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Classification and Reduction of Bifiltrations

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Abstract

In this paper, we define the classes of bifiltrations and we make their classification. We characterize (I, J)-good bifiltrations, a concept of β-reduction of bifiltrations is introduced.

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1. Introduction

Throughout this paper, A denotes a commutative ring.

The classes of filtrations were introduced in the theory of filtrations, we define these classes for the bifiltrations and we study the relations between these classes.

The concept of reduction of ideals was introduced by D.G. Northcott and D.Rees in [5]. It was actively studied in the literature since its introduction. Since powers of ideals are special filtrations, the concept of reduction of ideals was generalized to filtrations by J.S. Okon and L.J.Ratliff,Jr in [6], where the authors had given many important results on the subject.

Here we introduce a concept of β-reduction for rings bifiltrations similarly to the case of filtrations.

2. Bifiltrations

2.1. Filtrations

Let us recall the following definitions which will be used in the sequel.

(2.1.1) By a **filtration on the ring** A, we mean a family $f = (I_n)_{n\in\mathbb{Z}}$ of ideals of A such that $I_0 = A$, $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}$ and $I_pI_q \subseteq I_{p+q}$ for all $p, q \in \mathbb{Z}$. It follows that if $n \leq 0$, then $I_n = A$.

(2.1.2) If I is an ideal of A, then the filtration $f_I = (I^n)_{n\in\mathbb{Z}}$, where $I^n = A$ for all $n \leq 0$, is called the **I-adic filtration** of A

(2.1.3) Let $f = (I_n)_{n\in\mathbb{Z}}$ and $g = (J_n)_{n\in\mathbb{Z}}$ be filtrations on a ring A. The filtration f is called a **reduction of** g if $f \leq g$ and if there exists an integer $N \geq 1$ such that $J_n = \sum_{p=0}^{N} I_{n-p}J_p$ for all $n > N$, see [OR] for more information.

2.2. Bifiltrations.

The set $\mathbb{Z}^2$ is partially ordered as follows :

For all $m, n, p, q \in \mathbb{Z}$, $(m, n) \preceq (p, q)$ if and only if $m \leq p$ and $n \leq q$.

(2.2.1) A **bifiltration** on the ring A is a family $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of ideals of A such that

(i) $I_{0,0} = A$

(ii) For all $m, n \in \mathbb{Z}$, $I_{m+1,n} \subseteq I_{m,n}$ and $I_{m,n+1} \subseteq I_{m,n}$

(iii) $I_{m,n}I_{p,q} \subseteq I_{m+p,\, n+q}$ for all $m, n, p, q \in \mathbb{Z}$.

It follows that if $m \leq 0$, then $(m, 0) \preceq (0, 0)$, hence $A = I_{0,0} \subseteq I_{m,0}$, so $I_{m,0} = A$.

Similarly, if $n \leq 0$, then $I_{0,n} = A$.

In particular, if $(m, n) \preceq (0, 0)$, then $I_{m,n} = A$.

(2.2.2) Throughout this paper all the bifiltrations $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ are supposed to satisfy the following additional conditions :

(EPI) For all $(k, l) \in \mathbb{Z}^2$, $I_{k,l} = I_{k,0}$ if $k \geq 0$ and $l \leq 0$ and $I_{k,l} = I_{0,l}$ if $k \leq 0$ and $l \geq 0$.

Such a bifiltration is said to be of **Essentially Positive Indices type** (EPI type for short).

This definition will be extended to bifiltrations $F = (I_{m,n})_{(m,n)\in\mathbb{N}^2}$ with indices in $\mathbb{N}^2$ where negative sub-indices may occur in $I_{m,n}$.

(2.2.3) Let $F = (I_{m,n})_{(m,n)\in\mathbb{N}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be bifiltrations on A. We set $F \leq G$ if and only if $I_{m,n} \subseteq J_{m,n}$ for all $(m, n) \in \mathbb{Z}^2$.

3. Some classes of bifiltrations

3.1 (I, J)-good bifiltration

3.1.1 Definition

Let I and J be two ideals of a ring A . A bifiltration $F = (I_{m,n})_{m,n\in\mathbb{N}}$ is said (I, J)**-good** if :

$$(i)\quad \forall\, m, n \in \mathbb{N},\quad \begin{cases} II_{m,n} \subseteq I_{m+1,n} \\ JI_{m,n} \subseteq I_{m,n+1} \end{cases}$$

(ii) there exist m_0, $n_0 \in \mathbb{N}$ such that :

$$\forall\, m \geq m_0, n \geq n_0,\quad \begin{cases} II_{m,n} = I_{m+1,n} \\ JI_{m,n} = I_{m,n+1} \end{cases}$$

3.1.2 Remark

If $F = (I_{m,n})_{m,n\in\mathbb{N}}$ is (I, J)-good bifiltration then :

$$IJI_{m,n} \subseteq I_{m+1,n+1} \qquad \forall\, m, n \in \mathbb{N}$$

$$IJI_{m,n} = I_{m+1,n+1} \qquad \forall\, m \geq m_0, n \geq n_0\ .$$

3.1.3 Proposition

$F = (I_{m,n})_{m,n\in\mathbb{N}}$ is a (I, J)-good bifiltration if and only if F is $(I_{1,0}, I_{0,1})$-good .

Proof

Suppose F is (I, J)-good, then we have :

$$\begin{cases} II_{0,0} \subseteq I_{1,0} \\ JI_{0,0} \subseteq I_{0,1} \end{cases} \quad \text{and since} \quad I_{0,0} = A \quad \text{then} \quad \begin{cases} I \subseteq I_{1,0} \\ J \subseteq I_{0,1} \end{cases}$$

$$\text{So}\ \ \forall\, m, n \in \mathbb{N},\quad \begin{cases} II_{m,n} \subseteq I_{1,0}I_{m,n} \subseteq I_{m+1,n} \\ JI_{m,n} \subseteq I_{0,1}I_{m,n} \subseteq I_{m,n+1} \end{cases}$$

$$\text{and}\quad \forall\, m \geq m_0, n \geq n_0,\quad \begin{cases} II_{m,n} = I_{m+1,n} \\ JI_{m,n} = I_{m,n+1} \end{cases}$$

whence $\forall\ m \geq m_0, n \geq n_0,\ \begin{cases} I_{1,0}I_{m,n} = I_{m+1,n} \\ I_{0,1}I_{m,n} = I_{m,n+1} \end{cases}$

So F is $(I_{1,0}, I_{0,1})$-good.

The reciprocal is obvious.

3.1.4 Consequence

If F is (I,J)-good bifiltration then there exist $m_0,\ n_0 \in \mathbb{N}$ such that
$I^pJ^qI_{m_0,n_0} = I_{m_0+p,n_0+q} \qquad \forall p,q \in \mathbb{N}$

Indeed $\forall p,q \in \mathbb{N}$ we have

$I^pJ^qI_{m_0,n_0} = I^pJ^{q-1}JI_{m_0,n_0} = I^pJ^{q-1}I_{m_0,n_0+1} = I^pJ^{q-2}JI_{m_0,n_0+1} = I^pJ^{q-2}I_{m_0,n_0+2} = \cdots = I^pI_{m_0,n_0+q} = I^{p-1}II_{m_0,n_0+q} = I^{p-1}I_{m_0+1,n_0+q} = \cdots = I_{m_0+p,n_0+q}$

3.1.5 Definition

A bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ is good if there are two ideals I and J such that F is (I,J)-good.

3.2 EP Bifiltration

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *be bifiltration of EPI type of a ring* A.
The bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of a ring A is called **EP**
if there exist two integers $N_1 \geq 1$ and $N_2 \geq 1$ such that
$\forall\ m \geq N_1$ or $n \geq N_2$, $I_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}I_{p,q}$

3.3 Strongly EP Bifiltration

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ *be bifiltration of EPI type of a ring* A.
The bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of a ring A is said to be **strongly EP**
if there exist $k_1 \in \mathbb{N}^*$ and $k_2 \in \mathbb{N}^*$ such that
$I_{m+k_1,n+k_2} = I_{m,n}\ I_{k_1,k_2} \quad \forall m,n \in \mathbb{Z}$ with $m \geq k_1$ or $n \geq k_2$.

3.3.1 Proposition

If the bifiltration $F = (I_{m,n})_{m,n\in\mathbb{Z}}$ *is strongly EP, then* F *is EP.*

Proof

Suppose $F = (I_{m,n})_{m,n\in\mathbb{Z}}$ is strongly EP, then
there exist $k_1 \in \mathbb{N}^*$ and $k_2 \in \mathbb{N}^*$ such that
$I_{m+k_1,n+k_2} = I_{m,n}\ I_{k_1,k_2} \quad \forall m,n \in \mathbb{Z}$ with $m \geq k_1$ or $n \geq k_2$.

If $m \geq 2k_1$ or $n \geq 2k_2$, we have $I_{m,n} = I_{m-k_1,n-k_2}\ I_{k_1,k_2}$

so $I_{m,n} \subseteq \sum_{i=1}^{2k_1} \sum_{j=1}^{2k_2} I_{m-i,n-j} I_{i,j} \subseteq I_{m,n}$

whence $I_{m,n} = \sum_{i=1}^{2k_1} \sum_{j=1}^{2k_2} I_{m-i,n-j} I_{i,j}$

So F is EP bifiltration.

3.3.2 Proposition

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration of a ring A .

If F is $(I_{1,0}, I_{0,1})$-good then F is strongly EP.

Proof

If F is $(I_{1,0}, I_{0,1})$-good then there exist m_0 ,$n_0 \in \mathbb{N}^*$ such that

$I_{1,0}^p I_{0,1}^q I_{m,n} = I_{m+p,n+q}$ $\forall p, q \in \mathbb{N}$ and $\forall\ m \geq m_0,\ n \geq n_0$.(see consequence 3.1.4).

we know that $I_{1,0}^p I_{0,1}^q \subseteq I_{p,q}$ $\forall p, q \in \mathbb{N}$,

so $I_{1,0}^p I_{0,1}^q I_{m,n} \subseteq I_{p,q} I_{m,n}$

whence $\forall p, q \in \mathbb{N}$ and $\forall\ m \geq m_0$ or $n \geq n_0$,

$I_{m+p,n+q} = I_{1,0}^p I_{0,1}^q I_{m,n} \subseteq I_{p,q} I_{m,n} \subseteq I_{m+p,n+q}$.
By posing $k_1 = m_0$ and $k_2 = n_0$, we have

$I_{m+p,n+q} = I_{m,n}\ I_{p,q}$ with $m \geq k_1$ or $n \geq k_2$.

So F is strongly EP bifiltration.

3.4 Noetherian bifiltration

The bifiltration $F = (I_{m,n})_{m,n\in\mathbb{Z}}$ is said to be **noetherian** if it generalised Rees ring of F, $\Re(A, F) = \underset{m,n\,\in\,\mathbb{Z}}{\oplus} I_{m,n} X^m Y^n$ is a Noetherian ring.

In this case the ring A is noetherian ring.

3.5 Strongly Noetherian bifiltration

The bifiltration $F = (I_{m,n})_{m,n\in\mathbb{Z}}$ is said to be **strongly noetherian** if the ring A is noetherian and if F is strongly EP bifiltration.

3.6 Some examples of bifiltrations

Example 1

If $f=(I_m)_{m\in\mathbb{Z}}$ and $g=(J_n)_{n\in\mathbb{Z}}$ are two filtrations of a ring A then $h=(I_mJ_n)_{m,n\in\mathbb{Z}}$ is a bifiltration of A.

Let I and J be two ideals of A. We agree that $I^mJ^n=A$, $\forall\ m\leq 0$ et $n\leq 0$.We pose $f_{I,J}=(I^mJ^n)_{m,n\in\mathbb{Z}}$,

$f_{I,J}$ is a bifiltration of A called (I,J) -adic bifiltration of A.

Example 2

Let $A=\dfrac{\mathbb{Z}[X]}{\langle X^2\rangle}$ and let $x=X+\langle X^2\rangle$.

Consider the following family of ideals :

$I_{0,0}=A$, $I_{m,n}=A\ \ \forall\ \ m\leq 0$ and $n\leq 0$, $I_{m,n}=\langle x\rangle\ \forall\ \ 0\leq m\leq 2$ and $0\leq n\leq 2$ with $(m,n)\neq(0,0)$
$I_{m,n}=0\quad\forall\ \ m\geq 3$ or $n\geq 3$, $I_{m,n}=I_{0,n}\quad\forall\ \ m\leq 0$ and $n\geq 0$ and $I_{m,n}=I_{m,0}\quad\forall\ \ m\geq 0$ and $n\leq 0$
Then $F=(I_{m,n})_{m,n\in\mathbb{Z}}$ is a $(I_{1,0}I_{0,1})$-good bifiltration, but not $(I_{1,0}I_{0,1})$-adic bifiltration.

Example 3

Let $A=K[X]$ be the ring of polynomials at an indeterminate X over a field K.

We pose
$I_{0,0}=A$, $I_{m,n}=A\ \ \forall\ \ m\leq 0$ et $n\leq 0$, $I=X$, $I_{m,n}=I_{0,n}\quad\forall$ $m\leq 0$ et $n\geq 0$ et $I_{m,n}=I_{m,0}\quad\forall\ \ m\geq 0$ et $n\leq 0$ et

$$\forall\, m\geq 0 \text{ and } n\geq 0\ \ I_{m,n}=\begin{cases}(X^{\frac{m+n}{2}}) & \text{if } m \text{ and } n \text{ are of the same parity}\\ (X^{\frac{m+n+1}{2}}) & \text{if } m \text{ and } n \text{ are of different parities}\end{cases}$$

then $F=(\ I_{m,n})_{m,n\in\mathbb{Z}}$ is an EP bifiltration but not strongly EP.
Example 4

Let $A = K[X]$ be the ring of polynomials at an indeterminate X over a field K.

Let N_1 and N_2 be two integers such that $N_1 \geq 2$ and $N_2 \geq 2$.

We pose $I_{0,0} = A$, $I_{m,n} = A \ \forall \ m \leq 0$ and $n \leq 0$, $I_{m,n} = I_{0,n} \ \forall$ $m \leq 0$ and $n \geq 0$ and $I_{m,n} = I_{m,0} \ \forall \ m \geq 0$ and $n \leq 0$ and $I = (X)$

and

$$\forall\, m \geq 0 \text{ and } n \geq 0 \quad I_{m,n} = \begin{cases} (X^{m+n}) = I^{m+n} & \text{if } m \geq N_1 + 1 \text{ or } n \geq N_2 + 1 \\ (X^{N_1+N_2}) = I^{N_1+N_2} & \text{if } 0 \leq m \leq N_1 \text{ and } 0 \leq n \leq N_2 \end{cases}$$

then $F = (\ I_{m,n})_{m,n\in\mathbb{Z}}$ is a strongly EP bifiltration but not $(I_{1,0}, I_{0,1})$-good.

3.7 Classification

From all the above, we obtain the following classification:

$F \quad (I,J)$-adic $\overset{(1)}{\Rightarrow}$ $F \quad (I,J)$-good $\overset{(1)}{\Rightarrow}$ F strongly EP $\overset{(1)}{\Rightarrow}$ F EP

$\Updownarrow (2)$

$F\ (I,J)$-adic $\overset{(2)}{\Rightarrow}$ $F\ (I,J)$-good $\overset{(2)}{\Rightarrow}$ F strongly noethrian

Avec $\begin{cases} (1) : A \text{ is any ring} \\ (2) : A \text{ is a noetherian ring} \end{cases}$

4. Reduction of bifiltrations

4.1 Definition.

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be bifiltrations of EPI type on the ring A with $F \leq G$.

We call F a **reduction of** G if there exist integers $r \geq 1$ and $s \geq 1$ such that

(RED) $\quad J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q}$ for all $(m,n) \in \mathbb{N}^2$, with $m > r$ or $n > s$.

4.2 Example.

Each bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ is a reduction of itself. Indeed let $(m,n) \in \mathbb{N}^2$, with $m > r$ or $n > s$ where r and s are arbitrary positive integers. If $m > r$ and $n > s$, then we have

$$\sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q} = I_{m,n} + \sum_{\substack{0\leq p\leq r\\ 0\leq q\leq s\\ (p,q)\neq(0,0)}} I_{m-p,n-q}I_{p,q} = I_{m,n}$$

since

$I_{m-p,n-q}I_{p,q} \subseteq I_{m,n}$ for all p, q

If $m > r$ and $n \leq s$, then

$$\sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}I_{p,q} = \sum_{p=0}^{r}\sum_{q=0}^{n} I_{m-p,n-q}I_{p,q} + \sum_{p=0}^{r}\sum_{q=n+1}^{s} I_{m-p,n-q}I_{p,q}$$

We have

$\sum_{q=n+1}^{s} I_{m-p,n-q}I_{p,q} = \sum_{q=n+1}^{s} I_{m-p,0}I_{p,q} = I_{m-p,0}\sum_{q=n+1}^{s} I_{p,q}$

$= I_{m-p,0}I_{p,n+1} \subseteq I_{m,n+1} \subseteq I_{m,n}$

Since $\sum_{p=0}^{r}\sum_{q=0}^{n} I_{m-p,n-q}I_{p,q} = I_{m,n}$, then $\sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}I_{p,q} = I_{m,n}$

The case $m \leq r$ and $n > s$ is similar to the previous one.

5. $\beta-$ reduction

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

We say that F is a **β-reduction of** G if there exist $k, l \in \mathbb{N}^*$ such that $\forall m \geq k$ or $\forall\ n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l}$.

5.1 Consequence

If F is β-reduction of G then G is strongly EP bifiltration.

Sure enough, if F is β-reduction of G then

$\forall m \geq k$ or $\forall\ n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l} \subseteq J_{m,n}J_{k,l} \subseteq J_{m+k,n+l}$.

So $\forall m \geq k$ or $\forall\ n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l} = J_{m,n}J_{k,l}$.

Whence G is strongly EP bifiltration.

6. F-good bifiltrations

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

The bifiltration G is called F-**good** if there exist $N_1, N_2 \in \mathbb{N}^*$ such that

$$\forall m \geq N_1 \text{ or } \forall n \geq N_2, \quad J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}J_{p,q}.$$

7. F-fine bifiltrations

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

The bifiltration G is called F-**fine** if there exist $N_1, N_2 \in \mathbb{N}^*$ such that

$$\forall m \geq N_1 \text{ ou } \forall n \geq N_2,\ J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{p,q}J_{m-p,n-q}.$$

7.1 Remark

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ a bifiltration of A of EPI type.

Then F is F-good $\Leftrightarrow$ F is EP $\Leftrightarrow$ F-fine.

7.2 Proposition

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

Suppose G is F-good. So

i) G is G - good, so EP.

ii) F is a reduction of G.

Proof

i) Since G is F-good then there are $N_1, N_2 \in \mathbb{N}^*$ such that

$$\forall m \geq N_1 \text{ or } \forall n \geq N_2, \quad J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}J_{p,q}.$$

We know that $J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}J_{p,q} \subseteq \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} J_{m-p,n-q}J_{p,q} \subseteq J_{m,n}$

So $J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} J_{m-p,n-q}J_{p,q}$

Whence G is G-good so EP.

ii) For $m \geq N_1$ or $n \geq N_2$,

$$\sum_{p=0}^{N_1}\sum_{q=0}^{N_2} I_{m-p,n-q}J_{p,q} = I_{m,n} + \sum_{q=1}^{N_2} I_{m,n-q}J_{0,q} + \sum_{p=1}^{N_1} I_{m-p,n}J_{p,0} + \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}J_{p,q}$$
$$= I_{m,n} + J_{m,n} = J_{m,n}$$

because $I_{m,n-q}J_{0,q} \subseteq J_{m,n}$ and $I_{m-p,n}J_{p,0} \subseteq J_{m,n}$

So F is a reduction of G.

7.3 Theorem

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ et $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

If F is a β-reduction of G then :

i) G is strongly EP.
ii) G is F-good.
iii) G is G-good, so EP.
iv) F is a reduction of G
v) G is F-fine if F is EP.

Proof

i) Suppose that F is a β-reduction of G then $F \leq G$ and there exist $k, l \in \mathbb{N}^*$ such that $\forall m \geq k$ or $\forall n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l}$.

We have $\forall m \geq k$ or $\forall\ n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l} \subseteq J_{m,n}J_{k,l} \subseteq J_{m+k,n+l}$.

So $\forall m \geq k$ or $\forall\ n \geq l$, $J_{m+k,n+l} = J_{m,n}J_{k,l}$.

Therefore G is strongly EP.

ii) Let m, n such that $\forall m \geq 2k$ and $\forall n \geq 2l$, $J_{m+k,n+l} = I_{m,n}J_{k,l}$.

We have $\forall m \geq 2k$ or $\forall\ n \geq 2l$, $J_{m,n} = I_{m-k,n-l}J_{k,l}$

So $J_{m,n} \subseteq \sum_{p=1}^{2k}\sum_{q=1}^{2l} I_{m-p,n-q}J_{p,q} \subseteq \sum_{p=1}^{2k}\sum_{q=1}^{2l} J_{m-p,n-q}J_{p,q} \subseteq J_{m,n}$

Therefore $J_{m,n} = \sum_{p=1}^{2k}\sum_{q=1}^{2l} I_{m-p,n-q}J_{p,q}$

By putting $N_1 = 2k$ and $N_2 = 2l$, there exist $N_1, N_2 \in \mathbb{N}^*$ such that

$\forall m \geq N_1$ or $\forall n \geq N_2$, $J_{m,n} = \sum_{p=1}^{N_1}\sum_{q=1}^{N_2} I_{m-p,n-q}J_{p,q}$

So G is F-good.

iii) See proposition 7.2 (i)

iv) If F is a β-reduction of G then G is F-good according to (ii), F is a reduction of G (see Proposition 7.2 (ii))

v) Suppose F is EP bifiltration.

If F is a β-reduction of G then F is a reduction of G (from v), so there exist $r, s \in \mathbb{N}^*$ such that

$\forall m, n \in \mathbb{N}^*$, $J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q}$ with $m \geq r$ or $n \geq s$

Since F is EP bifiltration then there are $N_1, N_2 \in \mathbb{N}^*$ such that

$\forall\ m \geq N_1$ or $n \geq N_2$, $I_{m,n} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} I_{m-i,n-j}I_{i,j}$

We pose $N_1' = \max(r, N_1)$ and $N_2' = \max(s, N_2)$

We have $\forall m \geq N_1'$ or $\forall n \geq N_2'$, $I_{m-p,n-q} = \sum_{i=1}^{N_1'}\sum_{j=1}^{N_2'} I_{m-p-i,n-q-j}I_{i,j}$

$\forall m \geq N_1'$ or $\forall n \geq N_2'$, $J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q} = \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{i=1}^{N_1'}\sum_{j=1}^{N_2'} I_{m-p-i,n-q-j}I_{i,j}J_{p,q}$

then $\forall m \geq N'_1$ or $\forall n \geq N'_2$, $J_{m,n} = \sum_{i=1}^{N'_1}\sum_{j=1}^{N'_2} I_{i,j} \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p-i,n-q-j}J_{p,q}$

So $\forall m \geq N'_1$ or $\forall n \geq N'_2$, $J_{m,n} \subseteq \sum_{i=1}^{N'_1}\sum_{j=1}^{N'_2} I_{i,j}J_{m-i,n-j}$

Since $I_{i,j} \subseteq J_{i,j}$ then $I_{i,j}J_{m-i,n-j} \subseteq J_{i,j}J_{m-i,n-j} \subseteq J_{m,n}$

So $J_{m,n} \subseteq \sum_{i=1}^{N'_1}\sum_{j=1}^{N'_2} I_{i,j}J_{m-i,n-j} \subseteq J_{m,n}$

whence $\forall m \geq N'_1$ or $\forall n \geq N'_2$, $J_{m,n} = \sum_{i=1}^{N'_1}\sum_{j=1}^{N'_2} I_{i,j}J_{m-i,n-j}$

So G is F-fine.

7.4 Theorem

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

If F is strongly EP, then the following assertions are equivalent :

i) F is a β-reduction of G.

ii) F is a reduction of G.

iii) G is F-good.

Proof

i) $\Rightarrow$ ii) See the theorem 7.3 (iv) .

ii) $\Rightarrow$ i)

Suppose F is strongly EP then there are $k_1 \in \mathbb{N}^*$ and $k_2 \in \mathbb{N}^*$ such that $I_{m+k_1,n+k_2} = I_{m,n}\ I_{k_1,k_2}$ with $m \geq k_1$ or $n \geq k_2$.

Since F is a reduction of G then there exist $r, s \in \mathbb{N}^*$ such that

$\forall m,n \in \mathbb{N}^*$, $J_{m,n} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m-p,n-q}J_{p,q}$ with $m \geq r$ or $n \geq s$.

We pose $k = \max(k_1, r)$ and $l = \max(k_2, s)$ then

$\forall m \geq k$ or $\forall\, n \geq l$, $J_{m+k,n+l} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m+k-p,n+l-q}J_{p,q} = \sum_{p=0}^{r}\sum_{q=0}^{s} I_{m,n}I_{k-p,l-q}J_{p,q} =$
$I_{m,n} \sum_{p=0}^{r}\sum_{q=0}^{s} I_{k-p,l-q}J_{p,q} = I_{m,n}J_{k,l}$

So there exist $k, l \in \mathbb{N}^*$ such that $\forall m \geq k$ or $\forall\, n \geq l$, $J_{m+k,n+l} = I_{m,n}J_{k,l}$.

Whence F is a β-reduction of G.

i) $\Rightarrow$ iii) See Theorem 7.3 (ii).

iii) $\Rightarrow$ ii) See proposition 7.2 (ii).

7.5 Proposition

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

If G is $(I_{0,1}, I_{1,0})$ - good then F is a β-reduction of G.

Proof

If G is $(I_{0,1}, I_{1,0})$- good then there exist $m_0, n_0 \in \mathbb{N}$ such that $\forall\ m \geq m_0, n \geq n_0$,

$$\begin{cases} I_{1,0}J_{m,n} = J_{m+1,n} \\ \\ I_{0,1}I_{m,n} = J_{m,n+1} \end{cases}$$

We have $I^p_{1,0}I^q_{0,1}J_{m,n} = J_{m+p,n+q}\ \ \forall p, q \geq 1$ and $\forall\ m \geq m_0, n \geq n_0$ according to the consequence 3.1.4 .

We obtain $\forall p, q \geq 1$ and $\forall\ m \geq m_0, n \geq n_0$, $J_{m+p,n+q} = I^p_{1,0}I^q_{0,1}J_{m,n} \subseteq I_{p,q}J_{m,n} \subseteq J_{p,q}J_{m,n} \subseteq J_{m+p,n+q}$

So $J_{m+p,n+q} = I_{p,q}J_{m,n}\ \forall\ \ m \geq p$ or $n \geq q$ with $p = m_0$ or $q = n_0$.

Whence F is a β-reduction of G.

7.6 Theorem

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be two bifiltrations of EPI type of a ring A such that $F \leq G$. Assume that the ideal $J_{m,n}$ is finitely generated for all m, n.

Then F is a reduction of G if and only if $R(A, G)$ is a finitely generated $R(A, F)$-module.

Proof. See [3]

8. Bifiltration strongly integral over another

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ et $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of ring A of EPI type.

G is said to be strongly integral over F if $F \leq G$ and if $R(A, G)$ is a finitely generated $R(A, F)$-module.

8.1 Theorem

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of a ring A of EPI type such that $F \leq G$.

If A is noetherian ring then F is a reduction of G if and only if G is strongly integral over F.

Proof

Suppose that F is a reduction of G, since the ring A is noetherian then $J_{m,n}$ is of finite type; according to Theorem 7.9, we have the result.

8.2 Theorem

Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ and $G = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$ two bifiltrations of EPI type of a Noetherian ring A.

If F is strongly EP bifiltration, then the following assertions are equivalent:

i) F is a β-reduction of G.
ii) F is a reduction of G.
iii) G is F-good.
iv) G is strongly integral over F.

Proof

(i)$\Leftrightarrow$ (ii)$\Leftrightarrow$ (iii) according to Theorem 7.4

(ii)$\Leftrightarrow$ (iv) according to Theorem 8.1

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