ON BOUNDED OPERATORS ON BANACH SPACES

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♣ Dedication ♣

To my beloved parents NYONKEU Rene and TCHAPDA Rose

DJACHEUN NYONKEU Kevin.

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♣ sworn statement ♣

I hereby declare that this dissertation entitled "on bounded operators on Banach spaces" has been carry out by me, DJACHEUN NYONKEU Kevin, with registration number 11V0058 in the department of mathematics, University of Yaounde 1. External contributions have been duly mentioned and listed in the bibliography.

♣ Abstract ♣

We give sufficient conditions for a continuous linear mapping between (pre)-ordered Banach spaces to be order-bounded in the sens that its maps order-intervals of the type $[a, b]$ *to intervals of the same type.* The conditions that make continuous functions between (pre)-ordered Banach spaces order-bounded are generally related to the properties of the cones considered and are summarized in theorems 3.3.1 and 3.3.2 and in propositions 3.3.3, 3.3.4, 3.3.5. and 3.3.6.

key words: *ordered Banach spaces, Banach lattice, positive convex cone, normal cone, generating cone, proper cone, monotone norm, order-unit, order-bounded.*

♣ Résumé ♣

Le but de ce mémoire est de donner des conditions suffisantes pour qu'une application linéaire continue entre deux espaces de Banach ordonnés soit bornée au sens des relations d'ordre, c'est-à-dire qu'il transforme tout intervalle du type $[a, b]$ en un intervalle du même type. Les conditions rendant les applications linéaires continues bornées au sens des relations d'ordre sont en général liées au propriétés des cônes sous-jacents aux espaces de Banach (pré)-ordonnés considérés et sont résumées dans les theorèmes 3.3.1 et 3.3.2 et les propositions 3.3.3, 3.3.4 et 3.3.5 et 3.3.6

mots clés: *espaces de Banach ordonnés, treillis de Banach, cône convexe positif, cône normal, cône générateur, cône propre, norme monotone, unité d'ordre, borné au sens des relations d'ordre.*

♣ Some notations ♣

♠ $\mathbb{N}$ = the set of natural numbers

♠ $\exists$ = there exists

♠ $\forall$ = for every

♠ $\in$= element of

♠ $\cup$ = union

♠ $\cap$ = intersection

♠ $s.t.$ = such that

♠ iff=$\Longleftrightarrow$= if and only if

♠ map= mapping

♠ $\mathbb{R}$= the field of real numbers

♠ $\Longrightarrow$= imply

♠ $\sum$ = sum

♠ $\setminus$ = set minus

♠ $\mathbb{R}$= the field of real numbers

♠ $\mathbb{C}$= the field of complex numbers

♠ $\mathbb{F}$=the field $\mathbb{R}$ or the field $\mathbb{C}$

♠ If $(P, \leq)$ is a lattice, and $a, b \in P$, then $a \vee b$ denote the value of the supremum of the pair $\{a, b\}$

♠ $L^2(X; d\mu) = \{f | f : X \longrightarrow \mathbb{R}$ measurable and $\int_X |f|^2 d\mu < +\infty\}$

♠ $C(X) = \{f : X \longrightarrow \mathbb{R} | f$ continuous $\}$

♣ Contents ♣

♣ Introduction ♣

The theory of bounded operators on Banach spaces is directly related to the theory of ordered Banach spaces. The theory of Ordered Banach Spaces is the development of the structure associated with the classical Banach spaces of real functions. Each of these function spaces, for example, C(X) or $L^p(X; d\mu)$, can be ordered by setting $f \geq g$ whenever the function $f - g$ is pointwise positive. This ordering can, however, be des-
cribed in a more geometric manner which is more convenient for the introduction of other order relations.

The pointwise positive functions in each of the classical real Banach spaces form a convex cone $\mathscr{P}$ because if $f, g \geq 0$, then $\lambda f + \mu g \geq 0$ for all positive λ, μ. In terms of this cone the ordering $f \geq g$ is equivalent to the statement that $f - g \in \mathscr{P}$. More generally any cone $\mathscr{P}$ induces a pre-order by setting $f \geq g$ is equivalent to the statement that $f - g \in \mathscr{P}$. properties of the order relation are then determined by the geometric and topological properties of the cone $\mathscr{P}$. Duality properties are also of interest. If $\mathscr{P}$ is a convex cone in a real Banach space $\mathscr{B}$ one can define a cone $\mathscr{P}^\star$ in the dual $\mathscr{B}^\star$ as the elements of $\mathscr{B}^\star$ which are positive on $\mathscr{P}$, and then $\mathscr{B}^\star$ can be ordered by $\mathscr{P}^\star$. If, for example, $\mathscr{P}$ is the positive functions in $L^p(X, d\mu)$, then $\mathscr{P}^\star$ can be identified with the positive functions in $L^q(X; d\mu)$ whenever $1 \leq p < +\infty$ and $\frac{1}{q} = 1 - \frac{1}{p}$. There are two kinds of property of a cone $\mathscr{P}$ which are essential for an interesting order structure and for the theory of bounded operators on ordered Banach spaces. Firstly the cone must not be too large; typically this is expressed by some sort of pointedness condition, e.g., $\pm f \in \mathscr{P}$ if, and only if, $f = 0$. Secondly the cone must not be two small; for example a general f should be decomposable as a difference, $f = g - h$, of a positive g and a positive h, at least approximately.

These two types of restraint on $\mathscr{P}$ are referred to as normality and generation properties, respectively. Subsequently, we will discuss a whole hierarchy of such conditions. It is easy to see that the larger the cone $\mathscr{P}$, the smaller the cone $\mathscr{P}^{\star}$. Alternatively stated, a normality restriction on $\mathscr{P}$ is equivalent to a generation condition on $\mathscr{P}^{\star}$, and vice-versa. Duality results of this nature play a major role in the general analysis of ordered spaces and in particular in the theory of bounded operators on Banach spaces.

There are other more detailed properties of order relations which are also of interest. For example the classical function spaces are lattices with respect to the order defined by pointwise positivity. Moreover the norm of a function and the norm of its modulus coincide, and as the modulus increases so does the norm. Spaces with these properties are called Banach lattices. They have been extensively studied and possess a rich, well understood, structure. Unfortunately they do not describe all the commonly encountered examples of ordered Banach spaces.

However, the aim of this work is to study bounded operators on Banach spaces. Firstly we will recall some essential results both of the classical analysis and algebra. Secondly we will discuss the general structure of ordered Banach spaces and their ordered duals. We examine normality and generation properties of the cone of positive elements with particular emphasis on monotone properties of the norm. The special cases of ordered-unit spaces are also examined. Finally we will talk about the theory of bounded operators on ordered Banach spaces.

CHAPTER ONE

PRELIMINARIES

1.1 Preliminaries on the set theory

We will need some elements of the set theory to support our work on the bounded operators on Banach spaces.

We are going to define some concepts of the set theory which will be useful subsequently.

Definition 1.1.1. *(Relation)*

Let A be a set. A binary relation $\mathscr{R}$ on A is a subset of A^2.

Definition 1.1.2. *(pre-order)*

A binary relation $\leq$ defined on a set A is a pre-order on the set A if the following conditions hold:

(i) For all $a \in A, a \leq a$ *reflexivity.*

(ii) For all $a, b, c \in A$, $a \leq b$ and $b \leq c$ implies $a \leq c$ *transitivity.*

We also say that A is pre-ordered by $\leq$.

Definition 1.1.3. *(Partial order)*

A binary relation $\leq$ defined on a set A is a partial order on the set A if the following three conditions hold:

(i) For all $a \in A, a \leq a$ *reflexivity.*

(ii) For all $a, b \in A$, $a \leq b$ and $b \leq a$ imply $a = b$ *antisymmetry.*

(iii) For all $a, b, c \in A$, $a \leq b$ and $b \leq c$ imply $a \leq c$ *transitivity.*

If, in addition, for every $a, b \in A$ $a \leq b$ or $b \leq a$,
then we say $\leq$ is a total order on A. A set A with a partial order on it is called a partially ordered set, or more briefly a poset, an if the relation is a total order then we speak of a totally ordered set, or simply a chain. In a poset A we use the expression $a < b$ to mean $a \leq b$ but $a \neq b$.

Definition 1.1.4. *(upper bound, least upper bound, lower bound, the greatest lower bound) Let A be a subset of a poset P. An element p in P is an upper bound for A if $a \leq p$ for every a in A. An element p in P is the least upper bound of A (l.u.b. of A), or supremum of A (sup A) if p is an upper bound of A, and $a \leq b$ for every a in A implies $p \leq b$ (i.e., p is the smallest among the upper bounds of A). Similarly we can define what it means for p to be a lower bound of A, for p to be the greatest lower bound (g.l.b.) of A, also called the infimum of A (inf A).*

In the particular case where $P = \mathbb{R}$ and $\leq$ is the natural ordering on $\mathbb{R}$ we have the following result which will be very useful subsequently:

Proposition 1.1.1. *(The least upper bound property - The greatest lower bound property)*

(i) if a nonempty subset of $\mathbb{R}$ has an upper bound, then it has a unique least upper bound.

(ii) if a nonempty subset of $\mathbb{R}$ has a lower bound, then it has a unique greatest lower bound.

Proof . *See* [1], *page 36.*

Definition 1.1.5. *(Lattice) A nonempty poset L is a lattice iff for every $a, b \in L$ both sup $\{a, b\}$ and inf $\{a, b\}$ exist (in L).*
See [2] *for more details on lattices.*

1.2 Preliminaries on linear algebra

The sets on which we will work will be essentially the vector spaces. The concept of linear mapping (linear map) will be of interest as part of our work.

Definition 1.2.1. (*Linear map*) *Let V, W be two vector spaces over the same field $\mathbb{F}$. A mapping $T : V \longrightarrow W$ is called a linear mapping if the following two conditions are satisfied:*

(i) $T(x + y) = T(x) + T(y)$ *for all* $x, y \in V$.

(ii) $T(\lambda x) = \lambda T(x)$ *for all* $x \in V$ *and* $\lambda \in \mathbb{F}$.

Definition 1.2.2. (*Linear functional*) *A linear functional or linear form (also called a one-form or co-vector) is a linear map from a vector space to its field of scalars.*

Definition 1.2.3. (*Algebraic dual space*) *Let V be a vector space over a field $\mathbb{F}$. The (algebraic) dual space V' is defined as the set of all linear maps $\Phi : V \longrightarrow \mathbb{F}$.*

Definition 1.2.4. (*Norm on a vector space*) *A norm on a vector space V is a map $\| \,.\, \|: V \longrightarrow \mathbb{R}$ that satisfies the following conditions:*

(i) $\|v\| \geq 0, \forall\, v \in V$;

(ii) $\|v\| = 0 \iff v = 0$;

(i) $\|\alpha v\| = |\alpha|\|v\|, \forall\, \alpha \in \mathbb{F}, \forall\, v \in V$;

(iv) $\|u + v\| \leq \|v\| + \|v\|, \forall\, u, v \in V$.

Definition 1.2.5. (*Normed vector space*) *A vector space V equipped with a norm $\|.\|$ is called a normed vector space.*

In the situations where more than one vector space appear, we will frequently denote the norm on V by $\|.\|_V$.

Definition 1.2.6. (*Order-bounded mapping*) *A mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be an order-bounded mapping if it maps bounded intervals $[a, b]$ into intervals of same type, that is, for every $a, b \in \mathbb{R}, a < b$ there exists $\alpha, \beta \in \mathbb{R}, \alpha < \beta$ such that:*

$$T([a,b]) \subseteq [\alpha,\beta]$$

Example: every linear map from $\mathbb{R}$ to $\mathbb{R}$ is order-bounded.

Definition 1.2.7. *(Product of a subset by a scalar) Let E be a vector space over a field $\mathbb{F}$, A a subset of E and $\lambda \in \mathbb{F}$. The product of he set A by the scalar λ, denoted by λA is the subset of E defined by:*

$$\lambda A = \{\lambda a, a \in A\}$$

1.3 Metric spaces

As part of our work, we will focus on metric spaces in general and normed vector spaces in particular.

Definition 1.3.1. *(Metric) Let X be a nonempty set. A distance function or a metric on the set X is a mapping $d : X \times X \longrightarrow \mathbb{R}$ which assigns to each pair of elements $(x,y) \in X \times X$ a real number $d(x,y)$ having the following properties:*

d_1. $d(x,y) \geq 0, \forall\, x,y \in X$ *(Non-negativeness).*

d_2. $d(x,y) = 0 \iff x = y, \forall\, x,y \in X$ *(Identification).*

d_3. $d(x,y) = d(y,x), \forall\, x,y \in X$ *(Symmetry).*

d_4. $d(x,z) \leq d(x,y) + d(y,z), \forall\, x,y,z \in X$ *(Triangular inequality).*

Given a nonempty set X and a map $d : X \times X \longrightarrow \mathbb{R}$, we say that the pair (X,d) is a metric space if and only if d is a metric on X.

The following proposition gives a fundamental example of a metric space.

Proposition 1.3.1. *Let $(X, \|.\|)$ be a normed vector space and define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x,y) = \|x-y\|$*
Then (X,d) is a metric space.

Proof . *We need to prove each of the properties of a distance.*

Let x,y,z $\in X$.

$(d_1) : d(x,y) \geq 0$

$$\begin{aligned}(d_2) : d(x,y) = 0 &\iff \|x-y\| = 0\\ &\iff x-y=0\\ &\iff x=y\end{aligned}$$

$(d_3) : d(x,y) = \|x-y\| = \|-(x-y)\| = \|y-x\| = d(y,x)$

$$\begin{aligned}(d_4) : d(x,z) &= \|x-z\|\\ &\leq \|x-y\| + \|y-z\|\\ &= d(x,y)+d(y,z)\end{aligned}$$

The proposition is then proven.

Let x be a point of a metric space $(X;d)$, and assume that r is a real number, $r > 0$. We have the following definitions:

Definition 1.3.2.

(i) The open ball centered at x with radius r *is the set*
$B(x;r) = \{y \in X : d(x,y) < r\}$.

(ii) The closed ball centered at x with the radius r *is the set*
$B'(x;r) = \{y \in X : d(x,y) \leq r\}$.

(iii) The sphere centered at x with the radius r *is the set*
$S(x;r) = \{y \in X : d(x,y) = r\}$.

Let A be a subset of a metric space X.

Definition 1.3.3. *(Open sets, closed sets)*

(i) A *is an open subset of* X *if and only if for any* $x \in A$ *there is a* $r > 0$ *such that* $B(x;r) \subset A$.

(ii) A *is closed if the compliment of* A *in* X, $\complement_X(A)$ *is an open subset of* X.

Definition 1.3.4. *Let (X,d) be a metric space. If for all n $\in \mathbb{N}, x_n \in X$, we say that $(x_n)_n$ is a sequence of elements of X.*

Definition 1.3.5. *(convergence) A sequence $(x_n)_n$ in a metric space (X,d) converges to a point x $\in X$ if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n \in \mathbb{N}, (n \geq N) \Longrightarrow (d(x_n, x) < \varepsilon).$$

.

Definition 1.3.6. *(Cauchy sequence) A sequence $(x_n)_n$ in a metric space (X,d) is said to be a Cauchy sequence if:*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid \forall n, m \in \mathbb{N}, n, m \geq N \Longrightarrow d(x_n, x_m) < \varepsilon.$$

Definition 1.3.7. *(complete metric space)*
A metric space (X,d) is called a complete metric space if every Cauchy sequence in X is convergent.

Definition 1.3.8. *(Banach spaces)*
A Banach space is a complete, normed vector space.

Let (X,d) and (Y,d') be two metric spaces, $f : X \longrightarrow Y$ be a mapping, $a \in X$ and A be a nonempty subset of X.

(i) f is said to be continuous at the point a if:

$$\forall \varepsilon > 0, \exists \eta > 0 \mid \forall x \in X, d(a,x) < \eta \Longrightarrow d'(f(x), f(a)) < \varepsilon.$$

(ii) f is continuous on A if f is continuous at every point of A.

Proposition 1.3.2. *Let E, F be two normed vector spaces on the same field $\mathbb{F}$, $f : E \longrightarrow F$ be a linear mapping. Then the following conditions are equivalent:*

(i) f is continuous.

(ii) There exist a constant $c > 0$ such that :

$$\|f(x)\|_F \leq c\|x\|_E, \text{for every } x \in E.$$

Proof . *See [5], page 126.*

The set of all continuous linear mappings from E to F is denoted by $\mathscr{L}(E,F)$; but if $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, the set of all continuous linear functionals is denoted by $E^\star$ and is called the topological dual space of E.

It is laudable to state the various forms of the HAHN-BANACH theorem that will be very useful for us later.

Definition 1.3.9. *(sub-linear functional) Let E be a real vector space. A functional $p : E \longrightarrow]-\infty,+\infty]$ is sub-linear, if*

(i) $p(\lambda a)=\lambda p(a), \forall\, a\in E, \forall\, \lambda\in\mathbb{R}_+^\star$;

(ii) $p(a+b)\leq p(a)+p(b), \forall\, a,b\in E.$

This is the first form of the Hahn-Banach Theorem that will be of interest for us subsequently:

Theorem 1.3.1. *(A corollary of the Hahn-Banach theorem) Let $p : E \longrightarrow \mathbb{R}$ be a finite-valued sub-linear functional, and $a\in E$. There is a linear functional $\omega : E \longrightarrow \mathbb{R}$ with*
$\omega(a)=p(a)$ and $\omega(b)\leq p(b)$ for all $b\in E$.
Furthermore, for $c\in E$ and $\lambda\in\mathbb{R}$, there exists a linear functional $\omega : E\longrightarrow\mathbb{R}$ with $\omega(a)=p(a),\omega(c)=\lambda$, and $\omega(b)\leq p(b)$, for all $b\in E$ if, and only if,
$\frac{p(a)-p(a-tc)}{t}\leq\lambda\leq\frac{p(a+tc)-p(a)}{t}, t>0.$

Proof . *See [4] page 71-73.*

Theorem 1.3.2. *(Multiple Hahn-Banach theorem.) Let $p_i : E \longrightarrow]-\infty,+\infty]$ be sub-linear functionals $(1\leq i\leq n)$ with*
p_1 finite-valued, and suppose that if $a_i\in E$ $(1\leq i\leq n,)$ then:
$\sum_{i=1}^n a_i=0\Longrightarrow\sum_{i=1}^n p_i(a_i)\geq 0.$

Proof . *See [4] page 71-73.*

This is an important application of the multiple Hahn-Banach theorem.

Theorem 1.3.3. *Let $\mathscr{B}$ be a real Banach Space. If $\omega\in\mathscr{B}^\star$ and $\lambda,\mu\in\mathbb{R}$ then,*

$sup\{\lambda\omega(a)+(1-\lambda)\omega(b)+(\mu-\lambda)\omega(c); b,c\geq 0, \|a\|+\|a-b-c\|\leq 1\}$
$= inf\{\|\eta\|^\star \vee \|\eta-\lambda\omega\|^\star; \eta\in\mathscr{B}^\star, \eta\geq\omega, \eta\geq\mu\omega\}$

and the infimum is attained, whenever it's finite.

Proof . *In this proof, $\mathscr{B}_+$ is a non empty subset of $\mathscr{B}$ such that*

(i) $\mathscr{B}_+$ *is closed;*

(ii) $\forall\,(x,y)\in\mathscr{B}_+\times\mathscr{B}_+, \forall(\lambda,\mu)\in\mathbb{R}_+\times\mathbb{R}_+, \lambda x+\mu y\in\mathscr{B}_+;$
in other words,
$\lambda\mathscr{B}_+ + \mu\mathscr{B}_+ \subseteq \mathscr{B}_+, \forall\,\lambda,\mu\geq 0.$

and $a\leq b$ *iff* $b-a\geq 0$

Let S denote the value of the supremum and I the value of the infimum. Let show that $S\leq I$.

If $a,b,c\in\mathscr{B}, \|a\|+\|a-b-c\|\leq 1, \eta\geq\omega$, *and* $\eta\geq\mu\omega$, *then*

$\lambda\omega(a)+(1-\lambda)\omega(b)+(\mu-\lambda)\omega(c)$
$=\lambda\omega(a-b-c)+\omega(b)+\mu\omega(c)$
$\leq\lambda\omega(a-b-c)+\eta(b)+\eta(c)$, *since* $\eta\geq\omega$ *and* $\eta\geq\mu\omega$.
$=\lambda\omega(a-b-c)-\eta(-b)-\eta(-c)+\eta(a)-\eta(a)$
$=\lambda\omega(a-b-c)-\eta(a-b-c)+\eta(a)$
$=(\lambda\omega-\eta)(a-b-c)+\eta(a)$
$\leq|\lambda\omega-\eta)(a-b-c)+\eta(a)|$
$\leq|\lambda\omega-\eta)(a-b-c)|+|\eta(a)|$
$\leq\|\lambda\omega-\eta\|^\star\|a-b-c\|+\|\eta\|^\star\|a\|$
$\leq(\|\lambda\omega-\eta\|^\star\vee\|\eta\|^\star)(\|a-b-c\|+\|a\|)$
$\leq\|\lambda\omega-\eta\|^\star\vee\|\eta\|^\star$ *since* $\|a-b-c\|+\|a\|\leq 1$.

Thus $S\leq I$.

The converse inequality follows by application of the multiple Hahn-Banach theorem with $n=4$ and the p_i chosen such that:

$$p_1(b)=S\|b\|,\quad p_2(b)=S\|b\|+\lambda\omega(b),\quad p_3(b)=\begin{cases}\omega(b)\ \textit{if}\ -b\in\mathscr{B}_+\\ +\infty\ \textit{if}\ -b\in\mathscr{B}_+\end{cases}\qquad \textit{and}$$

$$p_4 = \begin{cases} \mu\omega(b) \text{ if } -b \in \mathscr{B}_+ \\ \\ +\infty \text{ if } -b \notin \mathscr{B}_+. \end{cases}$$

Thus, if $b_1 + b_2 + b_3 + b_4 = 0$*, one has:*

$\sum_{i=1}^{i=4} p_i(b_i) = +\infty$ *for* $-b_3 \notin \mathscr{B}_+$ *or* $-b_4 \notin \mathscr{B}_+$

and:

$\sum_{i=1}^{i=4} p_i(b_i) = S\|b_1\| + S\|b_2\| + \lambda\omega(b_2) + \omega(b_3) + \mu\omega(b_4) = S(\|b_1\| + \|b_2\|) - \{\lambda\omega(b_1) + (1-\lambda)\omega(-b_3) + (\mu-\lambda)\omega(-b_4) \geq 0$ *for* $-b_3 \in \mathscr{B}_+$ *and* $-b_4 \in \mathscr{B}_+$.

Hence by the multiple Hahn-Banach theorem, there exist a linear functional $\eta : \mathscr{B} \longrightarrow \mathbb{R}$ *satisfying:*

$\eta \leq p_i$ *for* $i = 1, 2, 3, 4$*. In particular,* $\eta \in \mathscr{B}^\star$, $\|\eta\|^\star \leq S$, $\|\eta - \lambda\omega\|^\star \leq S, \eta \geq \omega$*, and* $\eta \geq \lambda\omega$.

Thus $S \geq I$*. If comes that* $I = S$ *and also the infimum is attains at the point* η*.*

□

Notation 1.3.1. $S(\omega, \lambda, \mu) := sup\{\lambda\omega(a) + (1-\lambda)\omega(b) + (\mu-\lambda)\omega(c); \|a\| + \|a - b - c\| \leq 1\}$

and:

$I(\omega, \lambda, \mu) := inf\{\|\eta\|^\star \vee \|\eta - \lambda\omega\|^\star; \eta \in \mathscr{B}^\star, \eta \geq \omega, \eta \geq \mu\omega\}$,

for all $\omega \in \mathscr{B}^\star, \lambda, \mu \in \mathbb{R}$.

★ ★ CHAPTER TWO ★ ★

ORDERED BANACH SPACES

2.1 Normal generating cone

Definition 2.1.1. *(positive cone)*

*Let $\mathscr{B}$ be a normed vector space and $\mathscr{B}_+$ be a nonempty subset of $\mathscr{B}$. $\mathscr{B}_+$ is called **a positive cone** of $\mathscr{B}$ if the following conditions hold:*

(i) *$\mathscr{B}_+$ is closed;*

(ii) $\forall\,(x, y) \in \mathscr{B}_+ \times \mathscr{B}_+, \forall(\lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+, \lambda x + \mu y \in \mathscr{B}_+;$
in other words,
$\lambda\mathscr{B}_+ + \mu\mathscr{B}_+ \subseteq \mathscr{B}_+, \forall\,\lambda, \mu \geq 0.$

Definition 2.1.2. *A positive cone $\mathscr{B}_+$ of a Banach space $\mathscr{B}$ is defined to be **proper** or pointed if*

$$\mathscr{B}_+ \cap (-\mathscr{B}_+) = \{0_{\mathscr{B}}\}.$$

Definition 2.1.3. *(pre-ordered Banach space and ordered Banach space) A pre-ordered Banach space $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ is a real Banach space $\mathscr{B}$ with norm $\|.\|$ equipped with a positive cone $\mathscr{B}_+$ of $\mathscr{B}$. If in addition, $\mathscr{B}_+$ is proper, we say that $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ is an ordered Banach space.*

The elements of $\mathscr{B}$ will be denoted by $a, b, c, ...$ and for $\alpha \in \mathbb{R}_+^\star$,
$\mathscr{B}_\alpha$ will denote the open ball of radius α, i.e.,

$$\mathscr{B}_\alpha = \{a \in \mathscr{B} : \|a\| \ < \alpha\}$$

Associated to each pre-ordered Banach space $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ there is pre-ordered dual space $(\mathscr{B}^\star, \mathscr{B}_+^\star, \|.\|^\star)$ consisting of the real linear functionals $\mathscr{B}^\star = \{\omega, \xi, \eta, ...\}$ over $\mathscr{B}$, s.t. the dual norm:

$$\|\omega\|^{\star} = Sup\{| \omega(a) | \mid a \in \mathscr{B}_1\}$$

is finite, and the $weak^{\star} - closed$ dual cone $\mathscr{B}_+^{\star}$ which is defined by:

$$\mathscr{B}_+^{\star} = \{\omega \in \mathscr{B}^{\star} : \omega(a) \geq 0 \quad \forall a \in \mathscr{B}_+\}.$$

By the definition of the dual cone, one has:

$$\lambda \mathscr{B}_+^{\star} + \mu \mathscr{B}_+^{\star} \subseteq \mathscr{B}_+^{\star}, \forall\, \lambda, \mu \geq 0.$$

The norm closed ball of radius α in $\mathscr{B}^{\star}$ is denoted by $\mathscr{B}_{\alpha}^{\star}$.

Proposition 2.1.1. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be an ordered Banach space. The relation $\leq$ defined on $\mathscr{B}$ by setting $a \leq b \iff b - a \in \mathscr{B}_+$ is a partial order.*

Proof . *Let $a, b, c \in \mathscr{B}$.*

(i) $a - a = 0 \in \mathscr{B}_+$, *hence* $a \leq a$

(i) *If* $a \leq b$, $b \leq a$, *then,* $a - b \in \mathscr{B}_+ \cap (-\mathscr{B}_+) = \{0\}$. *Hence,* $a = b$

(i) *Assume* $a \leq b$ *and* $b \leq c$.
One has $c - a = (c - b) + (b - a)$ *and since* $a \leq b$, $b \leq c$, *it follows that* $(c - b), (b - a) \in \mathscr{B}_+$ *and* $c - a \in \mathscr{B}_+$.
hence, $a \leq b$.

□

Thanks to the preceding proposition, if the cones $\mathscr{B}_+$ and $\mathscr{B}_+^{\star}$ are proper, then partial order relations are defined on $\mathscr{B}$ and on $\mathscr{B}^{\star}$, by setting $a \geq b$ whenever $a - b \in \mathscr{B}_+$, and $\xi \geq \eta$ whenever $\xi - \eta \in \mathscr{B}_+^{\star}$. Thus $a \geq 0$ is equivalent to $a \in \mathscr{B}_+$, and $\xi \geq 0$ is equivalent to $\xi \in \mathscr{B}_+^{\star}$.

There are two deficiencies in this structure: there is no condition which ensures that $\mathscr{B}_+$ is large enough to introduce an interesting order relation and there is no condition which ensures that $\mathscr{B}_+$ is not too large. The aim of this chapter is to introduce and analyze such conditions. Further refinements of the conditions will be discussed. We begin with the weakest possible form of such conditions.

Definition 2.1.4. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be an ordered Banach space. The positive cone $\mathscr{B}_+$ is defined to be weakly generating if $\mathscr{B}_+ - \mathscr{B}_+$ is norm dense in $\mathscr{B}$, i.e., if each $a \in \mathscr{B}$ is the norm limit of a sequence$\{b_n - c_n\}_{n\geq 1}$ of differences of elements $b_n, c_n \in \mathscr{B}_+$.*

Remark 2.1.1. *The generation property has the tendency to make $\mathscr{B}_+$ 'large' and the dual cone $\mathscr{B}_+^\star$ 'small'. Conversely, the pointedness property requires $\mathscr{B}_+$ to be 'small' and $\mathscr{B}_+^\star$ to be 'large'. This elementary observation is at the root of a series of duality properties of which the following proposition is the simplest.*

Proposition 2.1.2. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be an ordered Banach space. The following conditions are equivalent:*

1. *$\mathscr{B}_+$ is weakly generating.*

2. *$\mathscr{B}_+^\star$ is proper.*

Proof . *1.$\Longrightarrow$ 2. Assume that $\mathscr{B}_+$ in weakly generating.*
Let $\omega \in \mathscr{B}^\star$. If $\omega \in \mathscr{B}_+^\star \cap \left(-\mathscr{B}_+^\star\right)$, then, there are $\omega_1, \omega_2 \in \mathscr{B}_+^\star$ such that $\omega = \omega_1 = -\omega_2$; this implies that $\omega(a) = 0$ for all $a \in \mathscr{B}_+$. Let $a \in \mathscr{B}$; since $\mathscr{B}_+$ is weakly generating, there are two sequences (b_n) and (c_n) of elements of $\mathscr{B}_+$ such that $a = b_n - c_n$ for all n. By the linearity of ω, one has: $\omega(a) = \omega(b_n) - \omega(c_n) = 0 - 0 = 0$.
2.$\Longrightarrow$ 1. This is an application of the Hahn-Banach theorem. (See [4])

Proposition 2.1.3. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be an ordered Banach space. The following conditions are equivalent:*

1. *$\mathscr{B}_+^\star$ is weakly generating.*

2. *$\mathscr{B}_+$ is proper.*

Proof . *See [4]*

now, we turn to the analysis of stronger properties of generation and pointedness.

Definition 2.1.5. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space. The cone $\mathscr{B}_+$ is defined to be generating if:*

$$\mathscr{B} = \mathscr{B}_+ - \mathscr{B}_+$$

Definition 2.1.6. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space. The cone $\mathscr{B}_+$ is defined to be normal if there is an $\alpha \geq 1$ such that:*

$$c \leq a \leq b \Longrightarrow \|a\| \leq \alpha\{\|b\| \vee \|c\|\}.$$

Proposition 2.1.4. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space. If $\mathscr{B}_+$ is **normal**, then $\mathscr{B}_+$ is **proper**.*

Proof . *Assume that $\mathscr{B}_+$ is normal. Let a be an element of $\mathscr{B}_+ \cap (-\mathscr{B}_+)$.*
One has $0 \leq a \leq 0$.
Since $\mathscr{B}_+$ is $\alpha-$normal for some $\alpha > 0$, it comes that
$\|a\| \leq \alpha\{\|0\| \vee \|0\|\} = 0.$
Hence $a = 0$. □

Remark 2.1.2. *Normality is a condition of compatibility between the order and the topology. Later, we will see that normality is equivalent to the requirement that **order-bounded** sets are **norm-bounded**.*

Normality and generation for $\mathscr{B}_+^\star$ are defined analogously. Again there is a duality between these properties for $\mathscr{B}_+$ and $\mathscr{B}_+^\star$. But before giving this, we first establish a **uniformity** of the generation property which allows a more precise indexation of generation and a subsequent closer comparison with normality.

Proposition 2.1.5. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space. The following conditions are equivalent:*
1. $\mathscr{B}_+$ is generating;
2. There is an $\alpha \geq 1$ such that each $a \in \mathscr{B}$ has a decomposition $a = b - c$ with $b, c \in \mathscr{B}_+$ and $\|b\| \vee \|c\| \leq \alpha\|a\|$.

Proof . $2. \Longrightarrow 1.$ *By definition.*
$1. \Longrightarrow 2.$ *condition 1. implies:*

$$\mathscr{B}_1 \subseteq \bigcup_{\alpha \geq 1} (\mathscr{B}_\alpha \cap \mathscr{B}_+ - \mathscr{B}_\alpha \cap \mathscr{B}_+)$$

Hence by the Baire category theorem there exists a $\beta \geq 1$ *such that:*
$\mathscr{B}_1 \subseteq \mathscr{B}_\beta \cap \mathscr{B}_+ - \mathscr{B}_\beta \cap \mathscr{B}_+$. *The proof is then completed by estimating that:*
$\mathscr{B}_1 \subseteq \mathscr{B}_\alpha \cap \mathscr{B}_+ - \mathscr{B}_\alpha \cap \mathscr{B}_+$ *for all* $\alpha > \beta$. *Since this type of estimate will be used several times in the sequel we present it in a suitably general form. But first recall that a subset* $C \subset \mathscr{B}$ *is said to be* $\sigma-$ *convex if the conditions* $c_n \in C, \lambda_n \geq 0, \sum_{n=1}^{\infty} \lambda_n = 1$, *together with the existence of* $c = \sum_{n=1}^{\infty} \lambda_n c_n$ *always imply that* $c \in C$. *For example* $\mathscr{B}_\beta, \mathscr{B}_\beta \cap \mathscr{B}_+, \ \mathscr{B}_\beta \cap \mathscr{B}_+ - \mathscr{B}_\beta \cap \mathscr{B}_+$ *are all* $\sigma-$*convex subsets of* $\mathscr{B}$.

Lemma 2.1.1. *If* $\mathcal{C} \subset \mathscr{B}$ *is a* $\sigma - convex$ *subset such that* $\mathscr{B}_1 \subset \overline{\mathcal{C}}$, *then* $\mathscr{B}_1 \subseteq \alpha\mathscr{C}$ *for all* $\alpha > 1$

Proof . *For* $a \in \mathscr{B}_1$ *and* $0 < \delta < 1$, *choose* $a_1 \in \mathcal{C}$ *such that* $\|a - a_1\| < \delta$. $\delta^{-1}(a - a_1) \in \mathscr{B}_1$ *and one can choose* $a_2 \in \mathcal{C}$ *such that* $\|\delta^{-1}(a - a_1) - a_2\| < \delta$. *By iteration one finds* $a_n \in \mathcal{C}$ *such that* $\|\delta^{-n+1}a - \sum_{m=1}^{n} \delta^{-n+m} a_m\| < \delta$. *Therefore setting* $\lambda_m = (1-\delta)\delta^{m-1}$ *one has*

$$\|a - (1-\delta)^{-1} \sum_{m=1}^{n} \lambda_m a_m\| < \delta^n \text{ for every } n \in \mathbb{N}^\star.$$

Passing to the limit as n tends to $+\infty$, *we find*

$$a = (1-\delta)^{-1} \sum_{m=1}^{+\infty} \lambda_m a_m.$$

Hence, the series $\sum_{m\geq 1} \lambda_m a_m$ *converges,* $\lambda_m \geq 0$ *for every* $m \geq 1, \sum_{m=1}^{+\infty} \lambda_m = 1, a_m \in \mathcal{C}$, *and* $\mathcal{C}$ *is* $\sigma-$*convex.*

Hence,
$b = \sum_{m\geq 1} \lambda_m a_m \in \mathcal{C}$.
Hence, $a \in (1-\delta)^{-1}\mathcal{C}$.
It remains to prove that $(1-\delta)^{-1}\mathcal{C} \subseteq \alpha\mathcal{C}$, *for every* $\alpha > 1$. *Let* $c \in \mathcal{C}$.
$(1-\delta)^{-1}c = \alpha\alpha^{-1}(1-\delta)^{-1}c$.
And,
$\alpha^{-1}(1-\delta)^{-1} = (1 - \alpha^{-1}(1-\delta)^{-1}))0 + \alpha^{-1}(1-\delta)^{-1}c \in \mathcal{C}$. □

In comparing normality and generation properties of $\mathscr{B}_+$, it is of interest to keep of the index α of uniformity.

Definition 2.1.7. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space and* $\alpha \geq 0$. *The cone* $\mathscr{B}_+$ *is defined to be* $\alpha_+ - generating$ *if each* $a \in \mathscr{B}$ *has a decomposition* $a = b - c$ *with* $b, c \in \mathscr{B}_+$ *and* $\|b\| + \|c\| \leq \alpha\|a\|$.

Definition 2.1.8. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be an ordered Banach space and* $\beta \geq 0$. *The cone* $\mathscr{B}_+$ *is defined to be* $\alpha_\vee - generating$ *if each* $a \in \mathscr{B}$ *has a decomposition* $a = b - c$ *with* $b, c \in \mathscr{B}_+$ *and* $\|b\| \vee \|c\| \leq \beta\|a\|$.

Remark 2.1.3. *If* $\mathscr{B}_+$ *is* $\alpha - generating$, *then* $\alpha \geq 1$, *by the triangle inequality.*

Definition 2.1.9. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space and* $\alpha, \beta \geq 0$. *The cone* $\mathscr{B}_+$ *is defined to be approximately* $\alpha_+ - generating$ *if it is* $\alpha'_+ - generating$ *for all* $\alpha' > \alpha$ *and approximately* $\beta_\vee - generating$ *if it is* $\beta'_\vee - generating$ *for all* $\beta' > \beta$.

Similarly we introduce two types of normality.

Definition 2.1.10. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space and* $\alpha, \beta \geq 0$.

(i) The cone $\mathscr{B}_+$ *is defined to be* $\alpha_\vee - normal$ *if* $c \leq a \leq b$ *always implies* $\|a\| \leq \alpha(\|a\| \vee \|c\|)$

(ii) The cone $\mathscr{B}_+$ *is defined to be* $\beta_+ - normal$ *if* $c \leq a \leq b$ *always implies* $\|a\| \leq \beta(\|a\| + \|c\|)$

To generalize the duality properties contained in propositions 2.1.2. and 2.1.3, we need theorem 1.3.3:

If $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *is a pre-ordered Banach space, then, the normality property of the cone* $\mathscr{B}_+$ *is characteristic of the generation property of its dual cone and vice versa as the following theorem is showing us.*

Theorem 2.1.1. *Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space. Then, the following conditions are equivalent:*

1.$\mathscr{B}_+$ is $\beta_+ - normal.$

2. $\mathscr{B}_+^\star$ is $\beta_\vee - generating.$

Proof . *One has:*

$$\begin{aligned} S(\omega,\lambda,0) &= sup\{\omega(a)-\omega(c); b,c\geq 0, \|a\|+\|a-b-c\|\leq 1\}, \\ &= sup\{\omega(a-c); b,c\geq 0, \|a\|+\|a-b-c\|\leq 1\}, \\ &= sup\{\omega(d); a\geq d, b\geq 0, \|a\|+\|d-b\|\leq 1\}, \\ &= sup\{\omega(d); a\geq d, d\geq h\geq 0, \|a\|+\|h\|\leq 1\}, \\ &= sup\{\omega(a); b\leq a\leq c, \|b\|+\|c\|\leq 1\}. \end{aligned}$$

and :

$$\begin{aligned} I(\omega,\lambda,0) &= inf\{\|\eta\|^\star \vee \|\eta-\omega\|^\star; \eta\in\mathscr{B}^\star, \eta\geq\omega, \eta\geq 0\} \\ &= inf\{\|\eta\|^\star \vee \|\xi\|^\star; \eta\in\mathscr{B}^\star, \eta\geq 0, \xi\geq 0\omega=\eta-\xi\} \\ &= inf\{\|\eta\|^\star \vee \|\xi\|^\star; \eta\in\mathscr{B}^\star, \eta,\xi\geq 0, \omega=\xi-\eta\} \end{aligned}$$

Hence:
$sup\{\omega(a); b\leq a\leq c, \|b\|+\|c\|\leq 1\} = inf\{\|\eta\|^\star \vee \|\xi\|^\star; \eta\in\mathscr{B}^\star, \eta,\xi\geq 0, \omega = \xi-\eta\}$ *and the infimum is attained. But this is just a statement of the equivalence of $\beta_+ - normality$ of $\mathscr{B}_+$ and $\beta_\vee - generation$ of $\mathscr{B}_+^\star$:*

-Let $\mathscr{B}_+$ be $\beta_+ - normal$. Let $\omega\in\mathscr{B}^\star$. As the infimum is attained, there exists $\eta,\xi\geq 0 : \omega=\xi-\eta$ and :

$$\begin{aligned} \|\eta\|^\star \vee \|\xi\|^\star &= sup\{\omega(a); b\leq a\leq c, \|b\|+\|c\|\leq 1\} \\ &\leq sup\{\omega(a) : \|a\|\leq 1\}[\mathscr{B}_+ is \beta_+ - normal] \\ &\leq \|\omega\|^\star \\ &\leq \beta\|\omega\|^\star since \beta\geq 1. \end{aligned}$$

Hence, $\mathscr{B}_+^\star$ *is* $\beta_\vee - generating$.

-Let the cone $\mathscr{B}_+^\star$ *be* $\beta_\vee-$*generating. Let* $a, b, c \in \mathscr{B}$ *such that* $c \leq a \leq b$*. We want to prove that* $\|a\| \leq \beta(\|b\| + \|c\|)$.

For all $\omega \in \mathscr{B}^\star$*, there exists* $\eta, \xi \in \mathscr{B}_+^\star$ *such that* $\omega = \|\eta\|^\star - \|\xi\|^\star$ *and* $\beta\|\omega\|^\star \geq \|\eta\|^\star \vee \|\xi\|^\star \geq I(\omega, 1, 0) = S(\omega, 1, 0)$.

For $\|b\| + \|c\| \neq 0, b \leq a \leq c$ *implies* $\frac{b}{\|b\|+\|c\|} \leq \frac{a}{\|b\|+\|c\|} \leq \frac{c}{\|b\|+\|c\|}$*. In addition,* $\|\frac{b}{\|b\|+\|c\|}\| + \|\frac{c}{\|b\|+\|c\|}\| \leq 1$.

So, $\omega(\frac{a}{\|b\|+\|c\|}) \leq \beta\|\omega\|$*. By the Hahn-Banach theorem, there is an* $\varphi \in \mathscr{B}^\star$ *such that* $\|\varphi\| = 1$ *and* $\varphi(a) = \|a\|$*. Taking* $\omega = \varphi, \|a\| \leq \beta(\|b\| + \|c\|)$*. Moreover, the implication is also true if* $\|b\| + \|c\| = 0$.

□

Theorem 2.1.2. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space. Then the following conditions are equivalent:*

1.$\mathscr{B}_+$ *is* $\alpha_\vee$*-normal.*

2. $\mathscr{B}_+^\star$ *is* α_+-*generating.*

Proof . *See [6] page 226.*

Theorem 2.1.3. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be an ordered Banach space. Then the following conditions are equivalent:*

1.$\mathscr{B}_+$ *is approximately* $\beta_\vee$*-generating.*

2. $\mathscr{B}_+^\star$ *is* β_+*-normal.*

Proof . $\Longrightarrow$)*. Assume that* $\mathscr{B}_+$ *is approximately* $\beta_\vee$*-generating. We need to show that* $\mathscr{B}_+^\star$ *is* β_+*-normal. Let a=b-c with* $b, c \in \mathscr{B}_+$ *and* $\|b\| \vee \|c\| \leq \beta'\|a\|$ *with* $\beta' > \beta$)*. Thus, if* $\xi \leq \omega \leq \eta$*, then,*

$\xi(b) - \eta(c) \leq \omega(a) \leq \eta(b) - \xi(c)$.

Therefore:

$$\begin{aligned}
|\omega(a)| &\leq |\xi(b)-\eta(c)| \vee |\eta(b)-\xi(c)| \\
&\leq (|\xi(b)|+|\eta(c)|) \vee (|\eta(b)|+|\xi(c)|) \\
&\leq (\|\xi\|^\star\|b\|+\|\eta\|^\star\|c\|) \vee (\|\eta\|^\star\|b\|+\|\xi\|^\star\|c\|) \\
&\leq [(\|\xi\|^\star+\|\eta\|^\star)(\|b\|\vee\|c\|)] \vee [(\|\xi\|^\star+\|\eta\|^\star)(\|b\|\vee\|c\|)] \\
&= (\|\xi\|^\star+\|\eta\|^\star)(\|b\|\vee\|c\|) \\
&\leq \beta^{'}(\|\xi\|^\star+\|\eta\|^\star)\|a\|
\end{aligned}$$

which implies that :

$\|\omega\|^\star \leq \beta^{'}(\|\xi\|^\star+\|\eta\|^\star$ *Since this is valid for all* $\beta^{'} > 0)$, *the cone* $\mathscr{B}_+^\star$ *is* β_+-*normal.*

$\Longleftarrow$.) *See [4]*

2.2 Monotone norms

Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space and* $\alpha \geq 0$

Definition 2.2.1. *The norm* $\|.\|$ *is defined to be* α *- monotone if* $0 \leq a \leq b$ *always implies that* $\|a\| \leq \alpha\|b\|$.

Proposition 2.2.1. *If* $\mathscr{B}_+ \neq \{0\}$ *and if the norm is* α *- monotone, then* $\alpha \geq 1$.

Proof . *Let* $\mathscr{B}_+ \neq \{0\}$ *and* $\|.\|$ $\alpha-$ *monotone. There exists* $a \in \mathscr{B}_+$ *such that* $a \neq 0$.
we have:
$0 \leq a \leq a.$
As $\|.\|$ *is* $\alpha-$*monotone, one has:*
$\|a\| \leq \alpha\|a\|.$
This implies $\alpha \geq 1$ *since* $a \neq 1.$

□

Proposition 2.2.2. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space*

(1) If $\mathscr{B}_+$ *is* α_+-*normal, then* $\|.\|$ *is* $\alpha-$*monotone*

(2) If $\mathscr{B}_+$ *is* α_V-*normal, then* $\|.\|$ *is* $\alpha-$*monotone.*

Proof . *(1) Suppose $\mathscr{B}_+$ is α_+-normal.*

If $0 \leq a \leq b$, then $\|a\| \leq \alpha(\|0\| + \|b\|)$ for $\mathscr{B}_+$ is α_+-normal.

In other words, $0 \leq a \leq b$ always implies $\|a\| \leq \alpha\|b\|$. Hence $\|.\|$ is $\alpha-$monotone.

(2) Suppose $\mathscr{B}_+$ is $\alpha_\vee-$ normal. One has $0 \leq a \leq b \Longrightarrow \|a\| \leq \alpha(\|0\| \vee \|b\|) = \alpha\|b\|$. Thus $\|.\|$ is α_+-normal

$\square$.

Proposition 2.2.3. *If $\|.\|$ is $\alpha-$monotone, then $\mathscr{B}_+$ is $(\alpha + \frac{1}{2})$-normal or $(2\alpha + 1)_\vee$-normal.*

Proof . *Suppose $\|.\|$ is α-monotone then $0 \leq a \leq b$ always implies that $\|a\| \leq \alpha\|b\|$. Let $c \leq a \leq b$; this implies $a - c \in \mathscr{B}_+$, $b - a \in \mathscr{B}_+$. Then, $a - c \in \mathscr{B}_+$, $b - a = (b - c) + (c - a) \in \mathscr{B}_+$*

ie $a - c \in \mathscr{B}_+$, $(b - c) - (a - c) \in \mathscr{B}_+$

ie $0 \leq a - c \leq b - c$; hence $\|a - c\| \leq \alpha\|b - c\|$.

If $\|b\| \leq \|c\|$, one has:

$$\begin{aligned}
\|a\| &= \|(a - c) + c\| \\
&\leq \|a - c\| + \|c\| \\
&\leq \alpha\|b - c\| + \|c\| \\
&\leq \alpha\|b\| + \alpha\|c\| + \|c\| \\
&\leq 2\alpha\|c\| + \|c\| \\
&= (2\alpha + 1)(\|b\| \vee \|c\|) \qquad (2.1)
\end{aligned}$$

If $\|c\| \leq \|b\|$, one has:

$$\begin{aligned}
\|a\| &\leq \alpha\|b\| + \|c\| + \|c\| \\
&\leq \alpha(\|b\| + \|b\|) + \|b\|
\end{aligned}$$

$$\leq (2\alpha + 1)\|b\| = (2\alpha + 1)(\|b\| \vee \|c\|). \qquad (2.2)$$

Hence, $c \leq a \leq b$ always implies $\|a\| \leq (2\alpha + 1)(\|b\| \vee \|c\|)$. $\square$

Proposition 2.2.4. *If $\mathscr{B}_+$ is normal, then there exists an equivalent monotone norm.*

Proof. Suppose $\mathscr{B}_+$ be normal. We are looking for an equivalent monotone norm. Since $\mathscr{B}_+$ is normal, there is an $\alpha \geq 1$ such that $c \leq a \leq b$ always implies

$$\|a\| \leq \alpha(\|b\| \vee \|c\|).$$

Now define the function $\|a\|_+ = inf\{\|b\| + \|c\|; c \leq a \leq b\}$ for each $a \in \mathscr{B}$.

We prove that $\|.\|_+$ is a norm.

$N_1.)$ Clearly, for every $a \in \mathscr{B}$ $\|a\|_+ \geq 0$.

$$\begin{aligned} N_2.)\|a\|_+ = 0 &\iff inf\{\|b\| + \|c\|; c \leq a \leq b\} = 0 \\ &\iff \forall \varepsilon, \exists c \leq a \leq b \ \ s.t. \ \ \|b\| + \|c\| < \varepsilon \\ &\iff \forall n \in \mathbb{N}^*, \exists c_n \leq a \leq b_n \ \ s.t. \ \ \|b_n\| + \|c_n\| < \frac{1}{n} \Longrightarrow b_n \longrightarrow 0, \ \ and \ \ c_n \longrightarrow 0, \ \ as \ \ n \end{aligned}$$

$b_n - a, a - c_n \in \mathscr{B}_+$ for every $n \in \mathbb{N}$. $b_n - a \longrightarrow -a$ in $\mathscr{B}$ $\ and \ $ $a - c_n \longrightarrow a$ in $\mathscr{B}$ imply $a = -a$, i.e. $a = 0$.

$N_3.)$ Let $a \in \mathscr{B}$ an $\lambda \in \mathbb{R}$.

$\|\lambda a\| = inf\{\|b\| + \|c\|; c \leq \lambda a \leq b\} = infA$ where $A = \{\|b\| + \|c\|; c \leq \lambda a \leq b\}$;

$\|a\| = infB$ where $B = \{\|b\| + \|c\|; c \leq a \leq b\}$.

If $c \leq \lambda a \leq b$ and $u = \|b\| + \|c\|$ $\lambda a - c, b - \lambda a \in \mathscr{B}_+$

If $\lambda > 0$, then $a - \frac{c}{\lambda}, \frac{b}{\lambda} - a \in \mathscr{B}_+$ implies $\frac{c}{\lambda} \leq a \leq \frac{b}{\lambda}$; implies $infB \leq \frac{1}{\lambda}(\|b\| + \|c\|)$ ie $\lambda infB \leq \|b\| + \|c\|$ and so $\lambda infB \leq infA$.

If $\lambda < 0$, $-a + \frac{c}{\lambda} \in \mathscr{B}_+, \frac{-b}{\lambda} + a \in \mathscr{B}_+$ implies $\frac{b}{\lambda} \leq a \leq \frac{c}{\lambda}$; $\Longrightarrow infB \leq \frac{1}{|\lambda|}(\|b\| + \|c\|)$;$\Longrightarrow -\lambda infB \leq \|b\| + \|c\|$; $\Longrightarrow -\lambda infB \leq infA$.

If $\lambda = 0$, $0 infB \leq infA$.

It comes that $|\lambda| infB \leq infA$, for each λ in $\mathbb{R}$.

If $c \leq a \leq b$, $a - c \in \mathscr{B}_+$, $b - a \in \mathscr{B}_+$

If $\lambda > 0$,

$$\begin{cases} \lambda a - \lambda c \in \mathscr{B}_+ \\ \lambda b - \lambda a \in \mathscr{B}_+ \end{cases}$$

ie $\lambda c \leq \lambda a \leq \lambda b$ implies $\|\lambda a\|_+ \leq |\lambda|(\|b\| + \|c\|)$ ie $\frac{1}{|\lambda|}\|\lambda a\|_+ \leq \|b\| + \|c\|$; $\frac{1}{|\lambda|}\|\lambda a\|_+ \leq \|a\|_+ \, \|\lambda a\|_+ \leq \|\lambda a\|_+$.
If $\lambda = 0$, $\|\lambda\|_+ \leq |\lambda|\|a\|_+$.
We have shown that $\|\lambda a\|_+ = |\lambda|\|a\|_+$.
N_4). Let $a, a^{'} \in \mathscr{B}_+$
If $c \leq a \leq b$, $c' \leq a' \leq b'$ then $a - b$, $b - a, a' - c', b' - a' \in \mathscr{B}_+$ implies $a + a' - (c + c') \in \mathscr{B}_+$, $b + b' - (a + a') \in \mathscr{B}_+$ $c + c' \leq a + a' \leq b + b'$ implies $\|a + a'\|_+ \leq \|b + b'\| + \|c + c'\| \leq \|b\| + \|c\| + \|b'\| + \|c'\|$ implies $\|a + a'\|_+ \leq \|a\| + \|a'\|$. We have shown that $\|.\|$ is a norm on $\mathscr{B}$.
Now, we prove that $\|\|_+$ is monotone.
$0 \leq a \leq b$ implies $\|a\| \leq \alpha\|b\| \leq \alpha^2(\|c\| + \|d\|)$ $\forall c \leq b \leq d \Longrightarrow \|a\| \leq \alpha^2\|b\|_+ \Longrightarrow$ $\|a\|_+ \leq \|a\| \leq \|b\|_+$.
Let's prove that $\|.\|$ and $\|.\|_+$ are equivalent. $\forall\, a \in \mathscr{B}_+$, $\|a\|_+ \leq 2\|a\| \leq 2\alpha\|a\|_+$ the proof is then finished. □

Definition 2.2.2. *A Banach lattice is an ordered Banach space which possesses an equivalent monotone norm.*

Definition 2.2.3. *The cone $\mathscr{B}_+$ is defined to be $\alpha-$ dominating if each $a \in \mathscr{B}$ has a decomposition $a = b - c$ with $b, c \in \mathscr{B}_+$ and $\|b\| \leq \alpha\|a\|$.*

Definition 2.2.4. *The cone $\mathscr{B}_+$ is defined to be approximately $\alpha-$ dominating if it is $\alpha'-$dominating for all $\alpha' > \alpha$. Subsequently we also use the terminology (approximately) dominating in place of (approximately) 1-dominating*

⋆ ⋆ CHAPTER THREE ⋆ ⋆

BOUNDED OPERATORS

3.1 Absolutely monotone norms

In this section we consider a more stringent notion of monotonicity for the norm, absolute monotonicity. This is of interest for several reasons. First, the norms and dual norms of the classical function spaces, and of $C^\star$-algebras, have this property. Second, absolute monotonicity of an equivalent norm and its dual norm are characteristics of normal generation cones. Third, the concept is useful in the theory of bounded operators on Banach spaces.
Let $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ be a pre-ordered Banach space and $\alpha \in \mathbb{R}_+$.

Definition 3.1.1. *The norm is defined to be α-absolutely monotone if for all $a, b \in \mathscr{B}_+$, $-b \leq a \leq b$ implies $\|a\| \leq \alpha\|b\|$.*

Note that $-b \leq a \leq b$ requires $b \geq 0$ and taking $a = b \in \mathscr{B}_+ - \{0\}$, one has $\alpha \geq 1$.
Next we define the dual concept.

Definition 3.1.2. *The cone $\mathscr{B}_+$ is defined to be $\alpha-$ absolutely dominating if for each $a \in \mathscr{B}$ there is a $b \geq 0$ such that $-b \leq a \leq b$ and $\|a\| \leq \alpha\|b\|$.*

More generally, one has:

Definition 3.1.3. *The cone $\mathscr{B}_+$ is defined to be approximately α-absolutely dominating if it's α'-absolutely dominating for all $\alpha' > \alpha$.*

Subsequently we use the simplified terminology absolutely monotone for 1-absolutely monotone and (approximately) absolutely dominating for (approximately) 1-absolutely dominating. The duality between these concepts is as follow.

Theorem 3.1.1. *The following conditions are equivalent:*

1.$\|.\|$ *is* $\alpha-$*absolutely monotone;*

2-$\mathscr{B}_+^\star$ *is* $\alpha-$*absolutely dominating.*

Moreover the following conditions are also equivalent:

1'-$\mathscr{B}_+$ *is approximately* $\alpha-$*absolutely dominating.*

2'-$\|.\|^\star$ *is* $\alpha-$*absolutely monotone.*

Proof . *See [6 page 233] for the proof.* □

3.2 Interior point and order-unit spaces

Definition 3.2.1. *An element* u *of the cone* $\mathscr{B}_+$ *is defined to be an interior point if* $\mathscr{B}_+$ *contains an open neighborhood of* u*, i.e., if there is an* $\varepsilon > 0$ *such that:*

$$\{a; \|u-a\| < \varepsilon\} \subseteq \mathscr{B}_+$$

The set of interior points of $\mathscr{B}_+$ *is denoted by* $int\mathscr{B}_+$*.*

Proposition 3.2.1. *If* $u \in int\mathscr{B}_+, b \in \mathscr{B}_+$ *and* $\lambda > 0,$ *then* $\lambda u + b \in int\mathscr{B}_+.$

Proof . *Let* $\varepsilon > 0$ *such that* $u + \mathscr{B}_{\frac{\varepsilon}{|\lambda|}} \subseteq \mathscr{B}_+$*.*
Let $a \in \mathscr{B}_\varepsilon$*.*
$\lambda u + b + a = \lambda(u + \lambda^{-1}a) + b$

$$\begin{aligned} \lambda^{-1}a \in \mathscr{B}_{\frac{\varepsilon}{\lambda}} \quad &\Longrightarrow \quad u + \lambda^{-1}a \in u + \mathscr{B}_{\frac{\varepsilon}{\lambda}} \subseteq \mathscr{B}_+ \\ &\Longrightarrow \quad u + \lambda^{-1}a, b \in \mathscr{B}_+. \end{aligned}$$

since this is true for all $a \in \mathscr{B}_\varepsilon$*, it follows that :*
$\lambda u + b + \mathscr{B}_\varepsilon \subseteq \mathscr{B}_+; \Longrightarrow \lambda u + b \in int\mathscr{B}_+.$ □

Definition 3.2.2. *Let* $u \in \mathscr{B}_+$*.* u *is defined to be an* ***order-unit*** *if*

$$\mathscr{B} = \bigcup_{\lambda \geq 0} [-\lambda u, \lambda u]$$

Where the ***order-interval*** $[c, b]$ *is defined by :*

$$[c, b] = \{a \in \mathscr{B} : c \leq a \leq b\}$$

Proposition 3.2.2. *If $u \in int\mathscr{B}_+$, then there is an $\varepsilon > 0$ such that for all $\lambda \in \mathbb{R}$, for all $a \in \mathscr{B}$,*

$$\lambda \geq \frac{\|a\|}{\varepsilon} \Longrightarrow a \in [-\lambda u, \lambda u].$$

Proof . $u \in int\mathscr{B}_+ \Longrightarrow \exists \varepsilon > 0 \mid$

$$\{a \in \mathscr{B} : \|u - a\| < \varepsilon\} \subseteq \mathscr{B}_+.$$

Assume $\lambda \in \mathbb{R}, a \in \mathscr{B}$ and $\lambda \geq \frac{\|a\|}{\varepsilon}$. We need to prove that $a \in [-\lambda u, \lambda u]$ i.e. $u \pm \dfrac{a}{\varepsilon} \in \mathscr{B}_+$.
On has:
$\|u - (u \pm \frac{a}{\lambda})\| = \frac{\|a\|}{\lambda} < \frac{1}{\lambda}\lambda\varepsilon = \varepsilon.$
Then $u \pm \frac{a}{\lambda} \in \{b \in \mathscr{B} : \|u - b\| < \varepsilon\} \subseteq \mathscr{B}_+; \Longrightarrow u \pm \frac{a}{\lambda} \in \mathscr{B}_+$.

□

This is a theorem of characterization of an order-unit.

Theorem 3.2.1. *Let $u \in \mathscr{B}$. The following conditions are equivalent:*

1. $u \in$ *int* $\mathscr{B}_+$;
2. u *is an order-unit.*

Proof . *(1.$\Longrightarrow$2.) This follows from proposition 2.3.2.*
(2.$\Longrightarrow$ 1.) Let u be an order-unit.
then,

$$\mathscr{B} = \bigcup_{\lambda \geq 0} [-\lambda u, \lambda u]$$

By the Baire category theorem, there is an $\lambda_0 > 0$ such that $\mathscr{B}_1 \subseteq [-\lambda_0 u, \lambda_0 u]$.
Let's prove that:

$$\{a; \|u - a\| < \tfrac{1}{\lambda_0}\} \subseteq \mathscr{B}_+$$

$$\begin{aligned}
b \in \{a; \|u-a\| < \frac{1}{\varepsilon}\} &\implies \|b-u\| < \frac{1}{\lambda_0} \\
&\implies \|\lambda_0(u-b)\| < 1 \\
&\iff \lambda_0(u-b) \in \mathscr{B}_1 \\
&\iff \lambda_0 b - \lambda_0 u \in \mathscr{B}_1 \\
&\implies \lambda_0 u - \lambda_0 b \in [-\lambda_0 u, \lambda_0 u] \\
&\iff \lambda_0 u - \lambda_0 b + \lambda_0 u, \lambda_0 u - \lambda_0 u + \lambda_0 b \in \mathscr{B}_+ \\
&\implies \lambda_0 b \in \mathscr{B}_+ \\
&\implies b \in \mathscr{B}_+
\end{aligned}$$

We have shown that :

$$B(u, \tfrac{1}{\lambda_0}) \subseteq \mathscr{B}_+$$

Hence $u \in int\mathscr{B}_+$

□

To define what we call an order-unit space, we need the following result:

Proposition 3.2.3. *If int $\mathscr{B}_+$ is non-empty, then $\mathscr{B}_+$ is generating and, for each $u \in$ int $\mathscr{B}_+$,*

$$\|a\|_u = inf\{\lambda > 0 : a \in [-\lambda u, \lambda u]\}$$

defines a semi-norm on $\mathscr{B}$.

Proof . *Let $u \in$ int $\mathscr{B}_+$.*

State 1. *Let's show that $\mathscr{B}_+$ is generating.*

As $u \in$ int $\mathscr{B}_+$, u is an order-unit. Hence,

$$\mathscr{B} = \bigcup_{\lambda \geq 0} [-\lambda u, \lambda u]$$

Therefore, if $a \in \mathscr{B}$*, then there is a* $\lambda_0 > 0$ *such that* $a \in [-\lambda_0 u, \lambda_0 u]$

It follows that $a + \lambda_0 u, \lambda_0 u \in \mathscr{B}_+$.

Furthermore, $a = a + \lambda_0 u - \lambda_0 u$

Hence $\mathscr{B}_+$ *is generating.*

State 2. *Let's show that the mapping*

$$\begin{array}{rcl} \|.\| : \quad \mathscr{B} & \longrightarrow & \mathbb{R}_+ \\ a & \longmapsto & \|a\|_u = inf\{\lambda > 0 : a \in [-\lambda u, \lambda u]\} \end{array}$$ *is a semi-norm on* $\mathscr{B}$.

$N_1)$ *Let* $a \in \mathscr{B}$ *and* $\alpha \in \mathbb{R}, \alpha \neq 0$. *For* $\lambda > 0$, *one has:*

$$\begin{aligned} \alpha a \in [-\lambda u, \lambda u] &\iff \lambda u \pm \alpha a \in \mathscr{B}_+ \\ &\Longrightarrow \lambda u \pm \|\alpha\| a \in \mathcal{B}_+ \\ &\Longrightarrow \frac{\lambda}{\|\alpha\|} u \pm a \in \mathscr{B}_+ \\ &\iff a \in [-\frac{\lambda}{\|\alpha\|} u, \frac{\lambda}{\|\alpha\|} u] \\ &\Longrightarrow \|a\|_u \leq \frac{\lambda}{\|\alpha\|} \end{aligned}$$

Hence $|\alpha| \|a\|_u \leq \|\alpha a\|_u$

and,

$$\begin{aligned} a \in [-\lambda u, \lambda u] &\iff \lambda u \pm a \in \mathscr{B}_+ \\ &\iff \|\alpha\| \lambda u \pm \alpha a \in \mathscr{B}_+ \\ &\iff \alpha a \in [-\|\alpha\| \lambda u, \|\alpha\| \lambda u] \\ &\Longrightarrow \|\alpha a\|_u \leq \|\alpha\| \lambda \\ &\Longrightarrow \frac{\|\alpha a\|_u}{\|\alpha\|} \leq \lambda \end{aligned}$$

Hence, $\|\alpha a\|_u \leq |\alpha| \|a\|_u$

It follows that

$$\|\alpha a\|_u = |\alpha| \|a\|_u \text{ and the equality is also satisfied for } \alpha = 0.$$

$N_2)$ *Let* $\lambda, \mu > 0, a \in [-\lambda u, \lambda u], b \in [-\mu u, \mu u]$

On has :

$$\begin{aligned}\lambda u \pm a, \mu u \pm b \in \mathscr{B}_+ &\Longrightarrow (\lambda+\mu)u \pm (a+b) \in \mathscr{B}_+\\ &\Longrightarrow a+b \in [-(\lambda+\mu)u, (\lambda+\mu)u]\\ &\Longrightarrow \|a+b\|_u \leq \lambda+\mu\end{aligned}$$

Hence, $\|a+b\|_u \leq \|a\|_u + \|b\|_u$

□

Remark 3.2.1. $\mathscr{B}_1 \subseteq [-\lambda_0 u, \lambda_0 u] \Longrightarrow \|a\|_u \leq \lambda_0 \|a\|$

Definition 3.2.3. *Now a pre-ordered Banach space* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *is defined to be an order-unit if* $\mathscr{B}_+$ *contains an interior point* u *and* $\|.\| = \|.\|_u$

This is a very good way to see an order-unit space.

Theorem 3.2.2. *Let* $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *be a pre-ordered Banach space. The following conditions are equivalent:*
1. $(\mathscr{B}, \mathscr{B}_+, \|.\|)$ *is a pre-ordered Banach Space.*
2. $\mathscr{B}_1 = [-u, u]$ *for some* $u \in \mathscr{B}_+$.

3.3 Ordered-bounded operators

In this section, $(\mathscr{A}, \mathscr{A}_+, \|.\|_{\mathscr{A}})$ *and* $(\mathscr{B}, \mathscr{B}_+, \|.\|_{\mathscr{B}})$ *will denote pre-ordered Banach spaces,* $\mathscr{L} = \mathscr{L}(\mathscr{A}, \mathscr{B})$ *the space of bounded linear operators* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *equipped with the norm* $\|.\|$ *defined by:*

$$\|u\| = sup\{\|u(x)\|_{\mathscr{B}}, x \in \mathscr{A}, \|x\|_{\mathscr{A}} \leq 1\}$$

for all $u \in \mathscr{L}$, *and,*

$$\mathscr{L}_+ = \{S; S \in \mathscr{L}, S\mathscr{A}_+ \subseteq \mathscr{B}_+\}$$

Definition 3.3.1. *An element* $S : \mathscr{A} \longrightarrow \mathscr{B} \in \mathscr{L}$ *is defined to be an* ***order-bounded operator*** *if for every* $a, b \in \mathscr{A}$ *such that* $a \leq b$, *there exists* $c, d \in \mathscr{B}$ *such that:*

$$S([a,b]) \subseteq [c,d]$$

with

$$S([a,b]) = \{s(x), x \in [a,b]\}.$$

We start our theory by defining a positive closed convex cone on the Banach space $\mathscr{L}$.

Proposition 3.3.1. *$\mathscr{L}_+$ is a closed subset of $\mathscr{L}$.*

Proof . ▶ *By the definition of $\mathscr{L}_+, \mathscr{L}_+ \subseteq \mathscr{L}$.*

▶ *Let $u \in \mathscr{L}, (u_n)_{n\geq 0}$ a sequence of elements of $\mathscr{L}_+$ which converges to $u \in \mathscr{L}$.*

We have to show that $u \in \mathscr{L}_+$.

$$\begin{aligned} u_n \xrightarrow{\mathscr{L}} u \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq N \Longrightarrow \|u_n - u\| < \varepsilon, \\ \iff & \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq N \Longrightarrow sup\{\|u_n(x) - u(x)\|_{\mathscr{B}}, \|x\|_{\mathscr{A}} \leq 1\} < \varepsilon, \\ \iff & u_n \longrightarrow u. \end{aligned}$$

(uniformly on I as $n \to \infty$ where I denote the closed ball of center 0 with radius 1.)

For all $n \geq 0, u_n$ is continuous on I and $u_n \longrightarrow u$ uniformly on I as $n \to \infty$. So u is continuous on I. In particular, u is continuous at the point 0. Since u is linear and continuous at 0, u is continuous on A. Hence $u \in \mathscr{L}$.

▶ *Let's show that $u\mathscr{A}_+ \subseteq \mathscr{B}_+$.*

Let $a \in \mathscr{A}_+$. We know that $(u_n(a))_{n\geq 0}$ converges to $u(a)$ in $\mathscr{B}$ since uniform convergence implies pointwise convergence. For all $n \geq 0, u_n(a) \in \mathscr{B}_+$ since (u_n) is a sequence of elements of $\mathscr{L}_+$. Since $\mathscr{B}_+$ is a closed subset of $\mathscr{B}, u(a) \in \mathscr{B}_+$. We have shown that $u(a) \in \mathscr{B}_+$, for all $a \in \mathscr{A}_+$. In other words, we have shown that $u\mathscr{A}_+ \subseteq \mathscr{B}_+$. Hence, $\mathscr{L}_+$ is a closed subset of $\mathscr{L}$.

□

Proposition 3.3.2. *$\mathscr{L}_+$ is a convex cone of $\mathscr{L}$*

Proof . *Let $\lambda, \mu \geq 0$.*

We have to show that :

$\lambda\mathscr{L}_+ + \mu\mathscr{L}_+ \subseteq \mathscr{L}_+$. Let $s_1, s_2 \in \mathscr{L}_+, s = \lambda s_1 + \mu s_2$.

.) Clearly, $s \in \mathscr{L}$.

.) For all $a \in \mathscr{A}_+$
$s(a) = \lambda s_1(a) + \mu s_2(a) \in \mathscr{B}_+$ *since* $s_1(a), s_2(a) \in \mathscr{B}_+$ *is a convex cone of* $\mathscr{B}$. *The belonging of* $s_1(a)$ *and* $s_2(a)$ *to* $\mathscr{B}_+$ *is dues to the fact that* $s_1, s_2 \in \mathscr{L}_+$.

□

Definition 3.3.2. *Operators* $S \in \mathscr{L}_+$ *will be referred to as positive operators.*

We now turn to some properties to control the size of the cone $\mathscr{L}_+$

Proposition 3.3.3. *If* $\mathscr{A}_+$ *is weakly generating and* $\mathscr{B}_+$ *is proper, then* $\mathscr{L}_+$ *is proper.*

Proof . *Assume that* $\mathscr{A}_+$ *is weakly generating and* $\mathscr{B}_+$ *is proper.*
We have to prove that $\mathscr{L}_+$ *is proper.*
Let $S \in \mathscr{L}_+ \cap (-\mathscr{L}_+)$ *that is* $S \in \mathscr{L}_+$ *and* $-S \in \mathscr{L}_+$. *Let* $a \in A$. *Since* $\mathscr{A}_+$ *is weakly generating, there is* $(b_n)_{n\geq 0}, (c_n)_{n\geq 0}$ *sequences of elements of* $\mathscr{A}_+$ *such that* $b_n - c_n \stackrel{\mathscr{A}}{\longrightarrow} a$.
Since S *is bounded,* $s(b_n - c_n) \stackrel{\mathscr{B}}{\longrightarrow} S(a)$ *i.e,* $S(b_n) - S(c_n) \stackrel{\mathscr{B}}{\longrightarrow} S(a)$ *because* S *is linear.*

$S(b_n) - S(c_n), -S(b_n) + S(c_n) \in \mathscr{B}_+$ *implies* $S(b_n) - S(c_n) = 0_A$, *because* $\mathscr{B}_+$ *is proper.*
Hence,
$S(b_n) - S(c_n) \stackrel{\mathscr{B}}{\longrightarrow} 0$ *and* $S(b_n) - S(c_n) \stackrel{\mathscr{B}}{\longrightarrow} S(a)$.
The uniqueness of the limit in $\mathscr{B}$ *implies* $S(a) = 0_{\mathscr{B}}$. *Since this is true for all* $a \in \mathscr{A}$, *we conclude that* $S = 0$. *We have proved that* $\mathscr{L}_+$ *is proper.*

Proposition 3.3.4. *If* $S_1, S_2 \in \mathscr{L}_+$ *and* $a_1 \leq a \leq a_2$, *then* $S_1a_1 - S_2a_2 \leq (S_1 - S_2)a \leq S_1a_2 - S_2a_1$.

Proof . *Let* $S_1, S_2 \in \mathscr{L}_+$ *and* $a_1 \leq a \leq a_2$. *One has:*
$(S_1 - S_2)a - (S_1a_1 - S_2a_2) = S_1(a - a_1) + S_2(a_2 - a) \in \mathscr{B}_+$ *since* $a - a_1, a_2 - a \in \mathscr{A}_+$ *and* $S_1, S_2 \in \mathscr{L}_+$.
one also has:
$(S_1a_2 - S_2a_1) - (S_1 - S_2)a = S_1(a_2 - a) + S_2(a - a_1) \in \mathscr{B}_+$ □

Definition 3.3.3. *A bounded linear operators* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is said to be order-bounded if it maps order-intervals* $[a, b]$ *into order-intervals of the same type, i.e. for every* $a, b \in A$, $a \leq b$ *there exists* $\alpha, \beta \in B$, $\alpha \leq \beta$ *such that*

$$S([a,b]) = [\alpha, \beta]$$

Remark 3.3.1. *The above proposition shows that the difference* $S = S_1 - S_2$ *of two positive operators is order-bounded. But there are many examples of bounded operators which are not order-bounded, even under fairly stringent conditions on* $\mathscr{A}$ *and* $\mathscr{B}$. *Our aim is to find sufficient conditions for a bounded operator to be order-bounded. One weak result in this direction is the following.*

Proposition 3.3.5. *If* $\mathscr{A}_+$ *is normal and* $int\mathscr{B}_+ \neq \emptyset$, *then every* $S \in \mathscr{L}(\mathscr{A}, \mathscr{B})$ *is order-bounded.*

Proof . *See [6] page 238*

Corollary 3.3.1. *The bounded operators on an order-unit space are order-bounded.*

Proposition 3.3.6. *If* $\mathscr{A}_+$ *is generating and* $\mathscr{B}_+$ *is normal, then every order-bounded linear operator* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is bounded.*

Proof . *If* S *is not bounded, there exists* $a_n \in \mathscr{A}_1$ *with* $\|Sa_n\| \geq 4^n$. *Since* $\mathscr{A}_+$ *is* $\alpha-$ *generating for some* α, *one has* $a_n = a_n^{'} - a_n^{''}$ *with* $a_n^{'}, a_n^{''} \in \mathscr{A}_+ \cap \mathscr{A}_\alpha$. *Replacing* a_n *by* $-a_n$ *if necessary, it may be assumed that* $\|Sa_n^{'}\| > 2^{2n-1}$.
Let $a = \sum 2^{-n} a_n^{'}$. *If* S *is order-bounded,* S *maps* $[0, a]$ *into an interval* $[b_1, b_2]$ *so that* $b_1 \leq 2^{-n} a_n^{'}$ *for all n. But this contradicts the normality of* $\mathscr{B}_+$: *since* $\mathscr{B}_+$ *is normal, there is an* $\beta > 0$ *such that*
$2^{-n}\|Sa_n^{'}\| \leq \beta(\|b_1\| \vee \|b_2\|)$ *for all* n. *So the sequence* $(2^{-n}\|Sa_n^{'}\|)_n$ *is bounded by* $\beta(\|b_1\| \vee \|b_2\|)$. *Contradiction because* $2^{-n}\|Sa_n^{'}\| > 2^{n-1}$ *for all* n. □

Corollary 3.3.2. *If* $\mathscr{A}_+$ *is generating, then every order-bounded linear functional over* $\mathscr{A}$, $S : \mathscr{A} \longrightarrow \mathbb{R}$ *is bounded. If particular, positive linear functionals over* $\mathscr{A}$ *are bounded.*

Next we discuss properties of the pre-ordered space $(\mathscr{L}, \mathscr{L}_+, \|.\|)$ *throughout we assume* $\mathscr{A}_+ \neq \mathscr{A}, \mathscr{B}_+ \neq \{0\}$.

Theorem 3.3.1. *The cone* $\mathscr{L}_+$ *is normal if, and only if,* $\mathscr{A}_+$ *is generating and* $\mathscr{B}_+$ *is normal.*

Proof . *Suppose* $\mathscr{L}_+$ *is* $\alpha_\vee-$ *normal* $(\alpha \geq 1)$. *For* $\omega \in \mathscr{A}^\star$ *and* $b \in \mathscr{B} - \{0_\mathscr{B}\}$, *let* $\omega \otimes b$ *be the rank-one operator with action* $(\omega \otimes b)(a) = \omega(a)b$.

If $\omega \in \mathscr{A}_+^\star$ *and* $b_1 \leq b \leq b_2 \in \mathscr{B}$, *one has:*

for all $a \in \mathscr{A}_+$,

$(\omega \otimes b)(a) - (\omega \otimes b_1)(a) = \omega(a)b - \omega(a)b_1 = \omega(a)(b - b_1) \in \mathscr{B}_+$ *for* $\omega(a) \in \mathbb{R}_+, b - b_1 \in \mathscr{B}_+$ *and* $\mathscr{B}_+$ *is a positive cone.*

$[\omega \otimes b_2 - \omega \otimes b](a) = \omega \otimes b_2(a) - \omega \otimes b(a) = \omega(a)b_2 - \omega(a)b = \omega(a)(b_2 - b) \in \mathscr{B}_+$.

Then $\omega \otimes b_1 \leq \omega \otimes b \leq \omega \otimes b_2$.

$$\begin{aligned} \|(\omega \otimes b)(a)\|_\mathscr{B} &= \|\omega(a)b\|_\mathscr{B} \\ &= |\omega(a)|\|b\|_\mathscr{B} \\ &\leq \|\omega\|^\star \|b\|_\mathscr{B} \|a\|_\mathscr{A} \end{aligned}$$

This implies $\|\omega \otimes (b)\|^\star \leq \|\omega\|^\star \|b\|_\mathscr{B}$

For $\|a\|_\mathscr{A} \leq 1, |\omega(a)|\|b\|_\mathscr{B} = \|\omega(a)b\|_\mathscr{B} = \|(\omega \otimes b)(a)\|_\mathscr{B}$. *This implies :*

$\|\omega\|^\star \|b\|_\mathscr{B} \leq \|\omega \otimes b\|^\star$

and then,

$$\begin{aligned} \|\omega\|^\star \|b\|_\mathscr{B} &= \|\omega \otimes b\|^\star \\ &\leq \alpha(\|\omega \otimes b_1\|^\star \vee \|\omega \otimes b_2\|^\star) \\ &= \alpha\|\omega\|^\star(\|b_1\| \vee \|b_2\|), \end{aligned}$$

This implies $\|b\|_\mathscr{B} \leq \alpha(\|b_1\| \vee \|b_2\|)$ *ie* $\mathscr{B}_+$ *is* $\alpha_\vee$*-normal. But if* $b \in \mathscr{B}_+$ *and* $\omega_1 \leq \omega \leq \omega_2$ *in* $\mathscr{A}^\star$ *then*

$\|\omega_1 \otimes b\|^\star \leq \|\omega \otimes b\|^\star \leq \|\omega_2 \otimes b\|^\star$.

Thus $\|\omega \otimes b\|^\star \leq \alpha(\|\omega_1 \otimes b\|^\star \vee \|\omega_2 \otimes b\|^\star)$ *ie* $\|\omega\|^\star \|b\| \leq \alpha\|b\|(\|\omega_1\|^\star \vee \|\omega_2\|^\star)$. *Thus* $\mathscr{A}_+^\star$ *is* $\alpha_\vee-$ *normal and* $\mathscr{A}_+$ *is approximately* α_+*-generating, according to the theorem 2.1.3.*

conversely suppose $\mathscr{A}_+$ *is* α_+*-generating and* $\mathscr{B}_+$ *is* $\beta_\vee$*-normal.*

Consider $S_1 \leq S \leq S_2$ *in* $\mathscr{L}$ *and* $a \in \mathscr{A}$. *Then* $a = a_1 - a_2, a_1, a_2 \in \mathscr{A}_+$ *and* $\|a_1\| + \|a_2\| \leq \alpha\|a\|$.

Now $S_1a_j \leq Sa_j \leq S_2a_j$.
So $\|Sa_j\| \leq \beta(\|S_1a_j\| \vee \|S_2a_j\|) \leq \beta\|a_j\|(\|S_1\|^\star \vee \|S_2\|^\star)$,
and consequently,

$$\begin{aligned}\|Sa\| &\leq \|Sa_1\| + \|Sa_2\| \\ &\leq \alpha\beta\|a\|(\|S_1\| \vee \|S_2\|); \Longrightarrow \|S\| \leq \alpha\beta(\|S_1\| \vee \|S_2\|)\end{aligned}$$

Thus $\mathscr{L}_+$ *is* $(\alpha\beta)_\vee-$ *normal.* □

Remark 3.3.2. *In the preceding theorem, if, for example, one chooses* $\mathscr{A} = \mathscr{B}$*, then the theorem states that the cone* $\mathscr{L}_+(\mathscr{B})$ *of positive bounded linear operators is normal, if and only if,* $\mathscr{B}_+$ *is normal and generating.*

The preceding theorem and proposition 3.3.6 lead us directly to the following theorem which is at the heart of our work.

Theorem 3.3.2. *If the cone* $\mathscr{L}_+(\mathscr{A}, \mathscr{B})$ *of all positive bounded linear operators* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is normal, then, every bounded operator* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is order-bounded.*

Proof . *Suppose that the cone* $\mathscr{L}_+(\mathscr{A}, \mathscr{B})$ *of all positive bounded linear operators* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is normal, then, thanks to the preceding theorem,* $\mathscr{A}_+$ *is generating and* $\mathscr{B}_+$ *is normal. It follows from proposition 3.3.6, every bounded operator* $S : \mathscr{A} \longrightarrow \mathscr{B}$ *is order-bounded.* □

♣ Pedagogical purpose ♣

As part of the drafting of the Dissertation of DIPES II, we have been asked to give the pedagogical interest of our work. Recalling that the theme submitted for our study is entitled "On Bounded Operators on Banach Spaces". It's an important topic for both the teacher and student:

3.4 for the teacher

- ♣ *It provides a new way to understand bounded operators which is not the classical ones.*
- ♣ *It allows to cultivate the spirit of creativity and research.*
- ♣ *It allows interdisciplinarity.*
- ♣ *It allows to become familiar with the latex programming software which makes a very good layout and is therefore very useful for typing of evaluation texts.*

3.5 for the students

- ♣ *It helps the students to notice that there is a theory which generalize order-bounded mappings.*
- ♣ *It provides students with experimented teachers.*
- ♣ *It accustoms students to the spirit of scientific research.*

♣ Conclusion ♣

Our work had the aim of finding sufficient conditions making a bounded operator between pre-(ordered) Banach spaces order-bounded. To achieve this, we gave in Chapter 1, the preliminaries needed to understand the problem. In Chapter 2, we defined, for every Banach space, an order relation associated with a proper cone of this space, which induces on the dual Banach space an order relation of the same type; we then studied the properties of the cone underlying pre-ordered Banach spaces. In chapter 3, we studied sufficient conditions for a linear operator bounded in the sense of the norm to be bounded in the sense of order relations. Those conditions are mostly conditions on the positive cones of the Banach spaces considered. The theory of pre-ordered bounded operators on Banach spaces is the generalization of the result of the classical analysis of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ that stipulate that every continuous function maps an interval $[a, b]$ to an interval of de same type. Unfortunately, the findings of this dissertation are limited to sufficient conditions of a continuous mapping to be order-bounded. Future research on bounded operators on Banach spaces should focus in particular on necessary and sufficient conditions for a continuous function between Banach spaces to be order-bounded.

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