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On a reduction criterion for bifiltrations in terms of their generalized Rees rings

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Abstract. In this paper, a concept of reduction of bifiltrations is introduced. Similarly to the case of filtrations, a reduction criterion for bifiltrations in some class in terms of their generalized Rees rings is given.

1. Introduction

Throughout this paper, A denotes a commutative ring.

The concept of reduction of ideals was introduced by D.G. Northcott and D.Rees in [**NR**]. It was actively studied in the literature since its introduction. Since powers of ideals are special filtrations, the concept of reduction of ideals was generalized to filtrations by J.S. Okon and L.J. Ratliff,Jr in [**OR**], where the authors had given many important results on the subject.

Here we introduce a concept of reduction for rings bifiltrations and we give a criterion of reduction for bifiltrations in some class in terms of their generalized Rees rings, similarly to the case of filtrations.

2. Bifiltrations

2.1. Filtrations.

Let us recall the following definitions which will be used in the sequel.

(2.1.1) By a **filtration on the ring** A, we mean a family $f = (I_n)_{n\in\mathbb{Z}}$ of ideals of A such that $I_0 = A$, $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}$ and $I_pI_q \subseteq I_{p+q}$ for all $p, q \in \mathbb{Z}$. It follows that if $n \leq 0$, then $I_n = A$.

(2.1.2) For each integer $k \geq 1$, let $f^{(k)} = (I_{nk})_{n\in\mathbb{Z}}$. Then $f^{(k)}$ is a filtration on A which is called **the filtration of index** k **extracted from the filtration** f.

(2.1.3) If I is an ideal of A, then the filtration $f_I = (I^n)_{n\in\mathbb{Z}}$,

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where $I^n = A$ for all $n \leq 0$, is called the **I-adic filtration** of A

(2.1.4) Let $f = (I_n)_{n\in\mathbb{Z}}$ and $g = (J_n)_{n\in\mathbb{Z}}$ be filtrations on a ring A. The filtration f is called a **reduction of** g if $f \leq g$ and if there exists an integer $N \geq 1$ such that $J_n = \sum_{p=0}^{N} I_{n-p}J_p$ for all $n > N$, see [OR] for more information.

2.2. Bifiltrations.

The set $\mathbb{Z}^2$ is partially ordered as follows :

For all $m, n, p, q \in \mathbb{Z}$, $(m, n) \preceq (p, q)$ if and only if $m \leq p$ and $n \leq q$.

(2.2.1) A **bifiltration** on the ring A is a family $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of ideals of A such that

(i) $I_{0,0} = A$

(ii) For all $m, n, p, q \in \mathbb{Z}$, if $(m, n) \preceq (p, q)$, then $I_{p,q} \subseteq I_{m,n}$

(iii) $I_{m,n}I_{p,q} \subseteq I_{m+p,\ n+q}$ for all $m, n, p, q \in \mathbb{Z}$.

Condition (ii) is equivalent to say that

$I_{m+1,n} \subseteq I_{m,n}$ and $I_{m,n+1} \subseteq I_{m,n}$ for all $m, n \in \mathbb{Z}$

It follows from condition (ii) that if $(m, n) \preceq (0, 0)$, then $A = I_{0,0} \subseteq I_{m,n}$, so $I_{m,n} = A$.

If $m \leq 0$, then $(m, 0) \preceq (0, 0)$, hence $A = I_{0,0} \subseteq I_{m,0}$, so $I_{m,0} = A$.

Similarly, if $n \leq 0$, then $I_{0,n} = A$.

(2.2.2) Throughout this paper all the bifiltrations $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ are supposed to satisfy the following additional conditions :

(EPI) For all $(k, l) \in \mathbb{Z}^2$, $I_{k,l} = I_{k,0}$ if $k \geq 0$ and $l \leq 0$ and $I_{k,l} = I_{0,l}$ if $k \leq 0$ and $l \geq 0$.

Such a bifiltration is said to be of **Essentially Positive Indices type** (EPI type for short).

This definition will be extended to bifiltrations $F = (I_{m,n})_{(m,n)\in\mathbb{N}^2}$ with indices in $\mathbb{N}^2$ where negative sub-indices may occur in $I_{m,n}$.

As an example, let $f = (I_n)_{n\in\mathbb{Z}}$ and $g = (J_n)_{n\in\mathbb{Z}}$ be filtrations on the ring A. Then $I_n = J_n = A$ for all $n \leq 0$. We will denote by $f \ltimes g$ the bifiltration F defined as $f \ltimes g = (F_{m,n})$, where $F_{m,n} = I_mJ_n$ for all m, n.

Then $f \ltimes g$ is a bifiltration which is of EPI type.

(2.2.3) Let $F = (I_{m,n})_{(m,n)\in\mathbb{N}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be bifiltrations on A. We set $F \leq G$ if and only if $I_{m,n} \subseteq J_{m,n}$ for all $(m, n) \in \mathbb{Z}^2$.

(2.2.4) Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration on the ring A. For integers $k, l \in \mathbb{N}^*$, we write $F^{(k,l)} = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$, where $J_{m,n} = I_{mk,nl}$ for all m, n. Then $F^{(k,l)}$ is a bifiltration on A which is called **the bifiltration of index** (k, l) **extracted from the bifiltration** F.

(2.2.5) Let $f = (I_n)_{n\in\mathbb{Z}}$ and $g = (J_n)_{n\in\mathbb{Z}}$ be filtrations on the ring A and let k, l be positive integers. Consider the filtrations $f^{(k)} = (I_{nk})_{n\in\mathbb{Z}}$ and $g^{(l)} = (J_{nl})_{n\in\mathbb{Z}}$. Then $f^{(k)} \ltimes g^{(l)} = (I_{mk}J_{nl}) = (f \ltimes g)^{(k,l)}$.

3. Bigraded rings

3.1. Definitions.

(3.1.1) The ring A is said to be $\mathbb{Z}^2$- **bigraded** if there exists a family $(A_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of subgroups of A such that $A = \bigoplus_{(m,n)\in\mathbb{Z}^2} A_{m,n}$ with $A_{m,n}A_{p,q} \subseteq A_{m+p,n+q}$ for all $m,n,p,q \in \mathbb{Z}$.

The elements of the subgroup $A_{m,n}$ are said to be **homogeneous of degree** (m,n).

(3.1.2) The concept of $\mathbb{N}^2$- **bigraded** ring is defined by replacing $\mathbb{Z}$ by $\mathbb{N}$ in the previous definition.

3.2. The Veronese subrings of a bigraded ring.

Let $A = \bigoplus_{(m,n)\in\mathbb{Z}^2} A_{m,n}$ be a bigraded ring. For integers $k,l \in \mathbb{N}^*$, we set $A^{(k,l)} = \bigoplus_{(m,n)\in\mathbb{Z}^2} A_{mk,nl}$.

Then $A^{(k,l)}$ is a bigraded subring of A which is called the **Veronese subring of index** (k,l) **of** A.

4. The Rees rings of a bifiltration

Definition 4.1. *Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration of EPI type on the ring A.*

(4.1.1) We define the ***Rees ring*** $R(A,F) = \bigoplus_{(m,n)\in\mathbb{N}^2} I_{m,n}X^mY^n$ *of F as a bigraded subring of the ring $A[X,Y]$ of polynomials in two indeterminates X and Y with coefficients in A, which is bigraded by taking as degrees, deg $X = (1,0)$ and deg $Y = (0,1)$.*

(4.1.2) ***The generalized Rees ring of*** *F is by definition the bigraded subring $\mathcal{R}(A,F) = \bigoplus_{(m,n)\in\mathbb{Z}^2} I_{m,n}X^mY^n$ of $A[u,v,X,Y]$ where $u = X^{-1}$ and $v = Y^{-1}$.*

(4.1.3) Notice that

$$R(A,F) = A[I_{1,0}X, I_{0,1}Y, I_{2,0}X^2, I_{1,1}XY, I_{0,2}Y^2, ...]$$

and that

$$\mathcal{R}(A,F) = A[u,v,I_{1,0}X, I_{0,1}Y, I_{2,0}X^2, I_{1,1}XY, I_{0,2}Y^2, ...] = R(A,F)[u,v].$$

(4.1.4) The ***bifiltration*** *F* ***is called Noetherian*** *if its generalized Rees ring $\mathcal{R}(A,F)$ is Noetherian.*

There is a close connexion between the generalized Rees ring $\mathcal{R}(A,F^{(k,l)})$ of the bifiltration $F^{(k,l)}$ and the Veronese subring $\mathcal{R}(A,F)^{(k,l)} = \bigoplus_{(m,n)\in\mathbb{Z}^2} I_{mk,nl}X^{mk}Y^{nl}$ of index (k,l) of the generalized Rees ring of F, as shown in the following Proposition.

Proposition 4.2. *Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration of EPI type on the ring A. Then the two bigraded rings $R(A,F)^{(k,l)}$ and $R(A,F^{(k,l)})$ are isomorphic for all $k,l \in \mathbb{N}^*$.*

PROOF. We have $\mathcal{R}(A,F)^{(k,l)} = \bigoplus_{(m,n)\in\mathbb{Z}^2} I_{mk,nl}X^{mk}Y^{nl}$

Consider the mapping φ from $\mathcal{R}(A,F^{(k,l)})$ into $\mathcal{R}(A,F)^{(k,l)}$ defined as

$$\varphi(z) = \sum_{m,n} a_{m,n}X^{mk}Y^{nl} \text{ for all } z = \sum_{m,n} a_{m,n}X^mY^n \in \mathcal{R}(A,F^{(k,l)}),$$

where $a_{m,n} \in I_{mk,nl}$ for all m,n. Then it is easily shown that φ is a ring isomorphism. □

REMARK 4.3. *(4.3.1) similarly $R(A,F)^{(k,l)} \simeq R(A,F^{(k,l)})$ for all $k,l \in \mathbb{N}^*$.*

(4.3.2) Let $f = (I_n)_{n\in\mathbb{Z}}$ and $g = (J_n)_{n\in\mathbb{Z}}$ be filtrations on the ring A. Consider the bifiltration $F = f \ltimes g$ defined as
$f \ltimes g = (F_{m,n})_{(m,n)\in\mathbb{Z}^2}$, where $F_{m,n} = I_mJ_n$ for all m,n. Let k,l be positive integers. Consider the filtrations $f^{(k)} = (I_{nk})_{n\in\mathbb{Z}}$ and $g^{(l)} = (J_{nl})_{n\in\mathbb{Z}}$. Then we have $f^{(k)} \ltimes g^{(l)} = (I_{mk}J_{nl}) = (\ f \ltimes g)^{(k,l)}$. Therefore by Proposition 4.2,

$$\mathcal{R}(A, f\ltimes g)^{(k,l)} \simeq \mathcal{R}(A,(f\ltimes g)^{(k,l)}) = \mathcal{R}(A, f^{(k)} \ltimes g^{(l)}).$$

5. Reduction of bifiltrations

DEFINITION 5.1. *Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be bifiltrations of EPI type on the ring A with $F \leq G$ and let $r \geq 1$ and $s \geq 1$ be arbitrary integers. Then if $m \leq 0$ and $n \leq 0$ or if $0 \leq m \leq r$ and $0 \leq n \leq s$, then it is easily checked that*

$$J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q}.$$

*We call F a **reduction of** G if there exist integers $r \geq 1$ and $s \geq 1$ such that*

(RED) $\displaystyle J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q}$ *for all $(m,n) \in \mathbb{Z}^2$, with $m > r$ or $n > s$.*

EXAMPLE 5.2. *(5.2.1) Each bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of EPI type is a reduction of itself.*

Indeed let $(m,n) \in \mathbb{Z}^2$, with $m > r = 1$ or $n > s = 1$. If $m > 1$ and $n > 1$, then

$$\sum_{p=0}^{1}\sum_{q=0}^{1} I_{m-p,n-q}I_{p,q} = I_{m,n} + I_{m,n-1}I_{0,1} + I_{m-1,n}I_{1,0} + I_{m-1,n-1}I_{1,1} = I_{m,n}$$

If $m > 1$ and $n \leq 1$, then

a) If $n < 0$, then $I_{m,n} = I_{m,0}$.

$$\sum_{p=0}^{1}\sum_{q=0}^{1} I_{m-p,n-q}I_{p,q} = I_{m,0} + I_{m,0}I_{0,1} + I_{m-1,0}I_{1,0} + I_{m-1,0}I_{1,1} = I_{m,0}$$

since $I_{m,0}I_{0,1} \subseteq I_{m,1} \subseteq I_{m,0}, I_{m-1,0}I_{1,0} \subseteq I_{m,0}, \qquad I_{m-1,0}I_{1,1} \subseteq I_{m,1} \subseteq I_{m,0}$

b) If $n \geq 0$, then $0 \leq n \leq 1$.

$\displaystyle\sum_{p=0}^{1}\sum_{q=0}^{1} I_{m-p,n-q}I_{p,q} = I_{m,n}$ since each component of the sum is contained in $I_{m,n}$ and one of them equals $I_{m,n}$.

The case $m \leq 1$ and $n > 1$ is similar to the previous one.

(5.2.2) Let $f = (I_n)_{n\in\mathbb{Z}}$, $g = (J_n)_{n\in\mathbb{Z}}$, $h = (H_n)_{n\in\mathbb{Z}}$ and $k = (K_n)_{n\in\mathbb{Z}}$ be filtrations on the ring A . Assume that f is a reduction of h and that g a reduction of k. Then $f \ltimes g$ is a reduction of $h \ltimes k$.
Indeed $f \ltimes g = (F_{m,n})$ with $F_{m,n} = I_m J_n$ and $h \ltimes k = (G_{m,n})$ where $G_{m,n} = H_m K_n$. We have $I_m J_n \subseteq H_m K_n$ for all m, n, hence $F \leq G$.

We know by (2.2.2) that $f \ltimes g$ and $h \ltimes k$ are bifiltrations of EPI type.

Let $N_1 \geq 1$ and $N_2 \geq 1$ be integers such that $H_m = \sum_{p=0}^{N_1} I_{m-p}H_p$ for all $m > N_1$ and $K_n = \sum_{q=0}^{N_2} J_{n-q}K_q$ for all $n > N_2$.

If $m > N_1$ and $n > N_2$, then

$$G_{m,n} = H_m K_n = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2} I_{m-p}J_{n-q}H_pK_q = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2} F_{m-p,n-q}G_{p,q}$$

If $m \leq N_1$ and $n > N_2$, then

a) If $m \leq 0$ then $H_m = A$, $I_{m-p}H_p = H_p$ for all $p = 0, 1, ..., N_1$.

$$\sum_{p=0}^{N_1} I_{m-p}H_p = I_m + H_1 + ... + H_{N_1} = A \text{ since } I_m = A$$

Therefore $H_m = \sum_{p=0}^{N_1} I_{m-p}H_p$

b) If $m > 0$, then

$$\begin{aligned} \sum_{p=0}^{N_1} I_{m-p}H_p &= I_m + I_{m-1}H_1 + ... + I_0H_m + I_{-1}I_{m+1} + .. + I_{m-N_1}H_{N_1} \\ &= I_m + I_{m-1}H_1 + ... + H_m + H_{m+1} + .. + H_{N_1} \\ &= H_m \end{aligned}$$

Hence $G_{m,n} = H_m K_n = \sum_{\substack{0\leq p\leq N_1 \\ 0\leq q\leq N_2}} F_{m-p,n-q}G_{p,q}$

If $m > N_1$ and $n \leq N_2$ then the result follows by similar arguments.

The following Theorem and its corollaries give some characterizations of the reduction of bifiltrations. They are the analogues for bifiltrations of Theorem 2.3 of [**OR**] and some of its Corollaries .

THEOREM 5.3. *Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be two bifiltrations of EPI type on the ring A such that $F \leq G$. Assume that the ideal $J_{m,n}$ is finitely generated for all m, n.*

Then F is a reduction of G if and only if $\mathcal{R}(A, G)$ is a finitely generated $\mathcal{R}(A, F)$-module.

PROOF. Suppose that F is a reduction of G. Then there exist integers $r \geq 1$ and $s \geq 1$ such that

$$(RED) \quad J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q} \quad \text{for all } (m,n) \in \mathbb{Z}^2 \text{ with } m > r \text{ or } n > s.$$

Let $(m,n) \in \mathbb{Z}^2$ and let $(x_{m,n,j})$ $1 \leq j \leq t_{m,n}$ be a system of generators of the ideal $J_{m,n}$.

Consider any homogeneous element z of degree (m,n) of $\mathcal{R}(A,G)$. Then z is of the form $z = d_{m,n}X^mY^n$, where $d_{m,n} \in J_{m,n}$.

Following equation (RED), $d_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{i} a_{i,p,q}b_{i,p,q}$, where $a_{i,p,q} \in I_{m-p,n-q}$ and $b_{i,p,q} \in J_{p,q}$ for all i, p and q. We may write

$$b_{i,p,q} = \sum_{j=1}^{t_{p,q}} c_{i,p,q,j}x_{p,q,j}, \quad \text{where } c_{i,p,q,j} \in A \text{ for all } i,p,q,j.$$

Therefore $d_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{i}\sum_{j=1}^{t_{p,q}} a_{i,p,q}c_{i,p,q,j}x_{p,q,j}$, where $a_{i,p,q} \in I_{m-p,n-q}$ and $c_{i,p,q,j} \in A$ for all i,p,q,j.

So

$$\begin{aligned} z &= \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{i}\sum_{j=1}^{t_{p,q}} a_{i,p,q}c_{i,p,q,j}x_{p,q,j}X^mY^n \\ &= \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{i}\sum_{j=1}^{t_{p,q}} a_{i,p,q}c_{i,p,q,j}X^{m-p}Y^{n-q}(x_{p,q,j}X^pY^q) \end{aligned}$$

Suppose that $m > r$ and $n > s$. Then $a_{i,p,q}c_{i,p,q,j}X^{m-p}Y^{n-q} \in \mathcal{R}(A,F)$ for all i,p,q,j and z is an $\mathcal{R}(A,F)$-linear combination of the elements $x_{p,q,j}X^pY^q$ of $\mathcal{R}(A,G)$, where $0 \leq p \leq r$, $0 \leq q \leq s$, $1 \leq j \leq t_{p,q}$.

Suppose now that $m \leq r$ and $n > s$. Consider an integer p such that $0 \leq p \leq r$.

a) If $p \leq m \leq r$ then z is an $\mathcal{R}(A,F)$- linear combination of the elements $x_{p,q,j}X^pY^q$ of $\mathcal{R}(A,G)$, where $0 \leq p \leq r$, $0 \leq q \leq s$, $1 \leq j \leq t_{p,q}$.

b) If $m \leq p \leq r$, then

$$\begin{aligned} a_{i,p,q}c_{i,p,q,j}X^{m-p}Y^{n-q} \in I_{m-p,n-q}X^{m-p}Y^{n-q} &= I_{0,n-q}u^{p-m}Y^{n-q} \\ &= (I_{0,n-q}X^0Y^{n-q})u^{p-m} \\ &= (I_{0,n-q}X^0Y^{n-q})u^k, \end{aligned}$$

with $k = p - m$.

If $m \geq 0$, then $k = p - m \leq p \leq r$

If $m < 0$, then $k = p - m = p + m'$ where $m' > 0$. So $u^{p-m} = u^pu^{m'}$. Therefore

$$a_{i,p,q}c_{i,p,q,j}X^{m-p}Y^{n-q} \in (I_{0,n-q}X^0Y^{n-q}u^{m'})u^p \subseteq \mathcal{R}(A,F)u^p$$

Hence z is an $\mathcal{R}(A,F)$-linear combination of the elements $u^kx_{p,q,j}X^pY^q$ of $\mathcal{R}(A,G)$, where $0 \leq k \leq r, 0 \leq p \leq r$, $0 \leq q \leq s$, $1 \leq j \leq t_{p,q}$.

If $m > r$ and $n \leq s$, then we have a similar conclusion with v^q in place of u^p. It follows that $\mathcal{R}(A,G)$ is generated as $\mathcal{R}(A,F)$-module by the set $\{ u^kv^lx_{p,q,j}X^pY^q\}$ where $o \leq p \leq r$, $o \leq q \leq s, 1 \leq j \leq t_{p,q}, 0 \leq k \leq r, 0 \leq l \leq s$, which is finite. Therefore $\mathcal{R}(A,G)$ is finitely generated as $\mathcal{R}(A,F)$-module.

Conversely suppose that $\mathcal{R}(A,G)$ is a finitely generated $\mathcal{R}(A,F)$-module. Since both rings are bigraded subrings of $A[u,v,X,Y]$, we may suppose that $\mathcal{R}(A,G)$

is generated as an $\mathcal{R}(A,F)$-module by finitely many homogeneous elements. Let $z_1, z_2, ..., z_t$ be the homogeneous generators of $\mathcal{R}(A,G)$ with nonnegative degrees. This set is obviously non empty. Let $(m_i, n_i) = deg\ z_i$ for $i = 1,2,...,t$. Then $z_i = b_i X^{m_i} Y^{n_i}$, where $b_i \in J_{m_i,n_i}$ for all i. Let $N_1 = \max\{m_i, i = 1,2,...,t\}$ and $N_2 = \max\{n_i, i = 1,2,...,t\}$.

Let $(m,n) \in \mathbb{Z}^2$. Suppose that $m > N_1$ and $n > N_2$. Let $b \in J_{m,n}$. Take $z = bX^mY^n \in \mathcal{R}(A,G)$. We have $z = \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_t z_t$, where $\alpha_i \in \mathcal{R}(A,F)$ for all i. For degree reasons, we may suppose that α_i is homogeneous with degree $(m - m_i,\ n - n_i)$. Write $\alpha_i = a_i X^{m-m_i} Y^{n-n_i}$ where $a_i \in I_{m-m_i,n-n_i}$. Then

$$b = a_1b_1 + a_2b_2 + ... + a_tb_t \in \sum_{i\,=\,1}^{t} I_{m-m_i,n-n_i} J_{m_i,n_i}$$

So

$$J_{m,n} \subseteq \sum_{i\,=\,1}^{t} I_{m-m_i,n-n_i} J_{m_i,n_i} \subseteq \sum_{p=\,0}^{N_1}\sum_{q=\,0}^{N_2} I_{m-p,n-q} J_{p,q}.$$

It follows that $J_{m,n} = \sum_{p=\,0}^{N_1}\sum_{q=\,0}^{N_2} I_{m-p,n-q} J_{p,q}$

Suppose now that $m \leq N_1$ and $n > N_2$.

a) If $m \leq 0$ then $J_{m,n} = J_{0,n}$

We have

$$\begin{aligned}
\sum_{p=\,0}^{N_1}\sum_{q=\,0}^{N_2} I_{m-p,n-q}J_{p,q} &= \sum_{p=\,0}^{N_1}\sum_{q=\,0}^{N_2} I_{0,n-q}J_{p,q} \\
&= \sum_{p=\,0}^{N_1} I_{0,n}J_{p,0} + I_{0,n-1}J_{p,1} + ... + I_{0,n-N_2}J_{p,N_2} \\
&= (I_{0,n}J_{0,0} + I_{0,n-1}J_{0,1} + ... + I_{0,n-N_2}J_{0,N_2}) + \\
&\quad +(I_{0,n}J_{1,0} + I_{0,n-1}J_{1,1} + ... + I_{0,n-N_2}J_{1,N_2}) + ... + \\
&\quad +(I_{0,n}J_{N_1,0} + I_{0,n-1}J_{N_1,1} + ... + I_{0,n-N_2}J_{N_1,N_2}) \\
&= I_{0,n}(J_{0,0} + J_{1,0} + ... + J_{N_1,0}) + \\
&\quad +I_{0,n-1}(J_{0,1} + J_{1,1} + ... + J_{N_1,1}) + ... + \\
&\quad +I_{0,n-N_2}(J_{0,N_2} + J_{1,N_2} + ... + J_{N_1,N_2}) \\
&= I_{0,n} + I_{0,n-1}J_{0,1} + ... + I_{0,n-N_2}J_{0,N_2}
\end{aligned}$$

So it suffices to show that $J_{0,n} \subseteq I_{0,n} + I_{0,n-1}J_{0,1} + ... + I_{0,n-N_2}J_{0,N_2}$

Let $b \in J_{m,n}$, where $m \leq 0$ and let $m' = -m$. Then $z = bX^mY^n \in \mathcal{R}(A,G)$.

Then $z = bu^{m'}Y^n$ where $b \in J_{m,n} = J_{0,n}$

We have $z = \alpha_1 z_1 + \alpha_2 z_2 + ... + \alpha_t z_t$, where $\alpha_i \in \mathcal{R}(A,F)$ for all i.

It suffices to solve the question in the case of one generator $t = 1$.

Assume that $z = bu^{m'}Y^n = \alpha_1 z_1$ where $\alpha_1 = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}X^iY^j$,

where $a_{i,j} \in I_{i,j}$ for all i,j. Then

$z = bu^{m'}Y^n$ where $b \in J_{m,n} = J_{0,n}$,

$$\alpha_1 z_1 = \left(\sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}X^iY^j\right)(b_1X^{m_1}Y^{n_1}) = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}b_1X^{m_1+i}Y^{n_1+j},$$

where $a_{i,j} \in I_{i,j}$ for all i, j and $b_1 \in J_{m_1,n_1}$

Since $z = bu^{m'}Y^n$ where $b \in J_{m,n} = J_{0,n}$, then

$$bu^{m'}Y^n = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}b_1X^{m_1+i}Y^{n_1+j}$$

If we multiply each member of the above equality by $X^{m'}$ then we get

$$bY^n = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}b_1X^{m'+m_1+i}Y^{n_1+j},$$

which gives $m' + m_1 + i = 0, n_1 + j = n$

Therefore $b = \sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j}b_1$, with $m' + m_1 + i = 0, n_1 + j = n$

So $i = -m_1 - m'$, $j = n - n_1$ and

$$b \in \sum_{(i,j)} I_{-m_1-m',\ n-n_1}J_{m_1,\ n_1} \subseteq I_{0,\ n-n_1}J_{0,n_1} \subseteq \sum_{q=0}^{n_1} I_{0,\ n-q}J_{0,q}$$

It follows that $J_{0,n} \subseteq \sum\limits_{q=0}^{n_1} I_{0,n-q}J_{0,q}$, hence $J_{0,n} = \sum\limits_{q=0}^{n_1} I_{0,n-q}J_{0,q}$

b) $0 \leq m \leq N_1$ et $n > N_2$

Apply the same arguments as the case $m > N_1$ and $n > N_2$.

The case $m > N_1$ and $n \leq N_2$ is similar to the previous one.

So $J_{m,n} = \sum\limits_{p=0}^{N_1}\sum\limits_{q=0}^{N_2} I_{m-p,n-q}J_{p,q}$ for all m, n with $m > N_1$ or $n > N_2$ and F is a reduction of G. □

Corollary 5.4. *Let A be a Noetherian ring and let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration of EPI type on the ring A. Then there exists a bijection between the set of bifiltrations $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of EPI type on the ring A such that F is a reduction of G and the bigraded subrings S of $A[u, v, X, Y]$ that are finite integral extensions of $\mathcal{R}(A,\ F)$.*

Proof. For each bifiltration $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of EPI type on the ring A such that F is a reduction of G, let $\theta(G) = S = \mathcal{R}(A, G)$. Then S is a bigraded subring of $A[u, v, X, Y]$ which is a finitely generated $\mathcal{R}(A, F)$-module by Theorem 5.3, hence a finite integral extension of $\mathcal{R}(A, F)$.

The mapping θ defined this way is a bijection between the set of bifiltrations $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of EPI type on the ring A such that F is a reduction of G into the set of bigraded subrings S of $A[u, v, X, Y]$ that are finite integral extensions of $\mathcal{R}(A, F)$. Indeed let S be a bigraded subring of $A[u, v, X, Y]$ which contains $\mathcal{R}(A, F)$ and which is a finite module over $\mathcal{R}(A, F)$. Let $J_{m,n} = u^m v^n S \cap A$ for all m, n and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$. Then one checks easily that G is a bifiltration on A and that $J_{m,n} \supseteq u^m v^n\mathcal{R}(A,\ F) \cap A = I_{m,n}$ for all m, n, so $F \leq G$. Furthermore we have $S = \mathcal{R}(A, G) = \theta(G)$. Consequently θ is surjective.

Let $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $H = (H_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be two bifiltrations of EPI type on the ring A such that F is a reduction of G and H. If $\theta(G) = \theta(H) = S$, then, as shown above, $J_{m,n} = u^m v^n S \cap A = H_{m,n}$ for all m, n and $G = H$. Therefore θ is injective hence bijective. □

Corollary 5.5. *Let* $F = (F_{m,n})_{(m,n)\in\mathbb{Z}^2}, G = (G_{m,n})_{(m,n)\in\mathbb{Z}^2}$, $H = (H_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *and* $K = (K_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *be bifiltrations of EPI type on a Noetherian ring* A*. Then :*

(5.5.1) If F *is a reduction of* G *and* G *is a reduction of* K*, then* F *is a reduction of* K

(5.5.2) If F *is a reduction of* G *and if* $F \leq H \leq G$*, then* H *is a reduction of* G*. In addition if* F *is Noetherian, then* F *is a reduction of* H*.*

Proof. (5.5.1) If F is a reduction of G and G is a reduction of K, then by Theorem 5.3, $\mathcal{R}(A,G)$ is a finitely generated $\mathcal{R}(A,F)$-module and $\mathcal{R}(A,K)$ is a finitely generated $\mathcal{R}(A,G)$-module. Therefore $\mathcal{R}(A,K)$ is a finitely generated $\mathcal{R}(A,F)$-module. It follows that F is a reduction of K.

(5.5.2) If F is a reduction of G and if $F \leq H \leq G$, then $\mathcal{R}(A,G)$ is a finitely generated $\mathcal{R}(A,F)$-module, hence a finitely generated $\mathcal{R}(A,H)$-module. So H is a reduction of G.

In addition if F is Noetherian, then, since $\mathcal{R}(A,G)$ is a finitely generated $\mathcal{R}(A,F)$-module and that $\mathcal{R}(A,F)$ is a Noetherian ring, then each submodule of $\mathcal{R}(A,G)$ and, in particular $\mathcal{R}(A,H)$, is a finitely generated $\mathcal{R}(A,F)$-module.

Therefore F is a reduction of H. □

Corollary 5.6. *Let* $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *and* $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *be bifiltrations of EPI type on a Noetherian ring* A *with* $F \leq G$*. Then*

(5.6.1) If F *is a reduction of* G*, then* F *is Noetherian if and only if* G *is Noetherian*

(5.6.2) If F *is Noetherian and is a reduction of* G*, then* $F^{(k,l)}$ *is Noetherian and is a reduction of* $G^{(k,l)}$ *for all integers* $k,l \in \mathbb{N}^*$*.*

Proof. (5.6.1) If F is a reduction of G, then $\mathcal{R}(A,G)$ is a finitely generated $\mathcal{R}(A,F)$-module. So, if F is Noetherian, then $\mathcal{R}(A,F)$ is a Noetherian ring.

Therefore $\mathcal{R}(A,G)$ is a Noetherian ring and the bifiltration G is Noetherian.

Conversely if G is Noetherian, then by Eakin's Theorem [Ea], F is Noetherian.

(5.6.2) Suppose that F is Noetherian and is a reduction of G. For integers $k,l \in \mathbb{N}^*$, consider the bifiltration $F^{(k,l)} = (I_{mk,nl})_{(m,n)\in\mathbb{Z}^2}$ of index (k,l) extracted from the bifiltration F as defined in (2.2.4).

Let $\mathcal{R}_{k,l} = \mathcal{R}(A,F)^{(k,l)} = \bigoplus\limits_{(m,n)\in\mathbb{Z}^2} I_{mk,nl}X^{mk}Y^{nl}$

In particular $\mathcal{R}_{1,1} = \bigoplus\limits_{(m,n)\in\mathbb{Z}^2} I_{m,n}X^mY^n = \mathcal{R}(A,F)$

Remark that $\mathcal{R}(A,F^{(k,l)}) = \bigoplus\limits_{(m,n)\in\mathbb{Z}^2} I_{mk,nl}X^mY^n \subseteq \mathcal{R}_{1,1}$.

Indeed, since $k,l \in \mathbb{N}^*$, if $m \geq 1$ and $n \geq 1$, then $mk \geq m$ and $nl \geq n$.

So $I_{mk,nl} \subseteq I_{m,n}$.

If $m \leq 0$ or $n \leq 0$, then $mk \leq 0$ or $nl \leq 0$.

Therefore suppose that $m \leq 0$ and $n \leq 0$. Then $I_{mk,nl} = A = I_{m,n}$

If $m \leq 0$ and $n > 0$, then $mk \leq 0$ and $nl > 0$.

Then by convention $I_{mk,nl} = I_{0,nl}$ and $I_{m,n} = I_{0,n}$

So $I_{mk,nl} \subseteq I_{m,n}$ since $I_{0,nl} \subseteq I_{0,n}$

The case $m > 0$ and $n \leq 0$ is similar.

This achieves to show that $I_{mk,nl} \subseteq I_{m,n}$ for all m, n.

Hence $I_{mk,nl}X^mY^n \subseteq I_{m,n}X^mY^n$.

It follows that $\mathcal{R}(A, F^{(k,l)}) \subseteq \mathcal{R}_{1,1} = \mathcal{R}(A, F)$

Similarly let $\mathcal{S}_{k,l} = \mathcal{R}(A, G)^{(k,l)} = \bigoplus_{(m,n)\in\mathbb{Z}^2} J_{mk,nl}X^{mk}Y^{nl}$.

In particular $\mathcal{S}_{1,1} = \bigoplus_{(m,n)\in\mathbb{Z}^2} J_{m,n}X^mY^n = \mathcal{R}(A, G)$

We have also $\mathcal{S}_{k,l} \simeq \mathcal{R}(A, G^{(k,l)}) = \bigoplus_{(m,n)\in\mathbb{Z}^2} J_{mk,nl}X^mY^n$

Assume that F is a reduction of G. Then $\mathcal{S}_{1,1} = \mathcal{R}(A, G)$ is a finitely generated $\mathcal{R}_{1,1} = \mathcal{R}(A, F)$-module. On the other hand

$$(I_{m,n}X^mY^n)^{kl} \subseteq I_{mkl,nkl}X^{mkl}Y^{nkl} = I_{(ml)k,(nk)l}X^{(ml)k}Y^{(nk)l} \subseteq \mathcal{R}(A, F)^{(k,l)} = \mathcal{R}_{k,l}$$

It follows that $\mathcal{R}_{1,1}$ is integral over $\mathcal{R}(A, F^{(k,l)}) \simeq \mathcal{R}_{k,l}$.

Similarly $\mathcal{S}_{1,1} = \mathcal{R}(A, G)$ is integral over $\mathcal{S}_{k,l}$ for all integers $k, l \in \mathbb{N}^*$

We have $A \subseteq \mathcal{R}_{k,l} \subseteq \mathcal{R}_{1,1} = \mathcal{R}(A, F)$.

Since F is noetherian then $\mathcal{R}_{1,1} = \mathcal{R}(A, F)$ is a finitely generated A-algebra by Theorem 1.1 of [GY].

Therefore $\mathcal{R}_{1,1} = \mathcal{R}(A, F)$ is finitely generated as $\mathcal{R}_{k,l}$-algebra and is integral over $\mathcal{R}_{k,l}$ for all integers $k, l \in \mathbb{N}^*$. So by Eakin's Theorem [Ea], $\mathcal{R}_{k,l}$ is Noetherian. Therefore $F^{(k,l)}$ is Noetherian for all integers $k, l \in \mathbb{N}^*$.

As for the last part of (5.6.2), it is clear that $F^{(k,l)} \leq G^{(k,l)}$ for all integers $k, l \in \mathbb{N}^*$.

On the other hand as already noted, $\mathcal{S}_{1,1}$ is a finitely generated $\mathcal{R}_{1,1}$-module. Since $A \subseteq \mathcal{R}_{k,l} \subseteq \mathcal{R}_{1,1} \subseteq \mathcal{S}_{1,1}$ then $\mathcal{S}_{1,1}$ is a finitely generated $\mathcal{R}_{k,l}$-module for all integers $k, l \in \mathbb{N}^*$. We have also $\mathcal{R}_{k,l} \subseteq \mathcal{S}_{k,l} \subseteq \mathcal{S}_{1,1}$. Since $\mathcal{R}_{k,l}$ is Noetherian, then $\mathcal{S}_{k,l}$ is a finitely generated $\mathcal{R}_{k,l}$-module for all integers $k, l \in \mathbb{N}^*$ and by Theorem 5.3, $F^{(k,l)}$ is a reduction of $G^{(k,l)}$ for all integers $k, l \in N^*$. □

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