lesssim \int\limits_{\mathbb{B}}\left(\int\limits_{\mathbb{B}} \chi_{B_{k'}(z)}(\zeta)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right) f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$
$$\lesssim \int\limits_{\mathbb{B}}\left(\int\limits_{\mathbb{B}} \chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c+(1-2p)(b-a)-b-Q+q-1-N}d\mu_a(z)\right) f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$

$$\lesssim \int_{\mathbb{B}} \left( \int_{\mathbb{B}} \chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{c-2p(b-a)-Q+q-1-N} d\mu(z) \right) f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)$$
$$\lesssim \int_{\mathbb{B}} (1-|\zeta|^2)^{c-2p(b-a)-(Q-q)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).$$

The proof is complete if we take

$$c = 2p(b-a) + (Q-q).$$

□

**Remark 5.21.** The result in Proposition 5.20 says that if we assume that the Sawyer type condition (5.14) holds, then the necessary condition $w \in (B_p^{a,b,q,Q})$ for the boundedness of $T_{a,b}$ from $L^p(wd\mu_q)$ to $L^p(wd\mu_Q)$ is also sufficient by the good lambda inequality in Theorem 1.13. Since we do not know if $w \in (B_p^{a,b,q,Q})$ implies the Sawyer type condition, we will provide in the sequel a testable sufficient condition for the boundedness of $T_{a,b}$ in this situation.

**Lemma 5.22.** *Let $k \in (0, \frac{1}{4})$. For $s+t > -1$, or for $-N-1 < s+t < -1$ with $s+t+\frac{Q-q}{p} > -1$, if $\omega \in D_p^{s,t,q,Q}$, we have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$.*

*Proof.* We have $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \in (A_{p,s+t+\frac{Q-q}{p}})$ if

$$V := \left( \frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \sigma(z) d\mu_{s+t+\frac{Q-q}{p}}(z) \right) \left( \frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B)} \int_B \sigma^{\frac{-1}{p-1}}(z) d\mu_{s+t+\frac{Q-q}{p}}(z) \right)^{p-1} \leq C_p(\omega).$$

Note that $\sigma(z) \simeq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(z)$ since $d\mu_b(B_{k'}(z)) \simeq (1-|z|^2)^{n+1+b}$. We consider two cases.

*First case:* $B := B(y,r)$ with $r \ll 1-|y|$.

In this case, since $R_{k'}^{a,b}f(x)$ is defined as in (5.13), Corollary 5.19 shows that there are two constants $0 < c < C$ such that

$$cR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y) \leq R_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(x) \leq CR_{k'}^{s+t+\frac{Q-q}{p},Q+pt}\omega(y)$$

for all $x \in B$. We then have

$$V \simeq \frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)} \left( \frac{\mu_{s+t+\frac{Q-q}{p}}(B)}{\mu_{s+t+\frac{Q-q}{p}}(B)} \right)^{p-1} = 1.$$

*Second case:* $B := B(y,r)$ touches the boundary.

Recall that our measures are homogeneous, and recall that if $z \in B$ and $x \in B_{k'}(z)$, then $z \in B_{k''}(x)$ and $x \in B' := B(y, 2k'Kr + Kr)$, where $k'' = \frac{k'}{1-k'}$. Let

$$VI = \int_B \left( \frac{1}{\mu_s(B_{k'}(z))} \int_{B_{k'}(z)} \omega(x)d\mu_{Q+pt}(x) \right) (1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(z).$$

 

Then by Fubini's theorem we have,

$$\begin{aligned}
VI &= \int\limits_{B} \left( \frac{1}{\mu_s(B_{k'}(z))} \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right) d\mu_s(z) \\
&= \int\limits_{B} \int\limits_{B_{k'}(z)} \frac{1}{\mu_s(B_{k'}(z))} \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&= \int\limits_{\mathbb{B}} \int\limits_{\mathbb{B}} \frac{1}{\mu_s(B_{k'}(z))} \chi_B(z) \chi_{B_{k'}(z)}(x) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int\limits_{\mathbb{B}} \int\limits_{\mathbb{B}} \frac{1}{\mu_s(B_{k''}(x))} \chi_{B'}(x) \chi_{B_{k''}(x)}(z) \omega(x) d\mu_s(z) d\mu_{Q+pt}(x) \\
&\lesssim \int\limits_{B'} \omega(x) d\mu_{Q+pt}(x).
\end{aligned}$$

So we have

$$VI = \int\limits_{B} R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(z) \lesssim \int\limits_{B'} \omega(x) d\mu_{Q+pt}(x). \tag{5.15}$$

Let us now control $(R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'}$. We have

$$\begin{aligned}
(R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}})^{1-p'} &= \left( \frac{1}{\mu_s(B_{k'}(z))} (1-|z|^2)^{-t-\frac{Q-q}{p}} \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{1-p'} \\
&\simeq \left[ \left( \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int\limits_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1}.
\end{aligned}$$

Setting

$$VII = \left[ \left( \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int\limits_{B_{k'}(z)} d\mu_{s+t+\frac{Q-q}{p}}(x) \right) \right]^{p'-1},$$

we have, by Hölder's inequality

$$\begin{aligned}
VII &= \left[ \left( \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int\limits_{B_{k'}(z)} \omega^{\frac{1}{p}}(x) \omega^{\frac{-1}{p}}(x) (1-|x|^2)^{s-\frac{q}{p}+\frac{Q+pt}{p}} d\mu(x) \right) \right]^{p'-1} \\
&\leq \left[ \left( \int\limits_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{\frac{1}{p}-1} \left( \int\limits_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p'}} \right]^{p'-1}
\end{aligned}$$

$$\leq \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \right]^{\frac{1}{p}}.$$

Let us set

$$VIII = \int_B \left( R_{k'}^{s,Q+pt} \omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}} \right)^{\frac{-1}{p-1}} d\mu_{s+t+\frac{Q-q}{p}}(z),$$

then we have

$$VIII \leq \int_B \left[ \left( \int_{B_{k'}(z)} \omega(x) d\mu_{Q+pt}(x) \right)^{-1} \left( \int_{B_{k'}(z)} \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right) \omega(z)(1-|z|^2)^{Q+pt} \right]^{\frac{1}{p}}$$
$$\omega^{\frac{-1}{p}}(z)(1-|z|^2)^{s-\frac{q}{p}} d\mu(z)$$

so that by Hölder's inequality and Fubini's theorem, we have

$$VIII \lesssim \left( \int_B \omega^{\frac{-p'}{p}}(z) d\mu_{q+p'(s-q)}(z) \right)^{\frac{1}{p'}}$$
$$\left( \int_{B'} \left[ \left( \int_{B_{k''}(x)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \right)^{-1} \left( \int_{B_{k''}(x)} \omega(z) d\mu_{Q+pt}(z) \right) \right] \omega^{\frac{-p'}{p}}(x) d\mu_{q+p'(s-q)}(x) \right)^{\frac{1}{p}}.$$

Finally, we obtain

$$VIII \lesssim \int_{B'} \omega^{\frac{-p'}{p}}(z) d\mu_{q+p'(s-q)}(z). \tag{5.16}$$

Where on the last but one inequality we used Fubini's theorem (as in the control of VI) and the fact that for $x \in B_{k'}(z)$ we have $z \in B_{k''}(x)$ and

$$\int_{B_{k''}(x)} \omega(\zeta) d\mu_{Q+pt}(\zeta) \lesssim \int_{B_{k'}(z)} \omega(\zeta) d\mu_{Q+pt}(\zeta).$$

This is a variant of Corollary 5.19 or simply the application of Lemma 5.17 for $f(\zeta) = (1-|\zeta|^2)^{t+\frac{Q-q}{p}}$ $1_{B_{k'}(z)}(\zeta)$ with $B := B(z, 2Kk'(1+k'')(1-|z|)) \supseteqq B_{k''}(x)$ (Lemma 5.9) and the fact that our measures are homogeneous. Since $\omega \in D_p^{s,t,q,Q}$, we use (5.15) and (5.16) to conclude that $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$. $\square$

**Theorem 5.23.** *In the case both $Q \geq q$ and $s+t > -1$ hold, and in the case both $s+t+\frac{Q-q}{p} > -1$ and $-1 > s+t > -N-1$ hold, if $\omega \in D_p^{s,t,q,Q}$, there is a constant $C_{s,t,p,q,Q} > 0$ such that $\forall f \in L^p(\omega d\mu_q)$,*

$$\int_{\mathbb{B}} (O_{s,t} f(z))^p \omega(z) d\mu_Q(z) \leq C_{s,t,p,q,Q} \int_{\mathbb{B}} |f(z)|^p \omega(z) d\mu_q(z).$$

 

*Proof.* Using in this order Lemma 5.11, Lemma 5.14, Hölder's inequality and Lemma 5.12 we have

$$\begin{aligned}\int\limits_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &= \int\limits_{\mathbb{B}}(m_{s+t,s}f(z))^p\omega(z)d\mu_{Q+pt}(z)\\ &\le C_k^p\int\limits_{\mathbb{B}}(m_{s+t,s}R_k^sf(z))^p\omega(z)d\mu_{Q+pt}(z)\\ &\le C_k^pA^p\int\limits_{\mathbb{B}}[R_k^s(m_{s+t,s}R_k^sf(z))]^p\omega(z)d\mu_{Q+pt}(z)\\ &\le C_k^pA^p\int\limits_{\mathbb{B}}R_k^s[(m_{s+t,s}R_k^sf(z))^p]\omega(z)d\mu_{Q+pt}(z)\\ &\le C_k^pA^pC\int\limits_{\mathbb{B}}(m_{s+t,s}R_k^sf(z))^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z).\end{aligned}$$

By Lemma 5.22, $R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}\in A_{p,s+t+\frac{Q-q}{p}}$. Using natural domination between our maximal operator defined by equation 5.4 and equation 5.6, we have

$$\begin{aligned}\int\limits_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\le C_k^pA^pC\int\limits_{\mathbb{B}}(m_{s+t,s}R_k^sf(z))^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z)\\ &\le C_k^pA^pC\int\limits_{\mathbb{B}}(M_{s+t,s}R_k^sf(z))^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z)\\ &= C_k^pA^pC\int\limits_{\mathbb{B}}(M_{s+t,s+t}[(1-|z|^2)^{-t}R_k^sf(z)])^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z).\end{aligned}$$

Because in each case we have $\frac{Q-q}{p}\ge 0$, using Theorem 2.13, we have that

$$\begin{aligned}\int\limits_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\lesssim \int\limits_{\mathbb{B}}(M_{s+t+\frac{Q-q}{p},s+t}[(1-|z|^2)^{-t}R_k^sf(z)])^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z)\\ &\lesssim \int\limits_{\mathbb{B}}(M_{s+t+\frac{Q-q}{p},s+t+\frac{Q-q}{p}}[(1-|z|^2)^{-t-\frac{Q-q}{p}}R_k^sf(z)])^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z)\\ &\lesssim \int\limits_{\mathbb{B}}(1-|z|^2)^{-pt-(Q-q)}(R_k^sf(z))^pR_{k'}^{s,Q+pt}\omega(z)d\mu_s(z). \qquad (5.17)\end{aligned}$$

Now let us control $IX=(R_k^sf(z))^pR_{k'}^{s,Q+pt}\omega(z)$. We have

$$\begin{aligned}IX &= \left(\frac{1}{\mu_s(B_k(z))}\int\limits_{B_k(z)}f(\zeta)d\mu_s(\zeta)\right)^p\left(\frac{1}{\mu_s(B_{k'}(z))}\int\limits_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\ &\lesssim \left(\frac{1}{\mu_s(B_{k'}(z))}\int\limits_{B_{k'}(z)}f(\zeta)d\mu_s(\zeta)\right)^p\left(\frac{1}{\mu_s(B_{k'}(z))}\int\limits_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\left(\frac{1}{\mu_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}\int\limits_{B_{k'}(z)} f(\zeta)d\mu_s(\zeta)\right)^p\left(\int\limits_{B_{k'}(z)}\omega(\zeta)d\mu_{Q+pt}(\zeta)\right)\\
&\lesssim C_{s,t,p,q,Q}\frac{\mu^p_{s+t+\frac{Q-q}{p}}(B_{k'}(z))}{\mu_s^{p+1}(B_{k'}(z))}\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim C_{s,t,p,q}(1-|z|^2)^{pt+(Q-q)-s-N-1}\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta),
\end{aligned}
$$

where for the last but one inequality we used Lemma 5.17. Hence using (5.17) and Fubini's theorem we have

$$
\begin{aligned}
\int\limits_{\mathbb{B}}(O_{s,t}f(z))^p\omega(z)d\mu_Q(z) &\lesssim \int\limits_{\mathbb{B}}\left(\int\limits_{B_{k'}(z)} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\right)(1-|z|^2)^{-1-N}d\mu(z)\\
&\lesssim \int\limits_{\mathbb{B}}\left(\int\limits_{\mathbb{B}}\chi_{B_{k''}(\zeta)}(z)(1-|z|^2)^{-1-N}d\mu(z)\right)f^p(\zeta)\omega(\zeta)d\mu_q(\zeta)\\
&\lesssim \int\limits_{\mathbb{B}} f^p(\zeta)\omega(\zeta)d\mu_q(\zeta).
\end{aligned}
$$

Here we used Lemma 5.8 with $k'' := (k')'$. The proof is complete. □

## 6. Good lambda inequality and sufficient conditions

In this section we will establish the good lambda inequality that allow us to provide sufficient conditions for the boundedness of our operators. We first need some preliminary results. The first result extends to the maximal function $m'_{s+t,s}$ the analogue result in [3]. For the sake of completeness, we include the proof.

**Proposition 6.1.** *Let $\beta > 0$ and $s+t > -1-N$ there is a constant $A > 0$ such that for all $\xi \in \mathbb{B}$ $R > 1-|z_0|$ and a positive locally integrable function $f$ and for all $z \in B(z_0, R)$ if $s+t > -1$, then*

$$
R^\beta \int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi) \leq Am_{s+t,s}f(z).
$$

*More generally if $-1-N < s+t$ then*

$$
R^\beta \int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi) \leq Am'_{s+t,s}f(z).
$$

*Proof.* Recall that if $s+t > -1$, by Lemma 2.3, there is a constant $a > 0$ such that for all $k \in \mathbb{N}$, we have $\mu_{s+t}(B(z_0, 2^{k+1}R)) \leq a(2^{k+1}R)^{N+1+s+t}$, so that setting

$$
X = R^\beta \int\limits_{d(z_0,\xi)\geq R}\frac{f(\xi)}{d(z_0,\xi)^{N+1+s+t+\beta}}\mathrm{d}\mu_s(\xi)
$$

 

we have

$$\begin{aligned}
X &\leq \sum_{k=0}^{+\infty} \frac{1}{2^{k(N+1+s+t+\beta)} R^{N+1+s+t}} \int\limits_{d(z_0,\xi)<2^{k+1}R} f(\xi)\mathrm{d}\mu_s(\xi) \\
&\leq a2^{N+1+s+t} \sum_{k=0}^{+\infty} 2^{-k\beta} \frac{1}{\mu_{s+t}(B(z_0, 2^{k+1}R))} \int\limits_{B(z_0,2^{n+1}R)} f(\xi)\mathrm{d}\mu_s(\xi) \\
&\leq a2^{N+1+s+t} m_{s+t,s} f(z) \sum_{n=0}^{+\infty} 2^{-k\beta} = \frac{a2^{N+1+s+t}}{1-2^{-\beta}} m_{s+t,s} f(z).
\end{aligned}$$

We can take $A = \dfrac{a2^{N+1+s+t}}{1-2^{-\beta}}$ to conclude. $\square$

**Proposition 6.2.** *Let $\omega \in (D_p^{s,t,q,Q})$, we set again $\sigma(z) = R_k^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$ with $k \in (0, 1/2)$. Set $B = B(z', r)$ with $1 - |z'| < cr$ and $L = \left\{z \in B : 1 - |z| < C_0' \gamma^{\frac{1}{N+1+s+t}} r\right\}$ where $C_0' > 0$, $0 < \gamma < 1$, $r > 0$ and $c > 0$ are constants. Then if we set $L' = \left\{z \in \bar{B} : 1 - |z| < 2C_0' \gamma^{\frac{1}{N+1+s+t}} r\right\}$ and $\bar{B} = B(z', ar)$ with $a = K(C_0' + 1)$, there are two constants $C_1$ and $C_2 > 0$ independent of $\gamma$ such that if $s + t + \frac{Q-q}{p} > -1$ then*

$$\omega d\mu_{Q+pt}(L) \leq C_1 \sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \quad \textit{and} \quad \mu_{s+t+\frac{Q-q}{p}}(L') \leq C_2 \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}} \mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{6.1}$$

*Proof.* Let $k_1 = \frac{k}{1+k}$, then for $z \in L$ and $\xi \in B_{k_1}(z)$ we have $z \in B = B(z', r)$, $1 - |z| < C_0' \gamma^{\frac{1}{N+1+s+t}} r$ and $z \in B_k(\xi)$ because $k_1' = k$. Then

$$d(z', \xi) \leq K(d(z', z) + d(z, \xi)) < K[r + k_1(1 - |z|)] < K[r + k_1 C_0' \gamma^{\frac{1}{N+1+s+t}} r] < (C_0' + 1) r K$$

because $0 < k_1, \gamma < 1$. Then $\xi \in \overline{B} = B(z', ar)$ with $a = K(C_0' + 1)$. Moreover,

$$1 - |\xi| < 1 - |z| + d(z, \xi) < (k_1 + 1)(1 - |z|) < 2C_0' \gamma^{\frac{1}{N+1+s+t}} r$$

because $0 < k_1 < 1$, so that $\xi \in L' = \left\{z \in \overline{B} : 1 - |z| < 2C_0' \gamma^{\frac{1}{N+1+s+t}} r\right\}$. Then we have

$$\chi_L(z) \chi_{B_{k_1}(z)}(\xi) \leq \chi_{L'}(\xi) \chi_{B_k(\xi)}(z).$$

Remember that

$$\mu_{s+t+\frac{Q-q}{p}}(B_k(\xi)) \simeq \mu_{s+t+\frac{Q-q}{p}}(B_{k_1}(z)).$$

Hence,

$$\begin{aligned}
\omega d\mu_{Q+pt}(L) &= \int\limits_L \omega(z) d\mu_{Q+pt}(z) \\
&= \int\limits_L \left( \frac{\omega(z)}{\mu_s(B_{k_1}(z))} \int\limits_{B_{k_1}(z)} d\mu_s(\xi) \right) d\mu_{Q+pt}(z)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{B}} \left( \int_{\mathbb{B}} \frac{\chi_L(z)\chi_{B_{k_1}(z)}(\xi)\omega(z)}{\mu_s(B_{k_1}(z))} d\mu_{Q+pt}(z) \right) d\mu_s(\xi) \\
&\lesssim \int_{\mathbb{B}} \left( \int_{\mathbb{B}} \frac{\chi_{L'}(\xi)\chi_{B_k(\xi)}(z)\omega(z)}{\mu_s(B_k(\xi))} d\mu_{Q+pt}(z) \right) d\mu_s(\xi) \\
&= \int_{\mathbb{B}} \left( \frac{\chi_{L'}(\xi)}{\mu_s(B_k(\xi))} \int_{B_k(\xi)} \omega(z) d\mu_{Q+pt}(z) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \int_{L'} \left( R_k^{s,Q+pt} w(\xi) \right) (1-|\xi|^2)^{-t-\frac{Q-q}{p}} d\mu_{s+t+\frac{Q-q}{p}}(\xi) \\
&= \sigma d\mu_{s+t+\frac{Q-q}{p}}(L').
\end{aligned}
$$

Let us show the second inequality. Let $z \in \overline{B}$, then, $d(z',z) = ||z'|-|z|| + \left|1 - \frac{\langle z',z\rangle}{|z'||z|}\right| < ar$. Then $||z'|-|z|| < ar$ and $\left|1 - \frac{\langle z',z\rangle}{|z'||z|}\right| < ar$. Moreover, as $1-|z'| < cr$ then $1-|z| = 1-|z'|+|z'|-|z| < cr+ar$. Then for $z \in L'$, setting $\beta_1 = 2C_0'\gamma^{\frac{1}{N+1+s+t}} r$ and $\beta_2 = (c+a)r$, we get $1-|z| < \beta_1$, $1-|z| < \beta_2$ and $\left|1 - \frac{\langle z',z\rangle}{|z'||z|}\right| < ar$. Then, $L' \subset \{z \in \mathbb{B} : 1-|z| < \min(\beta_1,\beta_2), \ \left|1 - \frac{\langle z',z\rangle}{|z'||z|}\right| < ar\}$. In spherical coordinates we have, for $s+t+\frac{Q-q}{p} > -1$

$$
\begin{aligned}
\mu_{s+t+\frac{Q-q}{p}}(L') &\lesssim (s+t+\frac{Q-q}{p}+1) \int_{1-\rho<\min(\beta_1,\beta_2)} (1-\rho^2)^{s+t+\frac{Q-q}{p}} \rho \mathrm{d}\rho \int_{|1-\frac{\langle z',z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\leq (s+t+\frac{Q-q}{p}+1) \int_{1-\min(\beta_1,\beta_2)<\rho<1} (1-\rho)^{s+t+\frac{Q-q}{p}} \mathrm{d}\rho \int_{|1-\frac{\langle z',z\rangle}{|z'||z|}|<ar} \mathrm{d}\sigma(\xi) \\
&\lesssim r^N [-(1-\rho)^{s+t+\frac{Q-q}{p}+1}]_{1-\min(\beta_1,\beta_2)}^1 = r^N (\min(\beta_1,\beta_2))^{s+t+\frac{Q-q}{p}+1}.
\end{aligned}
$$

Then,

$$
\mu_{s+t+\frac{Q-q}{p}}(L') \lesssim r^N \beta_1^{s+t+\frac{Q-q}{p}+1} \backsimeq r^{N+s+t+\frac{Q-q}{p}+1} \gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}}. \tag{6.2}
$$

On the other hand, as $1-|z'| < cr$, we have by Lemma 2.3

$$
\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \simeq (ar)^{N+1} (\max(1-|z'|, ar))^{s+t+\frac{Q-q}{p}} \lesssim r^{N+1+s+t+\frac{Q-q}{p}}.
$$

□

We are now ready to prove our good lambda inequality (Theorem 1.13) that we recall here for the reader convenience. It is used to show that $S_{s+t,s} f \in L^p(\omega d\mu_{Q+pt})$ when $m'_{s+t,s} f \in L^p(\omega d\mu_{Q+pt})$.

**Theorem 6.3** (Good lambda inequality)**.** *Let* $\omega \in (D_p^{s,t,q,Q})$ $(1 < p < +\infty)$ *in the case both* $s+t+\frac{Q-q}{p} > -1$ *and* $-1 > s+t > -N-1$ *hold, or both* $s+t > -1$ *and* $Q \geq q$ *hold. There are two positive constants*

*$C$ and $\beta$ such that for all $\gamma$ sufficiently small, $\lambda > 0$ and for all positive locally integrable functions $f$, we have*

$$\omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \le \gamma\lambda\}) \le \\ CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(\{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}). \quad (6.3)$$

*Proof.* Let $\lambda > 0$, $0 < \gamma < 1$ and $f$ a positive locally integrable function. Let $E_\lambda = \{z \in \mathbb{B} : S_{s+t,s}f(z) > \lambda\}$. By the Whitney decomposition Lemma (see [8]), there are a positive integer $J$, $\delta > 1$ and a sequence of pseudo-balls $\{B_j\}_{j=1}^\infty$, with $B_j = B(z_j, r_j)$, such that

- $E_\lambda = \bigcup_{j=1}^\infty B_j$;
- every point of $E_\lambda$ is at most in $J$ balls $B_j$;
- the balls $B'_j = B(z_j, \delta r_j)$ touch the complement of $E_\lambda$ in $\mathbb{B}$.

To obtain (6.3), it is then sufficient to show that

$$\omega\mu_{Q+pt}(\{z \in B : S_{s+t,s}f(z) > 2\lambda, m'_{s+t,s}f(z) \le \gamma\lambda\}) \le CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega\mu_{Q+pt}(B), \quad (6.4)$$

where $B = B(z', r)$ is a ball in the Whitney decomposition of $E_\lambda$. From the third property of the Whitney decomposition, there is $z_0 \in B' = B(z', \delta r)$ such that $S_{s+t,s}f(z_0) \le \lambda$. Without loss of generality, assume that there is $\xi_0 \in B$ such that $m'_{s+t,s}f(\xi_0) \le \gamma\lambda$. Let $\widetilde{B} = B(z_0, R)$ with $R = \max(1-|z_0|, C_0 r)$ where we choose $C_0 \ge \max(c_1 K(1+\delta), \delta)$ where $c_1$ is the constant $C_1$ in Lemma 2.15.

We set $f_1 = 1_{\widetilde{B}} f$ and $f_2 = 1_{\mathbb{B}\setminus\widetilde{B}} f$, then $f = f_1 + f_2$ and we have

$$S_{s+t,s}f_2(z) \le \int_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{f(\xi)}{|1-\langle z_0, \xi\rangle|^{N+1+s+t}} \right| d\mu_s(\xi) + \\ \int_{\mathbb{B}\setminus\widetilde{B}} \left| \frac{1}{|1-\langle z, \xi\rangle|^{N+1+s+t}} - \frac{1}{|1-\langle z_0, \xi\rangle|^{N+1+s+t}} \right| f(\xi) d\mu_s(\xi).$$

So that, by Lemma 2.15 and Proposition 6.1, we finally have, for $z \in B$

$$S_{s+t,s}f_2(z) \;\le\; S_{s+t,s}f(z_0) + A' m'_{s+t,s}f(\xi_0) \le \lambda + A'\gamma\lambda.$$

So, since

$$S_{s+t,s}f(z) \le S_{s+t,s}f_1(z) + S_{s+t,s}f_2(z) \le S_{s+t,s}f_1(z) + \lambda + A'\gamma\lambda,$$

we have that $S_{s+t,s}f(z) > 2\lambda$ implies that $S_{s+t,s}f_1(z) > (1-A'\gamma)\lambda$. Therefore, to prove (6.4), it will be enough to show that

$$\omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s}f_1(z) > b\lambda\}) \le CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(B), \quad (6.5)$$

where $b = 1 - A'\gamma$ with an appropriate choice of $\gamma$. We are going to discuss according to the values of the radius $R = \max(1-|z_0|, C_0 r)$ of $\widetilde{B} = B(z_0, R)$. Let $E'_\lambda = \{S_{s+t,s}f_1 \ge b\lambda\} \cap B$.

 

*First case:* $C_0 r \leq 1-|z_0|$.

Then, $\widetilde{B} = B(z_0, 1-|z_0|)$. Therefore for all $z \in B$ and $\xi \in \widetilde{B}$, $|1-\langle z,\xi\rangle| \geq 1-|z| > C'(1-|z_0|)$, so that for all $z \in B$,

$$S_{s+t,s}f_1(z) = \int_{\widetilde{B}} \frac{f(\xi)d\mu_s(\xi)}{|1-\langle z,\xi\rangle|^{N+1+s+t}} \leq \frac{1}{(C'(1-|z_0|))^{N+1+s+t}} \int_{\widetilde{B}} f(\xi)d\mu_s(\xi).$$

Then

$$S_{s+t,s}f_1(z) < C'' m'_{s+t,s} f(\xi_0) \leq C''\gamma\lambda.$$

Hence, if we take $0 < \gamma < \gamma_0 = \min(\frac{1}{A'}, \frac{b}{C''})$ then it remains only to prove the following case.

*Second case:* $1-|z_0| < C_0 r$.

Then $\widetilde{B} = B(z_0, C_0 r)$ and $E'_\lambda \subseteq L$ for $L$ defined in Proposition 6.2. In fact, if $z \in E'_\lambda$, then $z \in B$ and

$$\begin{aligned} b\lambda \leq S_{s+t,s}f_1(z) &= \int_{\widetilde{B}} \frac{f(\xi)d\mu_s(\xi)}{|1-\langle z,\xi\rangle|^{N+1+s+t}} \leq \frac{1}{(1-|z|)^{N+1+s+t}} \int_{\widetilde{B}} f(\xi)d\mu_s(\xi) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}} m'_{s+t,s} f(\xi_0) \\ &\leq \frac{(C_0 r)^{N+1+s+t}}{(1-|z|)^{N+1+s+t}} \gamma\lambda. \end{aligned}$$

For $\sigma(z) = R_{k'}^{s,Q+pt}\omega(z)(1-|z|^2)^{-t-\frac{Q-q}{p}}$, with $k' \in (0,\frac{1}{2})$, by Lemma 5.22 we have $\sigma \in (A_{p,s+t+\frac{Q-q}{p}})$ so that $\sigma \in (A_{\infty,s+t+\frac{Q-q}{p}})$ because $(A_{p,s+t+\frac{Q-q}{p}}) \subseteq (A_{\infty,s+t+\frac{Q-q}{p}})$.

Given the fact that $L'$ is a measurable subset of $\bar{B} = B(z', ar)$, we have by Proposition 6.2 and Lemma 2.12

$$\begin{aligned} \omega d\mu_{Q+pt}(L) &\leq C\sigma d\mu_{s+t+\frac{Q-q}{p}}(L') \\ &\leq C\left(\frac{\mu_{s+t+\frac{Q-q}{p}}(L')}{\mu_{s+t+\frac{Q-q}{p}}(\bar{B})}\right)^{\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \\ &\leq C\gamma^{\frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0} \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \end{aligned}$$

As $E'_\lambda$ is a subset of $L = \{z \in \bar{B} : 1-|z| < C'_0\gamma^{\frac{1}{N+1+s+t}} r\}$, it follows that for $\beta = \frac{s+t+\frac{Q-q}{p}+1}{N+1+s+t}\beta_0$ we have

$$\omega d\mu_{Q+pt}(E'_\lambda) \leq C\gamma^\beta \sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}). \tag{6.6}$$

One shows by Fubini's theorem that $\sigma d\mu_{s+t+\frac{Q-q}{p}}(\bar{B}) \leq C\omega d\mu_{Q+pt}(\tilde{\bar{B}})$ with $\tilde{\bar{B}} = B(z', (2k+1)arK)$. And by Corollary 5.19 we get $\omega d\mu_{Q+pt}(\tilde{\bar{B}}) \leq CD_p^{s,t,q,Q}(\omega)\omega d\mu_{Q+pt}(B)$. Then

$$\omega d\mu_{Q+pt}(E'_\lambda) = \omega d\mu_{Q+pt}(\{z \in B : S_{s+t,s}f_1(z) > b\lambda\}) \leq CD_p^{s,t,q,Q}(\omega)\gamma^\beta \omega d\mu_{Q+pt}(B).$$

This ends the proof. □

The following results appear as consequences of Theorem 6.3 and Lemma 2.14.

**Theorem 6.4.** *Let $p > 1$. For $Q \geq q$ and $s+t > -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s} f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m_{s+t,s} f(z))^p d\mu_{Q+pt}(z).$$

*Proof.* It is enough to apply Lemma 2.14 to $S_{s+t,s} f$ and $m_{s+t,s} f$ with $t = 2\lambda$, $c = \frac{\gamma}{2}$, $b = \frac{1}{2}$ and $a = C D_p^{s,t,q,Q}(\omega)\gamma^\beta$. We just need to take $\gamma$ small enough so that $a < b^p$. $\square$

**Theorem 6.5.** *Let $p > 1$. For $s + t + \frac{Q-q}{p} > -1$ and $-N - 1 < s + t < -1$, if $\omega \in D_p^{s,t,q,Q}$ there is a constant $C_{s,t,q,Q} > 0$ such that*

$$\int_{\mathbb{B}} (S_{s+t,s} f(z))^p d\mu_{Q+pt}(z) \leq C_{s,t,q,Q} \int_{\mathbb{B}} (m'_{s+t,s} f(z))^p d\mu_{Q+pt}(z).$$

## 7. Final remark and open question

This part is simply a direct application of the two preceding sections and Remark 1.12. Therefore for $1 < p < +\infty$ we have the following two corollaries.

**Corollary 7.1.** *Let $\omega$ be a weight on $\mathbb{B}$. Then for $s + t > -1$ and $q = Q$, the following assertions are equivalent.*

1. *$P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_q)$;*
2. *$T_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
3. *$S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{q+pt})$;*
4. *$\omega \in (K_p^{s,t,q,q})$.*

*Proof.* By Remark 1.12, we have that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that by Theorem 6.4, since $(1 - |z|^2)^t S_{s+t,s} f(z) = P_{s,t} f(z)$ and $O_{s,t} f(z) = (1 - |z|^2)^t m_{s+t,s} f(z)$ and Theorem 5.23, we have (4) implies (1). $\square$

**Corollary 7.2.** *Let $\omega$ be a weight on $\mathbb{B}$. In the case both $s + t + \frac{Q-q}{p} > -1$ and $-1 > s + t > -N - 1$ hold, and in the case both $s + t > -1$ and $Q \geq q$ hold, if $\omega \in (D_p^{s,t,q,Q})$ then $P_{s,t}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, so that $S_{s+t,s}$ is well defined and continuous from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_{Q+pt})$.*

In Theorem 5.5 and in Theorem 5.6 we show that being in $(K_p^{s,t,q,Q})$ is a necessary condition for the continuity of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$, while in Corollary 7.2, we have that being in $D_p^{s,t,q,Q}$ is a sufficient one. When $Q = q$, we find out that $(K_p^{s,t,q,q}) = (D_p^{s,t,q,q})$, so that we have a necessary and sufficient condition. But when $Q > q$ we have $(D_p^{s,t,q,Q}) \subseteq (K_p^{s,t,q,Q})$. It is an open question to find in this case a necessary and sufficient condition for the boundedness of $P_{s,t}$ from $L^p(\omega d\mu_q)$ to $L^p(\omega d\mu_Q)$.

 

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