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Abstract

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Some results on ideals of multilattices

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Abstract. We present several properties of the different type of ideals of an ordered multilattice in order to build the connection between them and between them and the different type of submultilattices of multilattices.

Keywords. Multilattice, Ideal, submultilattice.

I. Introduction

In 1934, during the eighth congress of Scandinavian mathematicians, Hyperstructures theory was firstly introduced by F. Marty in [10]. Nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics (see for example [4] and [6]). Particularly, The theory of Multilattices, introduced by Benado [1] and contrariwise to bilattices (which include two different orderings), are defined on just one order relation, and it is the requirement of the existence of suprema and infima what is relaxed. Much more recently, Cordero et al [9] proposed an alternative algebraic definition of multilattice which is more closely related to that of lattice, allowing for natural definitions of related structures such that multiseamilattices and, in addition, is better suited for applications; for instance, Medina et al [11] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic. Later, Medina et al [12] present several properties of the different type of ideals of an ordered multilattice in order to solve theoretical problems arisen from the use of multilattices as underlying set of truth-values for a generalized framework of logic programming.

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Our aim in this paper is to study the connection between the different types of ideals and the connection between the different types of submultilattices of multilattices.

In this work, after introducing the preliminary definitions in the first section, section 2 is the presentation of the alternative notions of ideals for multilattices and the Hasse diagrams of the collection of the set of those different types of ideals and the set of those different types of submultilattices are given. Finally, we draw some conclusions and prospects for future work.

II. Preliminaries

Let (P, \leq) be a poset. For any $A \subseteq P$, let $L(A) := \{y \in P; y \leq x, \forall x \in A\}$ (the set of lower bounds of A), and $U(A) := \{y \in P; y \geq x, \forall x \in A\}$ (the set of upper bounds of A). We will assume that $L(\emptyset) = U(\emptyset) = P$. We will write $L(x)$ and $U(x)$ for $L(\{x\})$ and $U(\{x\})$ respectively. Similarly $U(\{x, y\})$ and $L(\{x, y\})$ will be respectively denoted by $U(x, y)$ and $L(x, y)$.

A multisupremum (resp. multinfimum) of $A \subseteq P$ is a minimal (resp. maximal) element of the set $U(A)$ (resp. $L(A)$). We will denote by $\text{multisup}(A)$ (resp. $\text{multinf}(A)$) the set of multisuprema (resp. multinfima) of A . By $x \sqcup y$ and $x \sqcap y$ we shall mean $\text{multisup}\{x, y\}$ and $\text{multinf}\{x, y\}$. When $\{x, y\}$ has a unique multisupremum (resp. multinfimum), the set $x \sqcup y$ (resp. $x \sqcap y$) is denoted by $x \vee y$ or $\text{sup}(x, y)$, (resp. $x \wedge y$ or $\text{inf}(x, y)$).

Recall that L and U are defined from $\mathcal{P}(P)$ to $\mathcal{P}(P)$ (the power set of P). We shall write LU and UL instead of $L \circ U$ and $U \circ L$ respectively.

Here are some interesting properties of the operators L and U reformulated from [11] and improved for simplicity of some proofs.

Property II.1. *Let (P, \leq) be a poset. The following properties hold for all $A, B \subseteq P$ and for all $a, b \in P$:*

$$(P.1) \quad L(A \cup B) = L(A) \cap L(B); \quad U(A \cup B) = U(A) \cap U(B).$$

$$(P.2) \quad \text{If } A \subseteq B, \text{ then } L(B) \subseteq L(A) \text{ and } U(B) \subseteq U(A).$$

$$(P.3) \quad \text{If } a \leq b, \text{ then } L(a) \subseteq L(b) \text{ and } U(b) \subseteq U(a).$$

$$(P.4) \quad L(\text{multinf}(A)) \subseteq L(A) \subseteq L(\text{multisup}(A)) \text{ and } U(\text{multisup}(A)) \subseteq U(A) \subseteq U(\text{multinf}(A)).$$

$$(P.5) \quad LU(A) = L(\text{multisup}(A)) \text{ and } UL(A) = U(\text{multinf}(A)). \text{ Particularly, } LU(a) = L(a) \text{ and } UL(a) = U(a).$$

Definition II.2. [1] *A poset, (T, \leq) , is called a **lattice** if for all $x, y \in T$ the supremum and the infimum of $\{x, y\}$ exist.*

*T is said to be a **complete lattice** if every subset of T has a unique multisupremum and a unique multinfimum.*

We next recall the definition of a multilattice, where contrary to lattices the uniqueness condition is dropped in the sense that is more than one minimal (resp. maximal) element of the set of upper (resp. lower) bounds.

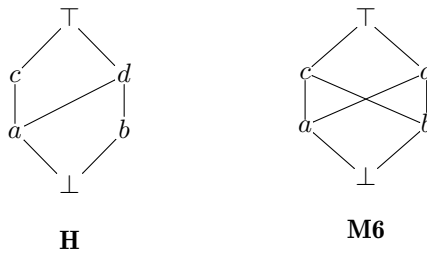
Definition II.3. [1] A poset, $(M; \leq)$, is a **multilattice** if for all $x, y, a \in M$ with $x \leq a$ and $y \leq a$, there exists $z \in x \sqcup y$ such that $z \leq a$ and its dual version for $x \sqcap y$.

A multilattice M is said to be a **complete multilattice** if every subset of M has at least one multisupremum and at least one multinfimum.

A \sqcup -**full multilattice** (resp. \sqcap -**full multilattice**) is a multilattice in which $x \sqcup y \neq \emptyset$ (resp. $x \sqcap y \neq \emptyset$), for all $x, y \in M$. A multilattice is full iff it is \sqcup -full multilattice and \sqcap -full multilattice

Here is an example illustrating the difference between lattice and multilattice.

Example II.4.



H is a complete lattice and **M6** is a complete multilattice, notice that **M6** is not a lattice since $\text{sup}(a, b)$ does not exist but $a \sqcup b = \{c, d\}$.

III. Types of ideals of multilattices

The extension of the concept of ideal to the framework of multilattices have been studied by several authors. In the literature one can find the definition of s-ideal given by Rachůnek [13], the definition of l-ideal by I. Jonhston [5], the definition of m-ideal due to Burgess [11].

Before recalling those types of ideals of multilattice, let us mention the definition of ideal in the framework of lattices.

Definition III.1. [11] A non-void subset X of a lattice T is said to be an ideal if it is downward closed and for all $x, y \in X$, $x \vee y \in X$.

Proposition III.2. A non-void subset X of a lattice T is an ideal if and only if for all $x, y \in X$, $L(x \vee y) \subseteq X$.

Proof. Let X be an ideal of a lattice T . Then, for every $x, y \in X$, $x \vee y \in X$ and X is downward closed. Thus, for all $x, y \in X$, $L(x \vee y) \subseteq X$.

Conversely, we suppose that $L(x \vee y) \subseteq X$, for all $x, y \in X$. Hence, it is sufficient to prove that X is downward closed. Let $x, y \in T$ such that $x \leq y$ and $y \in X$. Therefore, $x \in L(y) = L(y \vee y) \subseteq X$. Thus X is an ideal. ■

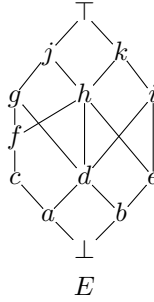
The following is the definition of some types of ideals in the framework of multilattices.

Definition III.3. [5] Let M be multilattice. A non-void subset X of M is

- An **s-ideal** if X is downward closed and for all $x, y \in X$, $U(x, y) \cap X \neq \emptyset$.
- An **l-ideal** if for all $x, y \in X$, $LU(x, y) \subseteq X$.
- An **m-ideal** if for all $x, y \in X$ such that $\sup(x, y)$ exists, $L(\sup(x, y)) \subseteq X$.

Now we illustrate those types of ideals by an example.

Example III.4. The following is the Hasse diagram of multilattice E



In the multilattice E , $A_s = \{\perp, a, b, c, d, e, f, h\}$ and $B_s = \{\perp, a, b, d, e, i\}$ are s-ideals; $A_l = \{\perp, a, b, d, e\}$ is an l-ideal and $A_m = \{\perp, a, b, c, d\}$ is an m-ideal.

Notation III.5. We will denote the set of all s, l, m-ideals of (M, \leq) by $\mathcal{I}_s(M)$, $\mathcal{I}_l(M)$ and $\mathcal{I}_m(M)$ respectively.

Remark III.6. 1. We have $\mathcal{I}_s(M) \subseteq \mathcal{I}_l(M) \subseteq \mathcal{I}_m(M)$ but those inclusions will be in general strict. For example in the multilattice E , $A_l \in \mathcal{I}_l$ but $A_l \notin \mathcal{I}_s$ and $A_m \in \mathcal{I}_m$ but $A_m \notin \mathcal{I}_l$.

2. The intersection of two l-ideals (resp. m-ideals) is an l-ideal (resp. m-ideal) but the intersection of two s-ideals is not in general an s-ideal (for more details see [5]). For example in the above multilattice E , A_s and B_s are s-ideals but $A_s \cap B_s = A_l$ is not an s-ideal.

In the particular case of a lattice, all the definitions above collapsed to the usual definition of ideal of a lattice.

The following theorem gives a characterization of s-ideal and l-ideal which is more computable and suitable for certain verifications.

Theorem III.7. *Let (M, \leq) be a multilattice, $X \subseteq M$ a non-empty subset of M . X is an*

1. *s-ideal if and only if for all $x, y \in X$, there is $z \in x \sqcup y$ such that $L(z) \subseteq X$.*
2. *l-ideal if and only if for all $x, y \in X$, we have $L(x \sqcup y) \subseteq X$.*

Proof. For the first point, let $x, y \in M$. Since X is an s-ideal we have $U(x, y) \cap X \neq \emptyset$ so there exists $a \in X$ such that $x \leq a, y \leq a$. Therefore there exists $z \in x \sqcup y$ such that $z \leq a$. Hence $L(z) \subseteq L(a) \subseteq X$.

(\Leftarrow) Suppose that for all $x, y \in X$, there exists $z \in x \sqcup y$ such that $L(z) \subseteq X$. Let $x, y \in M$, if $x \in X$ and $y \leq x$ then $x \sqcup y = \{x\} = x \sqcup x$, Therefore $y \in L(x) \subseteq X$. Hence X is downward closed.

Let $x, y \in X$, then by the hypothesis, there is $z \in (x \sqcup y) \cap X$. Since $x \sqcup y \subseteq U(x, y)$, we have $U(x, y) \cap X \neq \emptyset$

The second point is a consequence of P.6 of the Property II.1. ■

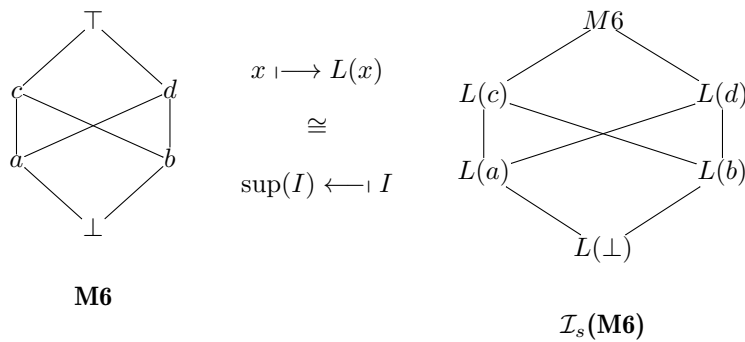
It had been proved in [12] that: if (M, \leq) is a **coherent multilattice** (i.e., multilattice without infinite antichain) then $(\mathcal{I}_s(M), \subseteq)$ is isomorphic to (M, \leq) .

Proposition III.8. [?] *If M is a coherent multilattice, then the map*

$$\begin{aligned} \varphi: (M, \leq) &\longrightarrow (\mathcal{I}_s(M), \subseteq) \\ x &\longmapsto \varphi(x) = L(x) \end{aligned} \quad \text{is an isomorphism of posets.}$$

From above proposition it is clear that each s-ideal of a coherent multilattice is principal. The following example is the illustration of the above isomorphism between the multilattice $M6$ and the set of its s-ideals

Example III.9.



Notice that $\mathcal{I}_l(M6) = \mathcal{I}_m(M6) = \{\{\perp\}, \{\perp, a\}, \{\perp, b\}, \{\perp, a, b\}, \{\perp, a, c\}, \{\perp, a, b, c\}, \{\perp, a, b, d\}, M6^*\}$.

The above notions of ideal are not suitable for congruence and homomorphism as stated by I.P. Cabrera et al. in [2] and where another definition of ideal is set up.

Definition III.10. [2] *A non-void subset X of multilattice M is an **ideal** if the following conditions hold:*

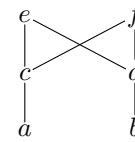
- (i) For all $x, y \in X$, $x \sqcup y \subseteq X$.
- (ii) For all $x \in X$ and for all $a \in M$, $x \sqcap a \subseteq X$.
- (iii) For all $x, y \in M$, if $(x \sqcap y) \cap X \neq \emptyset$ then $x \sqcap y \subseteq X$.

Notation III.11. We will denote by $\mathcal{I}(M)$ the set of all ideals of M .

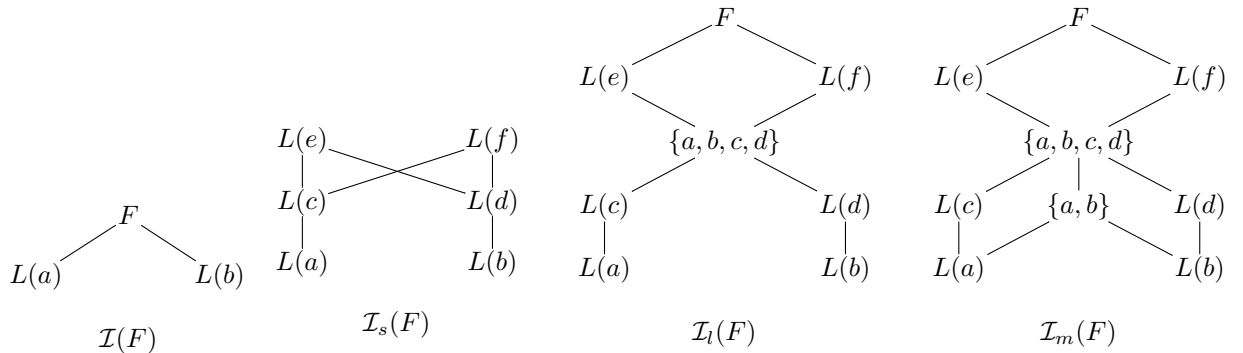
- Remark III.12.**
1. If a multilattice M is \sqcup -full, then any ideal of M is an s -ideal (i.e., $\mathcal{I} \subseteq \mathcal{I}_s$). But in general $\mathcal{I} \not\subseteq \mathcal{I}_s$ and $\mathcal{I}_s \not\subseteq \mathcal{I}$, nevertheless we always have $\mathcal{I} \subseteq \mathcal{I}_l$ (see [?]).
 2. If $|x \sqcup y| \geq 2$ then, for all $z \in x \sqcup y$, $L(z)$ is an s -ideal of M but not an ideal of M because we will have $x \sqcup y \not\subseteq L(z)$.
 3. For any multilattice M , the sets \mathcal{I} , \mathcal{I}_s , \mathcal{I}_l , \mathcal{I}_m are posets (ordered by the set inclusion).

In the example below we give a multilattice and the Hasse diagrams of its different type of ideal's set.

Example III.13. Consider the following multilattice:



F



One can observe in the above example that $\mathcal{I}(F) \not\subseteq \mathcal{I}_s(F)$, $\mathcal{I}_s(F) \not\subseteq \mathcal{I}(F)$, $\mathcal{I}_s(F) \subsetneq \mathcal{I}_l(F) \subsetneq \mathcal{I}_m(F)$, $\mathcal{I}_s(F) \subsetneq \mathcal{I}_l(F) \subsetneq \mathcal{I}_m(F)$ and $\mathcal{I}(F)$, $\mathcal{I}_l(F)$ and $\mathcal{I}_m(F)$ are semilattices which needs a bottom element to become a lattice.

Theorem III.14. [?] *Let (M, \leq) be a multilattice. Let $\mathcal{I}_0(M)$ denote the set $\mathcal{I}(M) \cup \{\emptyset\}$. The following statements are satisfied:*

- (i) $(\mathcal{I}(M), \subseteq)$ is a multilattice.
- (ii) $(\mathcal{I}_0(M), \subseteq)$ is a lattice.
- (iii) If M is bounded, then $(\mathcal{I}(M), \subseteq)$ is a lattice.

J. Medina et al. in [11] have similar result for $\mathcal{I}_l(M)$ and $\mathcal{I}_m(M)$. But $\mathcal{I}_S(M)$ will not always satisfy the above Theorem.

In [12], J. Medina, M. Ojeda-Aciego and J. Ruiz-Calviñó have defined two different types of submultilattices, namely : full submultilattice (or f-submultilattice) and restricted submultilattice (or r-submultilattice).

Definition III.15. [?] *A non-empty subset X of a multilattice M is said to be a*

- (1) *Full submultilattice, or a **f-submultilattice** if for every $x, y \in X$, all the multisuprema and multinfima of $\{x, y\}$ in M are in X (i.e., $\forall x, y \in X, (x \sqcap y) \cap X = x \sqcap y$ and $(x \sqcup y) \cap X = x \sqcup y$).*
- (2) *Restricted submultilattice, or a **r-submultilattice** if for every $x, y \in X$, at least one multisupremum and at least one multinfimum of $\{x, y\}$ in M is in X (i.e., $\forall x, y \in X, (x \sqcap y) \cap X \neq \emptyset$ and $(x \sqcup y) \cap X \neq \emptyset$).*

Example III.16. *In the multilattice E of Example III.4, $A_f = \{b, d, e, h, i, k\}$ is a f-submultilattice and $A_r = \{b, d, e, i\}$ is a r-submultilattice.*

Clearly, any f-submultilattice is a r-submultilattice but the converse does not hold, for example the above A_r is not a f-submultilattice since $(b \sqcap d) \cap A_r = \{e\} \neq b \sqcap d$.

Here are some relations between those types of submultilattices and the different types of ideals of multilattices.

Proposition III.17. [?]

- a) *Let X be a r-submultilattice of a multilattice M . X is an m -ideal of M if and only if for all $x \in X$ and for all $a \in M$, $a \sqcap x \subseteq X$.*
- b) *Every s -ideal of a coherent multilattice M is a r-submultilattice of M but the converse is not true.*

Notice that every ideal of M is a f-submultilattice and a f-submultilattice X is an ideal of M if and only if it is downward closed and for all $x, y \in M$, $(x \sqcap y) \cap X \neq \emptyset \Rightarrow (x \sqcap y) \subseteq X$.

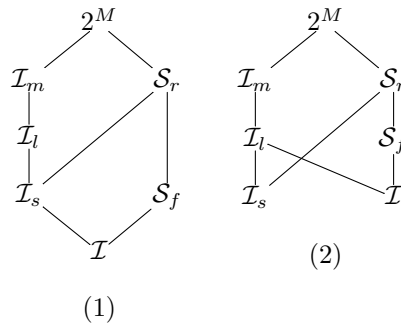
Notation III.18. Let us denote by $\mathcal{S}_f(M)$ and $\mathcal{S}_r(M)$ the set all f - and r -submultilattices of M respectively.

Remark III.19. (1) Clearly, any f -submultilattice is a r -submultilattice (i.e., $\mathcal{S}_f(M) \subseteq \mathcal{S}_r(M)$) but the converse does not hold;

(2) Every ideal of M is a f -submultilattice (i.e., $\mathcal{I}(M) \subseteq \mathcal{S}_f(M)$) and a f -submultilattice X is an ideal of M if and only if it is downward closed and for all $x, y \in M$, $(x \sqcap y) \cap X \neq \emptyset \Rightarrow (x \sqcap y) \subseteq X$;

(3) Any s -ideal of M is a r -submultilattice of M , i.e., $\mathcal{I}_s(M) \subseteq \mathcal{S}_r(M)$.

The relations between the different types of ideals and submultilattices can briefly be summarized by the following Hasse diagrams (with set inclusion as order): (1) when the multilattice M is \sqcup -full and (2) when it is not \sqcup -full



IV. Conclusion and future research

Our goal in this paper was to study some connections between different types of ideals and the connections between different notion of ideal and the notion of submultilattice in the framework of multilattices. The lattice of those connection is constructed and illustrated by some examples.

The systematic generalization of crisp concepts to the fuzzy case has proven to be an important theoretical tool for the development of new methods of reasoning under uncertainty, imprecision and lack of information. In [2] it have been proved that the quotient of a full bounded multilattice by a congruence is also a multilattice. As future work, we will focus on the corresponding fuzzifications of concepts such as ideal and congruences over multilattices, in the line of [3], [7] and [8].

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