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UNIVERSITE DE YAOUNDE I
FACULTÉ DES SCIENCES
DEPARTEMENT DE MATHÉMATIQUES
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CENTRE DE RECHERCHE ET DE FORMATION DOCTORALE EN SCIENCES, TECHNOLOGIE ET GEOSCIENCES
Laboratoire d'Analyse et Applications

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UNIVERSITY OF YAOUNDE I
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
****
POSTGRADUATE SCHOOL OF SCIENCE, TECHNOLOGY AND GEOSCIENCES
Laboratory of Analysis and Application

The inhomogeneous relativistic Boltzmann equation in a Bianchi type I space-time for soft potentials, hard potentials and with Israel particles

THESIS
Submitted in partial fulfilment of the requirements for the degree of Doctorat /PhD in Mathematics

Par : **TCHUENGUE KAMDEM Emmanuel**
Master degree

Sous la direction de
TAKOU Etienne
Associate Professor
University of Yaoundé I
NOUTCHEGUEME Norbert
Professor
University of Yaoundé I

Année Académique : 2021

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DÉPARTEMENT DE MATHÉMATIQUES
DEPARTMENT OF MATHEMATICS

ATTESTATION DE CORRECTION DE LA THESE DE DOCTORAT / Ph.D

Nous soussignés, **AYISSI Raoul Domingo, Pr., UYI, Président du jury; TAKOU Etienne, Pr., Rapporteur; NOUNDJEU Pierre, Pr., UYI, Examinateur,** membres du jury de la thèse de Doctorat / Ph.D présentée par **M. TCHUENGUE KAMDEM Emmanuel, Matricule 97R051**, intitulée : **«THE INHOMOGENEOUS RELATIVISTIC BOLTZMANN EQUATION IN A BIANCHI TYPE I SPACE-TIME FOR SOFT POTENTIALS, HARD POTENTIALS AND WITH ISRAEL PARTICLES»** et soutenue en vue de l'obtention du diplôme de Doctorat / Ph.D **en Mathématiques**, Spécialité : **Equations aux Dérivées Partielles,** Option : **Analyse**, attestons que toutes les corrections demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

En foi de quoi la présente attestation lui est délivrée pour servir et valoir ce que de droit.

Président	**Rapporteur**	**Examinateurs**
AYISSI Raoul Domingo, Pr., UYI	**TAKOU Etienne, Pr., UYI**	**NOUNDJEU Pierre, Pr., UYI**

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THE UNIVERSITY OF YAOUNDE I
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****
RESEARCH AND TRAINING UNIT FORDOCTORATE
IN MATHEMATICS, INFORMATIC, BIOINFORMATICSAND APPLICATIONS

DEPARTMENT OF MATHEMATICS
DÉPARTEMENT DE MATHÉMATIQUES

Laboratory of Analysis and Application
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The inhomogeneous relativistic Boltzmann equation in a Bianchi type I space-time for soft potentials, hard potentials and with Israel particles

THESIS

Submitted in partial fulfilment of the requirements for the degree of Doctorat /PhD in Mathematics

Option: Analysis

Speciality: Partial Differential Equations

By:

TCHUENGUE KAMDEM Emmanuel
Register number: **97R051**
Master degree

Under the direction of:

TAKOU Etienne
Associate Professor
University of Yaoundé I

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ANNÉE ACADEMIQUE 2019/2020
(Par Département et par Grade)
DATE D'ACTUALISATION 12 Juin 2020

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38	WILFRIED ANGIE Abia	Assistante	En poste

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28	DONFACK Mireille	Chargée de Cours	En poste
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32	LEKEUFACK FOLEFACK Guy B.	Chargé de Cours	En poste
33	MAHOB Raymond Joseph	Chargé de Cours	En poste
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35	MOUNGANG Luciane Marlyse	Chargée de Cours	En poste
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40	NOAH EWOTI Olive Vivien	Chargée de Cours	En poste
41	TADU Zephyrin	Chargé de Cours	En poste
42	TAMSA ARFAO Antoine	Chargé de Cours	En poste
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46	KOGA MANG DOBARA	Assistant	En poste
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21	MAHBOU SOMO TOUKAM. Gabriel	Chargé de Cours	En poste
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23	NGOUO Lucas Vincent	Chargé de Cours	En poste
24	NNANGA MEBENGA Ruth Laure	Chargé de Cours	En poste
25	NOUKEU KOUAKAM Armelle	Chargé de Cours	En poste
26	ONANA JEAN MICHEL	Chargé de Cours	En poste
27	GODSWILL NTSOMBAH NTSEFONG	Assistant	En poste
28	KABELONG BANAHO Louis-Paul-Roger	Assistant	En poste
29	KONO Léon Dieudonné	Assistant	En poste
30	LIBALAH Moses BAKONCK	Assistant	En poste
31	LIKENG-LI-NGUE Benoit C	Assistant	En poste
32	TAEDOUNG Evariste Hermann	Assistant	En poste
33	TEMEGNE NONO Carine	Assistant	En poste

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7	NGOMO Horace MANGA	Professeur	*Vice Chancelor/UB*
8	NDIKONTAR Maurice KOR	Professeur	*Vice-Doyen Univ. Bamenda*
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23	MESSI NGUELE Thomas	Assistant	En poste
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4	MBANG Joseph	Maître de Conférences	En poste
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24	TETSADJIO TCHILEPECK M. E.	Chargée de Cours	En poste
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4	KOFANE Timoléon Crépin	Professeur	En poste
5	NANA ENGO Serge Guy	Professeur	En poste
6	NDJAKA Jean Marie Bienvenu	Professeur	Chef de Département
7	NOUAYOU Robert	Professeur	En poste
8	NJANDJOCK NOUCK Philippe	Professeur	*Sous Directeur/ MINRESI*
9	PEMHA Elkana	Professeur	En poste
10	TABOD Charles TABOD	Professeur	Doyen Univ/Bda
11	TCHAWOUA Clément	Professeur	En poste
12	WOAFO Paul	Professeur	En poste

13	BIYA MOTTO Frédéric	Maître de Conférences	DG/HYDRO Mekin
14	BODO Bertrand	Maître de Conférences	En poste
15	DJUIDJE KENMOE épouse ALOYEM	Maître de Conférences	En poste
16	EYEBE FOUDA Jean sire	Maître de Conférences	En poste
17	FEWO Serge Ibraïd	Maître de Conférences	En poste

18	HONA Jacques	Maître de Conférences	En poste
19	MBANE BIOUELE César	Maître de Conférences	En poste
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DÉPARTEMENT	**Professeurs**	**Maîtres de Conférences**	**Chargés de Cours**	**Assistants**	**Total**
BCH	9 (1)	13 (09)	14 (06)	3 (2)	**39 (18)**
BPA	13 (1)	09 (06)	19 (05)	05 (2)	**46 (14)**
BPV	06 (0)	11 (02)	9 (06)	07 (01)	**33 (9)**
CI	10 (1)	9 (02)	12 (02)	03 (0)	**34 (5)**
CO	7 (0)	17 (04)	09 (03)	02 (0)	**35(7)**
IN	2 (0)	1 (0)	13 (01)	09 (01)	**25 (2)**
MAT	1 (0)	5 (0)	19 (01)	05 (02)	**30 (3)**
MIB	1 (0)	5 (02)	06 (01)	06 (02)	**18 (5)**
PHY	12 (0)	15 (02)	10 (03)	03 (0)	**40 (5)**
ST	8 (1)	14 (01)	19 (05)	02 (0)	**43(7)**
Total	**69 (4)**	**99 (28)**	**130 (33)**	**45 (10)**	**343 (75)**

Soit un total de **344 (**75**)** dont :

- Professeurs **68 (4)**
- Maîtres de Conférences **99 (**28**)**
- Chargés de Cours **130 (33)**
- Assistants **46** (10**)**

( ) = Nombre de Femmes **75**

Dedication

THis work is dedicated to Holy Virgin Mary for her love and her assistance.

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Contents

Abstract

In this work, we consider the Cauchy problem for the spatially homogeneous and for the inhomogeneous Boltzmann equations with small initial data. The collision kernels considered are respectively for Israel particles, for hard and for soft potentials. The background space-time in which the study is done is the Bianchi type I space-time. Under certain conditions made on the scattering kernel and on the metric, a unique global (in time) solution is obtained in a suitable weighted functional space; that are for Israel particles, the hard potentials and the soft potentials. The clue of this work is to establish the derivatives of the post-collisional momenta and their technical control in order to overcome the problem of singularities and also to construct the energy estimates that are crucial in the global existence theorem application. Prior to the global existence theorem we have established the local existence theorem by taking the small initial data that enable to bound the first order derivatives; this allows to extend the local-in-time solution to a global-in-time solution. To build the local existence theorem we introduced a recursive sequence and proved that it converges and is uniformly bounded.

Keywords : relativistic Boltzmann equation – Bianchi type I space-time – hard potentials – soft potentials – Israel Particles – homogeneous solutions – inhomogeneous solutions – L^2-stability.

Résumé

DAns ce travail nous considérons le problème de Cauchy pour l'equation de Boltzmann d'abord spatialement homogène et ensuite l'équation de Boltzmann non homogène avec des conditions initiales petites. L'espace-temps dans lequel nous travaillons est l'espace-temps de Bianchi de type I. Avec des hypothèses sur le noyau de collision et sur la métrique de l'espace-temps, une solution unique et globale (dans le temps) est obtenue dans un espace fonctionnel à poids convenable; notamment pour les particules d'Israel, les potentiels durs et les potentiels mous. Le grand enjeu dans ce travail est d'une part, le calcul des dérivées des variables d'impulsion et leurs estimations selon une forme spécifique qui permettrait de surmonter les problèmes générés par les singularités, et d'autres parts l'élaboration des inégalités d'énergie qui sont cruciales dans la construction des théorèmes d'existence des solutions. Pour établir les théorèmes d'existence globale nous avons commencé par établir le théorème d'existence locale en prenant des conditions initiales petites qui permettent de borner la solution locale et ses dérivées d'ordre 1; cela nous permet d'étendre cette solution locale en une solution globale. Pour construire le théorème d'existence locale nous construisons une suite de solutions de l'équation linéarisée et nous démontrons qu'elle est convergence et bornée uniformement.

Mots Clés : équation de Boltzmann relativiste – espace-temps de Bianchi de type I– potentiels durs – potentiels mous – particules d'Israel – solutions homogènes – solutions inhomogènes – stabilité dans L^2.

Introduction

THe expression "Boltzmann equation" used in a more general sense refers to any kinetic equation that describes the change of a macroscopic quantity in a thermodynamic system, such as energy, charge or particle number. The equation arises not by statistical analysis of all the individual positions and momenta of each particle in the fluid; rather by considering the probability that a number of particles occupy a very small region of space centered at the tip of the position vector, and have very nearly equal small changes in momenta from a momentum vector, at an instant of time. In such situation, one assumes that the particles interact only via binary and elastic collisions. This occurs when the mean free time is much shorter than the characteristic length time associated with the system. The Boltzmann equation can be used to determine how physical quantities, such as heat energy and momentum, change when a fluid is in transport. Other characteristic properties to fluids such as viscosity, thermal conductivity, etc. can be derived.

Due to its importance in the kinetic theory, several authors have studied and proved local and global in time existence theorems for the Boltzmann equation, in both the non-relativistic case, that considers particles with low velocities, and the full-relativistic case, which includes the case of fast moving particles with arbitrarily high velocities, such as, particles of ionized gas in some media at a very high temperature like: burning reactors, solar winds, nebular galaxies.

In the non-relativistic case, the first original global result is due to T. Carleman in [7]; R. J. Diperna and P.L. Lions proved global existence and weak stability in [12]. R. Illner and M. Shinbrot proved a global result in [23], in the case of small initial data and without symmetry assumption. For more details in the non-relativistic Boltzmann equation, we refer to [7, 23, 12] and references therein.

In the full-relativistic case, let $\Gamma^{\gamma}_{\mu\nu}$ denote the Christoffel symbols of the metric tensor ds^2 and $\tilde{Q}(f, f)$ denote the collisional operator, if we adopt the Einstein summation convention as indicated below, the Boltzmann equation reads

$$p^{\mu}\frac{\partial f}{\partial x^{\mu}} - \Gamma^{i}_{\mu\nu}p^{\mu}p^{\nu}\frac{\partial f}{\partial p^{i}} = \widetilde{Q}(f, f). \tag{1}$$

In their pioner work [3], Daniel Bancel and Yvonne Choquet Bruhat defined the concept of $\mu - N$ regularity on the collision kernel. A cross-section S which appears in (1.16) and (1.17) below is $\mu - N$ regular if $\frac{1}{p^0}\widetilde{Q}$ is a bounded quadratic form in a suitable weighted Sobolev space. More precisely, it exists a constant C such that

$$\|\frac{1}{p^0}\widetilde{Q}(f, f)(t)\|_{H^{\mu,N}} \leq C\|f(t)\|^2_{H^{\mu,N}} \tag{2}$$

where $H^{\mu,N}$ is a convenient Sobolev space of order N. Under assumptions of $\mu - N$ regularity, several authors proved local existence theorems. This was done considering this equation alone, as K. Bichteler in [6], D. Bancel in [2], or coupling it to other fields equations as D. Bancel and Y. Choquet-Bruhat did in [3].

With Bianchi type 1 space-time as background and under assumption close to $\mu - N$ regularity, N. Noutchegueme, E. Takou and D. Dongo proved in [31] the existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data. About the coupled Einstein-Boltzmann equation, N. Noutchegueme and E. Takou [32] proved a global in time existence theorem with positive cosmological constant and arbitrarily large initial data in the spatially homogeneous case in a Robertson-Walker space-time; N. N. Noutchegueme and D. Dongo [30] proved a global in time existence theorem for arbitrarily large initial data, in the spatially homogeneous case in a Bianchi type 1 space-time. About the properties of solutions, E. Takou [38] proved that at late times in the future, the solution of Einstein-Boltzmann system with positive cosmological constant for the Robertson-Walker space time is asymptotically dust-like.

Unfortunately, the assumption of $\mu - N$ regularity on the collision kernel used in [31, 30, 32, 38] is not physically well-motivated. In fact, this does not allow a good interpretation of the type of collisions between particles. The scattering kernel is a quantity that determines the nature of collisions between particles. The scattering cross-section depends strongly on the kind of interaction between the molecules of the gas. In the non-relativistic (Newtonian mechanics) case, several different types of scattering kernel have been found to be of interest. For instance, the inverse power law gives the best-known types of scattering kernel, and they are further classified into hard and soft potentials cases. In the relativistic setting, it is not very clear which types of the scattering kernel should be of interest. But a classification of (special) relativistic called short range interactions (hard and soft potentials) has been proposed in [14] by applying arguments similar to those used in the non-relativistic case. This classification was recently reformulated to the full-relativistic case by R. Strain in [35]. Beside the class of short range interactions, it exists several kinds of differential cross-sections. Below are some examples:

- Møller scattering which is used as an approximation of electron-electron scattering. In this case photon-photon scattering is often neglected because the size of the cross-section is "negligible".
- Compton scattering which is an approximation of photon-electron scattering.
- Neutrino gas for which the differential cross-section is independent of the scattering angle.
- The Israel particles which is one of the scattering kernel used in the present work, [24] are the analogue of the "Maxwell molecules" cross-section in the Newtonian theory. With this cross-section, Israel derives eigenfunctions for the linearized relativistic Boltzmann collision operator. Note that it converges to the Maxwell molecules cross-section in the Newtonian limit.

In this work, we consider separately the collision kernels for the class of short range interactions (hard potentials and soft potentials) and collision kernel generated by Israel particles.

As in the non-relativistic case, the scattering kernel depends only on the relative momentum and the scattering the angle of two colliding particles. The generalisation of these two concepts (relative momentum and scattering angle) in general relativity was done by Glassey in [16]. This will be specified in chapter 1.

For the homogeneous relativistic Boltzmann equation and with the scattering kernel formulated in [14, 35], H. Lee proved [25, 27], a global existence of solution in the Robertson-Walker space-time (FRW) with near vacuum initial data.

For the inhomogeneous relativistic case, several authors studied this problem by taking the Minkowski space-time as background. Most of results available concern the study of mild solutions. A *mild* solution to the initial-value problem associated to the Boltzmann equation is a continuous function f such that the function denoted $f^{\#}$ defined in (4.53) satisfies the time-integrated form of the Boltzmann equation. Even though the Boltzmann equation is an integro-differential equation, with the differential part being described by a first order partial differential operator, for the mild solutions the differentiability in each variable is not required. Glassey [16] studied for some appropriate classes of scattering cross-section a global mild solution to the Cauchy problem for the relativistic Boltzmann equation with small data. E. Takou and F. L. Ciake Ciake extended in [42] the Glassey's result to the Robertson-Walker (FRW) space-time. E. Takou and F. L. Ciake Ciake later also proved in [43] the existence of mild solution of the relativistic Boltzmann equation on a spherically symmetric gravitational field.

Now for classical solutions, one of the most interesting aspect while working in the Minkowski space-time is the existence of an equilibrium solution. The steady state of this model is the well known Jüttner solution, also known as the relativistic Maxwellian. This allows to define the perturbation of the distribution function to the relativistic Maxwellian. Using this splitting, Strain [35] proved that in the case of special relativity, unique classical solutions to the relativistic Boltzmann equation exists for all time and decay with any polynomial rate towards their steady state. This result was carried out in the case of a spatially periodic box and with collision kernel for soft potentials. The main technique in [35] is to study the linearized equation and then the equation.

One interesting question when dealing with the relativistic Boltzmann equation is the possibility of finding analytic solutions. Such solutions are possible only under very restrictive assumptions of relaxation time approximation (RTA); the total distribution function is then split into one symmetric term (which is generally large) and one asymmetric term (which is small). Recently, analytic solutions of the RTA Boltzmann equation for a system with Gubser flow; i.e a flow pattern that combines boost-invariant longitudinal expansion with fast azimuthally symmetric transverse flow were presented in [4, 5, 13, 22].

Unlike FRW space-time which has the same scale factor for each of the three spatial directions, Bianchi Type I space-time has different scale factor in each direction, thereby introducing an anisotropy to the system. It is natural to try to see what happens in the relativistic Boltzmann equation when this metric is taken into account.

The purpose of this thesis is to obtain analogous results of [25, 42, 43, 40, 41] in the Bianchi type 1 space-time in which the metric generalizes that of Robertson-Walker. One of the most important point to note here is the form of parametrization of the post-collisional momenta. The presence of a second factor in the metric imposes another formulations and proofs of several estimates used in [25, 42, 43, 40, 41]. The aim of this work is then to establish an existence theorem for classical solutions for the Boltzmann equation in the Bianchi type I space-time for short range interactions and for a collision kernel generated by Israel particles.

So in this thesis, we use Bianchi type 1 space-time as back ground. Firstly, we consider the homogeneous relativistic Boltzmann equation and we look for the L^∞ and L^2-solutions . For the L^∞-solution, we use the fixed point theorem in an appropriate framework and for the L^2-solution, we formulate energy estimates which allow us to construct an appropriate sequence which converges to the solution of the problem. Secondly, we study the inhomogeneous Boltzmann equation in the same space-time. To our knowledge nothing is known in the literature about the inhomogeneous equation in the Bianchi type 1 space-time; we start by obtaining an unique global (in time) mild solution in a suitable weighted space. This is done using the fixed point theorem. Next unique global (with respect to the direction of time corresponding to the expansion of the universe) classical solution is obtained. All of the results of the present thesis are obtained by taking separately the collision kernel for hard potentials, soft potentials or generated by the Israel particles.

The thesis is organized as follows:

- In chapter 1, we introduce some notations, the relativistic Boltzmann equation, the three parameterizations of the post-collisional momenta and we describe the three forms of scattering kernel used in this work . The clue of this part is the change of variables that enable us to take up the covariant variables and to get the reduced form of the relativistic Boltzmann equation.

- In chapter 2, we firstly establish some fundamental results that are useful in the sequel, secondly we control the derivatives of the post-collisional momenta and at the end we establish the L^∞-existence theorem for classical solutions of the homogeneous equation, in the cases of Israel particles, hard potentials and soft potentials. The computation of the derivatives and their estimations are difficult and crucial to handle. We are then forced to use two parameterizations to overcome such difficulties.

- In chapter 3, we establish the energy estimates and the L^2-existence theorem for classical solution of the homogeneous equation, in the cases of Israel particles, hard potentials and soft potentials. The energy estimates are technically obtained both by the use of the Cauchy-Schwartz inequality and the specific properties on the momentum variables. At the end of this part, we establish the L^2-stability of the solutions.

- In chapter 4, we establish some fundamental results on the differential characteristic system. This allows us to prove the existence of mild solutions for the inhomogeneous equation, in the cases of Israel particles, hard potentials and soft potentials. Here we extend the results on some crucial estimates in the Robertson-Walker space-time.

- In chapter 5, we establish some fundamental results both on the scattering kernel and the post-collisional momenta. We then prove L^∞-existence theorem for classical solutions of the inhomogeneous equation, in the cases of Israel particles, hard potentials and soft potentials. We first establish the energy estimates.

CHAPTER ONE

THE EQUATION AND SPECIFICATION OF THE COLLISION OPERATOR

Contents

IN this chapter we introduce some notations, the relativistic Boltzmann equation, the three parameterizations of the post-collisional momenta and we describe the three forms of scattering kernel used in this work.

1.1 Choice of the space-time and notations

In this section, we collect some notations which are used in this work. Unless otherwise specified, Greek indices run from 0 to 3 while Latin indices run from 1 to 3. The spatial variable x denotes a four-vector, while the momentum variable p denotes a three-dimensional vector. That is

$$x = (x^0, x^1, x^2, x^3) = (t, x^1, x^2, x^3) \qquad and \qquad p = (p^1, p^2, p^3). \tag{1.1}$$

We also adopt the Einstein summation convention $a_\alpha b^\alpha = \sum a_\alpha b^\alpha$.
As indicated in the introduction, we consider the spatially Bianchi type I space-time where the metric is defined by

$$ds^2 = -dt^2 + a^2(t)(dx)^2 + b^2(t)\left[(dy)^2 + (dz)^2\right] \tag{1.2}$$

where a and b are two positive real numbers, x, y and z the variables of space.
Throughout this thesis, the speed of light and the mass of the particles are assume to be unity. Hence the momentum $p^\alpha = (p^0, p)$ lies on a hyper surface defined by the equation

$$g_{\alpha\beta}p^\alpha p^\beta = -1 \tag{1.3}$$

which is called the mass shell condition. Due to the mass shell condition, p^0 reads

$$p^0 = \sqrt{1 + a^2(p^1)^2 + b^2((p^2)^2 + (p^3)^2)}. \tag{1.4}$$

Henceforward, due to the form of the metric and for certain conveniences, for a three vector (d^1, d^2, d^3), we sometimes let

$$\bar{d} = (d^2, d^3) \qquad and \qquad |\bar{d}| = \sqrt{(d^2)^2 + (d^3)^2}. \tag{1.5}$$

Unless otherwise specified, we use the euclidian norm in $\mathbb{R}^n$, that is
for $p = (p^1, ..., p^n) \in \mathbb{R}^n$

$$|p| = \sqrt{\sum_{k=1}^{n}(p^i)^2}. \tag{1.6}$$

We denote by . the usual inner product in $\mathbb{R}^n$, that is for $p, q \in \mathbb{R}^n$

$$p.q = p^1q^1 + ... + p^nq^n. \tag{1.7}$$

$\dot{a}$ denotes the derivative of a with respect to t.

We consider the collisional evolution of a kind of uncharged particles in the time-oriented curved space-time $(\mathbb{R}^4, ds^2)$. An essential tool to describe the dynamic of such particles is their distribution function that we denote by f, and that is a non-negative real-valued function of both the position x^α, the 4-momentum $p^\alpha = (p^0, p) = (p^0, p^1, p^2, p^3)$ of the particles. More precisely, we have

$$f : T(\mathbb{R}^4) \cong \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}_+; \ \ (x^\alpha, p^\alpha) \ \ \to \ \ f(x^\alpha, p^\alpha). \tag{1.8}$$

We let C, and sometimes c denote generic positive inessential constants whose values may change from line to line.

The notation $A \lesssim B$ will imply that a positive constant exists such that $A \leq CB$ holds uniformly over the range of parameters which are present in the inequality and moreover that the precise magnitude of the constant is unimportant.

1.2 The relativistic Boltzmann equation in the Bianchi type I space-time

In this section, we present the relativistic Boltzmann equation in the Bianchi type I space-time.
Let $p^\alpha = (p^0, p^1, p^2, p^3)$ such that p^α satisfies

$$-(p^0)^2 + a^2(t)(p^1)^2 + b^2(t)|\bar{p}|^2 = -1.$$

We denote by $\Gamma^\lambda_{\alpha\beta}$ the Christoffel symbols defined by

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}[\partial_{x^\alpha} g_{\mu\beta} + \partial_{x^\beta} g_{\alpha\mu} - \partial_{x^\mu} g_{\alpha\beta}] \tag{1.9}$$

and we consider the vector field $X = (p^\alpha, -\Gamma^\alpha_{\lambda\nu} p^\lambda p\nu)$.
If we denote by $L_X f$ the Lie derivative of f along the vector field X, the relativistic Boltzmann equation reads

$$L_X f = \widetilde{Q}(f, f) \tag{1.10}$$

that is

$$p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^i} = \widetilde{Q}(f, f)$$

where $\widetilde{Q}(f, f)$ stands for the collision operator that will be specified soon.
With the Bianchi type I space-time, the Christoffel symbols were computed in [30]. All of them vanish except

$$\begin{cases} \Gamma^0_{11} &= a\dot{a} \\ \Gamma^0_{22} &= \Gamma^0_{33} = b\dot{b} \end{cases} \tag{1.11}$$

and

$$\begin{cases} \Gamma^1_{10} = \Gamma^1_{01} &= \frac{\dot{a}}{a} \\ \Gamma^2_{20} = \Gamma^2_{02} &= \Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{b}}{b} \end{cases} \tag{1.12}$$

Taking into account (1.11) and (1.12) the Lie derivative of f is given by

$$L_X f = p^0 \frac{\partial f}{\partial t} + p^1 \frac{\partial f}{\partial x^1} + p^2 \frac{\partial f}{\partial x^2} + p^3 \frac{\partial f}{\partial x^3} - 2p^0 \left(\frac{\dot{a}}{a} p^1 \frac{\partial f}{\partial p^1} + \frac{\dot{b}}{b} p^2 \frac{\partial f}{\partial p^2} + \frac{\dot{b}}{b} p^3 \frac{\partial f}{\partial p^3} \right).$$

Here with the Bianchi type I space-time, (1.10) reduces as follows

$$\frac{\partial f}{\partial t} + \frac{1}{p^0} \left(p^1 \frac{\partial f}{\partial x^1} + p^2 \frac{\partial f}{\partial x^2} + p^3 \frac{\partial f}{\partial x^3} \right) - 2\frac{\dot{a}}{a} p^1 \frac{\partial f}{\partial p^1} - 2\frac{\dot{b}}{b} p^2 \frac{\partial f}{\partial p^2} - 2\frac{\dot{b}}{b} p^3 \frac{\partial f}{\partial p^3} = \frac{1}{p^0} \widetilde{Q}(f, f).$$

In the sequel, we let

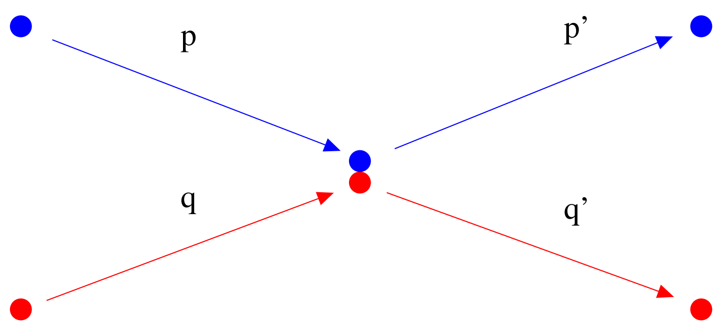
$$Q(f, f) = \frac{1}{p^0} \widetilde{Q}(f, f)$$

and the relativistic Boltzmann equation reduces to

$$\frac{\partial f}{\partial t} + \frac{1}{p^0} \left(p^1 \frac{\partial f}{\partial x^1} + p^2 \frac{\partial f}{\partial x^2} + p^3 \frac{\partial f}{\partial x^3} \right) - 2\frac{\dot{a}}{a} p^1 \frac{\partial f}{\partial p^1} - 2\frac{\dot{b}}{b} p^2 \frac{\partial f}{\partial p^2} - 2\frac{\dot{b}}{b} p^3 \frac{\partial f}{\partial p^3} = Q(f, f). \tag{1.13}$$

1.3 Collision operator

In the instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov [29], we consider that at a given point (t, x), only two particles collide instantaneously without destroying each other. The collision affects only the momenta of the two particles that change after the collision; only the sum of the two momenta is preserved.



Collision between two particles.

Let's suppose p^α and q^α stand for the momenta of two particles before their collision, p'^α and q'^α stand for their momenta after the collision. By the energy-momentum conservation principle, we have

$$p^\alpha + q^\alpha = p'^\alpha + q'^\alpha. \tag{1.14}$$

The expressions of the post-collisional momenta p'^α and q'^α in function of the pre-collisional momenta p^α and q^α will be specified later. The collision operator $\widetilde{Q}$ that acts only on the momentum variable, is defined by

$$\widehat{Q}(f, h) = \widetilde{Q}_{gain}(f, h) - \widetilde{Q}_{loss}(f, h). \tag{1.15}$$

In the definition (1.15) of $\widetilde{Q}$, $\widetilde{Q}_{gain}$ and $\widetilde{Q}_{loss}$ represent respectively the gain and the loss term. They are defined by

$$\widetilde{Q}_{loss}(f, h)(t, p) = \int_{\mathbb{R}^3}\int_{S^2} S(t, p, q, \omega) f(t, p) h(t, q) d\omega \frac{|det(g_{\alpha\beta})|^{\frac{1}{2}}}{-q_0} dq, \tag{1.16}$$

and

$$\widetilde{Q}_{gain}(f, h)(t, p) = \int_{\mathbb{R}^3}\int_{S^2} S(t, p, q, \omega) f(t, p') h(t, q') d\omega \frac{|det(g_{\alpha\beta})|^{\frac{1}{2}}}{-q_0} dq. \tag{1.17}$$

In (1.16) and (1.17)

- $-q_0 = g_{00} q^0 = q^0$.

• $|det(g_{\alpha\beta})|^{\frac{1}{2}}$, easily computed is ab^2.

• $S(t,p,q,\omega)$ is called the collision cross-section and is a non-negative function. It is defined by

$$S(t,p,q,\omega) = g\sqrt{s}\sigma. \tag{1.18}$$

In (1.18), the terms g, s and σ are defined as follows

Definition 1.1.

$$g = g(p^\alpha, q^\alpha) = \sqrt{(p^\alpha - q^\alpha)(p_\alpha - q_\alpha)}, \tag{1.19}$$

and

$$s = s(p^\alpha, q^\alpha) = -(p^\alpha + q^\alpha)(p_\alpha + q_\alpha). \tag{1.20}$$

s is called energy in the center of momentum system and g is called relative momentum.

Definition 1.2. $\sigma = \sigma(g,\theta)$ is called the scattering kernel. It measures interaction effects between particles and determines their nature. σ depends on the relative momentum g defined by (1.19) and on the scattering angle θ.

Definition 1.3. The scattering angle in the case of the relativistic Boltzmann equation is defined by

$$\cos\theta = \frac{(p^\alpha - q^\alpha)(p'_\alpha - q'_\alpha)}{g^2}. \tag{1.21}$$

For more details, we refer interested readers to [17].

Remark 1.1. Note that the scattering angle θ and the parameter ω along the unit sphere are linked through the relation (1.14).

Definition 1.4. As usual in the relativistic Boltzmann equation, we define the Møller velocity ϑ_ϕ for two colliding particles by

$$\vartheta_\phi = \frac{g\sqrt{s}}{p^0 q^0}. \tag{1.22}$$

With the above notations, the relativistic Boltzmann in the Bianchi type I space-time reads

$$\frac{\partial f}{\partial t} + \frac{1}{p^0}\left(p^1\frac{\partial f}{\partial x^1} + p^2\frac{\partial f}{\partial x^2} + p^3\frac{\partial f}{\partial x^3}\right) - 2\frac{\dot{a}}{a}p^1\frac{\partial f}{\partial p^1} - 2\frac{\dot{b}}{b}p^2\frac{\partial f}{\partial p^2} - 2\frac{\dot{b}}{b}p^3\frac{\partial f}{\partial p^3} = Q(f,f)(t,p)$$

with

$$Q(f,f)(t,x,p) = ab^2\int_{\mathbb{R}^3}\int_{S^2}\vartheta_\phi\sigma(g,\omega)\left[f(p')f(q') - f(p)f(q)\right]d\omega dq \tag{1.23}$$

where for simplicity we abbreviate $f(t,x,p)$, $f(t,x,q)$, $f(t,x,p')$ and $f(t,x,q')$ by $f(p)$, $f(q)$, $f(p')$ and $f(q')$ respectively.

Remark 1.2. Since the scattering angle θ and the parameter ω along the unit sphere of $\mathbb{R}^3$ are linked, we will note $\sigma = \sigma(g,\omega)$ in the sequel instead of $\sigma(g,\theta)$.

1.4 Parametrization of the post-collisional momenta

One of the main problem while dealing with the relativistic Boltzmann equation is the choice of the parametrization of the post-collisional momenta. In fact our main goal in this work is to obtain classical solutions. So the derivatives of post-collisional momenta are not trivial to compute and to control. Sometimes their dependence on the relative momentum provides singularities and sometimes leads to some integrals which are not trivial to estimate. To circumvent such difficulties we will work with the following three kinds of parametrization.

1.4.1 First parametrization

We consider a parametrization of post-collisional momenta introduced in [26].
Suppose that p^α and q^α are given, and consider the following four-vectors

$$n^\alpha = p^\alpha + q^\alpha \quad and \quad t^\alpha = (n_i\omega^i, n^0\omega), \quad (\omega \in S^2). \tag{1.24}$$

p'^α and q'^α can be parameterized by

$$p'^\alpha = \frac{p^\alpha + q^\alpha}{2} + \frac{g}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \tag{1.25}$$

$$q'^\alpha = \frac{p^\alpha + q^\alpha}{2} - \frac{g}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}}. \tag{1.26}$$

This parametrization has an advantage that it looks like the usual parametrization in classical Boltzmann equation.
From (1.25) and (1.26), we express easily p'^0 and q'^0 as a function of p^0 and q^0 as

$$\begin{cases} p'^0 &= \frac{p^0+q^0}{2} + \frac{g}{2}\frac{a^2n^1\omega^1 + b^2n^2\omega^2 + b^2n^3\omega^3}{\sqrt{t_\beta t^\beta}}, \\ q'^0 &= \frac{p^0+q^0}{2} - \frac{g}{2}\frac{a^2n^1\omega^1 + b^2n^2\omega^2 + b^2n^3\omega^3}{\sqrt{t_\beta t^\beta}}. \end{cases} \tag{1.27}$$

1.4.2 Second parametrization

By using the Minkowski space-time, Strain has found in [35] the following parametrization of post-collisional momenta: denoting by V and U the pre-collisional momenta and by V' and U' the post-collisional momenta, we obain

$$\begin{cases} V' = \frac{V+U}{2} + \frac{g}{2}\left(\omega + (\gamma-1)\frac{(V+U).\omega}{|V+U|^2}\right), \\ U' = \frac{V+U}{2} - \frac{g}{2}\left(\omega + (\gamma-1)\frac{(V+U).\omega}{|V+U|^2}\right), \quad \omega \in S^2 \end{cases} \tag{1.28}$$

where $\gamma = (V^0 + U^0)/\sqrt{s}$.
In this work, we adapt it to the Bianchi type I space-time by setting

$$V^1 = a^2p^1, \qquad U^1 = a^2q^1, \tag{1.29}$$

$$V^i = b^2p^i, \qquad U^i = b^2q^i, \qquad for\ i = 2, 3. \tag{1.30}$$

$$\begin{aligned} V'^1 &= a^2 p'^1, \qquad U'^1 = a^2 q'^1, & (1.31)\\ V'^i &= b^2 p'^i, \qquad U'^i = b^2 q'^i, \qquad for\ i = 2,3. & (1.32)\end{aligned}$$

We easily express p'^α and q'^α in function of p^α and q^α as follows

$$p'^1 = \frac{p^1+q^1}{2} + \frac{g}{2a^2}\left(\omega^1 + (\frac{n^0}{\sqrt{s}} - 1)\frac{(a^2n^1, b^2\bar{n}).\omega}{|(a^2n^1, b^2\bar{n})|^2}\right), \qquad (1.33)$$

$$p'^k = \frac{p^k+q^k}{2} + \frac{g}{2b^2}\left(\omega^k + (\frac{n^0}{\sqrt{s}} - 1)\frac{(a^2n^1, b^2\bar{n}).\omega}{|(a^2n^1, b^2\bar{n})|^2}\right), \qquad for\ k = 2,3, \qquad (1.34)$$

$$q'^1 = \frac{p^1+q^1}{2} - \frac{g}{2a^2}\left(\omega^1 + (\frac{n^0}{\sqrt{s}} - 1)\frac{(a^2n^1, b^2\bar{n}).\omega}{|(a^2n^1, b^2\bar{n})|^2}\right), \qquad (1.35)$$

$$q'^k = \frac{p^k+q^k}{2} - \frac{g}{2a^2}\left(\omega^k + (\frac{n^0}{\sqrt{s}} - 1)\frac{(a^2n^1, b^2\bar{n}).\omega}{|(a^2n^1, b^2\bar{n})|^2}\right), \qquad for\ k = 2,3. \qquad (1.36)$$

1.4.3 Third parametrization

Suppose that p^α and q^α are given. Let us define the following four-vectors

$$n^\alpha = p^\alpha + q^\alpha \qquad and \qquad t^\alpha = (n_i\omega^i, -n_0\omega), \qquad for\ \omega \in S^2. \qquad (1.37)$$

Lemma 1.1. The vectors t^α and n^α defined by (1.37) are orthogonal.

Proof. Using the metric $(g_{\alpha\beta})$, one has

$$\begin{aligned} t_\alpha n^\alpha &= g_{\alpha\beta} t^\beta n^\alpha \\ &= g_{00} t^0 n^0 + g_{ij} t^i n^j \\ &= g_{00} t^0 n^0 + g_{ij}(-n_0\omega^i) n^j \\ &= g_{00} n^0 (n_i\omega^i) - n_0 g_{ij} n^j \omega^i \\ &= n_0 n_i \omega^i - n_0 n_i \omega^i \\ &= 0. \end{aligned}$$

□

Lemma 1.2. The post-collisional momenta p'^α and q'^α are parameterized by

$$p'^1 = p^1 - \frac{2p^0q^0n^0\left[a^2(\widehat{p^1} - \widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}} - \widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right] - \left[a^2(p^1+q^1)\omega^1 + b^2(\bar{p} - \bar{q}).\bar{\omega}\right]}\omega^1. \qquad (1.38)$$

For $i = 2, 3$

$$p'^i = p^i - \frac{2p^0q^0n^0\left[a^2(\widehat{p^1} - \widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}} - \widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right] - \left[a^2(p^1+q^1)\omega^1 + b^2(\bar{p} - \bar{q}).\bar{\omega}\right]}\omega^i. \qquad (1.39)$$

$$q'^1 = q^1 + \frac{2p^0q^0n^0\left[a^2(\widehat{p^1} - \widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}} - \widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right] - \left[a^2(p^1+q^1)\omega^1 + b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^1. \tag{1.40}$$

For $i = 2, 3$

$$q'^i = q^i + \frac{2p^0q^0n^0\left[a^2(\widehat{p^1} - \widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}} - \widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right] - \left[a^2(p^1+q^1)\omega^1 + b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^i, \tag{1.41}$$

where $\widehat{p^i} = \frac{p^i}{p^0}$ $(i = 1, 2, 3)$, $\widehat{q^i} = \frac{q^i}{q^0}$ $(i = 1, 2, 3)$, $\widehat{\bar{p}} = \frac{\bar{p}}{p^0}$, and $\widehat{\bar{q}} = \frac{\bar{q}}{q^0}$.

Proof. Using the relation $t_\alpha n^\alpha = 0$, we have by the energy-momentum conservation law

$$p'^\alpha = p^\alpha - \frac{t_\beta(p^\beta - q^\beta)}{t_\beta t^\beta}t^\alpha = p^\alpha + 2\frac{t_\beta q^\beta}{t_\beta t^\beta}t^\alpha, \tag{1.42}$$

$$q'^\alpha = q^\alpha - \frac{t_\beta(q^\beta - p^\beta)}{t_\beta t^\beta}t^\alpha = q^\alpha - 2\frac{t_\beta q^\beta}{t_\beta t^\beta}t^\alpha. \tag{1.43}$$

Recalling that

$$\begin{aligned} t^\alpha &= (n_i\omega^i, -g_{00}n^0\omega) \\ &= (n_i\omega^i, n^0\omega) \end{aligned}$$

we have

$$\begin{aligned} t_\alpha t^\alpha &= g_{\alpha\beta}t^\beta t^\alpha \\ &= g_{00}(t^0)^2 + g_{ij}t^it^j \\ &= -(n_i\omega^i)^2 + a^2(t^1)^2 + b^2\left[(t^2)^2 + (t^3)^2\right] \\ &= -(g_{ij}n^i\omega^j)^2 + a^2(t^1)^2 + b^2\left[(t^2)^2 + (t^3)^2\right] \\ &= -\left[a^2n^1\omega^1 + b^2\bar{n}.\bar{\omega}\right]^2 + a^2(p^0+q^0)^2(\omega^1)^2 + b^2(p^0+q^0)^2|\bar{\omega}|^2 \\ &= -\left[a^2n^1\omega^1 + b^2\bar{n}.\bar{\omega}\right]^2 + (p^0+q^0)^2(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2). \end{aligned} \tag{1.44}$$

Using the same computation as above, we have

$$\begin{aligned} t_\alpha q^\alpha &= g_{\alpha\beta}t^\beta q^\alpha \\ &= g_{00}t^0q^0 + g_{ij}t^iq^j \\ &= -q^0n_i\omega^i + a^2t^1q^1 + b^2\left(t^2\omega^2 + t^3\omega^3\right) \\ &= -q^0\left[a^2n^1\omega^1 + b^2\bar{n}.\bar{\omega}\right] + a^2(p^0+q^0)\omega^1q^1 + b^2(p^0+q^0)\bar{\omega}.\bar{q} \\ &= -q^0\left[a^2p^1\omega^1 + b^2\bar{p}.\bar{\omega}\right] + p^0\left[a^2q^1\omega^1 + b^2\bar{q}.\bar{\omega}\right] \\ &= -p^0q^0\left[(a^2\frac{p^1}{p^0}\omega^1 + b^2\frac{\bar{p}}{p^0}.\bar{\omega}) - (a^2\frac{q^1}{q^0}\omega^1 + b^2\frac{\bar{q}}{q^0}.\bar{\omega})\right] \\ &= -p^0q^0\left[(a^2\widehat{p^1}\omega^1 + b^2\widehat{\bar{p}}.\bar{\omega}) - (a^2\widehat{q^1}\omega^1 + b^2\widehat{\bar{q}}.\bar{\omega})\right] \end{aligned} \tag{1.45}$$

where

$$\widehat{p^i} = \frac{p^i}{p^0}, \qquad \widehat{q^i} = \frac{q^i}{q^0}, \qquad for\ i = 1, 2, 3.$$

Using (1.42), (1.44) and (1.45) we have

$$p'^1 = p^1 - \frac{2p^0q^0n^0\left[a^2(\widehat{p^1}-\widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}}-\widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[a^2(p^1+q^1)\omega^1+b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^1,$$

and for $i = 2, 3$

$$p'^i = p^i - \frac{2p^0q^0n^0\left[a^2(\widehat{p^1}-\widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}}-\widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[a^2(p^1+q^1)\omega^1+b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^i.$$

Using (1.43), (1.44) and (1.45) we have

$$q'^1 = q^1 + \frac{2p^0q^0n^0\left[a^2(\widehat{p^1}-\widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}}-\widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[a^2(p^1+q^1)\omega^1+b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^1,$$

and for $i = 2, 3$

$$q'^i = q^i + \frac{2p^0q^0n^0\left[a^2(\widehat{p^1}-\widehat{q^1})\omega^1 + b^2(\widehat{\bar{p}}-\widehat{\bar{q}}).\bar{\omega}\right]}{(p^0+q^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[a^2(p^1+q^1)\omega^1+b^2(\bar{p}-\bar{q}).\bar{\omega}\right]}\omega^i.$$

□

Remark 1.3. The second and third parametrization are very similar to those of the non-relativistic case.

1.5 The relativistic Boltzmann equation in covariant variables

Now we are going to introduce a change of variables so that the relativistic Boltzmann equation in the Bianchi type I space-time is written in a simple form. In our context (where we use the Bianchi type I space-time), the relativistic Boltzmann equation is written in a simple form if we use covariant variables. So, the distribution function f will be considered as a function of t, $x = (x^1, x^2, x^3)$ and $v = (v^1, v^2, v^3) = (v^1, \bar{v})$ where

$$\begin{cases} v^1 &= p_1 = g_{1i}p^i = a^2p^1, \\ v^2 &= p_2 = g_{2i}p^i = b^2p^2, \\ v^3 &= p_3 = g_{3i}p^i = b^2p^3. \end{cases} \tag{1.46}$$

We also observe that if we set

$$v^0 = \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2} \tag{1.47}$$

we obtain

$$v^0 = p^0. \tag{1.48}$$

Using the variables t, x, v, we change the unknown function as

$$\widetilde{f}(t, x, v) = f(t, x, p). \tag{1.49}$$

In order to have a good description of the relativistic Boltzmann equation in the new variables, it is necessary to express the collision operator in term of new variables.
If we let $v = (a^2p^1, b^2\bar{p})$ and $u = (a^2q^1, b^2\bar{q})$ the momenta of the incoming particles, we may write

$$\begin{cases} v'^1 &= a^2 p'^1 \\ v'^i &= b^2 p'^i, \qquad for\ i = 2,3 \\ v'^0 &= p'^0 \end{cases} \qquad and \qquad \begin{cases} u'^1 &= a^2 q'^1 \\ u'^i &= b^2 q'^i, \qquad for\ i = 2,3 \\ u'^0 &= q'^0 \end{cases} \tag{1.50}$$

in the similar way the post-collisional momenta. So, the collision operator becomes

$$\begin{aligned} Q(\widetilde{f},\widetilde{f})(t,x,v) &= a^{-1}b^{-2}\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{g\sqrt{s}}{v^0u^0}\sigma(g,\omega)\left[\widetilde{f}(v')\widetilde{f}(u') - \widetilde{f}(v)\widetilde{f}(u)\right] du \\ &= Q_{gain}(\widetilde{f},\widetilde{f})(t,x,v) - Q_{loss}(\widetilde{f},\widetilde{f})(t,x,v), \end{aligned} \tag{1.51}$$

where for the sake of simplicity we have abbreviated $\widetilde{f}(t,x,v')$, $\widetilde{f}(t,x,u')$, $\widetilde{f}(t,x,v)$ and $\widetilde{f}(t,x,u)$ by $\widetilde{f}(v')$, $\widetilde{f}(u')$, $\widetilde{f}(v)$ and $\widetilde{f}(u)$ respectively.
About the left-hand side of the relativistic Boltzmann equation, we have

$$\partial_{x^i}\widetilde{f} = \partial_{x^i} f, \qquad \text{for } i = 1,2,3, \tag{1.52}$$

and

$$\begin{aligned} \frac{\partial \widetilde{f}}{\partial t} &= \frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a^3}v^1\frac{\partial f}{\partial p^1} - 2\frac{\dot{b}}{b^3}v^2\frac{\partial f}{\partial p^2} - 2\frac{\dot{b}}{b^3}v^3\frac{\partial f}{\partial p^3} \\ &= \frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a}p^1\frac{\partial f}{\partial p^1} - 2\frac{\dot{b}}{b}p^2\frac{\partial f}{\partial p^2} - 2\frac{\dot{b}}{b}\frac{\partial f}{\partial p^3}. \end{aligned} \tag{1.53}$$

The left-hand side of the relativistic Boltzmann equation becomes

$$\partial_t\widetilde{f} + \frac{1}{a^2}\frac{v^1}{v^0}\partial_{x^1}\widetilde{f} + \frac{1}{b^2}\frac{v^2}{v^0}\partial_{x^2}\widetilde{f} + \frac{1}{b^2}\frac{v^3}{v^0}\partial_{x^3}\widetilde{f}.$$

In the sequel, by abuse of notation we will write f instead of $\widetilde{f}$. So the equation in new variables writes

$$\partial_t f + \frac{1}{a^2}\frac{v^1}{v^0}\partial_{x^1} f + \frac{1}{b^2}\frac{v^2}{v^0}\partial_{x^2} f + \frac{1}{b^2}\frac{v^3}{v^0}\partial_{x^3} f = Q(f,f)(t,x,v). \tag{1.54}$$

1.6 Parametrization of the post-collisional momenta in new variables

To complete the description of the equation in term of new variables, we may write the parametrization of post-collisional momenta with the new variables. With the new variables defined in (1.50), the first parametrization given by (1.25)-(1.26)-(1.27) is stated as follows:

$$v'^0 = \frac{v^0 + u^0}{2} + \frac{g}{2}\frac{n.\omega}{\sqrt{-(n.\omega)^2 + (n^0)^2(a^2(\omega^1)^2 + b^2|\overline{\omega}|^2)}}, \tag{1.55}$$

$$v'^1 = \frac{v^1+u^1}{2} + \frac{g}{2}\frac{n^0\omega^1}{\sqrt{-(n.\omega)^2+(n^0)^2(a^2(\omega^1)^2+b^2|\overline{\omega}|^2)}}, \tag{1.56}$$

$$v'^k = \frac{v^k+u^k}{2} + \frac{g}{2}\frac{n^0\omega^k}{\sqrt{-(n.\omega)^2+(n^0)^2(a^2(\omega^1)^2+b^2|\overline{\omega}|^2)}}, \qquad k=2,3, \tag{1.57}$$

$$u'^0 = \frac{v^0+u^0}{2} - \frac{g}{2}\frac{n.\omega}{\sqrt{-(n.\omega)^2+(n^0)^2(a^2(\omega^1)^2+b^2|\overline{\omega}|^2)}}, \tag{1.58}$$

$$u'^1 = \frac{v^1+u^1}{2} - \frac{g}{2}\frac{n^0\omega^1}{\sqrt{-(n.\omega)^2+(n^0)^2(a^2(\omega^1)^2+b^2|\overline{\omega}|^2)}}, \tag{1.59}$$

$$u'^k = \frac{v^k+u^k}{2} - \frac{g}{2}\frac{n^0\omega^k}{\sqrt{-(n.\omega)^2+(n^0)^2(a^2(\omega^1)^2+b^2|\overline{\omega}|^2)}}, \qquad k=2,3. \tag{1.60}$$

With the new variables defined in (1.50), the second parametrization defined by (1.33), (1.34), (1.35) and (1.36) is stated as follows:
for the parameter $\omega \in S^2$, if we let

$$\Omega^i = (w^i - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^i,$$

we obtain

$$v'^1 = \frac{v^1+u^1}{2} + \frac{ag}{2}\Omega^1, \tag{1.61}$$

$$v'^k = \frac{v^k+u^k}{2} + \frac{bg}{2}\Omega^k, \qquad k=2,3, \tag{1.62}$$

$$u'^1 = \frac{v^1+u^1}{2} - \frac{ag}{2}\Omega^1, \tag{1.63}$$

$$u'^k = \frac{v^k+u^k}{2} - \frac{bg}{2}\Omega^k, \qquad k=2,3. \tag{1.64}$$

With the new variables defined in (1.50), the third parametrization defined by (1.38), (1.39), (1.40) and (1.41) is stated as follows:

$$v'^1 = v^1 - \frac{2a^2v^0u^0n^0\left[(\widehat{v^1}-\widehat{u^1})\omega^1+(\widehat{\bar{v}}-\widehat{\bar{u}}).\bar{\omega}\right]}{(v^0+u^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[(v^1+u^1)\omega^1+(\bar{v}-\bar{u}).\bar{\omega}\right]}\omega^1, \tag{1.65}$$

$$v'^k = v^k - \frac{2b^2v^0u^0n^0\left[(\widehat{v^1}-\widehat{u^1})\omega^1+(\widehat{\bar{v}}-\widehat{\bar{u}}).\bar{\omega}\right]}{(v^0+u^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[(v^1+u^1)\omega^1+(\bar{v}-\bar{u}).\bar{\omega}\right]}\omega^k, \qquad k=2,3, \tag{1.66}$$

$$u'^1 = u^1 + \frac{2a^2v^0u^0n^0\left[(\widehat{v^1}-\widehat{u^1})\omega^1+(\widehat{\bar{v}}-\widehat{\bar{u}}).\bar{\omega}\right]}{(v^0+u^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[(v^1+u^1)\omega^1+(\bar{v}-\bar{u}).\bar{\omega}\right]}\omega^1, \tag{1.67}$$

$$u'^k = u^k + \frac{2b^2v^0u^0n^0\left[(\widehat{v^1}-\widehat{u^1})\omega^1+(\widehat{\bar{v}}-\widehat{\bar{u}}).\bar{\omega}\right]}{(v^0+u^0)^2\left[a^2(\omega^1)^2+b^2|\bar{\omega}|^2\right]-\left[(v^1+u^1)\omega^1+(\bar{v}-\bar{u}).\bar{\omega}\right]}\omega^k, \qquad k=2,3. \tag{1.68}$$

1.7 Specification of scattering kernels

1.7.1 Scattering kernel generated by Israel particles

One of the scattering kernel used in this work is generated by Israel particles. The Israel particles are analogue of the Maxwell molecules cross-section in the Newtonian theory. Then $\sigma(g,\omega)$ is defined by

$$\sigma(g,\omega)=\frac{4\sigma_0(\omega)}{g(4+g^2)}. \tag{1.69}$$

With this scattering kernel, Israel derives eigenfunctions for the linearized relativistic Boltzmann collision operator. Note that it converges to the Maxwell molecule cross-section in the Newtonian limit.

1.7.2 Scattering kernel for hard potentials

One assumes that the scattering kernel $\sigma(g,\omega)$ satisfies the following growth/decay estimates

$$\frac{g}{\sqrt{s}}g^{\beta}\sigma_0(\omega)\lesssim\sigma(g,\omega)\lesssim(g^{\alpha}+g^{-\beta})\sigma_0(\omega). \tag{1.70}$$

The angular factors are such that $\sigma_0(\omega)\geq 0$ and $\sigma_0(\omega)\lesssim sin^{\gamma}\theta$ with $\gamma>-2$.
α and β are such that $0\leq\alpha\leq 2+\gamma$ and $0\leq\beta<min(4,4+\gamma)$.

1.7.3 Scattering kernel for soft potentials

One assumes that the scattering kernel $\sigma(g,\omega)$ satisfies the following growth/decay estimates

$$\frac{g}{\sqrt{s}}g^{-\beta}\sigma_0(\omega)\lesssim\sigma(g,\omega)\lesssim g^{-\beta}\sigma_0(\omega). \tag{1.71}$$

In addition to the previous angular factors defined for the hard potentials case,
we consider $0<\beta<min(4,4+\gamma)$.

CHAPTER TWO

L^∞-EXISTENCE THEOREM OF THE RELATIVISTIC BOLTZMANN EQUATION IN THE BIANCHI TYPE I SPACE-TIME

Contents

In this chapter, we provide some L^∞-classical solutions for the homogeneous Boltzmann equation in Bianchi type I space-time for a hard potential, a soft potential and with the Israel particles respectively.

Homogeneity means that the unknown function in the equation, which generally depends on time, spatial variables and momentum variables, is restricted to depend only on time and momentum variables.

Let's consider the set Λ that will be defined in the sequel, the relativistic Boltzmann equation (1.54) in f with initial data $f_0 \in \Lambda$ then reads in term of variables (t, v)

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{2.1}$$

$$f(0,v) = f_0(v). \tag{2.2}$$

So, f is the solution of the homogeneous relativistic Boltzmann equation with initial data f_0 if f is regular and is the solution of the following integral equation:

$$f(t,v) = f_0(v) + \int_0^t Q(f,f)(\tau,v)d\tau. \tag{2.3}$$

We assume that the coefficients a and b of the Bianchi type I metric are given increasing functions of the time t and are such that:

$$a(0) \geq 1, \quad a \leq b \leq \sqrt{2}a, \quad \lim_{t \longrightarrow \infty} a(t) = \infty, \tag{2.4}$$

$$\int_{\mathbb{R}_+} a^{-2}(\tau)d\tau < \infty. \tag{2.5}$$

2.1 Estimates of the terms allowing to define the collision kernel

Lemma 2.1. The relative momentum and the energy in the center of momentum system enjoy the following estimates:

$$s = 4 + g^2, \qquad 2 \leq \sqrt{s}, \qquad g \leq \sqrt{s}, \tag{2.6}$$

$$g \leq \sqrt{s} \leq 2\sqrt{v^0 u^0}. \tag{2.7}$$

Proof. The relative momentum and the energy in the center of momentum system are given respectively by

$$g = \sqrt{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)} \qquad and \qquad s = -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha).$$

Taking into account the mass shell condition $p_\alpha p^\alpha = -1$, we have

$$s = -p_\alpha p^\alpha - q_\alpha q^\alpha - p_\alpha q^\alpha - q_\alpha p^\alpha = 2 - 2p_\alpha q^\alpha, \tag{2.8}$$

$$g^2 = p_\alpha p^\alpha + q_\alpha q^\alpha - p_\alpha q^\alpha - q_\alpha p^\alpha = -2 - 2p_\alpha q^\alpha. \tag{2.9}$$

Combining (2.8) and (2.9) we obtain

$$s = 4 + g^2.$$

This implies

$$s \geq 4, \qquad \sqrt{s} \geq 2 \qquad and \qquad \sqrt{s} \geq g.$$

Using (2.8) and the Bianchi type I metric, and since $v^0 = p^0$ and $u^0 = q^0$, we obtain

$$\begin{aligned} s &= 2 - 2[-p^0q^0 + a^2p^1q^1 + b^2p^2q^2 + b^2p^3q^3] \\ &= 2p^0q^0 + 2[1 - a^2p^1q^1 - b^2p^2q^2 - b^2p^3q^3] \\ &\leq 2p^0q^0 + 2[1 + a^2|p^1||q^1| + b^2|p^2||q^2| + b^2|p^3||q^3|] \\ &= 2p^0q^0 + 2(1, a|p^1|, b|p^2|, b|p^3|).(1, a|q^1|, b|q^2|, b|q^3|) \\ &\leq 2p^0q^0 + 2\sqrt{1 + a^2(p^1)^2 + b^2(p^2)^2 + b^2(p^3)^2}\sqrt{1 + a^2(q^1)^2 + b2(q^2)^2 + b^2(q^3)^2} \\ &= 4p^0q^0. \end{aligned}$$

So $s \leq 4v^0u^0$ and then $\sqrt{s} \leq 2\sqrt{v^0u^0}$. □

Lemma 2.2. s and g enjoy the following estimates

$$g \leq \sqrt{s} \leq 2\sqrt{v'^0u'^0}. \tag{2.10}$$

Proof. Let's consider the conservation law

$$p'^\alpha + q'^\alpha = p^\alpha + q^\alpha.$$

We have

$$\begin{aligned} s &= -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha) \\ &= -(p'_\alpha + q'_\alpha)(p'^\alpha + q'^\alpha) \\ &= 2 - 2p'_\alpha q'^\alpha. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} s &= 2 - 2p'_\alpha q'^\alpha \\ &= 2 - 2\left[-p'^0q'^0 + a^2p'^1q'^1 + b^2p'^2q'^2 + b^2p'^3q'^3\right] \\ &= 2p'^0q'^0 + 2\left[1 - a^2p'^1q'^1 - b^2p'^2q'^2 - b^2p'^3q'^3\right] \\ &\leq 4p'^0q'^0. \end{aligned}$$

So $s \leq 4v'^0u'^0$ and then $\sqrt{s} \leq 2\sqrt{v'^0u'^0}$. □

Lemma 2.3. The relative momentum fulfills the estimates:

$$\frac{|v-u|}{\sqrt{v^0u^0}} \leq bg \qquad and \qquad ag \leq |v-u|. \tag{2.11}$$

Proof. For the first inequality in (2.11), by direct computation, we have

$$\begin{aligned} g^2 &= 2p^0q^0 - 2[1 + a^2p^1q^1 + b^2p^2q^2 + b^2p^3q^3] \\ &= 2p^0q^0 - 2[1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})] \\ &= 2\frac{(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2}{p^0q^0 + [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]}. \end{aligned}$$

Denoting by dot the usual inner product in $\mathbb{R}^3$, by Cauchy-Schwartz inequality, we ontain

$$(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2 \geq |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2 \geq 0 \tag{2.12}$$

and we notice that if we set $\Delta = (p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2$, We have

$$\begin{aligned} \Delta &= 1 + (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2)(a^2(q^1)^2 + b^2|\bar{q}|^2) - \Delta_1 \\ &= (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) - 2(ap^1, b\bar{p}).(aq^1, b\bar{q}) + \Delta_2 \\ &\geq (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) - 2(ap^1, b\bar{p}).(aq^1, b\bar{q}) \\ &= [(ap^1, b\bar{p}) - (aq^1, b\bar{q})].[(ap^1, b\bar{p}) - (aq^1, b\bar{q})] \\ &= |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2 \end{aligned}$$

where Δ_1 and Δ_2 are defined by

$$\begin{aligned} \Delta_1 &= 1 + 2(ap^1, b\bar{p})(aq^1, b\bar{q}) + [(ap^1, b\bar{p})(aq^1, b\bar{q})]^2, \\ \Delta_2 &= [(a^2(p^1)^2 + b^2|\bar{p}|^2)(a^2(q^1)^2 + b^2|\bar{q}|^2) - [(ap^1, b\bar{p}).(aq^1, b\bar{q})]^2]. \end{aligned}$$

Thus

$$(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2 \geq |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2.$$

By (2.12) we have

$$\begin{aligned} p^0q^0 &\geq |1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})| \geq 1 + (ap^1, b\bar{p}).(aq^1, b\bar{q}) \\ 2p^0q^0 &\geq p^0q^0 + 1 + (ap^1, b\bar{p}).(aq^1, b\bar{q}). \end{aligned}$$

$$g^2 = 2\frac{(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2}{p^0q^0 + [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]} \geq 2\frac{|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2}{2p^0q^0}.$$

Let us observe that

$$|v - u|^2 = a^4(p^1 - q^1)^2 + b^4|\bar{p} - \bar{q}|^2.$$

Since $a \leq b$, we obtain

$$\begin{aligned} |v - u|^2 &\leq b^2[a^2(p^1 - q^1)^2 + b^2(p^2 - q^2)^2 + b^2(p^3 - q^3)^2] \\ &\leq b^2|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2. \end{aligned}$$

This leads to

$$g^2 \geq \frac{|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2}{p^0q^0} \geq \frac{|v - u|^2}{b^2p^0q^0}$$

and then

$$bg \geq \frac{|v-u|}{\sqrt{v^0 u^0}}.$$

This proves the first relation.
For the second inequality in (2.11), we have

$$(v^0)^2 - (u^0)^2 = a^{-2}(v^1 - u^1)n^1 + b^{-2}(\bar{v} - \bar{u}).\bar{n}$$

and

$$\begin{aligned} |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2 - (p^0 - q^0)^2 &= |(ap^1, b\bar{p})|^2 + |(aq^1, b\bar{q})|^2 - 2(ap^1, b\bar{p}).(aq^1, b\bar{q}) \\ &\quad - (p^0)^2 - (q^0)^2 + 2p^0 q^0 \\ &= 2p^0 q^0 - 2 - 2(ap^1, b\bar{p}).(aq^1, b\bar{q}) \\ &= 2p^0 q^0 - 2\left[1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})\right] \\ &= g^2. \end{aligned}$$

Let us denote by θ_0 the angle between the two vectors $(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))$ and $(a^{-1}n^1, b^{-1}(\bar{n}))$.

$$\begin{aligned} g^2 &= a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 - (v^0 - u^0)^2 \\ &= a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 - \left[\frac{a^{-2}(v^1 - u^1)n^1 + b^{-2}(\bar{v} - \bar{u}).\bar{n}}{v^0 + u^0}\right]^2 \\ &= a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 - \left[\frac{|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))||(a^{-1}n^1, b^{-1}\bar{n})| \cos\theta_0}{v^0 + u^0}\right]^2 \\ &= |(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2 \left[1 - (\frac{|(a^{-1}n^1, b^{-1}\bar{n})| \cos\theta_0}{v^0 + u^0})^2\right]. \end{aligned} \tag{2.13}$$

Thus

$$g^2 \leq a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 \leq a^{-2}|v - u|^2.$$

□

Lemma 2.4. For the increasing functions $a = a(t)$ and $b = b(t)$ such that $a(0) \geq 1$ and $a(t) \leq b(t)$, the following inequalities hold

$$|v| \leq bv^0 \qquad \textit{and} \qquad v^0 \leq \sqrt{1 + |v|^2}. \tag{2.14}$$

Proof. Since $a \leq b$, we have

$$\begin{aligned} (v^0)^2 =& 1 + a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2 \\ &\geq a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2 \\ &\geq b^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2 \\ &\geq b^{-2}|v|^2. \end{aligned}$$

Since $a \geq 1$ and then $b \geq 1$, we have

$$\begin{aligned}(v^0)^2 =& 1 + a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2\\ &\leq 1 + (v^1)^2 + (v^2)^2 + (v^3)^2\\ &\leq 1 + |v|^2.\end{aligned}$$

So we obtain

$$\begin{aligned}(v^0)^2 \geq b^{-2}|v|^2 \quad and \quad v^0 \geq b^{-1}|v|,\\ (v^0)^2 \leq 1 + |v|^2 \quad and \quad v^0 \leq \sqrt{1 + |v|^2}.\end{aligned}$$

□

Lemma 2.5. For the unit vector $\omega \in S^2$, using the adopted notation $\bar{\omega} = (\omega^2, \omega^3)$, the four vector $t^\alpha = (n_i\omega^i, n^0\omega)$ enjoys the estimate:

$$\sqrt{t_\beta t^\beta} \geq \sqrt{s}(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)^{\frac{1}{2}}. \tag{2.15}$$

Proof. By using elementary algebra properties, we have

$$\begin{aligned}t_\beta t^\beta &= -(t^0)^2 + a^2(t^1)^2 + b^2|\bar{t}|^2\\ &= -(a^2n^1\omega^1 + b^2\bar{n}.\bar{w})^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2]\\ &= -[(an^1, b\bar{n}).(a\omega^1, b\bar{w})]^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2]\\ &\geq -|(an^1, b\bar{n})|^2|(a\omega^1, b\bar{w})|^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2]\\ &\geq -[a^2(n^1)^2 + b^2|\bar{n}|^2][a^2(\omega^1)^2 + b^2|\bar{w}|^2] + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2]\\ &\geq [a^2(\omega^1)^2 + b^2|\bar{w}|^2][(n^0)^2 - a^2(n^1)^2 - b^2|\bar{n}|^2]\\ &\geq [a^2(\omega^1)^2 + b^2|\bar{w}|^2]s\end{aligned}$$

and then the desired result. □

Lemma 2.6. The energy s defined by (1.20) enjoys the estimate

$$\sqrt{s} \geq max\left(\sqrt{\frac{v^0}{u^0}}, \sqrt{\frac{u^0}{v^0}}\right). \tag{2.16}$$

Proof. From the definition of s, we have

$$\begin{aligned}
s &= (v^0)^2 + 2v^0u^0 + (u^0)^2 - a^{-2}(v^1)^2 - a^{-2}(u^1)^2 - 2a^{-2}v^1u^1 - b^{-2}|\bar{v}|^2 - b{-}2|u|^2 - 2b^{-2}\bar{v}.\bar{u} \\
&= 2 + 2\sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2}\sqrt{1 + a^{-2}(u^1)^2 + b^{-2}|\bar{u}|^2} - 2(a^{-1}v^1, b^{-1}\bar{v}).(a^{-1}u^1, b^{-1}\bar{u}) \\
&\geq 2 + 2\sqrt{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1 + |(a^{-1}u^1, b^{-1}\bar{u})|^2} - 2|(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})| \\
&\geq 2 + 2\frac{(1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2)(1 + |(a^{-1}u^1, b^{-1}\bar{u})|^2) - |(a^{-1}v^1, b^{-1}\bar{v})|^2|(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1 + |(a^{-1}u^1, b^{-1}\bar{u})|^2} + |(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})|} \\
&\geq 2 + 2\frac{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2 + |(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1 + |(a^{-1}u^1, b^{-1}\bar{u})|^2} + |(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})|} \\
&\geq 2 + \frac{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2 + |(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1 + |(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1 + |(a^{-1}u^1, b^{-1}\bar{u})|^2}} \\
&\geq 2 + \frac{(v^0)^2 + (u^0)^2 - 1}{v^0u^0} \\
&\geq \frac{(v^0)^2 + (u^0)^2 + 2v^0u^0 - 1}{v^0u^0} \\
&\geq \frac{(v^0)^2 + (u^0)^2}{v^0u^0} \geq \frac{v^0}{u^0} + \frac{u^0}{v^0}.
\end{aligned}$$

□

Lemma 2.7. If the pre-collisional momenta v and u are such that $|v| < 2|u|$, we have

$$v^0 \leq 2\sqrt{2}u^0. \tag{2.17}$$

Proof. Using the relation between a and b we have

$$\begin{aligned}
v^0 &= \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2} \\
&\leq \sqrt{1 + 2b^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2} \\
&\leq \sqrt{1 + 2b^{-2}|v|^2} \\
&\leq \sqrt{1 + 8b^{-2}|u|^2} \\
&\leq \sqrt{8}\sqrt{1 + b^{-2}|u|^2} \\
&\leq 2\sqrt{2}u^0.
\end{aligned}$$

□

Lemma 2.8. The following estimate holds

$$v^0 \leq 2v'^0u'^0. \tag{2.18}$$

Proof. By the energy conservation law

$$v^0 + u^0 = v'^0 + u'^0$$

we have

$$v^0 \leq \sqrt{1 + a^{-2}(v'^1)^2 + b^{-2}|\bar{v'}|^2} + \sqrt{1 + a^{-2}(u'^1)^2 + b^{-2}|\bar{u'}|^2}.$$

Hence

$$\begin{aligned}(v^0)^2 &\le 2[1 + a^{-2}(v'^1)^2 + b^{-2}|\bar{v'}|^2 + 1 + a^{-2}(u'^1)^2 + b^{-2}|\bar{u'}|^2]\\ &\le 4[1 + a^{-2}(v'^1)^2 + b^{-2}|\bar{v'}|^2][1 + a^{-2}(u'^1)^2 + b^{-2}|\bar{u'}|^2]\\ &= 4(v'^0)^2(u'^0)^2.\end{aligned}$$

So we have: $(v^0)^2 \le 2v'^0u'^0$. □

Lemma 2.9. For $m \in \mathbb{N}$ and $\xi \in \mathbb{R}^3$:

$$(1 + |\xi|^2)^m \le 2^m(1 + |\xi|^{2m}) \qquad \text{and} \qquad (1 + |\xi|^{2m}) \le (1 + |\xi|^2)^m. \tag{2.19}$$

Proof. The proof of this lemma is obvious.

□

Lemma 2.10. For $m \in \mathbb{Z}$:

$$\int_{\mathbb{R}^3} (1 + |u|^2)^m e^{-|u|^2} du < \infty. \tag{2.20}$$

This proof is done here because we have seen none in the literature.

Proof. **Case 1:** $m < 0$

Here $(1 + |u|^2)^m < 1$ and since $(\int_{\mathbb{R}^3} e^{-|u|^2} du < \infty)$, we have

$$\int_{\mathbb{R}^3} (1 + |u|^2)^m e^{-|u|^2} du \le \int_{\mathbb{R}^3} e^{-|u|^2} du < \infty.$$

Case 2: $m \in \mathbb{N}$

According to (2.19) we have $(1 + |u|^2)^m \le 2^m(1 + |u|^{2m})$.

So we can state that

$$\begin{aligned}\int_{\mathbb{R}^3} (1 + |u|^2)^m e^{-|u|^2} du &\le 2^m \int_{\mathbb{R}^3} (1 + |u|^{2m}) e^{-|u|^2} du\\ &\le 2^m \int_{\mathbb{R}^3} e^{-|u|^2} du + 2^m \int_{\mathbb{R}^3} |u|^{2m} e^{-|u|^2} du\\ &\le 2^m \int_{\mathbb{R}^3} e^{-|u|^2} du + 2^m \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\varphi \int_0^{\infty} r^{2m+2} e^{-r^2} dr\\ &\le 2^m \int_{\mathbb{R}^3} e^{-|u|^2} du + 4\pi 2^m \int_0^{\infty} r^{2m+2} e^{-r^2} dr.\end{aligned}$$

Let the sequence $(I_m)_{m\in\mathbb{N}}$ be defined as follows

$$I_m = \int_0^{\infty} r^{2m+2} e^{-r^2} dr, \qquad m \ge 0.$$

We can integrate this term as follows

$$\begin{aligned}I_m &= \int_0^{\infty} r^{2m+2} e^{-r^2} dr\\ &= -\frac{1}{2} \int_0^{\infty} r^{2m+1}(-2r) e^{-r^2} dr\\ &= -\frac{1}{2}([r^{2m+1} e^{-r^2}]_0^{\infty} - \int_0^{\infty} r^{2m} e^{-r^2} dr)\\ &= \frac{1}{2} \int_0^{\infty} r^{2m} e^{-r^2} dr.\end{aligned}$$

Then we have the following relation

$$I_m = \frac{1}{2} I_{m-1}, \qquad for\ all\ m \geq 0.$$

This leads to

$$I_m = \frac{1}{2^m} I_0 = \frac{1}{2^m}\frac{\sqrt{\pi}}{4}.$$

Finally, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (1+|u|^2)^m e^{-|u|^2} du &\leq 2^m \int_{\mathbb{R}^3} e^{-|u|^2} du + 4\pi 2^m \frac{1}{2^m}\frac{\sqrt{\pi}}{4} \\ &\leq 2^m \int_{\mathbb{R}^3} e^{-|u|^2} du + \pi\sqrt{\pi} \\ &< \infty.\end{aligned}$$

□

Lemma 2.11. For $0 \leq \alpha < 3$ and $v \in \mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} |v-u|^{-\alpha} e^{-|u|^2} du \leq C_\alpha (1+|v|^2)^{-\frac{\alpha}{2}}. \tag{2.21}$$

Proof. The complete proof is done here because we have seen none in the literature.

$$\int_{\mathbb{R}^3} |v-u|^{-\alpha} e^{-|u|^2} du = \int_{|v-u|\leq \frac{|v|}{2}} |v-u|^{-\alpha} e^{-|u|^2} du + \int_{|v-u|\geq \frac{|v|}{2}} |v-u|^{-\alpha} e^{-|u|^2} du.$$

Let

$$\begin{aligned}I &= \int_{|v-u|\leq \frac{|v|}{2}} |v-u|^{-\alpha} e^{-|u|^2} du, \\ J &= \int_{|v-u|\geq \frac{|v|}{2}} |v-u|^{-\alpha} e^{-|u|^2} du.\end{aligned}$$

The relation $|v-u| \geq \frac{|v|}{2}$ leads to $|v-u|^{-\alpha} \leq (\frac{|v|}{2})^{-\alpha}$ in J definition.
Then

$$\begin{aligned}J &= \int_{|v-u|\geq \frac{|v|}{2}} |v-u|^{-\alpha} e^{-|u|^2} du \\ &\leq \int_{|v-u|\geq \frac{|v|}{2}} (\frac{|u|}{2})^{-\alpha} e^{-|u|^2} du \\ &\leq \left(\frac{|v|}{2}\right)^{-\alpha} \int_{|v-u|\geq \frac{|v|}{2}} e^{-|u|^2} du.\end{aligned}$$

Let's estimate I: The relation

$$|v-u| \leq \frac{|v|}{2} \qquad leads \qquad to \qquad |u| \geq \frac{|v|}{2}$$

because $|v| \leq |v-u| + |u| \leq \frac{|v|}{2} + |u|$.
We let

$$E = \{(x,y,z) \in \mathbb{R}^3,\ \ x^2+y^2+z^2 \leq \frac{|v|^2}{4}\}$$

where

$$u^1 = x + v^1 \qquad and \qquad u^2 = y + v^2 \qquad and \qquad u^3 = z + u^3.$$

Then

$$\begin{aligned} I &= \int_{\{|v-u| \leq \frac{|v|}{2}\}} |v-u|^{-\alpha} e^{-|u|^2} du \\ &\leq e^{-\frac{|v|^2}{4}} \int_E (x^2+y^2+z^2)^{-\frac{\alpha}{2}} dxdydz. \end{aligned}$$

We use the spherical coordinates $x = rsin(\theta)cos(\rho)$, $y = rsin(\theta)sin(\rho)$ and $z = rcos(\theta)$ where $r \in [0, \frac{|v|}{2}]$, $\theta \in [0, \pi]$ and $\rho \in [0, 2\pi]$.

$$I \leq e^{\frac{-|v|^2}{4}} \int_0^{\pi} sin(\theta) d\theta \int_0^{2\pi} d\rho \int_0^{\frac{|v|}{2}} r^{2-\alpha} dr.$$

For $0 \leq \alpha < 3$

$$\begin{aligned} I &\leq 4\pi \left[\frac{1}{3-\alpha} r^{3-\alpha} \right]_0^{\frac{|v|}{2}} e^{-\frac{|v|^2}{4}} \\ &\leq C_\alpha |v|^{3-\alpha} e^{-\frac{|v|^2}{4}}. \end{aligned}$$

Summing up I and J, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |v-u|^{-\alpha} e^{-|u|^2} du &\leq C_\alpha |v|^{-\alpha} + C_\alpha |v|^{3-\alpha} e^{-\frac{|v|^2}{4}} \\ &\leq C_\alpha |v|^{-\alpha} + C_\alpha |v|^{-\alpha} |v|^3 e^{-\frac{|v|^2}{4}}. \end{aligned}$$

Let

$$\psi(x) = x^3 e^{-\frac{x^2}{4}}, \qquad with \qquad x \in \mathbb{R}_+.$$

$\psi(0) = 0$ and $\psi(x)$ goes to infinity as x goes to infinity.
The derivative

$$\psi'(x) = \frac{1}{2} x^2 (6 - x^2) e^{-\frac{x^4}{4}}$$

vanishes as $x = \sqrt{6}$.

So for a non-negative x, we have

$$0 \leq \psi(x) \leq \psi(\sqrt{6}).$$

We return to the estimate of $\int_{\mathbb{R}^3} |v-u|^{-\alpha} e^{-|u|^2} du$. That is

$$\begin{aligned} \int_{\mathbb{R}^3} |v-u|^{-\alpha} e^{-|u|^2} du &\leq C_\alpha |v|^{-\alpha} + C_\alpha |v|^{-\alpha} \psi(\sqrt{6}) \\ &\leq C_\alpha |v|^{-\alpha} \\ &\leq C_\alpha (1 + |v|^{-\alpha}) \\ &\leq C_\alpha (1 + |v|^{2(-\frac{\alpha}{2})}) \\ &\leq C_\alpha (1 + |v|^2)^{-\frac{\alpha}{2}}. \end{aligned}$$

□

Lemma 2.12. For $0 \leq \beta < 4$, we have the following estimates:

$$\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2} du \leq C \quad for \;\; 0 \leq \beta \leq 1, \tag{2.22}$$

$$\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2} du \leq C_\beta b^{\beta-1} \quad for \;\; 1 \leq \beta < 4, \tag{2.23}$$

where C_β is a constant that depends only on β.

Proof. The proof of this lemma is similar to that of [25] in the Robertson-Walker space-time. We present it for the reader convenience.

Proof of the first inequality (2.22).

Let β such that $0 \leq \beta \leq 1$:

$$\begin{aligned}
\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2} du &= \int_{\mathbb{R}^3} \frac{g^{1-\beta}\sqrt{s}}{v^0 u^0} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} \frac{(v^0u^0)^{\frac{1-\beta}{2}} (v^0u^0)^{\frac{1}{2}}}{v^0u^0} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} (v^0u^0)^{\frac{-\beta}{2}} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} e^{-|u|^2} du \\
&\leq C.
\end{aligned}$$

Proof of the second inequality (2.23).

• Let β such that $1 \leq \beta \leq 2$:

$$\begin{aligned}
\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2} du &= \int_{\mathbb{R}^3} \frac{\sqrt{s}}{v^0u^0} \frac{1}{g^{\beta-1}} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} \frac{1}{v^0u^0} \frac{b^{\beta-1}(v^0u^0)^{\frac{\beta-1}{2}}}{|v-u|^{\beta-1}} e^{-|u|^2} du \\
&\leq C b^{\beta-1} \int_{\mathbb{R}^3} \frac{1}{|v-u|^{\beta-1}} \frac{1}{(v^0u^0)^{\frac{2-\beta}{2}}} e^{-|u|^2} du \\
&\leq C b^{\beta-1} \int_{\mathbb{R}^3} \frac{1}{|v-u|^{\beta-1}} e^{-|u|^2} du \\
&\leq C_\beta b^{\beta-1} (1+|v|^2)^{-\frac{\beta-1}{2}} \\
&\leq C_\beta b^{\beta-1}.
\end{aligned}$$

• Let β such that $2 \leq \beta < 4$:

$$\begin{aligned}
\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2} du &= \int_{\mathbb{R}^3} \frac{\sqrt{s}}{v^0 u^0} \frac{1}{g^{\beta-1}} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} \frac{1}{\sqrt{v^0 u^0}} \frac{b^{\beta-1}(v^0 u^0)^{\frac{\beta-1}{2}}}{|v-u|^{\beta-1}} e^{-|u|^2} du \\
&\leq C \int_{\mathbb{R}^3} e^{-|u|^2} \frac{1}{|v-u|^{\beta-1}} (v^0 u^0)^{\frac{\beta-2}{2}} du \\
&\leq C b^{\beta-1} \int_{\mathbb{R}^3} \frac{(1+|v|^2)^{\frac{\beta-2}{4}} (1+|u|^2)^{\frac{\beta-2}{4}}}{|v-u|^{\beta-1}} e^{-|u|^2} du \\
&\leq C_\beta b^{\beta-1} (1+|v|^2)^{\frac{\beta-2}{4}} (1+|v|^2)^{-\frac{\beta-1}{2}} \\
&\leq C_\beta b^{\beta-1} (1+|v|^2)^{-\frac{\beta}{4}} \\
&\leq C_\beta b^{\beta-1}.
\end{aligned}$$

□

Lemma 2.13. For $0 \leq \beta < 3$ and $m \geq 0$, we have the following estimate:

$$\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} (1+|u|^2)^m e^{-|u|^2} du \leq C b^{\beta-1} \tag{2.24}$$

where C is a positive constant depending on β and m.

Proof. Let recall that $\vartheta_\phi = \frac{g\sqrt{s}}{v^0 u^0}$ and by (2.11), $\frac{1}{g} \leq \frac{b\sqrt{v^0 u^0}}{|v-u|}$.
By (2.11), we have

$$\int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} (1+|u|^2)^m e^{-|u|^2} du \leq \int_{\mathbb{R}^3} \frac{\sqrt{s}}{v^0 u^0} \frac{b^{\beta-1}(\sqrt{v^0 u^0})}{|v-u|^{\beta-1}} (1+|u|^2)^m e^{-}|u|^2 du.$$

Denoting by I the left hand side of (2.24), we obtain using (2.23)

$$\begin{aligned}
I &\leq C \int_{\mathbb{R}^3} \frac{b^{\beta-1}(v^0 u^0)^{\frac{\beta-2}{2}}}{|v-u|^{\beta-1}} (1+|u|^2)^m e - |u|^2 du \\
&\leq C \int_{\mathbb{R}^3} \frac{b^{\beta-1}(1+|v|^2)^{\frac{\beta-2}{4}} (1+|u|^2)^{\frac{\beta-2}{4}}}{|v-u|^{\beta-1}} (1+|u|^2)^m e^{-|u|^2} du \\
&\leq C b^{\beta-1} (1+|v|^2)^{\frac{\beta-2}{4}} \int_{\mathbb{R}^3} \frac{(1+|u|^2)^{\frac{\beta-2+4m}{4}}}{|v-u|^{\beta-1}} e^{-|u|^2} du \\
&\leq C b^{\beta-1 (1+|v|^2)^{\frac{\beta-2}{4}}} C_\beta (1+|v|^2)^{-\frac{\beta-1}{2}} \\
&\leq C_\beta b^{\beta-1} (1+|v|^2)^{-\frac{\beta}{4}} \\
&\leq C_\beta b^{\beta-1}.
\end{aligned}$$

□

Lemma 2.14. For $0 \leq \beta < 3$ and $m \geq 0$, we have the following estimate:

$$\int_{\mathbb{R}^3} g^{-\beta} (1+|u|^2)^m e^{-|u|^2} du \leq C b^{\beta} \tag{2.25}$$

where C is a positive constant depending only on β and m.

Proof. Following the proof of (2.24), we have

$$\begin{aligned}\int_{\mathbb{R}^3}(1+|u|^2)^m g^{-\beta}e^{-|u|^2}du &\leq \int_{\mathbb{R}^3}(1+|u|^2)^m\frac{b^\beta(v^0u^0)^{\frac{\beta}{2}}}{|v-u|^\beta}e^{-|u|^2}du\\ &\leq b^\beta(1+|v|^2)^{\frac{\beta}{4}}\int_{\mathbb{R}^3}\frac{(1+|u|^2)^{m+\frac{\beta}{4}}}{|v-u|^\beta}e^{-|u|^2}du\\ &\leq Cb^\beta(1+|v|^2)^{\frac{\beta}{4}}(1+|v|^2)^{-\frac{\beta}{2}}\\ &\leq Cb^\beta(1+|v|^2)^{\frac{-\beta}{2}}\\ &\leq Cb^\beta.\end{aligned}$$

□

2.2 Cutoff on the unit sphere

Let C be a positive real number, we defined a cutoff S_{ab} of the unit sphere S^2 by

$$S_{ab}=\{w\in S^2,\ \frac{|\omega^1||\bar{v}-\bar{u}|}{|\bar{\omega}||v^1-u^1|}\leq 1,\ \frac{1}{2}\frac{|v-u|}{s}\frac{|n\times w|^2+3|n|^2}{a^2(\omega^1)^2+b^2|\bar{w}|^2}\leq C\} \tag{2.26}$$

since $b^2(t)$ goes to ∞ as t goes to ∞. For fixed v and u, the restriction

$$\frac{1}{2}\frac{|v-u|}{s}\frac{|n\times w|^2+3|n|^2}{a^2(\omega^1)^2+b^2|\bar{w}|^2}\leq C$$

disappears for a large t. This cutoff depends on t and on pre-collisional momenta v and u.
We note that since $a^2(\omega^1)^2+b^2|\bar{\omega}|^2\leq b^2$

$$\frac{1}{2}\frac{|v-u|}{s}\frac{|n\times w|^2+3|n|^2}{a^2(\omega^1)^2+b^2|\bar{w}|^2}\geq\frac{1}{2}\frac{|v-u|}{s}\frac{|n\times w|^2+3|n|^2}{b^2}.$$

In the sequel, we will use the cutoff S_{ab} on the angular part of the scattering kernel. Henceforth, unless otherwise specified, the parameter ω will always belong to S_{ab}.

Lemma 2.15. Let v and u be given. Suppose that v' and u' are post-collisional momenta with a parameter $\omega\in S_{ab}$. If $a^2\leq b^2\leq 2a^2$, We have:

$$|v|^2+|u|^2-|v'|^2-|u'|^2\leq C. \tag{2.27}$$

Proof. We let $A=|v|^2+|u|^2-|v'|^2-|u'|^2$. Using the parametrization (1.56)-(1.57)-(1.59)-(1.60),

we have

$$
\begin{aligned}
A &= |v|^2 + |u|^2 - |v'|^2 - |u'|^2 \\
&= [(v^1)^2 + (v^2)^2 + (v^3)^2] + [(u^1)^2 + (u^2)^2 + (u^3)^2] \\
&- \left[(v'^1)^2 + (v'^2)^2 + (v'^3)^2\right] - \left[(u'^1)^2 + (u'^2)^2 + (u'^3)^2\right] \\
&= [(v^1)^2 + (v^2)^2 + (v^3)^2] + [(u^1)^2 + (u^2)^2 + (u^3)^2] \\
&- [\frac{(v^1)^2 + 2v^1u^1 + (u^1)^2}{4} + \frac{(v^1 + u^1)(a^2gn^0\omega^1)}{2r} + \frac{a^4g^2(n^0)^2(\omega^1)^2}{4r^2} \\
&+ \frac{(v^2)^2 + 2v^2u^2 + (u^2)^2}{4} + \frac{(v^2 + u^2)(b^2gn^0\omega^2)}{2r} + \frac{b^4g^2(n^0)^2(\omega^2)^2}{4r^2} \\
&+ \frac{(v^3)^2 + 2v^3u^3 + (u^3)^2}{4} + \frac{(v^3 + u^3)(b^2gn^0\omega^3)}{2r} + \frac{b^4g^2(n^0)^2(\omega^3)^2}{4r^2}] \\
&- [\frac{(v^1)^2 + 2v^1u^1 + (u^1)^2}{4} - \frac{(v^1 + u^1)(a^2gn^0\omega^1)}{2r} + \frac{a^4g^2(n^0)^2(\omega^1)^2}{4r^2} \\
&+ \frac{(v^2)^2 + 2v^2u^2 + (u^2)^2}{4} - \frac{(v^2 + u^2)(b^2gn^0\omega^2)}{2r} + \frac{b^4g^2(n^0)^2(\omega^2)^2}{4r^2} \\
&+ \frac{(v^3)^2 + 2v^3u^3 + (u^3)^2}{4} - \frac{(v^3 + u^3)(b^2gn^0\omega^3)}{2r} + \frac{b^4g^2(n^0)^2(\omega^3)^2}{4r^2}] \\
&= \frac{1}{2}(v^1)^2 + \frac{1}{2}(v^2)^2 + \frac{1}{2}(v^3)^2 - v^1u^1 - v^2u^2 - v^3u^3 + \frac{1}{2}(u^1)^2 + \frac{1}{2}(u^2)^2 + \frac{1}{2}(u^3)^2 \\
&- \frac{a^4g^2(n^0)^2(\omega^1)^2}{2r^2} - \frac{b^4g^2(n^0)^2(\omega^2)^2}{2r^2} - \frac{b^4g^2(n^0)^2(\omega^3)^2}{2r^2} \\
&= \frac{1}{2}|v-u|^2 - \frac{g^2(n^0)^2\left[a^4(\omega^1)^2 + b^4(\omega^2)^2 + b^4(\omega^3)^2)\right]}{2r^2} \\
&= \frac{1}{2}(v-u)^2 - \frac{1}{2}\frac{g^2(n^0)^2}{r^2}|(a^2\omega^1, b^2\bar{\omega})|^2. \qquad (2.28)
\end{aligned}
$$

Let us compute $(v^0 - u^0)^2$ and $(n^0)^2g^2$.

$$
\begin{aligned}
(v^0 - u^0)^2 &= \left(\sqrt{1 + a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2} - \sqrt{1 + a^{-2}(u^1)^2 + b^{-2}(u^2)^2 + b^{-2}(u^3)^2}\right)^2 \\
&= \left(\frac{a^{-2}((v^1)^2 - (u^1)^2) + b^{-2}(|\overline{v}|^2 - |\overline{u}|^2)}{n^0}\right)^2 \\
&= \left[\frac{(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))\,.\,(a^{-1}(u^1 + v^1), b^{-1}(\bar{v} + \bar{u}))}{n^0}\right]^2 \\
&= \frac{|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2|(a^{-1}(u^1 + v^1), b^{-1}(\bar{v} + \bar{u}))|^2 \cos^2\theta_0}{(n^0)^2}.
\end{aligned}
$$

By the relation (2.13) of g^2, we have

$$
\begin{aligned}
(n^0)^2g^2 &= -|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2 \cos^2\theta_0 \\
&+ (n^0)^2|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2. \qquad (2.29)
\end{aligned}
$$

By (2.28) and (2.29), we obtain

$$
\begin{aligned}
A &= \frac{1}{2r^2}[r^2|v-u|^2 - (n^0)^2|(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2 \\
&+ |(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2 \cos^2\theta_0 \\
&= \frac{1}{2r^2}[A_1 + |(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2 \cos^2\theta_0].
\end{aligned}
$$

Let us set

$$A_1 = r^2|v-u|^2 - (n^0)^2|(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2.$$

Since

$$r^2 = -(n.\omega)^2 + (n^0)^2\left(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right)$$

we have

$$\begin{aligned} A_1 &= \left[-(n.\omega)^2 + (n^0)^2\left(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right)\right]|v-u|^2 \\ &- (n^0)^2|(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2 \\ &= -(n.w)^2|v-u|^2 + (n^0)^2A_2 \end{aligned}$$

where

$$\begin{aligned} A_2 &= \left(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right)|v-u|^2 \\ &- |(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2 \\ &= \left(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2\right)\left(|v^1-u^1|^2 + |\bar{v}-\bar{u}|^2\right) \\ &- \left(a^4(\omega^1)^2 + b^4|\bar{\omega}|^2\right)\left(a^{-2}|v^1-u^1|^2 + b^{-2}|\bar{v}-\bar{u}|^2\right) \\ &= a^2\left(1-\left(\frac{a}{b}\right)^2\right)(\omega^1)^2|\bar{v}-\bar{u}|^2 + b^2\left(1-\left(\frac{b}{a}\right)^2\right)|\bar{\omega}|^2|v^1-u^1|^2. \end{aligned}$$

If we let $t = (\frac{a}{b})^2$, then $t \in]0,1[$. Since the parameter $\omega \in S_{ab}$, one has

$$\begin{aligned} A_2 &= b^2\left(t\,(1-t)\,(\omega^1)^2|\bar{v}-\bar{u}|^2 + \left(1-\frac{1}{t}\right)|\bar{\omega}|^2|v^1-u^1|^2\right) \\ &= b^2\frac{1-t}{t}\left[t^2(\omega^1)^2|\bar{v}-\bar{u}|^2 - |\bar{\omega}|^2|v^1-u^1|^2\right] \\ &\leq b^2\frac{1-t}{t}\left[(\omega^1)^2|\bar{v}-\bar{u}|^2 - |\bar{\omega}|^2|v^1-u^1|^2\right] \\ &\leq 0. \end{aligned}$$

Since $A_2 \leq 0$, we have

$$A_1 \leq -(n.\omega)^2|v-u|^2.$$

Thus

$$\begin{aligned} A &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[-(n.w)^2 + a^{-4}|n|^2b^4\cos^2\theta_0] \\ &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[-(n.w)^2 + (\frac{b}{a})^4|n|^2|w|^2] \\ &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[-(n.w)^2 + 4|n|^2|w|^2] \\ &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[|n\times\omega|^2 + 3|n|^2] \\ &\leq C \end{aligned}$$

since

$$-(n.\omega)^2 = |n\times\omega|^2 - |n|^2|\omega|^2 = |n\times\omega|^2 - |n|^2.$$

□

2.3 Estimates of the derivatives of the energy and the relative momentum

Lemma 2.16. The derivatives of v^0 with respect to v^i fulfill the following estimates:

$$|\partial_{v^1} v^0| \leq \frac{1}{a}, \tag{2.30}$$

$$|\partial_{v^i} v^0| \leq \frac{1}{b}, \quad for\ i = 2, 3. \tag{2.31}$$

Proof. Since $v^0 = \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|v|^2}$, we have

$$a^{-2}(v^1)^2 \leq (v^0)^2 \qquad and \qquad b^{-2}(v^i)^2 \leq (v^0)^2.$$

We compute the derivatives of v^0 as follows

$$\partial_{v^1} v^o = \frac{v^1}{a^2 v^0} \quad and \quad \partial_{v^i} v^0 = \frac{v^i}{b^2 v^0}, \ for\ i = 2, 3.$$

By the mass-shell assumption

$$|\frac{v^1}{a v^0}| \leq 1 \qquad and \qquad |\frac{v^i}{b v^0}| \leq 1, \qquad for\ i = 2, 3.$$

Thus

$$|\partial_{v^1} v^0| \leq \frac{1}{a} \qquad and \qquad |\partial_{v^i} v^0| \leq \frac{1}{b}, \qquad for\ i = 2, 3.$$

□

Lemma 2.17. The derivatives of g and $\sqrt{s}$ with respect to v^1 enjoy the following estimates:

$$|\partial_{v^1} g| \leq \frac{2u^0}{ag}, \tag{2.32}$$

$$|\partial_{v^1} \sqrt{s}| \leq \frac{2u^0}{a\sqrt{s}}. \tag{2.33}$$

Proof. We split g^2 as

$$g^2 = -2 + 2v^0 u^0 - 2\left[a^{-2} v^1 u^1 + b^{-2} v^2 u^2 + b^{-2} v^3 u^3\right]. \tag{2.34}$$

Since $\partial_{v^1} g^2 = 2g \partial_{v^1} g$, we get

$$\partial_{v^1} g = \frac{u^0}{ag}[\frac{v^1}{av^0} - \frac{u^1}{au^0}] \qquad then \qquad |\partial_{v^1} g| \leq \frac{2u^0}{ag}.$$

We split s as

$$s = 2 + 2v^0 u^0 - 2\left[a^{-2} v^1 u^1 + b^{-2} v^2 u^2 + b^{-2} v^3 u^3\right]. \tag{2.35}$$

Since $\partial_{v^1} s = \partial_{v^1}(\sqrt{s})^2 = 2\sqrt{s}\partial_{v^1}\sqrt{s}$, we get

$$\partial_{v^1}\sqrt{s} = \frac{u^0}{a\sqrt{s}}[\frac{v^1}{av^0} - \frac{u^1}{au^0}] \qquad then \qquad |\partial_{v^1}\sqrt{s}| \leq \frac{2u^0}{a\sqrt{s}}.$$

□

Lemma 2.18. The derivatives of g and $\sqrt{s}$ with respect to $v^i,\ i=2,3$ enjoy the following estimates:

$$|\partial_{v^i} g| \leq \frac{2u^0}{bg}, \qquad for\ i=2,3, \tag{2.36}$$

$$|\partial_{v^i} \sqrt{s}| \leq \frac{2u^0}{b\sqrt{s}}, \qquad for\ i=2,3. \tag{2.37}$$

Proof. Using the relation (2.34)

$$\partial_{v^i} g = \frac{u^0}{bg}[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}] \qquad then \qquad |\partial_{v^i} g| \leq \frac{2u^0}{bg}, \qquad for\ i=2,3.$$

Using the relation (2.35)

$$\partial_{v^i} \sqrt{s} = \frac{u^0}{b\sqrt{s}}[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}] \qquad then \qquad |\partial_{v^i} \sqrt{s}| \leq \frac{2u^0}{b\sqrt{s}}, \qquad for\ i=2,3.$$

□

Lemma 2.19. If we let $G := G(\omega,a,b) = a^2(\omega^1)^2 + b^2|\bar{\omega}|^2$ and $t^\alpha = (n_i\omega^i, n^0\omega)$ $\omega \in S^2$, then the derivative of $r = \sqrt{t_\alpha t^\alpha}$ with respect to v^1 satisfies

$$|\partial_{v^1} r| \leq \frac{(\frac{b^2}{a}+b)(v^0+u^0)}{\sqrt{(n^0)^2 G - (n.w)^2}}. \tag{2.38}$$

Proof. After expanding r, we have

$$\begin{aligned} \partial_{v^1} r = & \frac{u^0}{a\sqrt{(n^0)^2G(\omega,a,b)-(n.w)^2}}(\frac{v^1}{av^0}G(\omega,a,b) - \frac{(u.w)\omega^1}{u^0}a) \\ & + \frac{v^0}{a\sqrt{(n^0)^2G(\omega,a,b)-(n.w)^2}}(\frac{v^1}{av^0}G(\omega,a,b) - \frac{(v.w)\omega^1}{v^0}a). \end{aligned} \tag{2.39}$$

It is easy to see that

$$a^2 \leq G(\omega,a,b) \leq b^2.$$

By (2.39), We have

$$\begin{aligned} |\partial_{v^1} r| & \leq \frac{u^0}{a\sqrt{(n^0)^2G(\omega,a,b)-(n.w)^2}}(b^2+ba) + \frac{v^0}{a\sqrt{(n^0)^2G(\omega,a,b)-(n.w)^2}}(b^2+ba) \\ & \leq (\frac{b^2}{a}+b)\frac{u^0+v^0}{\sqrt{(n^0)^2G(\omega,a,b)-(n.w)^2}}. \end{aligned}$$

□

Lemma 2.20. The derivatives of $r = \sqrt{t_\alpha t^\alpha}$ with respect to v^i for $i=2,3$ satisfy

$$|\partial_{v^i} r| \leq \frac{2b(v^0+u^0)}{\sqrt{(n^0)^2G-(n.w)^2}} \qquad i=2,3. \tag{2.40}$$

Proof.

$$\partial_{v^i} r = \frac{u^0}{b\sqrt{(n^0)^2 G(\omega,a,b) - (n.w)^2}}\left(\frac{v^i}{bv^0}G(\omega,a,b) - \frac{(u.w)\omega^i}{u^0}b\right) + \frac{v^0}{b\sqrt{(n^0)^2 G(\omega,a,b) - (n.w)^2}}\left(\frac{v^i}{bv^0}G(\omega,a,b) - \frac{(v.w)\omega^i}{v^0}b\right), \quad i = 2,3. \tag{2.41}$$

Using the same method as in Lemma 2.19, we obtain

$$|\partial_{v^i} r| \leq 2b\frac{u^0 + v^0}{sqrt(n^0)^2 G(\omega,a,b) - (n.w)^2}, \qquad i = 2,3.$$

□

Lemma 2.21. The pre-collisional momenta v and u satisfy

$$|\frac{v^1}{av^0} - \frac{u^1}{au^0}| \leq \frac{1}{a}(1 + \frac{b^2}{a^2})|v-u|, \tag{2.42}$$

$$|\frac{v^i}{bv^0} - \frac{u^i}{bu^0}| \leq \frac{1}{b}(1 + \frac{b^2}{a^2})|v-u|, \quad i = 2,3. \tag{2.43}$$

Proof. In order to have (2.42), we can write

$$|\frac{v^1}{v^0} - \frac{u^1}{u^0}| = \frac{1}{v^0u^0}|u^0(v^1-u^1) + u^1(u^0-v^0)| \leq \frac{1}{v^0u^0}\left[|v-u|u^0 + |u||u^0-v^0|\right].$$

We now try to control $|u^0 - v^0|$.

One has

$$\begin{aligned} |(u^0)^2 - (u^0)^2| &= |(a^{-1}(u^1-v^1), b^{-1}(\bar{u}-\bar{v})).(a^{-1}n^1, b^{-1}\bar{n})| \\ &\leq |(a^{-1}(u^1-v^1), b^{-1}(\bar{u}-\bar{v}))||(a^{-1}n^1, b^{-1}\bar{n})| \\ &\leq a^{-2}|u-v||u+v|. \end{aligned}$$

In another hand, we easily have

$$v^0 + u^0 \geq b^{-1}(|v| + |u|).$$

Thus

$$|u^0 - v^0| = \frac{|(u^0)^2 - (v^0)^2|}{n^0} \leq \frac{a^{-2}|v-u||v+u|}{b^{-1}(|v|+|u|)} \leq \frac{b}{a^2}|v-u|,$$

$$|\frac{v^1}{v^0} - \frac{u^1}{u^0}| \leq |v-u|\left[\frac{u^0}{v^0u^0} + \frac{b}{a^2}\frac{|u|}{v^0u^0}\right].$$

Since $u^0 \geq b^{-1}|u|$ and $v^0 \geq 1$. These estimates lead to

$$|\frac{v^1}{v^0} - \frac{u^1}{u^0}| \leq |v-u|\left[\frac{u^0}{v^0u^0} + \frac{b^2}{a^2}\frac{u^0}{v^0u^0}\right] \leq (1 + \frac{b^2}{a^2})|v-u|.$$

Using the same method as in the proof of (2.42), we obtain the relation (2.43). □

Lemma 2.22. The partial derivatives of g and $\sqrt{s}$ with respect to v^1 satisfy the following estimates

$$|\partial_{v^1} g| \leq \frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}, \tag{2.44}$$

$$|\partial_{v^1}\sqrt{s}| \leq \frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}. \tag{2.45}$$

Proof. Using (2.11)-(2.42)-(2.43), we deduce that

$$\partial_{v^1} g = \frac{u^0}{ag}[\frac{v^1}{av^0} - \frac{u^1}{au^0}] \ and \ |\partial_{v^1} g| \leq \frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0},$$

$$\partial_{v^1}\sqrt{s} = \frac{u^0}{a\sqrt{s}}[\frac{v^1}{av^0} - \frac{u^1}{au^0}] \ and \ |\partial_{v^1}\sqrt{s}| \leq \frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}.$$

□

Lemma 2.23. The partial derivatives of g and $\sqrt{s}$ with respect to v^i, $i = 2, 3$ satisfy the following estimates

$$|\partial_{v^i} g| \leq \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}, \quad i = 2, 3, \tag{2.46}$$

$$\partial_{v^i}\sqrt{s}| \leq \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}, \quad i = 2, 3. \tag{2.47}$$

Proof. Using (2.11) and (2.42)-(2.43), we deduce that

$$\partial_{v^i} g = \frac{u^0}{bg}[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}] \ and \ |\partial_{v^i} g| \leq \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}, \qquad for \ i = 2, 3,$$

$$\partial_{v^i}\sqrt{s} = \frac{u^0}{b\sqrt{s}}[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}] \ and \ |\partial_{v^i}\sqrt{s}| \leq \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}, \qquad for \ i = 2, 3.$$

□

2.4 Estimates of the derivatives of the scattering kernel

2.4.1 Estimates of the derivatives of the scattering kernel generated by the Israel particles

Lemma 2.24. : We have the following estimates

$$|\partial_{v^i}(\frac{1}{v^0u^0\sqrt{s}})| \leq 2, \qquad for \ i = 1, 2, 3. \tag{2.48}$$

Proof. For $i = 1,\ 2$ or 3, a direct computation leads to

$$\begin{aligned}\partial_{v^i}(\frac{1}{v^0u^0\sqrt{s}}) &= \partial_{v^i}(\frac{1}{v^0})\frac{1}{u^0\sqrt{s}} + \frac{1}{v^0}\partial_{v^i}(\frac{1}{u^0})\frac{1}{\sqrt{s}} + \frac{1}{v^0u^0}\partial_{v^i}(\frac{1}{\sqrt{s}}) \\ &= -\frac{\partial_{v^i}(v^0)}{(v^0)^2}\frac{1}{u^0\sqrt{s}} - \frac{1}{v^0\sqrt{s}}\frac{\partial_{v^i}(u^0)}{(u^0)^2} - \frac{\partial_{v^i}(\sqrt{s})}{v^0u^0s}.\end{aligned} \tag{2.49}$$

For $i = 1$, by (2.6)-(2.30)-(2.33) we have

$$\begin{aligned}
|\partial_{v^1}(\frac{1}{v^0u^0\sqrt{S}})| &\leq \frac{1}{(v^0)^2u^0\sqrt{s}}|\partial_{v^1}(v^0)| + \frac{1}{v^0(u^0)^2\sqrt{s}}|\partial_{v^1}(u^0)| + \frac{1}{v^0u^0s}|\partial_{v^1}(\sqrt{s})| \\
&\leq \frac{1}{a(v^0)^2u^0\sqrt{s}} + \frac{2}{av^0s\sqrt{s}} \\
&\leq \frac{1}{av^0\sqrt{s}}\left[\frac{1}{v^0u^0} + \frac{2}{s}\right] \\
&\leq 2.
\end{aligned}$$

For $i = 2, 3$, by (2.6)-(2.31)-(2.37) we have

$$\begin{aligned}
|\partial_{v^i}(\frac{1}{v^0u^0\sqrt{s}})| &\leq \frac{1}{(v^0)^2u^0\sqrt{s}}|\partial_{v^i}(v^0)| + \frac{1}{v^0(u^0)^2\sqrt{s}}|\partial_{v^i}(u^0)| + \frac{1}{v^0u^0s}|\partial_{v^i}(\sqrt{s})| \\
&\leq \frac{1}{(v^0)^2u^0\sqrt{s}}\frac{1}{b} + \frac{1}{v^0u^0s}\frac{2u^0}{b\sqrt{s}} \\
&\leq \frac{1}{bv^0\sqrt{s}}\left[\frac{1}{v^0u^0} + \frac{2}{s}\right] \\
&\leq 2.
\end{aligned}$$

□

2.4.2 Estimates of the derivatives of the scattering kernel for hard potentials

In this part we take $\alpha = 0$ in (1.70) and we make an additional assumption

$$|\partial_g(\sigma(g,\omega))| \lesssim g^{-1-\beta}\sigma_0(\omega) \qquad with \qquad \beta \in [0,3]. \tag{2.50}$$

Lemma 2.25. Under assumptions (1.70)-(2.50) on the scattering kernel, we have the following estimate

$$|\partial_{v^1}[v_\phi\sigma(g,\omega)]| \leq ca^{-1}u^0(1 + g^{-\beta})\sigma_0(\omega). \tag{2.51}$$

Proof. For $i = 1, 2$ *or* 3, a direct computation leads to

$$\begin{aligned}
\partial_{v^i}(\vartheta_\phi\sigma(g,\omega)) &= \partial_{v^i}(g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\sigma)) \\
&= \partial_{v^i}(g)\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) + g\partial_{v^i}(\sqrt{s})\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) \\
&+ g\sqrt{s}\partial_{v^i}(\frac{1}{v^0})\frac{1}{u^0}\sigma(g,\omega) + g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\partial_{v^i}(\sigma(g,\omega)) \\
&= \partial_{v^i}(g)\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) \\
&+ g\partial_{v^i}(\sqrt{s})\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) \\
&- g\sqrt{s}\partial_{v^i}(v^0)\frac{1}{(v^0)^2}\frac{1}{u^0}\sigma(g,\omega) + g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\partial_{v^i}(g)\partial_g(\sigma(g,\omega)) \\
&= \left[(\partial_{v^i}g)\frac{\sqrt{s}}{v^0u^0} + (\partial_{v^i}\sqrt{s})\frac{g}{v^0u^0} - (\partial_{v^i}v^0)\frac{g\sqrt{s}}{v^0u^0}\right]\sigma(g,\omega) \\
&+ \frac{g\sqrt{s}}{v^0u^0}(\partial_{v^i}g)(\partial_g\sigma(g,\omega)).
\end{aligned} \tag{2.52}$$

And so we have

$$
\begin{aligned}
|\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))| \leq\, & |\partial_{v^i}(g)|\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) \\
& + g|\partial_{v^i}(\sqrt{s})|\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega) \\
& + g\sqrt{s}|\partial_{v^i}(v^0)|\frac{1}{(v^0)^2}\frac{1}{u^0}\sigma(g,\omega) \\
& + g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}|\partial_{v^i}(g)||\partial_g(\sigma(g,\omega))|. \qquad (2.53)
\end{aligned}
$$

We let $i = 1$ in (2.53).
By (2.7)-(2.30)-(2.44)-(2.45) and since $u^0 \geq 1$, $v^0 \geq 1$ and $|\partial_g(\sigma(g,\omega))| \leq g^{-1-\beta}\sigma_0(\omega)$,
the derivative of $\vartheta_\phi\sigma(g,\omega)$ with respect to v^1 is estimated as follows

$$
\begin{aligned}
|\partial_{v^1}[v_\phi\sigma(g,\omega)]| &\leq \frac{b}{a^2}(1+\frac{b^2}{a^2})\left[\frac{u^0}{\sqrt{v^0u^0}}(\sqrt{s}+g)\sigma(g,\omega) + |\partial_g\sigma(g,\omega)|\frac{u^0g\sqrt{s}}{\sqrt{v^0u^0}}\right] + \frac{1}{a}\frac{g\sqrt{s}}{(v^0)^2u^0}\sigma(g,\omega) \\
&\leq \frac{cu^0}{a}(\sigma(g,\omega) + g|\partial_g\sigma(g,\omega)|) \\
&\leq ca^{-1}u^0(1+g^{-\beta})\sigma_0(\omega).
\end{aligned}
$$

□

Lemma 2.26. Under assumptions (1.70)-(2.50) on the scattering kernel, we have the following estimates

$$
|\partial_{v^i}[v_\phi\sigma(g,\omega)]| \leq cb^{-1}u^0(1+g^{-\beta})\sigma_0(\omega), \qquad for\ i = 2,3. \qquad (2.54)
$$

Proof. The relation (2.52) leads to

$$
\begin{aligned}
\partial_{v^i}(\vartheta_\phi\sigma(g,\omega)) &= \partial_{v^i}(g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\sigma)) \\
&= \left[(\partial_{v^i}g)\frac{\sqrt{s}}{v^0u^0} + (\partial_{v^i}\sqrt{s})\frac{g}{v^0u^0} - (\partial_{v^i}v^0)\frac{g\sqrt{s}}{v^0u^0}\right]\sigma(g,\omega) \\
&+ \frac{g\sqrt{s}}{v^0u^0}(\partial_{v^i}g)(\partial_g\sigma(g,\omega)).
\end{aligned}
$$

By (2.53)-(2.7)-(2.31)-(2.46)-(2.47) and since $u^0 \geq 1$, $v^0 \geq 1$ and $|\partial_g(\sigma(g,\omega))| \leq g^{-1-\beta}\sigma_0(\omega)$, the derivatives of $\vartheta_\phi\sigma(g,\omega)$ with respect to v^i $i = 2,3$ are estimated as follows

$$
\begin{aligned}
|\partial_{v^i}[v_\phi\sigma(g,\omega)]| &\leq \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\frac{\sqrt{s}}{v^0u^0}\sigma(g,\omega) + \frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\frac{g}{v^0u^0}\sigma(g,\omega) \\
&+ \frac{1}{b}\frac{g\sqrt{s}}{(v^0)^2u^0}\sigma(g,\omega) + \frac{1}{b}(1+\frac{b^2}{a^2})\frac{g\sqrt{s}}{v^0u^0}|\partial_g\sigma(g,\omega)|u^0\sqrt{v^0u^0} \\
&\leq \frac{cu^0}{b}(\sigma(g,\omega) + g|\partial_g\sigma(g,\omega)|) \\
&\leq cb^{-1}u^0(1+g^{-\beta})\sigma_0(\omega).
\end{aligned}
$$

□

2.4.3 Estimates of the derivatives of the scattering kernel for soft potentials

In this part we consider the additional assumption (2.50).

Lemma 2.27. Under assumptions (1.71)-(2.50) on the scattering kernel, we have the following result

$$|\partial_{v^1}(\vartheta_\phi \sigma(g,\omega))| \leq Ca^{-1}u^0 g^{-\beta}\sigma_0(\omega). \tag{2.55}$$

Proof. By (2.52) we have

$$\begin{aligned}
\partial_{v^i}(\vartheta_\phi \sigma(g,\omega)) &= \partial_{v^i}(g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega)) \\
&= \left[(\partial_{v^i} g)\frac{\sqrt{s}}{v^0u^0} + (\partial_{v^i}\sqrt{s})\frac{g}{v^0u^0} - (\partial_{v^i} v^0)\frac{g\sqrt{s}}{v^0u^0}\right]\sigma(g,\omega) \\
&+ \frac{g\sqrt{s}}{v^0u^0}(\partial_{v^i} g)(\partial_g \sigma(g,\omega)).
\end{aligned}$$

By (2.7)-(2.30)-(2.44)-(2.45) and since $u^0 \geq 1$, $v^0 \geq 1$ and $|\partial_g(\sigma(g,\omega))| \leq g^{-1-\beta}\sigma_0(\omega)$, the derivative of $\vartheta_\phi\sigma(g,\omega)$ with respect to v^1 is estimated as follows

$$\begin{aligned}
|\partial_{v^1}(\vartheta_\phi \sigma(g,\omega))| &\leq 2\frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega) \\
&+ 2\sqrt{v^0u^0}\frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega) \\
&+ 4\sqrt{v^0u^0}\sqrt{v^0u^0}\frac{1}{a}\frac{1}{(v^0)^2}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega) \\
&+ 2\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}\frac{b}{a^2}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}g|\partial_g(\sigma(g,\omega))| \\
&\leq Cu^0a^{-1}g^{-\beta}\sigma_0(\omega) + Cu^0a^{-1}g^{-\beta}\sigma_0(\omega) \\
&+ Ca^{-1}g^{-\beta}\sigma_0(\omega) + Cu^0a^{-1}g|\partial_g(\sigma(g,\omega))| \\
&\leq Cu^0a^{-1}g^{-\beta}\sigma(\omega).
\end{aligned}$$

□

Lemma 2.28. Under assumptions (1.71)-(2.50) of the scattering kernel, we have the following results

$$|\partial_{v^i}(\vartheta_\phi \sigma(g,\omega))| \leq Cb^{-1}u^0 g^{-\beta}\sigma_0(\omega), \qquad for\ i = 2,3. \tag{2.56}$$

Proof. The relation (2.52) leads to

$$\begin{aligned}
\partial_{v^i}(\vartheta_\phi \sigma(g,\omega)) &= \partial_{v^i}(g\sqrt{s}\frac{1}{v^0}\frac{1}{u^0}\sigma(g,\omega)) \\
&= \left[(\partial_{v^i} g)\frac{\sqrt{s}}{v^0u^0} + (\partial_{v^i}\sqrt{s})\frac{g}{v^0u^0} - (\partial_{v^i} v^0)\frac{g\sqrt{s}}{v^0u^0}\right]\sigma(g,\omega) \\
&+ \frac{g\sqrt{s}}{v^0u^0}(\partial_{v^i} g)(\partial_g \sigma(g,\omega)).
\end{aligned}$$

By (2.53)-(2.7)-(2.31)-(2.46)-(2.47) and since $u^0 \geq 1$, $v^0 \geq 1$ and $|\partial_g(\sigma(g,\omega))| \leq g^{-1-\beta}\sigma_0(\omega)$, the derivatives of $\vartheta_\phi\sigma(g,\omega)$ with respect to v^1 are estimated as follows

$$\begin{aligned}
|\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))| &\leq 2\frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega)\\
&+ 2\sqrt{v^0u^0}\frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega)\\
&+ 4\sqrt{v^0u^0}\sqrt{v^0u^0}\frac{1}{b}\frac{1}{(v^0)^2}\frac{1}{u^0}g^{-\beta}\sigma_0(\omega)\\
&+ 2\sqrt{v^0u^0}\frac{1}{v^0}\frac{1}{u^0}\frac{1}{b}(1+\frac{b^2}{a^2})u^0\sqrt{v^0u^0}g|\partial_g(\sigma(g,\omega))|\\
&\leq Cu^0b^{-1}g^{-\beta}\sigma_0(\omega) + Cu^0b^{-1}g^{-\beta}\sigma_0(\omega)\\
&+ Cb^{-1}g^{-\beta}\sigma_0(\omega) + Cu^0b^{-1}g|\partial_g(\sigma(g,\omega))|\\
&\leq Cu^0b^{-1}g^{-\beta}\sigma(\omega).
\end{aligned}$$

□

2.5 Estimates of the derivatives of the post-collisional momenta

2.5.1 For the first parametrization

We consider the parametrization of post-collisional momenta (1.56)-(1.57) introduced in [26].
Let's recall that $r \geq \sqrt{s}(G(\omega,a,b))^{\frac{1}{2}}$ and $\sqrt{s} \geq max\left(\sqrt{\frac{v^0}{u^0}},\sqrt{\frac{u^0}{v^0}}\right)$, with
$G(\omega,a,b) = a^2(w^1)^2 + b^2|\overline{w}|^2$ and $r = \sqrt{t_\alpha t^\alpha}$.
We recall that the first coordinate of the post-collisional momentum v' reads

$$v'^1 = \frac{v^1+u^1}{2} + \frac{a^2g}{2}\frac{n^0w^1}{r}.$$

For $i = 1, 2$ or 3, straightforward computations lead to following relations

$$\partial_{v^i}v'^1 = \frac{\delta^{i1}}{2} + \frac{a^2(\partial_{v^i}g)}{2}\frac{n^0\omega^1}{r} + \frac{a^2g}{2}\frac{(\partial_{v^i}v^0)\omega^1}{r} - \frac{a^2g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i}r). \tag{2.57}$$

We bound the main terms of (2.57) as follows

$$|\frac{a^2g}{2}\frac{(\partial_{v^1}v^0)\omega^1}{r}| \leq \frac{a\sqrt{v^0u^0}}{r},$$

$$|\frac{a^2g}{2}\frac{(\partial_{v^i}v^0)\omega^1}{r}| \leq \frac{a^2\sqrt{v^0u^0}}{br}, \qquad i=2,3,$$

$$|\frac{a^2(\partial_{v^1}g)}{2}\frac{n^0\omega^1}{r}| \leq \frac{a^2}{2}\frac{b}{a^2}(1+3\frac{b^2}{a^2})u^0\sqrt{v^0u^0}(v^0+u^0)\frac{1}{r},$$

$$|\frac{a^2(\partial_{v^i}g)}{2}\frac{n^0\omega^1}{r}| \leq \frac{a^2}{2}\frac{1}{b}(1+3\frac{b^2}{a^2})u^0\sqrt{v^0u^0}(v^0+u^0)\frac{1}{r}, \qquad i=2,3,$$

$$|\frac{a^2g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^1}r)| \leq \frac{a^2}{r^3}\sqrt{v^0u^0}(n^0)^2(\frac{b^2}{a}+b),$$

$$|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i}r)| \leq \frac{2ba^2}{r^3}\sqrt{v^0u^0}(n^0)^2.$$

The second and the third coordinates of the post-collisional momentum v' read

$$v'^k = \frac{v^k+u^k}{2} + \frac{b^2 g}{2}\frac{n^0 w^k}{r}, \;\; (k=2,3).$$

For $i = 1, 2$ or 3, straightforward computations lead to following relations

$$\partial_{v^i}v'^k = \frac{\delta^{ik}}{2} + \frac{b^2(\partial_{v^i}g)}{2}\frac{n^0\omega^k}{r} + \frac{b^2 g}{2}\frac{(\partial_{v^i}v^0)\omega^k}{r} - \frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i}r), \;\; (k=2,3). \tag{2.58}$$

We bound the main terms of (2.58) as follows

$$|\frac{b^2(\partial_{v^1}g)}{2}\frac{n^0\omega^k}{r}| \leq \frac{b^2}{2}\frac{b}{a^2}(1+3\frac{b^2}{a^2})u^0\sqrt{v^0u^0}(v^0+u^0)\frac{1}{r},$$

$$|\frac{b^2(\partial_{v^i}g)}{2}\frac{n^0\omega^k}{r}| \leq \frac{b^2}{2}\frac{1}{b}(1+3\frac{b^2}{a^2})u^0\sqrt{v^0u^0}(v^0+u^0)\frac{1}{r}, \qquad i=2,3,$$

$$|\frac{b^2 g}{2}\frac{(\partial_{v^1}v^0)\omega^k}{r}| \leq \frac{b^2}{ar}\sqrt{v^0u^0},$$

$$|\frac{b^2 g}{2}\frac{(\partial_{v^i}v^0)\omega^k}{r}| \leq \frac{b}{r}\sqrt{v^0u^0}, \qquad i=2,3,$$

$$|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^1}r)| \leq \frac{b^2}{r^3}\sqrt{v^0u^0}(u^0+v^0)^2(\frac{b^2}{a}+b)$$

$$|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i}r)| \leq \frac{2b^3}{r^3}\sqrt{v^0u^0}(n^0)^2 \qquad i=2,3.$$

In the propositions below, we are going to collect estimates on each of the following terms:

$$|\partial_{v^1}v'^1|, \qquad |\partial_{v^i}v'^1|, \; (i=2,3), \qquad |\partial_{v^1}v'^k|, \; (k=2,3), \qquad and \qquad |\partial_{v^i}v'^k|, \; (i=2,3 \; and \; k=2,3).$$

Proposition 2.1. Consider the first parametrization (1.56)-(1.57). We have the following estimate:

$$|\partial_{v^1}v'^1| \leq Cv^0(u^0)^4 \tag{2.59}$$

where C does not depend on a or b.

Proof. The following estimates holds; thanks to (2.30)-(2.32)-(2.38),

$$|\frac{a^2(\partial_{v^1}g)}{2}\frac{n^0\omega^1}{r}| \leq \frac{b}{a^2}(1+3\frac{b^2}{a^2})u^0\sqrt{v^0u^0}(v^0+u^0)\frac{1}{r} \leq \frac{b}{2a}(1+3\frac{b^2}{a^2})(u^0)^2(v^0+u^0),$$

$$|\frac{a^2 g}{2}\frac{(\partial_{v^1}v^0)\omega^1}{r}| \leq \frac{a}{r}\sqrt{v^0u^0} \leq \frac{a}{\sqrt{G(\omega,a,b)}}u^0 \leq u^0,$$

$$|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^1}r)| \leq \frac{a^2}{2}2\sqrt{v^0u^0}(v^0+u^0)(\frac{b^2}{a}+b)(v^0+u^0)\frac{1}{r}\frac{1}{r^2} \leq (\frac{b^2}{a^2})\frac{(u^0)^2}{v^0}(v^0+u^0)^2.$$

We combine the above estimates to obtain the desired result. □

Proposition 2.2. Consider the first parametrization (1.56)-(1.57). We have the following estimates:

$$|\partial_{v^i} v'^1| \leq C v^0 (u^0)^4 \qquad for\ i = 2, 3 \tag{2.60}$$

where C does not depend on a or b.

Proof. The following estimates holds; thanks to (2.31)-(2.36)-(2.40),

$$|\frac{a^2(\partial_{v^i} g)}{2}\frac{n^0\omega^1}{r}| \leq \frac{a^2}{2}\frac{1}{b}(1+3\frac{b^2}{a^2})u^0\sqrt{u^0v^0}(u^0+v^0)\frac{1}{r} \leq \frac{a}{2b}(u^0)^2 n^0,$$

$$|\frac{a^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^1}{r}| \leq \frac{a^2}{br}\sqrt{v^0u^0} \leq \frac{a}{b}u^0 \leq u^0,$$

$$|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i} r)| \leq \frac{2a^2 b}{r^3}\sqrt{v^0u^0}(n^0)^2 \leq \frac{2b}{a}\frac{(u^0)^2}{v^0}(n^0)^2.$$

We combine the above estimates to obtain the desired result. □

Proposition 2.3. Consider the first parametrization (1.56)-(1.57). We have the following estimates:

$$|\partial_{v^1} v'^k| \leq C v^0 (u^0)^4, \qquad for\ k = 2, 3 \tag{2.61}$$

where C does not depend on a or b.

Proof. The following estimates holds; thanks to (2.30)-(2.32)-(2.38),

$$|\frac{b^2(\partial_{v^1} g)}{2}\frac{n^0\omega^1}{r}| \leq \frac{b^2}{2}\frac{b}{a^2}(1+3\frac{b^2}{a^2})u^0\sqrt{u^0v^0}(v^0+u^0) \leq \frac{2b^3(u^0)^2(n^0)}{2a\sqrt{G(\omega,a,b)}} \leq \frac{b^3}{2a^3}(u^0)^2(n^0),$$

$$|\frac{b^2 g}{2}\frac{(\partial_{v^1} v^0)\omega^k}{r}| \leq \frac{b^2}{ar}\sqrt{v^0u^0} \leq \frac{b^2}{a\sqrt{G(\omega,a,b)}}u^0 \leq \frac{b^2}{a^2}u^0 \leq u^0,$$

$$|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^1} r)| \leq \frac{b^2}{r^3}\sqrt{v^0u^0}(n^0)^2(\frac{b^2}{a}+b) \leq (\frac{b^4}{a^4}+\frac{b^3}{a^3})\frac{(u^0)^2}{v^0}(n^0)^2.$$

We combine the above estimates to obtain the desired result. □

Proposition 2.4. Consider the first parametrization (1.56)-(1.57). We have the following estimates:

$$|\partial_{v^i} v'^k| \leq C v^0 (u^0)^4, \qquad for\ i = 2, 3 \qquad and \qquad for\ k = 2, 3 \tag{2.62}$$

where C does not depend on a or b.

Proof. The following estimates holds; thanks to (2.31)-(2.36)-(2.40),

$$|\frac{b^2(\partial_{v^i} g)}{2}\frac{n^0\omega^k}{r}| \leq \frac{b^2}{2}\frac{1}{b}(1+3\frac{b^2}{a^2})u^0\sqrt{u^0v^0}(u^0+v^0)\frac{1}{r} \leq \frac{b}{2a}(1+3\frac{b^2}{a^2})(u^0)^2(v^0+u^0),$$

$$|\frac{b^2 g}{2}\frac{(\partial_{v^i} n^0)\omega^k}{r}| \leq \frac{b}{r}\sqrt{v^0u^0} \leq \frac{b}{\sqrt{G(\omega,a,b)}}u^0 \leq \frac{b}{a}u^0,$$

$$|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i} r)| \leq \frac{2b^3}{r^3}\sqrt{v^0u^0}(v^0+u^0)^2 \leq \frac{2b^3}{a^3}\frac{(u^0)^2}{v^0}(u^0+v^0)^2.$$

We combine the above estimates to obtain the desired result. □

2.5.2 For the second parametrization

We consider the parametrization (1.61)-(1.62) of the post-collisional momenta and we respectively compute the quantities

$$\partial_{v^1}v'^1, \qquad \partial_{v^i}v'^1,\ (i=2,3), \qquad \partial_{v^1}v'^k,\ (k=2,3), \qquad and \qquad \partial_{v^i}v'^k,\ (i=2,3;\ k=2,3)$$

and control each of them afterwards.
First of all, we make some helpful and obvious statements for further application.
For $\omega=(\omega^1,\bar{w})\in S_{ab}$, we have

$$|\omega^i|\le|\omega|=1, \qquad for\ i=1,2,3. \tag{2.63}$$

For $n=(n^i)=(v^i+u^i)$, we have

$$|n^i|=|v^i+u^i|\le|v+u|, \qquad for\ i=1,2,3. \tag{2.64}$$

The following relation holds

$$|(a^{-1}n^1,b^{-1}\bar{n}).\omega|\le|(a^{-1}n^1,b^{-1}\bar{n})||\omega|\le a^{-1}|n|=a^{-1}|v+u|, \qquad for\ i=1,2,3. \tag{2.65}$$

Since $|(a^{-1}n^1,b^{-1}\bar{n})|\ge b^{-1}|n|$, we have

$$\frac{1}{|(a^{-1}n^1,b^{-1}\bar{n})|}\le\frac{1}{b^{-1}|v+u|}. \tag{2.66}$$

Proposition 2.5. With the parametrization (1.61)-(1.62) of v', we have the following estimate

$$|\partial_{v^1}v'^1|\le C\left(\frac{bv^0}{|v-u|}+\frac{bv^0}{|v+u|}+\frac{b^2(v^0)^2}{|v-u|^2}\right)(u^0)^3 \tag{2.67}$$

where the constants C do not depend on a and b.

Proof. By a direct computation we have

$$\begin{aligned}
\partial_{v^1}v'^1 &= \frac{1}{2}+\frac{a}{2}(\partial_{v^1}g)[(w^1-\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1+\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1]\\
&+\frac{ag}{2}[-a^{-1}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}-\frac{a^{-2}n^1w^1}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}+\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-3}(n^1)^2\\
&+\frac{\partial_{v^1}v^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1-\frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1\\
&+\frac{n^0}{\sqrt{s}}\frac{a^{-2}n^1w^1}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}-\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-3}(n^1)^2+a^{-1}\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}].
\end{aligned}$$

We can state that

$$|\partial_{v^1}v'^1|\le\frac{1}{2}+|J_1|+|J_2|+|J_3|+|J_4|+|J_5|+|J_6|+|J_7|+|J_8|+|J_9|+|J_{10}|+|J_{11}|$$

where

$$
\begin{aligned}
J_1 &= \frac{a}{2}(\partial_{v^1} g) w^1, \\
J_2 &= \frac{1}{2}(\partial_{v^1} g)\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1), \\
J_3 &= \frac{1}{2}(\partial_{v^1} g)\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1), \\
J_4 &= \frac{g}{2}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}, \\
J_5 &= \frac{g}{2}\frac{a^{-1}(v^1+u^1)w^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}, \\
J_6 &= g\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}a^{-2}(v^1+u^1)^2, \\
J_7 &= \frac{g}{2}\frac{\partial_{v^1}v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1), \\
J_8 &= \frac{g}{2}\frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1), \\
J_9 &= \frac{g}{2}\frac{n^0}{\sqrt{s}}\frac{a^{-1}(v^1+u^1)w^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}, \\
J_{10} &= g\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}a^{-2}(v^1+u^1)^2, \\
J_{11} &= \frac{g}{2}\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}.
\end{aligned}
$$

Let's control each of the eleven terms.

Using (2.32), (2.63) and (2.11), J_1 is controlled by

$$|J_1| \leq b\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq \frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.32), (2.11), (2.64),(2.65) and (2.66), J_2 is controlled by

$$|J_2| \leq \frac{b^3}{a^2}\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq 2\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.32),(2.64),(2.65), (2.66) and (2.11), J_3 is controlled by

$$|J_3| \leq \frac{b^4}{a^2}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} \leq 4\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.7), (2.65) and (2.66), J_4 is controlled by

$$|J_4| \leq \frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.63), (2.64) and (2.66), J_5 is controlled by

$$|J_5| \leq \frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7),(2.64), (2.65) and (2.66), J_6 is controlled by

$$|J_6| \leq \frac{b^4}{a^3}\frac{\sqrt{v^0u^0}}{|v+u|} \leq 2\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using the relation (2.7), (2.30), (2.11), (2.64), (2.65) and (2.66), J_7 is controlled by

$$|J_7| \leq \frac{b^3}{a^2}\frac{v^0u^0}{|v-u|} \leq 2\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.6), (2.33), (2.11), (2.64), (2.65) and (2.66), J_8 is controlled by

$$|J_8| \leq \frac{b^2}{a^2}\frac{u^0(v^0+u^0)}{s} \leq 4\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.6), (2.63), (2.64) and (2.66), J_9 is controlled by

$$|J_9| \leq \frac{1}{2}\frac{b^2}{a}\frac{v^0+u^0}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.64), (2.65) and (2.66), J_{10} is controlled by

$$|J_{10}| \leq \frac{b^4}{a^3}\frac{v^0+u^0}{|v+u|} \leq 4\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.65) and (2.66), J_{11} is controlled by

$$|J_{11}| \leq \frac{1}{2}\frac{b^2}{a}\frac{v^0+u^0}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

In virtue of the above estimates, after some rearrangements, and the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result. □

Proposition 2.6. With the parametrization (1.61)-(1.62) of v', we have the following estimates

$$|\partial_{v^i}v'^1| \leq C(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2})(u^0)^3, \qquad for\ i=2,3 \tag{2.68}$$

where the constants C do not depend on a and b.

Proof. By a direct computation we have

$$\begin{aligned}
\partial_{v^i}v'^1 &= \frac{a}{2}(\partial_{v^i}g)[(w^1 - \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1] \\
&+ \frac{ag}{2}[0 - \frac{a^{-1}b^{-1}n^1w^i}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-1}b^{-2}n^1n^i + \frac{\partial_{v^i}v^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1 \\
&- \frac{n^0}{s}(\partial_{v^i}\sqrt{s})\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}a^{-1}n^1 + \frac{n^0}{\sqrt{s}}\frac{a^{-1}b^{-1}n^1w^i}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} \\
&- \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-1}b^{-2}n^1n^i].
\end{aligned}$$

We can state that

$$|\partial_{v^i} v'^1| \leq |M_1| + |M_2| + |M_3| + |M_4| + |M_5| + |M_6| + |M_7| + |M_8| + |M_9|$$

where

$$\begin{aligned}
M_1 &= \frac{a}{2}(\partial_{v^i} g) w^1,\\
M_2 &= \frac{1}{2}(\partial_{v^i} g)\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1),\\
M_3 &= \frac{1}{2}(\partial_{v^i} g)\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1),\\
M_4 &= \frac{g}{2}\frac{b^{-1}(v^1+u^1)w^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2},\\
M_5 &= g\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}b^{-2}(v^1+u^1)(v^i+u^i),\\
M_6 &= \frac{g}{2}\frac{\partial_{v^i} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1),\\
M_7 &= \frac{g}{2}\frac{n^0}{s}(\partial_{v^i}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^1+u^1),\\
M_8 &= \frac{g}{2}\frac{n^0}{\sqrt{s}}\frac{b^{-1}(v^1+u^1)w^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2},\\
M_9 &= g\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}b^{-2}(v^1+u^1)(v^i+u^i).
\end{aligned}$$

Let's control each of the nine terms.

Using (2.36), (2.63) and (2.11), M_1 is controlled by

$$|M_1| \leq a\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq \frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.36), (2.11), (2.64),(2.65) and (2.66), M_2 is controlled by

$$|M_2| \leq \frac{b^2}{a}\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.36), (2.11), (2.64),(2.65) and (2.66), M_3 is controlled by

$$|M_3| \leq \frac{b^3}{a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} \leq 2\sqrt{2}\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.7), (2.63), (2.64) and (2.66), M_4 is controlled by

$$|M_4| \leq b\frac{\sqrt{v^0u^0}}{|v+u|} \leq \frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.64), (2.65) and (2.66), M_5 is controlled by

$$|M_5| \leq 2\frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} \leq 2\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.11), (2.64), (2.65), (2.66) and (2.31), M_6 is controlled by

$$|M_6| \leq \frac{b^2}{a}\frac{v^0u^0}{|v-u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.6), (2.37), (2.64), (2.65) and (2.66), M_7 is controlled by

$$|M_7| \leq \frac{b}{a}\frac{u^0(v^0+u^0)}{s} \leq 2\sqrt{2}\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.6), (2.63), (2.64) and (2.66), M_8 is controlled by

$$|M_8| \leq \frac{1}{2}\frac{b(v^0+u^0)}{|v+u|} \leq \frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.64), (2.65) and (2.66), M_9 is controlled by

$$|M_9| \leq \frac{b(v^0+u^0)}{|v+u|} \leq 2\frac{bv^0(u^0)^3}{|v+u|}.$$

In virtue of the above estimates, after some rearrangements, and the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result. □

Proposition 2.7. With the parametrization (1.61)-(1.62) of v', we have the following estimates

$$|\partial_{v^1}v'^k| \leq C(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2})(u^0)^3, \qquad k=2,3 \tag{2.69}$$

where the constants C do not depend on a and b.

Proof. By a direct computation we have

$$\begin{aligned}
\partial_{v^1}v'^k &= \frac{b}{2}(\partial_{v^1}g)[(w^k - \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k] \\
&+ \frac{bg}{2}[0 - \frac{a^{-1}b^{-1}n^kw^1}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} \\
&+ \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-2}b^{-1}n^1n^k + \frac{\partial_{v^1}v^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k \\
&- \frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k + \frac{n^0}{\sqrt{s}}\frac{a^{-1}b^{-1}n^kw^1}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} \\
&- \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2a^{-2}b^{-1}n^1n^k + 0].
\end{aligned}$$

We can state that

$$|\partial_{v^1}v'^k| \leq |D_1| + |D_2| + |D_3| + |D_4| + |D_5| + |D_6| + |D_7| + |D_8| + |D_9|$$

where

$$\begin{aligned}
D_1 &= \frac{b}{2}(\partial_{v^1} g) w^k, \\
D_2 &= \frac{1}{2}(\partial_{v^1} g)\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k + u^k), \\
D_3 &= \frac{1}{2}(\partial_{v^1} g)\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k + u^k), \\
D_4 &= \frac{g}{2}\frac{a^{-1}(v^k + u^k)w^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}, \\
D_5 &= g\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}a^{-2}(v^1 + u^1)(v^k + u^k), \\
D_6 &= \frac{g}{2}\frac{\partial_{v^1} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k + u^k), \\
D_7 &= \frac{g}{2}\frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k + u^k), \\
D_8 &= \frac{g}{2}\frac{n^0}{\sqrt{s}}\frac{a^{-1}(v^k + u^k)w^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}, \\
D_9 &= g\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}a^{-2}(v^1 + u^1)(v^k + u^k)),
\end{aligned}$$

Let's control each of the nine terms.

Using (2.32), (2.11) and (2.63), D_1 is controlled by

$$|D_1| \leq \frac{b^2}{a}\frac{u^0\sqrt{v^0 u^0}}{|v-u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.32), (2.11), (2.64), (2.65) and (2.66), D_2 is controlled by

$$|D_2| \leq \frac{b^3}{a^2}\frac{u^0\sqrt{v^0 u^0}}{|v-u|} \leq 2\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.32), (2.11), (2.64), (2.65) and (2.66), D_3 is controlled by

$$|D_3| \leq \frac{b^4}{a^2}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} \leq 4\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.7), (2.63), (2.64) and (2.66), D_4 is controlled by

$$|D_4| \leq \frac{b^2}{a}\frac{\sqrt{v^0 u^0}}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.64), (2.65) and (2.66), D_5 is controlled by

$$|D_5| \leq 2\frac{b^4}{a^3}\frac{\sqrt{v^0 u^0}}{|v+u|} \leq 4\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.11), (2.30), (2.64), (2.65) and (2.66), D_6 is controlled by

$$|D_6| \leq \frac{b^3}{a^2}\frac{v^0 u^0}{|v-u|} \leq 2\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.6), (2.11), (2.33), (2.64), (2.65) and (2.66), D_7 is controlled by

$$|D_7| \leq \frac{b^2}{a^2}\frac{u^0(v^0+u^0)}{s} \leq 4\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.6), (2.63), (2.64) and (2.66), D_8 is controlled by

$$|D_8| \leq \frac{1}{2}\frac{b^2}{a}\frac{v^0+u^0}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.64), (2.65) and (2.66), D_9 is controlled by

$$|D_9| \leq \frac{b^4}{a^3}\frac{v^0+u^0}{|v+u|} \leq 4\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

In virtue of the above estimates, after some rearrangements, and the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result. □

Proposition 2.8. With the parametrization (1.61)-(1.62) of v', we have the following estimates

$$|\partial_{v^i}v'^k| \leq C(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2})(u^0)^3, \quad for\ i=2,3\ \ and\ \ for\ k=2,3 \tag{2.70}$$

where the constants C do not depend on a and b.

Proof. By a direct computation we have

$$\begin{aligned}
\partial_{v^i}v'^k =& \frac{\delta^{ik}}{2} + \frac{b}{2}(\partial_{v^i}g)[(w^k - \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k] \\
&+ \frac{bg}{2}[-\delta^{ik}b^{-1}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} - \frac{b^{-2}n^kw^i}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2b^{-3}n^kn^i \\
&+ \frac{\partial_{v^i}v^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k - \frac{n^0}{s}(\partial_{v^i}\sqrt{s})\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}b^{-1}n^k + \frac{n^0}{\sqrt{s}}\frac{b^{-2}n^kw^i}{|(a^{-1}n^1,b^{-1}\bar{n})|^2} \\
&- \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^4}2b^{-3}n^kn^i + \delta^{ik}b^{-1}\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1,b^{-1}\bar{n}).w}{|(a^{-1}n^1,b^{-1}\bar{n})|^2}].
\end{aligned}$$

We can state that

$$|\partial_{v^i}v'^k| \leq \frac{\delta^{ik}}{2} + |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| + |A_7| + |A_8| + |A_9| + |A_{10}| + |A_{11}|$$

where

$$
\begin{aligned}
A_1 &= \frac{b}{2}(\partial_{v^i} g)w^k,\\
A_2 &= \frac{1}{2}(\partial_{v^i} g)\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k+u^k),\\
A_3 &= \frac{1}{2}(\partial_{v^i} g)\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k+u^k),\\
A_4 &= \frac{g}{2}\delta^{ik}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2},\\
A_5 &= \frac{g}{2}\frac{b^{-1}(v^k+u^k)w^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2},\\
A_6 &= g\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}b^{-2}(v^k+u^k)(v^i+u^i),\\
A_7 &= \frac{g}{2}\frac{\partial_{v^i} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k+u^k),\\
A_8 &= \frac{g}{2}\frac{n^0}{s}(\partial_{v^i}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}(v^k+u^k),\\
A_9 &= \frac{g}{2}\frac{n^0}{\sqrt{s}}\frac{b^{-1}(v^k+u^k)w^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2},\\
A_{10} &= g\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}b^{-2}(v^k+u^k)(v^i+u^i),\\
A_{11} &= \frac{g}{2}\delta^{ik}\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}.
\end{aligned}
$$

Let's control each of the eleven terms.

Using (2.36), (2.63) and (2.11), A_1 is controlled by

$$|A_1| \leq b\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq \frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.36), (2.11), (2.64),(2.65) and (2.66), A_2 is controlled by

$$|A_2| \leq \frac{b^2}{a}\frac{u^0\sqrt{v^0u^0}}{|v-u|} \leq \frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.36), (2.11), (2.64),(2.65) and (2.66), A_3 is controlled by

$$|A_3| \leq \frac{b^3}{a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} \leq 2\sqrt{2}\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.7), (2.65) and (2.66), A_4 is controlled by

$$|A_4| \leq \frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7), (2.63), (2.64) and (2.66), A_5 is controlled by

$$|A_5| \leq b\frac{\sqrt{v^0u^0}}{|v+u|} \leq \frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.7),(2.64), (2.65) and (2.66), A_6 is controlled by

$$|A_6| \leq 2\frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} \leq 2\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using the relation (2.7), (2.31), (2.11), (2.64), (2.65) and (2.66), A_7 is controlled by

$$|A_7| \leq \frac{b^2}{a}\frac{v^0u^0}{|v-u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v-u|}.$$

Using (2.6), (2.37), (2.11), (2.64), (2.65) and (2.66), A_8 is controlled by

$$|A_8| \leq \frac{b}{a}\frac{u^0(v^0+u^0)}{s} \leq 2\sqrt{2}\frac{b^2(v^0)^2(u^0)^3}{|v-u|^2}.$$

Using (2.6), (2.63), (2.64) and (2.66), A_9 is controlled by

$$|A_9| \leq \frac{1}{2}\frac{b(v^0+u^0)}{|v+u|} \leq \frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.64), (2.65) and (2.66), A_{10} is controlled by

$$|A_{10}| \leq \frac{b^2}{a}\frac{v^0+u^0}{|v+u|} \leq 2\sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

Using (2.6), (2.65) and (2.66), A_{11} is controlled by

$$|A_{11}| \leq \frac{1}{2}\frac{b^2}{a}\frac{v^0+u^0}{|v+u|} \leq \sqrt{2}\frac{bv^0(u^0)^3}{|v+u|}.$$

In virtue of the above estimates, after some rearrangements, and the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result. □

Remark 2.1. Similar estimates on the derivatives of post-collisional momenta could be obtained while using the third parametrization (1.65)-(1.66). We did not present it because we don't use them in the present thesis.

2.6 L^∞-existence theorem of classical solutions

2.6.1 Functional space

We choose $e^{|v|^2}$ as weight.
We define the appropriate framework as

$$\Lambda = \{f \in C^1([0,\infty[\times\mathbb{R}^3), A(f) < \infty\} \tag{2.71}$$

where

$$A(f) = Sup\{e^{|v|^2}|\partial^j_{v^k} f(t,v)|,\ \ t\in[0,\infty[,\ \ v\in\mathbb{R}^3,\ \ j=0,1,\ \ k=1,2,3\}. \tag{2.72}$$

Λ is not an empty set. In fact, the function $\rho : [0, \infty[\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $\rho(t, v) = e^{-2|v|^2}$ belongs to Λ.
For $f \in \Lambda$, we let

$$\|f(t)\| = A(f)(t), \qquad for\ t \in [0, \infty[$$

with

$$A(f)(t) = Sup\{e^{|v|^2}|\partial^j_{v^k} f(t, v)|, \ \ v \in \mathbb{R}^3, \ \ j = 0, 1, \ \ k = 1, 2, 3\}$$

and

$$|||f||| := \underset{t\in\mathbb{R}_+}{Sup}\|f(t)\|. \tag{2.73}$$

$|||\ |||$ is a norm on Λ and $(\Lambda, |||\ |||)$ is a Banach space.

2.6.2 L^∞-existence theorem for the homogeneous equation with Israel particles case

2.6.2.1 Estimates of the loss and gain terms

Proposition 2.9. For any $t \geq 0$ and $f \in \Lambda$, there is a constant C which does not depend on $t,\ x,\ v$ such that:

$$\left|\int_0^t Q_{loss}(f, f)(\tau, v)d\tau\right| \leq Ce^{-|v|^2}|||f|||^2, \tag{2.74}$$

$$\left|\partial_{v^k}\left(\int_0^t Q_{loss}(f, f)(\tau, v)d\tau\right)\right| \leq Ce^{-|v|^2}|||f|||^2, \qquad for\ k = 1, 2, 3. \tag{2.75}$$

Proof. For the proof of (2.74), we recall that the loss term generated by Israel particles is given by

$$Q_{loss}(f, f)(t, v) = a^{-1}(t)b^{-2}(t)\int_{S^2}\int_{\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(u)||f(v)|d\omega du.$$

We have the following control

$$\begin{aligned}
\left|e^{|v|^2}\int_0^t Q_{loss}(f, f)(\tau, v)d\tau\right| &\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(u)||f(v)|e^{|v|^2}d\omega du\\
&\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}e^{|u|^2}|f(u)|e^{|v|^2}|f(v)|e^{-|u|^2}d\omega du\\
&\leq |||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}e^{-|u|^2}d\omega du\\
&\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3}e^{-|u|^2}du.
\end{aligned}$$

Since $\sigma_0(\omega)$ is bounded on S^2, $v^0 \geq 1$, $u^0 \geq 1$ and $\sqrt{s} \geq 2$.
Recalling that

$$\int_{\mathbb{R}^3}e^{-|u|^2}du < \infty \qquad and \qquad \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau < \infty,$$

then we have

$$|e^{|v|^2}\int_0^t Q_{loss}(f, f)(\tau, v)d\tau| \leq C|||f|||^2.$$

We multiply the previous expression by $e^{-|v|^2}$ to establish (2.74).
Let's establish the proof of (2.75).

$|e^{|v|^2}\partial_{v^i}(\int_0^t Q_{loss}(f,f)(\tau,v)d\tau)|$

$\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}|\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||f(u)||f(v)|d\omega du$

$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|(\partial_{v^i}f)(v)||f(u)|d\omega du$

$\leq 8\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \sigma_0(\omega)e^{|v|^2}|f(u)|e^{|u|^2}|f(u)|e^{-|u|^2}d\omega du$

$+C\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \sigma_0(\omega)e^{|v|^2}|(\partial_{v^i}f)(v)|e^{|u|^2}|f(u)|e^{-|u|^2}d\omega du$

$\leq C|||f|||^2.$

We multiply the previous expression by $e^{-|v|^2}$, we obtain the desired result. □

Proposition 2.10. For any $t\geq 0$ and $f\in\Lambda$, there exists a constant C which does not depend on $t,\ x,\ v$ such that:

$$\left|\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right| \leq Ce^{-|v|^2}|||f|||^2, \tag{2.76}$$

$$\left|\partial_{v^k}\left(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right)\right| \leq Ce^{-|v|^2}|||f|||^2, \qquad for\ k=1,2,3. \tag{2.77}$$

Proof. For the proof of (2.76) we recall that the gain term generated by Israel particles is given by

$$Q_{gain}(f,f)(t,v) = a^{-1}(t)b^{-2}(t)\iint_{S^2\times\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f(u')f(v')d\omega du.$$

Here we recall (2.27) and we have

$$\begin{aligned}\left|e^{|v|^2}\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right| &\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{S}}e^{|v|^2}|f(v')||f(u')|d\omega du\\ &\leq C\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3} e^{-|u'|^2-|v'|^2+|u|^2+|v|^2}\\ &e^{|v'|^2}|f(v')|e^{|u'|^2}|f(u')|e^{-|u|^2}d\omega du\\ &\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3}e^{-|u|^2}du\\ &\leq C|||f|||^2.\end{aligned}$$

We multiply the previous expression by $e^{-|v|^2}$ to obtain the desired result.
Let's establish the second estimate (2.77).

$$|e^{|v|^2}\partial_{v^i}(\int_0^t Q_{gain}(f,f))(\tau,v)d\tau)|$$

$$\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}|\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||f(v')||f(u')|d\omega du$$

$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(f(v'))||f(u')|d\omega du$$

$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(v')||\partial_{v^i}(f(u'))|d\omega du.$$

By the expressions (3.18), we can state that

$$|e^{|v|^2}\partial_{v^i}(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau)|$$
$$\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|u|^2+|v|^2-|v'|^2-|u'|^2}\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})e^{|v'|^2}|f(v')|e^{|u'|^2}|f(u')|e^{-|u|^2}d\omega du$$

$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\sum_{j=1}^3(\partial_{v^i}v'^j)(\partial_{v'^j}f)(v')||f(u)|d\omega du$$

$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(v')||\sum_{j=1}^3(\partial_{v^i}u'^j)(\partial_{u'^j}f)(u')|d\omega du.$$

Hence

$$|e^{|v|^2}\partial_{v^i}(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau)|$$

$$\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{-|u|^2}du$$

$$+C\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\frac{4}{v^0u^0\sqrt{s}}e^{|v'|^2}$$

$$\sum_{j=1}^3|\partial_{v^i}v'j||(\partial_{v'^j}f)(v')|e^{|u'|^2}|f(u')|e^{-|u|^2}d\omega du$$

$$+C\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\frac{4}{v^0u^0\sqrt{s}}e^{|u'|^2}$$

$$\sum_{j=1}^3|\partial_{v^i}u'j||(\partial_{u'^j}f)(u')|e^{|v'|^2}|f(v')|e^{-|u|^2}d\omega du.$$

Since $u^0\leq\sqrt{1+|u|^2}$ and $\sqrt{s}\geq 2$, we obtain

$$|e^{|v|^2}\partial_{v^i}(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau)|$$

$$\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{-|u|^2}du$$

$$+C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3}(\sqrt{1+|u|^2})^3e^{-|u|^2}du$$

$$+C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3}(\sqrt{1+|u|^2})^3e^{-|u|^2}du$$

$$\leq C|||f|||^2.$$

We multiply this expression by $e^{-|v|^2}$ to obtain the desired result.

□

2.6.2.2 L^∞-existence theorem for the Israel particles

Theorem 2.1. Consider the relativistic Boltzmann equation in the Bianchi type I space-time in the form of (2.3). Suppose that the scattering kernel satisfies (1.69), and let the coefficients a and b be given and satisfy (2.4)-(2.5). Let $f_0 \in \Lambda$ be an initial data such that it is differentiable and satisfies $\|f_0\| \leq r_0$ for some positive constant r_0. If the constant r_0 is sufficiently small, then there exists a unique non-negative classical solution of the relativistic Boltzmann equation (2.3) such that $\underset{t\in\mathbb{R}_+}{Sup}\|f(t)\| \leq C_{r_0}$ where C_{r_0} is some positive constant depending on r_0.

Proof. Proving the main theorem is equivalent to prove the existence and uniqueness solution of the integral equation (2.3). In order to do so, we are going to use the fixed point theorem. We define the map Υ from Λ by

$$\Upsilon(f)(t,v) = f_0(v) + \int_0^t Q(f,f)(\tau,v)d\tau. \tag{2.78}$$

Let

$$\Lambda_{r_0} = \{f \in \Lambda, |||f||| \leq r_0\}.$$

Λ_{r_0} is a closed subset of the Banach space $(\Lambda, ||||||)$.
Let's suppose that $|||f_0||| \leq \frac{r_0}{2}$.
For $f \in \Lambda_{r_0}$, from (2.78) and the relation

$$\partial_{v^i}\Upsilon(f)(t,v) = \partial_{v^i} f_0 + \partial_{v^i}\int_0^t Q(f,f)(\tau,v)d\tau,$$

we have the following two inequalities for any (t,v):

$$|\Upsilon(f)(t,v)| \leq e^{-|v|^2}|||f_0||| + Ce^{-|v|^2}|||f|||^2 \leq e^{-|v|^2}[\frac{r_0}{2} + Cr_0^2], \tag{2.79}$$

$$|\partial_{v^i}\Upsilon(f)(t,v)| \leq e^{-|v|^2}|||f_0||| + Ce^{-|v|^2}|||f|||^2 \leq e^{-|v|^2}[\frac{r_0}{2} + Cr_0^2]. \tag{2.80}$$

Thus, if

$$\frac{r_0}{2} + Cr_0^2 < r_0 \qquad i.e \qquad r_0 < \frac{1}{2C},$$

after multiplying (2.79) and (2.80) by $e^{|v|^2}$ we have

$$\begin{aligned} e^{|v|^2}|\Upsilon(f)(t,v)| &< r_0, \\ e^{|v|^2}|\partial_{v^i}\Upsilon(f)(t,v)| &< r_0. \end{aligned}$$

Taking the supremum with respect to t and v, we have

$$\begin{aligned} |||\Upsilon(f)||| &< r_0, \\ |||\partial_{v^i}\Upsilon(f)||| &< r_0. \end{aligned}$$

So Υ maps Λ_{r_0} into itself.
On the other hand, using the bilinearity of Q, we prove in such situation that Υ is a contraction.
In fact, if $\|f_0\| \leq \frac{r_0}{2}$ and $f, g \in \Lambda_{r_0}$, then

$$|\Upsilon f(t,v) - \Upsilon g(t,v)| \leq Ce^{-|v|^2}(|||f||| + |||g|||)|||f-g||| \leq 2Cr_0 e^{-|v|^2}|||f-g|||, \tag{2.81}$$

$$|\partial_{v^i}\Upsilon f(t,v) - \partial_{v^i}\Upsilon g(t,v)| \leq Ce^{-|v|^2}(|||f||| + |||g|||)|||f-g||| \leq 2Cr_0 e^{-|v|^2}|||f-g|||. \tag{2.82}$$

The desired result is obtained if $r_0 < \frac{1}{2C}$.
In fact after multiplying (2.81) and (2.82) by $e^{|v|^2}$ and taking the supremum with respect to t and v, we have:

$$\begin{aligned} |||\Upsilon(f) - \Upsilon(g)||| &< |||f-g|||, \\ |||\partial_{v^i}\Upsilon(f) - \partial_{v^i}\Upsilon(g)||| &< |||f-g|||. \end{aligned}$$

It follows that Υ is a contraction.
So using the fixed point theorem, we claim that the desired result is proved. □

2.6.3 L^∞-existence theorem for the homogeneous equation for hard potentials case

We assume that the coefficient b of the metric tensor enjoys the following condition

$$\int_{\mathbb{R}_+} b^{\beta-3}(\tau)d\tau < \infty, \qquad \beta \in [0,3] \tag{2.83}$$

2.6.3.1 Estimates of the loss and gain terms

Proposition 2.11. Under assumptions (1.70) with $\alpha = 0$ and (2.50) on the scattering kernel and the assumptions (2.4), (2.5) and (2.83) on the metric tensor, for any $t \geq 0$ and $f \in \Lambda$, there is a constant c independent on t, x, v, for which

$$\begin{aligned} \left|\int_0^t Q_{loss}(f,f)(\tau,v)d\tau\right| &\leq ce^{-|v|^2}|||f|||^2, & (2.84)\\ \left|\partial_{v^k}\left(\int_0^t Q_{loss}(f,f)(\tau,v)d\tau\right)\right| &\leq ce^{-|v|^2}|||f|||^2, \qquad for\ k=1,2,3. & (2.85) \end{aligned}$$

Proof. For the first inequality (2.84), we have

$$\begin{aligned}
\left|e^{|v|^2}\int_0^t Q_{loss}(f,f)(\tau,v)d\tau\right| &= \int_0^t a^{-1}b^{-2}d\tau \iint v_\phi\sigma(g,w)(e^{|v|^2}f(v))(e^{|u|^2}f(u))e^{-|u|^2}d\omega du\\
&\leq |||f|||^2\int_0^t a^{-1}b^{-2}d\tau\iint v_\phi\sigma(g,w)e^{-|u|^2}d\omega du\\
&\leq c|||f|||^2\int_0^t a^{-1}b^{-2}d\tau(\int_{\mathbb{R}^3} v_\phi e^{-|u|^2}du+\int_{\mathbb{R}^3} v_\phi g^{-\beta}e^{-|u|^2}du)\\
&\leq c|||f|||^2\int_0^t (a^{-1}b^{-2}+a^{-1}b^{\beta-3})d\tau\\
&\leq c|||f|||^2.
\end{aligned}$$

We multiply this previous relation by $e^{-|v|^2}$ to obtain the desired result.
For the second inequality (2.85), we have

$$|e^{|v|^2}\partial_{v^i}\left(\int_0^t Q_{loss}(f,f)(\tau,x,v)d\tau\right)|\leq I_1+I_2$$

where

$$\begin{cases}I_1=\int_0^t a^{-1}b^{-2}\iint|\partial_{v^i}[v_\phi\sigma(g,\omega)]|e^{|v|^2}f(v)f(u)d\omega dud\tau,\\ I_2=\int_0^t a^{-1}b^{-2}\iint v_\phi\sigma(g,\omega)e^{|v|^2}|\partial_{v^i}(f(v))|f(u)d\omega dud\tau.\end{cases}$$

The estimate of I_2 is obvious. That gives

$$I_2\leq c|||f|||^2\int_0^t(a^{-1}b^{-2}+a^{-1}b^{\beta-3})d\tau\leq c|||f|||^2.$$

For the estimate of I_1, we separate it into two cases and we use the same argument as in the estimates (2.51)-(2.54).

Case i = 1: From the estimate (2.51) of $\partial_{v^1}[v_\phi\sigma(g,\omega)]$, we have

$$\begin{aligned}
I_1 &\leq \int_0^t d\tau a^{-2}b^{-2}\iint u^0(1+g^{-\beta})\sigma_0(\omega)e^{|v|^2}f(v)f(u)d\omega du\\
&\leq c|||f|||^2\int_0^t d\tau a^{-2}b^{-2}\iint(1+g^{-\beta})\sigma_0(\omega)\sqrt{1+|u|^2}e^{-|u|^2}d\omega du\\
&\leq c|||f|||^2\int_0^t(a^{-2}b^{-2}+a^{-2}b^{\beta-3})d\tau\\
&\leq c|||f|||^2.
\end{aligned}$$

Case i = 2, 3: From the estimate (2.54) of $\partial_{v^k}[v_\phi\sigma(g,\omega)]$, we have for $i=2,3$

$$\begin{aligned}
I_1 &\leq \int_0^t d\tau a^{-1}b^{-3}\iint u^0(1+g^{-\beta})\sigma_0(\omega)e^{|v|^2}f(v)f(u)d\omega du\\
&\leq c|||f|||^2\int_0^t d\tau a^{-1}b^{-3}\iint(1+g^{-\beta})\sigma_0(\omega)\sqrt{1+|u|^2}e^{-|u|^2}d\omega du\\
&\leq c|||f|||^2\int_0^t(a^{-1}b^{-3}+a^{-1}b^{\beta-4})d\tau\\
&\leq c|||f|||^2.
\end{aligned}$$

Hence

$$I_1 + I_2 \leq C|||f|||^2.$$

We multiply the previous relation by $e^{-|v|^2}$ to obtain the desired result.

□

Proposition 2.12. Under assumptions (1.70) with $\alpha = 0$ and (2.50) on the scattering kernel and the assumptions (2.4), (2.5) and (2.83) on the metric tensor, for any $t \geq 0$ and $f \in \Lambda$, there is a constant c independent on t, x, v, for which

$$\left|\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right| \leq ce^{-|v|^2}|||f|||^2, \quad (2.86)$$

$$\left|\partial_{v^i}\left(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right)\right| \leq ce^{-|v|^2}|||f|||^2, \qquad for\ i = 1,2,3. \quad (2.87)$$

Proof. About the inequality (2.86), we recall (2.27) and (2.23). Let

$$I_{gain} = \left|e^{|v|^2}\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right|.$$

By direct computation, we have

$$\begin{aligned}
I_{gain} &\leq \int_0^t d\tau a^{-1}b^{-2}|||f|||^2 \iint v_\phi\sigma(g,w)e^{|v|^2+|u|^2-|v'|^2-|u'|^2}e^{-|u|^2}d\omega du\\
&\leq \int_0^t d\tau a^{-1}b^{-2}|||f|||^2 \iint v_\phi\sigma(g,w)e^{-|u|^2}d\omega du\\
&\leq c\int_0^t d\tau a^{-1}b^{-2}|||f|||^2 (\int_{\mathbb{R}^3} v_\phi e^{-|u|^2}du + \int_{\mathbb{R}^3} v_\phi g^{-\beta}e^{-|u|^2}du)\\
&\leq c|||f|||^2\int_0^t (a^{-1}b^{-2} + a^{-1}b^{\beta-3})d\tau\\
&\leq c|||f|||^2.
\end{aligned}$$

As expected, the derivatives of the gain term is much more difficult to handle.
First, we have

$$\left|e^{|v|^2}\partial_{v^i}\left(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right)\right| \leq J_1 + J_2$$

where J_1 and J_2 are defined as:

$$\begin{aligned}
J_1 &= \int_0^t d\tau a^{-1}b^{-2}\iint |\partial_{v^i}[v_\phi\sigma(g,\omega)]|e^{|v|^2}f(v')f(u')d\omega du,\\
J_2 &= \int_0^t d\tau a^{-1}b^{-2}\iint v_\phi\sigma(g,\omega)e^{|v|^2}|\partial_{v^i}[f(v')f(u')]|d\omega du.
\end{aligned}$$

We recall (2.51)-(2.54) and the estimate of J_1 is done easily, following the estimate of $\left|\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right|$.
About the estimate of J_2, we recall (3.18) to observe that

$$\partial_{v^i}[f(v')f(u')] = \sum_{k=1}^3(\partial_{v^i}v'^k)(\partial_{v^k}f)(v')f(u') + f(v')\sum_{k=1}^3(\partial_{v^i}u'^k)(\partial_{v^k}f)(u').$$

We let

$$j_2(t) = a^{-1}b^{-2}\iint v_\phi \sigma(g,\omega)e^{|v|^2}|\partial_{v^i}[f(v')f(u')]|d\omega du$$

and we fix a momentum v.

We recall that $a(t)$ and $b(t)$ are increasing functions with $a(0) = 1$. For a fixed v there exists a finite time t_0 such that for $t \geq t_0$, we have $|v| \leq a(t)$.

We break up the estimate of $j_2(t)$ into a number of steps.

Step 1: $t \geq t_0$.

From the relations $|v| \leq a(t)$ and (2.62) allowing the estimate of derivatives of the post-collisional momenta, we have

$$\begin{aligned}|\partial_{v^i}v'^k| &\leq c\sqrt{1+a^{-2}(v^1)^2+b^{-2}|\bar{v}|^2}(u^0)^4\\ &\leq c\sqrt{1+a^{-2}|v|^2}\\ &\leq c(u^0)^4.\end{aligned}$$

In this case, to control $j_2(t)$ we use the same argument which allowed us to control I_{gain}. This leads to

$$|j_2(t)| \leq c\|f(t)\|^2(a^{-1}b^{-2} + a^{-1}b^{\beta-3}).$$

Step 2: $t < t_0$ **and** $|v| \leq 2|u|$.

In this case, by (2.17) and from the relations (2.59)-(2.60)-(2.61)-(2.62) of the first parametrization, all the terms $|\partial_{v^i}v'^k|$ are controlled by $c(u^0)^5$ and $|j_2(t)|$ is exactly controlled as in the first step.

Step 3: $t < t_0$ **and** $|v| \geq 2|u|$.

In this case, instead of the first parametrization (1.56)-(1.57), we use the second parametrization (1.61)-(1.62).

From the relation $|v| \geq 2|u|$, it follows that

$$|v-u| \geq \frac{1}{2}|v| \quad and \quad |v+u| \geq \frac{1}{2}|v|.$$

From the estimates (2.67)-(2.68)-(2.69)-(2.70), using the assumption $a(t) \leq b(t) \leq \sqrt{2}a(t)$, a straightforward computation allows us to control all the terms $|\partial_{v^i}v'^k|$ by $c(u^0)^3$.

Finally $|j_2(t)|$ is exactly controlled as in the first step.

To end the proof, we integrate $j_2(\tau)$ from 0 to t and this leads to the estimate of J_2. □

2.6.3.2 L^∞-existence theorem for hard potentials

Theorem 2.2. Consider the relativistic Boltzmann equation in the Bianchi type I space-time in the form of (2.3). Suppose that the scattering kernel satisfies (1.70) with $\alpha = 0$ and (2.50), and let the coefficients a and b be given and satisfy (2.4), (2.5) and (2.83). Let $f_0 \in \Lambda$ be an initial data such that it is differentiable and satisfies $\|f_0\| \leq r_0$ for some positive constant r_0. If the constant r_0 is sufficiently small, then there exists a unique non-negative classical solution of the Boltzmann equation (2.3) such that $\underset{t\in\mathbb{R}_+}{Sup}\|f(t)\| \leq C_{r_0}$ where C_{r_0} is some positive constant depending on r_0.

Proof. This proof is done exactly as in Theorem 2.1. □

2.6.4 L^∞-existence theorem for the homogeneous equation for soft potentials case

We assume that the coefficient b of the metric tensor enjoys the condition (2.83).

2.6.4.1 Estimates of the loss and gain terms

Proposition 2.13. Under assumptions (1.71) on the collisional cross section $\sigma(g, \omega)$ and the assumption on a and b, for any $t \geq 0$ and $f \in \Lambda$, there is a constant C not dependent on t, x and v for which:

$$\left| \int_0^t Q_{loss}(f,f)(\tau, v) d\tau \right| \leq C e^{-|v|^2} |||f|||^2, \tag{2.88}$$

$$\left| \partial_{v^k} (\int_0^t Q_{loss}(f,f)(\tau, v) d\tau) \right| \leq C e^{-|v|^2} |||f|||^2, \qquad for\ k = 1,2,3. \tag{2.89}$$

Proof. We recall that the loss term is given by

$$Q_{loss}(f,f)(t,v) = a^{-1}(t) b^{-2}(t) \iint_{S^2 \times \mathbb{R}^3} \vartheta_\phi \sigma(g, \omega) f(t,v) f(t,u) d\omega du.$$

For the first inequality (2.88) we have

$$\begin{aligned}
\left| e^{|v|^2} \int_0^t Q_{loss}(f,f)(\tau, v) d\tau \right| &\leq \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau \iint_{S^2 \times \mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{|v|^2} |f(u)||f(u)| d\omega du \\
&\leq \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau \iint_{S^2 \times \mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{|v|^2} |f(v)| e^{|u|^2} |f(u)| e^{-|u|^2} d\omega du \\
&\leq |||f|||^2 \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau \iint_{S^2 \times \mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{-|u|^2} d\omega du \\
&\leq C |||f|||^2 \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau \iint_{S^2 \times \mathbb{R}^3} \vartheta_\phi g^{-\beta} \sigma_0(\omega) e|u|^2 d\omega du \\
&\leq C |||f||| \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{|u|^2} d\omega du \\
&\leq C |||f||| \int_0^t a^{-1}(\tau) b^{\beta-3}(\tau) d\tau \\
&\leq C |||f|||^2,
\end{aligned}$$

since

$$\int_0^t a^{-1}(\tau) b^{\beta-3}(\tau) d\tau < \infty,$$

and

$$\int_{\mathbb{R}^{\not\models}} \vartheta_\phi g^{-\beta} e^{-|u|^2} du \leq C b^{\beta-1}.$$

For the second inequality (2.89) we have

$$\left|e^{|v|^2}\partial_{v^i}\left(\int_0^t Q_{loss}(f,f)(\tau,v)d\tau\right)\right| \leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau$$
$$\iint_{S^2\times\mathbb{R}^3}|\partial_{v^i}(\vartheta_\phi\sigma(g,\omega)||f(u)||f(v)|e^{|v|^2}d\omega d\omega$$
$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\sigma(g,\omega)|\partial_{v^i}(f(v))||f(u)|e^{|v|^2}d\omega du.$$

Case i=1:

$$|e^{|v|^2}\partial_{v^1}(\textstyle\int_0^t Q_{loss}(f,f)(\tau,v)d\tau)| \ \leq$$

$$C\textstyle\int_0^t a^{-1}u^0g^{-\beta}\sigma_0(\omega)e^{|v|^2}|f(v)|e^{|u|^2}|f(u)|e^{-|u|^2}d\omega du$$

$$+\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)e^{|v|^2}|\partial_{v^1}(f(v))|e^{|u^2|}d\omega du$$

$$\leq C|||f|||^2\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}a^{-1}u^0g^{-\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$+C|||f|||^2\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$\leq C|||f|||^2\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3}a^{-1}\sqrt{1+|u|^2}g^{-\beta}e^{-|u|^2}du$$

$$+C|||f|||^2\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du.$$

By (2.23) and since

$$\int_{\mathbb{R}^3}a^{-1}\sqrt{1+|u|^2}g^{-\beta}e^{-|u|^2}du \leq b^{\beta-1},$$

we have the following estimate

$$|e^{|v|^2}\partial_{v^1}(\int_0^t Q_{loss}(f,f)(\tau,v)d\tau)| \leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$
$$\leq C|||f|||^2.$$

We multiply the above relation by $e^{-|v|^2}$ to have the desired result.

Case i=2,3:

$$|e^{|v|^2}\partial_{v^i}(\textstyle\int_0^t Q_{loss}(f,f)(\tau,v)d\tau)| \ \leq$$

$$C\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}b^{-1}u^0g^{-\beta}\sigma_0(\omega)e^{|v|^2}|f(v)|e^{|u|^2}|f(u)|e^{-|u|^2}|d\omega du$$

$$+\textstyle\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)e^{|v|^2}|\partial_{v^i}(f(v))|e^{|u|^2}|f(u)|e^{-|u|^2}d\omega du.$$

Taking the same argument as in the case 1, we have

$$\begin{aligned}\left|e^{|v|^2}\partial_{v^i}\Big(\int_0^t Q_{loss}(f,f)(\tau,v)d\tau\Big)\right| &\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{\beta-3}d\tau\\ &\leq C|||f|||^2.\end{aligned}$$

We multiply this expression with $e^{-|v|^2}$ to obtain the desired result. ☐

Proposition 2.14. Under the assumptions (1.71) on the collisional cross section $\sigma(g,\omega)$ and the assumptions on a and b, for any $t\geq 0$ and $f\in\Lambda$, there is a constant C independent on t, x, and v for which:

$$\left|\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\right| \leq Ce^{-|v|^2}|||f|||^2, \tag{2.90}$$

$$\left|\partial_{v^k}\Big(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\Big)\right| \leq Ce^{-|v|^2}|||f|||^2, \qquad for\ k=1,2,3. \tag{2.91}$$

Proof. We recall that the gain term is given by

$$Q_{gain}(f,f)(t,v) = a^{-1}(t)b^{-2}(t)\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)f(v')f(u')d\omega du.$$

For the first inequality (2.90) we have

$$\begin{aligned}\left|e^{|v|^2}\int_0^t Q_g(f,f)(\tau,v)d\tau\right| &\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma g,\omega e^{|v|^2}|f(v')||f(u')|d\omega du\\ &\leq \int_0^t\lambda(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}h(v',u',\omega,g,s)e^{|v'|^2}|f(v')|e^{|u'|^2}|f(u')|e^{-|u|^2}d\omega du\\ &\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)e^{-|u|^2}d\omega du\\ &\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du\\ &\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau\\ &\leq C|||f|||^2.\end{aligned}$$

where $h(v',u',\omega,g,s)=\vartheta_\phi\sigma(g,\omega)e^{|v|^2+|u|^2-|u'|^2-|v'|^2}$
and $\lambda(\tau)=a^{-1}(\tau)b^{-2}(\tau)$.
Since $|v|^2+|u|^2-|v'|^2-|u'|^2\leq C$, we multiply this expressions with $e^{-|v|^2}$ to obtain the desired result.
For the second inequality (2.91), let

$$I=\left|e^{|v|^2}\partial_{v^k}\Big(\int_0^t Q_{gain}(f,f)(\tau,v)d\tau\Big)\right|.$$

Then we have

$$I \leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \partial_{v^k}(\vartheta_\phi\sigma(g,\omega))e^{|v|^2}|f(v')||f(u')|d\omega du$$

$$+\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} e^{|v|^2}\vartheta_\phi\sigma(g,\omega)(\textstyle\sum_{j=1}^3 |\partial_{v^k}(v'^j)||\partial_{v'^j}(f)(v')||f(u')|$$

$$+\textstyle\sum_{j=1}^3 |\partial_{v^k}(u'^j)||\partial_{u'j}(f)(u')||f(v')|)d\omega du.$$

Case k=1:

$$I \leq C\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} a^{-1}u^0g^{-\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2+|u|^2-|v'|^2-|u'|^2}e^{|v'|^2}|f(v')|e^{|u'|^2}|f(u')|d\omega du$$

$$+\int_0^t \lambda(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)e^{-|u|^2}(e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\textstyle\sum_{J=1}^3 |\partial_{v^1}(v'^j)|e^{|v'|^2}|\partial_{v'^j}(f)(v')|e^{|u'|^2}|f(u')|$$

$$+e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\textstyle\sum_{j=1}^3 |\partial_{v^1}(u'^j)|e^{|u'|^2}|\partial_{u'^j}(f)(u')|e^{|v'|^2}|f(v')|)d\omega du$$

$$\leq |||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} a^{-1}u^0g^{-\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$+c|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)e^{-|u|^2}(\textstyle\sum_{j=1}^3 |\partial_{v^1}(v'^j)| + \sum_{j=1}^3 |\partial_{v^1}(u'^j))d\omega du.$$

We recall that a is an increasing function and $a(0)=1$. For a fixed v there exists a finite time t_0 such that for $t\geq t_0$, we have $|v|\leq a(t)$.

So we can state that

$$|\partial_{v^1}(v'^j)| \leq Cv^0(u^0)^4 \leq C\sqrt{1+a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^4.$$

Hence,

$$I \leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} a^{-1}u^0g^{-\beta}e{-}|u|^2du$$

$$+C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta}(u^0)^4e^{-|u|^2}du$$

$$\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} a^{-1}\sqrt{1+|u|^2}e^{-|u|^2}g^{-\beta}du$$

$$+C|||f|||^2\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3}(\sqrt{1+|u|^2})^4e^{-|u|^2}\vartheta_\phi g^{-\beta}du$$

$$\leq C|||f|||^2\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$

$$\leq C|||f|||^2.$$

We multiply this expression by $e^{-|v|^2}$ to obtain the desired result.

For $t<t_0$, allowing $|v|\geq a(t)$, we will consider two cases:

$$|v|<2|u| \quad and \quad |v|\geq 2|u|.$$

Case 1**:**

$$|v| \geq a(t) \quad and \quad |v| \leq 2|u|.$$

In this case, by (2.17) we have

$$|\partial_{v^1}(v'^j)| \leq Cv^0(u^0)^4 \leq C\sqrt{1+a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^5,$$

$$\begin{aligned} I &\leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} a^{-1}\sqrt{1+|u|^2}e^{-|u|^2}g^{-\beta}du \\ &+ C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} (\sqrt{1+|u|^2})^5 e^{-|u|^2}\vartheta_\phi g^{-\beta}du. \end{aligned}$$

Case 2**:**

$$|v| \geq a(t) \qquad and \qquad |v| \geq 2|u|.$$

It follows that

$$|v-u| \geq \frac{1}{2}|v| \qquad and \qquad |v+u| \geq \frac{1}{2}|v|,$$

and then

$$|\partial_{v^1}(v'^j)| \leq (\frac{bu^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2}) \leq C(u^0)^3.$$

So we can state that

$$\begin{aligned} I &\leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} a^{-1}\sqrt{1+|u|^2}e^{-|u|^2}g^{-\beta}du \\ &+ C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} (\sqrt{1+|u|^2})^3 e^{-|u|^2}\vartheta_\phi g^{-\beta}du \\ &\leq C|||f|||^2. \end{aligned}$$

We multiply this expression by $e^{-|v|^2}$ to obtain the desired result.

Case k=2,3:

$$I \leq C \int_0^t b^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} a^{-1}u^0 g^{-\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2+|u|^2-|v'|^2-|u'|^2}e^{|v'|^2}|f(v')|e^{|u'|^2}|f(u')|d\omega du$$

$$+ \int_0^t \lambda(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma g,\omega e^{-|u|^2}(e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\sum_{J=1}^3 |\partial_{v^k}(v'^j)|e^{|v'|^2}|\partial_{v'^j}(f)(v')|e^{|u'|^2}|f(u')|$$

$$+ e^{|v|^2+|u|^2-|v'|^2-|u'|^2}\sum_{j=1}^3 |\partial_{v^k}(u'^j)|e^{|u'|^2}|\partial_{u'^j}(f)(u')|e^{|v'|^2}|f(v')|)d\omega du$$

$$\leq |||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} b^{-1}u^0 g^{-\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$+ c|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)e^{-|u|^2}(\sum_{j=1}^3 |\partial_{v^k}(v'^j)| + \sum_{j=1}^3 |\partial_{v^k}(u'^j))d\omega du.$$

We recall that a is an increasing function and $a(0) = 1$. For a fixed v there exists a finite time t_0 such that for $t \geq t_0$, we have $|v| \leq a(t)$.

So we can state that

$$|\partial_{v^k}(v'^j)| \leq Cv^0(u^0)^4 \leq C\sqrt{1+a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^4.$$

Hence,

$I \leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} b^{-1}u^0 g^{-\beta} e{-}|u|^2 du$

$+C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta}(u^0)^4 e^{-|u|^2} du$

$\leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} b^{-1}\sqrt{1+|u|^2} e^{-|u|^2} g^{-\beta} du$

$+C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} (\sqrt{1+|u|^2})^4 e^{-|u|^2} \vartheta_\phi g^{-\beta} du$

$\leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$

$\leq C|||f|||^2.$

We multiply this expression by $e^{-|v|^2}$ to obtain the desired result.
For $t < t_0$, meaning $|v| \geq a(t)$, we will consider two cases:

$$|v| < 2|u| \quad and \quad |v| \geq 2|u|.$$

Case 1:

$$|v| \geq a(t) \quad and \quad |v| \leq 2|u|.$$

In this case, by (2.17) we have

$$|\partial_{v^k}(v'^j)| \leq Cv^0(u^0)^4 \leq C\sqrt{1+a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^5.$$

$I \leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} b^{-1}\sqrt{1+|u|^2} e^{-|u|^2} g^{-\beta} du$

$+C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} (\sqrt{1+|u|^2})^5 e^{-|u|^2} \vartheta_\phi g^{-\beta} du.$

Case 2:

$$|v| \geq a(t) \qquad and \qquad |v| \geq 2|u|.$$

It follows that

$$|v-u| \geq \frac{1}{2}|v| \qquad and \qquad |v+u| \geq \frac{1}{2}|v|,$$

and then

$$|\partial_{v^k}(v'^j)| \leq (\frac{bu^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2}) \leq C(u^0)^3.$$

So we can state that

$$\begin{aligned} I &\leq C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} b^{-1}\sqrt{1+|u|^2} e^{-|u|^2} g^{-\beta} du \\ &+ C|||f|||^2 \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} (\sqrt{1+|u|^2})^3 e^{-|u|^2} \vartheta_\phi g^{-\beta} du \\ &\leq C|||f|||^2. \end{aligned}$$

We multiply this expression by $e^{-|v|^2}$ to obtain the desired result.

□

2.6.4.2 L^∞-existence theorem for soft potentials

Theorem 2.3. Consider the relativistic Boltzmann equation in the Bianchi type I space-time in the form of (2.3). Suppose that the scattering kernel satisfies (1.71)-(2.50), and let the coefficients a and b be given and satisfy (2.4), (2.5) and (2.83). Let $f_0 \in \Lambda$ be an initial data such that it is differentiable and satisfies $\|f_0\| \leq r_0$ for some positive constant r_0. If the constant r_0 is sufficiently small, then there exists a unique non-negative classical solution of the Boltzmann equation (2.3) such that $\underset{t\in\mathbb{R}_+}{Sup}\|f(t)\| \leq C_{r_0}$ where C_{r_0} is some positive constant depending on r_0.

Proof. This proof is done exactly as in Theorem 2.1.

□

CHAPTER THREE

L^2-EXISTENCE THEOREM OF THE HOMOGENEOUS RELATIVISTIC BOLTZMANN EQUATION IN THE BIANCHI TYPE I SPACE-TIME

Contents

IN this chapter, we study the L^2-existence theorem as well as the L^2-stability of solution of the relativistic Boltzmann equation. In order to do so, we carry the study of uniqueness existence of solution with some L^2-weighted norm rather than L^∞-norm.

Let's consider the set $L^2_1(t),\ t \in [0,\infty[$ that we will define in the sequel, the Cauchy problem of the homogeneous relativistic Boltzmann equation in f with initial data $f_0 \in L^2_1(t),\ t \in [0,\infty[$ reads in term of variables (t,x,v)

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{3.1}$$
$$f(0,v) = f_0(v). \tag{3.2}$$

We assume that the coefficients a and b of the Bianchi type I metric are given increasing functions of the time t and are such that:

$$a(0) \geq 1, \quad a \leq b \leq \sqrt{2}a, \quad \lim_{t\longrightarrow\infty} a(t) = \infty, \tag{3.3}$$

$$\int_{\mathbb{R}_+} a^{-2}(\tau)d\tau < \infty. \tag{3.4}$$

3.1 The functional space

The space in which we will look for the existence of solutions is

$$L^2_1 = \{f : [0,\infty[\times\mathbb{R}^3 \longrightarrow \mathbb{R}, \int e^{|v|^2}|f(t,v)|^2 dv < \infty, \int e^{|v|^2}|\partial_{v^i} f(t,v)|dv < \infty$$
$$\forall i = 1,2,3 \qquad and \qquad \forall t \in [0,\infty[\}. \tag{3.5}$$

L^2_1 is not an empty set. In fact $\rho(t,v) = e^{-2|v|^2}$ belong to L^2_1.
For $t \in [0,\infty[$, we let

$$L^2_1(t) = \{f \in L^2_1, \int e^{|v|^2}|f(t,v)|^2 dv < \infty\}. \tag{3.6}$$

We endow $L^2_1(t)$ with the norm defined by

$$\|f(t)\|_e = \left(\int_{\mathbb{R}^3} e^{|v|^2}|f(t,v)|^2 dv\right)^{\frac{1}{2}}. \tag{3.7}$$

With this norm, $L^2_1(t)$ is a Banach space.
We define the norm of $L^2_1(t)$ by

$$|||f(t)|||^2_e = \|f(t)\|^2_e + \sum_{k=1}^{3} \|\partial_{v^k} f(t)\|^2_e. \tag{3.8}$$

3.2 L^2-energy estimates of the homogeneous equation

Remark 3.1. For the sake of simplicity, we will sometimes denote by $\iint$ the integral over $\mathbb{R}^3 \times S_{ab}$ and by $\iiint$ the integral over $\mathbb{R}^3 \times \mathbb{R}^3 \times S_{ab}$. In this section we study the energy estimates for the equation by using the weigh function $e^{\frac{1}{2}|v|^2}$.

3.2.1 L^2-energy estimates of the homogeneous equation with Israel particles

Lemma 3.1. Let f be a solution to the Cauchy problem (3.1)-(3.2). Then f satisfies

$$\|f(t)\|_e^2 \leq \|f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(\|f(s)\|_e^3) \tag{3.9}$$

for some constant C which does not depend on t.

Proof. We multiply the equation (3.1) by $2f(t,v)$ and integrate from 0 to t to obtain

$$f^2(t,v) = f^2(0,v) - 2\int_0^t f(s,v)Q_{gain}(f,f)(s,v)ds - 2\int_0^t f(s,v)Q_{loss}(f,f)(s,v)ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2}f^2(t,v) = e^{|v|^2}f^2(0,v) - 2\int_0^t e^{|v|^2}f(s,v)Q_{gain}(f,f)(s,v)ds - 2\int_0^t e^{|v|^2}f(s,v)Q_{loss}(f,f)(s,v)ds.$$

Integrating the above equation with respect to v yields

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\textstyle\int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2}f(v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f(v')f(u')d\omega dudv$$

$$-2\textstyle\int_0^t a^{-1}b^{-2}ds \iiint f(v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f(v)f(u)d\omega dudv.$$

Since the function f is non-negative, we can ignore the loss term and have

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\int_0^t a^{-1}b^{-2}A(s)ds, \tag{3.10}$$

where

$$A(s) = \iiint e^{|v|^2}f(v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f(v')f(u')d\omega dudv.$$

By (2.27) as well as the Cauchy-Schwartz inequality, and taking into account that $\vartheta_\phi \leq 4$, we have

$$\begin{aligned}
A(s) &= \iiint e^{\frac{1}{2}|v|^2}f(v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega dudv \\
&\leq C\iiint e^{\frac{1}{2}|v|^2}f(v)\sigma_0\frac{1}{v^0u^0}(\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega dudv \\
&\leq C\iiint [\sqrt{\sigma_0(\omega)\frac{1}{v^0u^0}}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}f(v)][\sqrt{\sigma_0(\omega)\frac{1}{v^0u^0}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')]d\omega dudv \\
&\leq C(\iiint \sigma_0(\omega)\frac{1}{v^0u^0}e^{-|u|^2}e^{|v|^2}f^2(v)d\omega dudv)^{\frac{1}{2}} \\
&\quad\times(\iiint \sigma_0(\omega)\frac{1}{v^0u^0}e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega dudv)^{\frac{1}{2}} \\
&\leq C(\iiint \sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du)^{\frac{1}{2}} \\
&\quad\times(\iint \sigma_0(\omega)\frac{1}{v'^0u'^0}e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega du'dv')^{\frac{1}{2}} \\
&\leq C(\int_{\mathbb{R}^3} e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}e^{|v|^2}f^2(v)dv)^{\frac{1}{2}} \\
&\quad\times(\int_{\mathbb{R}^3} e^{|v'|^2}f^2(v')dv'\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}e^{|u'|^2}f^2(u')du')^{\frac{1}{2}} \\
&\leq C\|f(s)\|_e^3,
\end{aligned} \tag{3.11}$$

where we have used the relation

$$\frac{dvdu}{v^0u^0} = \frac{dv'du'}{v'^0u'^0}. \tag{3.12}$$

Inserting (3.11) into (3.10) yields

$$\|f(t)\|_e^2 \leq \|f(0)\|_e^0 + C\, \underset{s\in[0,t]}{Sup}\,(\|f(s)\|_e^3) \int_0^t a^{-1}(s)b^{-2}(s)ds.$$

The desired result is obtained because $a^{-1}b^{-2}$ is integrable over $\mathbb{R}_+$. □

Lemma 3.2. Let f be a solution to the Cauchy problem (3.1)-(3.2). Then for $k \in \{1, 2, , 3\}$, f satisfies the following estimates

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^2 + C\, \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^2) \tag{3.13}$$

for some constant C which does not depend on t.

Proof. We take the partial derivative of the Boltzmann equation (3.1) with respect to v^k

$$\partial_t\partial_{v^k} f(t, v) = \partial_{v^k} Q_{gain}(f, f)(t, v) - \partial_{v^k} Q_{loss}(f, f)(t, v).$$

We multiply the above equation by $2\partial_{v^k} f(t, v)$ and integrate from 0 to t to obtain

$$(\partial_{v^k} f)^2(t, v) = (\partial_{v^k} f)^2(0, v) - 2\int_0^t \partial_{v^k} f(s, v)\partial_{v^k} Q_{gain}(f, f)(s, v)ds - 2\int_0^t \partial_{v^k} f(s, v)\partial_{v^k} Q_{loss}(f, f)(s, v)ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2}(\partial_{v^k} f)^2(t, v) = e^{|v|^2}(\partial_{v^k} f)^2(0, v) - 2\int_0^t e^{|v|^2}\partial_{v^k} f(s, v)\partial_{v^k} Q_{gain}(f, f)(s, v)ds$$

$$-2\int_0^t e^{|v|^2}\partial_{v^k} f(s, v)\partial_{v^k} Q_{loss}(f, f)(s, v)ds.$$

Integrating the above equation with respect to v yields

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^k + J_{1k}(t) + J_{2k}(t) + J_{3k}(t) + J_{4t}(t) \tag{3.14}$$

where $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ are defined as follows

$$J_{1k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}(\partial_{v^k} f)^2(v)dvdud\omega,$$

$$J_{2k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)f(v)|\partial_{v^k} f(v)\partial_{v^k}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})|dvdud\omega,$$

$$J_{3k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^k}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})\partial_{v^k} f(v)|f(v')f(u')dvdud\omega,$$

$$J_{4k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^k} f(v)\partial_{v^k}[f(v')f(u')]|dvdud\omega.$$

Let's estimate $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ for $k \in \{1,2,3\}$.
Estimate of $J_{1k}(t)$: For any $k \in \{1,2,3\}$ we have

$$\begin{aligned}
J_{1k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\sigma_0(\omega)(\partial_{v^k} f)^2(v)dvdud\omega \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k} f)^2(v)dv \iint \sigma_0(\omega)f(u)d\omega du \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds \iint \sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|u|^2}\sqrt{\sigma_0(\omega)}f(u)d\omega du \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds \left(\iint \sigma_0(\omega)e^{-|u|^2}d\omega du\right)^{\frac{1}{2}} \left(\iint \sigma_0(\omega)(f(u))^2 e^{|u|^2}d\omega du\right)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds \left(\int_{\mathbb{R}^3} e^{-|u|^2}du \int_{S_{ab}}\sigma_0(\omega)d\omega\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathbb{R}^3} e^{|u|^2}(f(u))^2 du \int_{S_{ab}}\sigma_0(\omega)d\omega\right)^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}\left(\|\partial_{v^k} f(s)\|_e^2 \|f(s)\|_e\right)\int_0^t a^{-1}b^{-2}ds. \qquad (3.15)
\end{aligned}$$

Estimate of $J_{2k}(t)$: By (2.48)

$$\begin{aligned}
J_{2k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)f(u)f(v)|\partial_{v^k} f(v)|dvdud\omega \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint [e^{\frac{1}{2}|v|^2}\sqrt{\sigma_0(\omega)}|\partial_{v^k} f(v)|e^{-\frac{1}{2}|u|^2}] \\
&\times [e^{\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}\sqrt{\sigma_0(\omega)}f(u)f(v)]dvdud\omega \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\iiint e^{|v|^2}\sigma_0(\omega)(\partial_{v^k} f(v))^2 e^{-|u|^2}dvdud\omega)^{\frac{1}{2}} \\
&\times (\iiint e^{|u|^2}e^{|v|^2}\sigma_0(\omega)f^2(u)f^2(v)dvdud\omega)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k} f(v))^2 dv \int_{\mathbb{R}^3} e^{-|u|^2}du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times (\int_{\mathbb{R}^3} e^{|u|^2}f^2(u)du \int_{\mathbb{R}^3} e^{|v|^2}f^2(v)dv \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}\left(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2\right)\int_0^t a^{-1}b^{-2}ds. \qquad (3.16)
\end{aligned}$$

Estimate of $J_{3k}(t)$: By (2.48), (2.18), (2.20), (3.12) and (2.27) as well as the Cauchy-Schwartz inequality

$$
\begin{aligned}
J_{3k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f(v)|f(v')f(u')d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint \frac{v^0u^0}{v^0u^0}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint [\sqrt{\sigma_0(\omega)}\frac{\sqrt{u^0}}{\sqrt{v^0u^0}}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)] \\
&\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0u^0}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}]d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\iiint \frac{u^0}{v^0u^0}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^k}f(v))^2d\omega dudv)^{\frac{1}{2}} \\
&\times(\iiint \frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|v'|^2}(f^2(v'))^2e^{|u'|^2}(f^2(u'))^2d\omega dudv)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^k}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{1}{2}}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times(\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(f(u'))^2d\omega du'dv')^{\frac{1}{2}} \\
&\leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e^2)\int_0^t a^{-1}b^{-2}ds. \qquad (3.17)
\end{aligned}
$$

Estimate of $J_{4k}(t)$: Let us observe that

$$
\begin{aligned}
\partial_{v^k[f(v')f(u')]} &= f(u')\partial_{v^k}(f(v'))+f(v')\partial_{v^k}(f(u')) \\
&= f(u')\sum_{j=1}^3\partial_{v^k}(v'^j)\partial_{v'^j}(f(v'))+f(v')\sum_{j=1}^3\partial_{v^k}(u'^j)\partial_{u'^j}(f(u')) \\
&= f(u')\sum_{j=1}^3\partial_{v^k}(v'^j)(\partial_{v^j}f)(v')+f(v')\sum_{j=1}^3\partial_{v^k}(u'^j)(\partial_{v^j}f)(u'). \qquad (3.18)
\end{aligned}
$$

By (3.18), (2.62) and since $|\partial_{v^i}u'^k|\leq Cv^0(u^0)^4\quad i=1,2,3\ \ and\ \ k=1,2,3$, we deduce

$$|\partial_{v^k[f(v')f(u')]}|\leq Cv^0(u^0)^4[f(u')\sum_{j=1}^3|(\partial_{v^j}f)(v')|+f(v')\sum_{j=1}^3|(\partial_{v^j}f)(u')|], \qquad (3.19)$$

$$J_{4k}(t)=\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^k}f(v)\partial_{v^k}[f(v')f(u')]|dvdud\omega.$$

Taking into account (3.19) and knowing that $(x+y+z)^2\leq 3(x^2+y^2+z^2)$ and $\sqrt{a+b+c}\leq\sqrt{a}+\sqrt{b}+\sqrt{c}$, we control $J_{4k}(t)$ exactly as we have done for $J_{3k}(t)$.
We have

$$J_{4k}(t)\leq Z_1(t)+Z_2(t) \qquad (3.20)$$

where

$$Z_1(t)=C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^3e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f(v)|f(u')\sum_{j=1}^3|(\partial_{v^j}f)(v')|d\omega dudv,$$

$$Z_2(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^3 e^{|v|^2}\sigma_0(\omega)|\partial_{v^k} f(v)| f(v') \sum_{j=1}^{3} |(\partial_{v^j} f)(u')| d\omega du dv.$$

We now estimate $Z_1(t)$ and $Z_2(t)$.

$$\begin{aligned}
Z_1(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^3 \frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^k} f(v)|e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2} \\
&\times \sum_{j=1}^{3}|(\partial_{v^j} f)(v')|e^{-\frac{1}{2}|u|^2}d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint [(u^0)^3\sqrt{u^0}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^k} f(v)|] \\
&\times [\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0v^0}}e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}\sum_{j=1}^{3}|(\partial_{v^j} f)(v')|]d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds (\iiint (u^0)^7\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^k} f(v))^2 d\omega du dv)^{\frac{1}{2}} \\
&\times (\iiint \frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2 e^{|v'|^2}(\sum_{j=1}^{3}|(\partial_{v^j} f)(v')|)^2 d\omega du dv)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds (\int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k} f(v))^2 \int_{S_{ab}}\sigma_0(\omega)d\omega \int_{\mathbb{R}^3}(1+|u|^2)^{\frac{7}{2}}e^{-|u|^2}du)^{\frac{1}{2}} \\
&\times (\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2 e^{|v'|^2}(\sum_{j=1}^{3}|(\partial_{v^j} f)(v')|)^2 d\omega du' dv')^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k} f(s)\|_e ds \\
&\times (\int_{\mathbb{R}^3} e^{|u'|^2}(f(u'))^2 du' \int_{S_{ab}}\sigma_0(\omega)d\omega \int_{\mathbb{R}^3} e^{|v'|^2}(\sum_{j=1}^{3}|(\partial_{v^j} f)(v')|)^2 dv')^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e \|f(s)\|_e ds \\
&\times (\sum_{j=1}^{3}\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j} f)(v'))^2 dv')^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e \|f(s)\|_e ds \\
&\times \sum_{j=1}^{3}(\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j} f)(v'))^2 dv')^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3}\|\partial_{v^j} f(s)\|_e)\int_0^t a^{-1}b^{-2}ds.
\end{aligned}$$

With the same steps as we estimate $Z_1(t)$, we have

$$Z_2(t) \leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3}\|\partial_{v^j} f(s)\|_e)\int_0^t a^{-1}b^{-2}ds.$$

By (3.20), it follows that

$$J_{4t} \leq C \underset{s\in[0,t]}{Sup} (\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e) \int_0^t a^{-1}b^{-2} ds. \tag{3.21}$$

By (3.15), (3.16), (3.17) and (3.21) we obtain

$$\begin{aligned}
\|\partial_{v^k} f(t)\|_e^2 \leq\ & \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (\|\partial_{v^k} f(s)\|_e^2 \|f(s)\|_e) \int_0^t a^{-1}b^{-2} ds \\
& + C \underset{s\in[0,t]}{Sup} (\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2) \int_0^t a^{-1}b^{-2} ds + C \underset{s\in[0,t]}{Sup} (\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2) \int_0^t a^{-1}b^{-2} ds \\
& + C \underset{s\in[0,t]}{Sup} (\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e) \int_0^t a^{-1}b^{-2} ds \\
& \leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (\||f(s)\||_e^3) \int_0^\infty a^{-1}b^{-2} ds \\
& + C \underset{s\in[0,t]}{Sup} (\||f(s)\||_e^3) \int_0^\infty a^{-1}b^{-2} ds + C \underset{s\in[0,t]}{Sup} (\||f(s)\||_e^3) \int_0^\infty a^{-1}b^{-2} ds \\
& + C \underset{s\in[0,t]}{Sup} (\||f(s)\||_e^3) \int_0^\infty a^{-1}b^{-2} ds \\
& \leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (\||f(s)\||_e^3).
\end{aligned}$$

□

3.2.2 L^2-energy estimates of the homogeneous equation for hard potentials

In this part we take $\alpha = 0$ in (1.70) and we work on the additional assumption (2.50).
We also consider that the coefficient b of the metric tensor satisfies

$$\int_{\mathbb{R}_+} b^{\beta-\frac{3}{2}}(\tau) d\tau < \infty, \qquad \beta \in [0, \frac{3}{2}]. \tag{3.22}$$

Lemma 3.3. Let f be a solution to the Cauchy problem (3.1)-(3.2). Then f satisfies

$$\|f(t)\|_e^2 \leq \|f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (\|f(s)\|_e^3) \tag{3.23}$$

for some constant C which does not depend on t.

Proof. We multiply the equation (3.1) by $2f(t,v)$ and integrate from 0 to t to obtain

$$f^2(t,v) = f^2(0,v) - 2\int_0^t f(s,v) Q_{gain}(f,f)(s,v) ds - 2\int_0^t f(s,v) Q_{loss}(f,f)(s,v) ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2} f^2(t,v) = e^{|v|^2} f^2(0,v) - 2\int_0^t e^{|v|^2} f(s,v) Q_{gain}(f,f)(s,v) ds - 2\int_0^t e^{|v|^2} f(s,v) Q_{loss}(f,f)(s,v) ds.$$

Integrating the above equation with respect to v yields

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2} f(v)\vartheta_\phi\sigma(g,\omega)f(v')f(u')d\omega du dv$$

$$-2\int_0^t a^{-1}b^{-2}ds \iiint f(v)\vartheta_\phi\sigma(g,\omega)f(v)f(u)d\omega du dv.$$

Since the function f is non-negative, we can ignore the loss term and have

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\int_0^t a^{-1}b^{-2}A(s)ds, \tag{3.24}$$

where

$$A(s) = \iiint e^{|v|^2} f(v)\vartheta_\phi\sigma(g,\omega)f(v')f(u')d\omega du dv.$$

By (2.43), (2.18), (2.7), (2.23), (3.12) and (2.27) as well as the Cauchy-Schwartz inequality, and since $\vartheta_\phi \leq 4$, we can state that

$$\begin{aligned}
A(s) &= \iiint e^{\frac{1}{2}|v|^2} f(v)\vartheta_\phi\sigma(g,\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega du dv\\
&\leq C\iiint e^{\frac{1}{2}|v|^2} f(v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega du dv\\
&+ C\iiint e^{\frac{1}{2}|v|^2} f(v)\vartheta_\phi \sigma_0(\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega du dv\\
&\leq C\iiint [g^{-\beta}\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}f(v)][\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')]d\omega du dv\\
&+ C\iiint [\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}f(v)][\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')]d\omega du dv\\
&\leq C(\iiint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du dv)^{\frac{1}{2}}\\
&\times(\iiint \vartheta_\phi\sigma_0(\omega)e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega du dv)^{\frac{1}{2}}\\
&+ C(\iiint \vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du dv)^{\frac{1}{2}}\\
&\times(\iiint \vartheta_\phi\sigma_0(\omega)e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega du dv)^{\frac{1}{2}}\\
&\leq C(\iiint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du)^{\frac{1}{2}}\\
&+ C(\iiint \vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du)^{\frac{1}{2}}\\
&\times(\iint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega du'dv')^{\frac{1}{2}}.
\end{aligned}$$

Then we have

$$\begin{aligned} A(s) &\leq C\Big(\int_{\mathbb{R}^3} g^{-2\beta}\vartheta_\phi e^{-|u|^2}du \int_{S_{ab}} \sigma_0(\omega)d\omega \int_{\mathbb{R}^3} e^{|v|^2} f^2(v)dv\Big)^{\frac{1}{2}} \\ &\times \Big(\int_{\mathbb{R}^3} e^{|v'|^2} f^2(v')dv' \int_{S_{ab}} \sigma_0(\omega)d\omega \int_{\mathbb{R}^3} e^{|u'|^2} f^2(u')du'\Big)^{\frac{1}{2}} \\ &+ C\Big(\int_{\mathbb{R}^3} \vartheta_\phi e^{-|u|^2}du \int_{S_{ab}} \sigma_0(\omega)d\omega \int_{\mathbb{R}^3} e^{|v|^2} f^2(v)dv\Big)^{\frac{1}{2}} \\ &\times \Big(\int_{\mathbb{R}^3} e^{|v'|^2} f^2(v')dv' \int_{S_{ab}} \sigma_0(\omega)d\omega \int_{\mathbb{R}^3} e^{|u'|^2} f^2(u')du'\Big)^{\frac{1}{2}} \\ &\leq C\|f(s)\|_e^3\Big(\int_{\mathbb{R}^3} g^{-2\beta}\vartheta_\phi e^{-|u|^2}du\Big)^{\frac{1}{2}} \\ &+ C\|f(s)\|_e^3\Big(\int_{\mathbb{R}^3} e^{-|u|^2}du\Big)^{\frac{1}{2}} \\ &\leq Cb^{\beta-\frac{1}{2}}\|f(s)\|_e^3 + C\|f(s)\|_e^3 \\ &\leq C(1+b^{\beta-\frac{1}{2}}). \end{aligned} \tag{3.25}$$

Inserting (3.25) into (3.24) yields

$$\|f(t)\|_e^2 \leq C \underset{s\in[0,t]}{Sup}(\|f(s)\|_e^3) \int_0^t (a^{-1}(s)b^{-2}(s) + a^{-1}(s)b^{\beta-\frac{5}{2}}(s))ds.$$

The desired result is obtained because $a^{-1}b^{-2} + a^{-1}b^{\beta-\frac{5}{2}}$ is integrable over $\mathbb{R}_+$. □

Lemma 3.4. Let f be a solution to Cauchy problem (3.1)-(3.2). Then for $k \in \{1, 2, , 3\}$, f satisfies the following estimate

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(\||f(s)\||_e^2) \tag{3.26}$$

for some constant C which does not depend on t.

Proof. We take the partial derivative of the Boltzmann equation (3.1) with respect to v^k

$$\partial_t\partial_{v^k} f(t,v) = \partial_{v^k} Q_{gain}(f,f)(t,v) - \partial_{v^k} Q_{loss}(f,f)(t,v).$$

We multiply the above equation by $2\partial_{v^k} f(t,v)$ and integrate from 0 to t to obtain

$$(\partial_{v^k} f)^2(t,v) = (\partial_{v^k} f)^2(0,v) - 2\int_0^t \partial_{v^k} f(s,v)\partial_{v^k} Q_{gain}(f,f)(s,v)ds - 2\int_0^t \partial_{v^k} f(s,v)\partial_{v^k} Q_{loss}(f,f)(s,v)ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2}(\partial_{v^k} f)^2(t,v) = e^{|v|^2}(\partial_{v^k} f)^2(0,v) - 2\int_0^t e^{|v|^2}\partial_{v^k} f(s,v)\partial_{v^k} Q_{gain}(f,f)(s,v)ds$$

$$-2\int_0^t e^{|v|^2}\partial_{v^k} f(s,v)\partial_{v^k} Q_{loss}(f,f)(s,v)ds.$$

Integrating the above equation with respect to v yields

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^k + J_{1k}(t) + J_{2k}(t) + J_{3k}(t) + J_{4t}(t) \tag{3.27}$$

where $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ are defined as follows:

$$J_{1k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\vartheta_\phi \sigma(g,\omega)(\partial_{v^k} f)^2(v)dvdud\omega,$$

$$J_{2k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)f(v)|\partial_{v^k} f(v)\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))|dvdud\omega,$$

$$J_{3k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))\partial_{v^k} f(v)|f(v')f(u')dvdud\omega,$$

$$J_{4k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi\sigma(g,\omega)|\partial_{v^k} f(v)\partial_{v^k}[f(v')f(u')]|dvdud\omega.$$

Let's estimate $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ for $k \in \{1,2,3\}$.

Estimate of $J_{1k}(t)$: For any $k \in \{1,2,3\}$ and knowing that $\vartheta_\phi \leq 4$, we have

$$\begin{aligned}
J_{1k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(\partial_{v^k} f)^2(v)dvdud\omega \\
&+ C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\vartheta_\phi\sigma_0(\omega)(\partial_{v^k} f)^2(v)dvdud\omega \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k} f)^2(v)dv \iint g^{-\beta}\vartheta_\phi\sigma_0(\omega)f(u)d\omega du \\
&+ C\int_0^t 2a^{-1}b^{-2}ds \int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k} f)^2(v)dv \iint \vartheta_\phi\sigma_0(\omega)f(u)d\omega du \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds \iint g^{-\beta}\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|u|^2}\sqrt{\vartheta_\phi\sigma_0(\omega)}f(u)d\omega du \\
&+ C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds \iint \sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|u|^2}\sqrt{\vartheta_\phi\sigma_0(\omega)}f(u)d\omega du \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds(\iint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}d\omega du)^{\frac{1}{2}}(\iint \vartheta_\phi\sigma_0(\omega)(f(u))^2 e^{|u|^2}d\omega du)^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds(\iint \vartheta_\phi\sigma_0(\omega)e^{-|u|^2}d\omega du)^{\frac{1}{2}}(\iint \vartheta_\phi\sigma_0(\omega)(f(u))^2 e^{|u|^2}d\omega du)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds(\int_{\mathbb{R}^3} g^{-2\beta}\vartheta_\phi e^{-|u|^2}du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times(\int_{\mathbb{R}^3}\vartheta_\phi e^{|u|^2}(f(u))^2 du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f(s)\|_e^2 ds(\int_{\mathbb{R}^3}\vartheta_\phi e^{-|u|^2}du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times(\int_{\mathbb{R}^3}\vartheta_\phi e^{|u|^2}(f(u))^2 du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e^2\|f(s)\|_e)(\int_0^t a^{-1}b^{\beta-\frac{5}{2}}ds + \int_0^t a^{-1}b^{-2}ds). \qquad (3.28)
\end{aligned}$$

Estimate of $J_{2k}(t)$: For $k=1$, by (2.55) we have

$$
\begin{aligned}
J_{21}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}u^0 g^{-\beta}\sigma_0(\omega)f(u)f(v)|\partial_{v^1}f(v)|dvdud\omega \\
&+C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}u^0\sigma_0(\omega)f(u)f(v)|\partial_{v^1}f(v)|dvdud\omega \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint [e^{\frac{1}{2}|v|^2}u^0 g^{-\beta}\sqrt{\sigma_0(\omega)}|\partial_{v^1}f(v)|e^{-\frac{1}{2}|u|^2}] \\
&\times [e^{\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}\sqrt{\sigma_0(\omega)}f(u)f(v)]dvdud\omega \\
&+C\int_0^t 2a^{-2}b^{-2}ds\iiint [e^{\frac{1}{2}|v|^2}u^0\sqrt{\sigma_0(\omega)}|\partial_{v^1}f(v)|e^{-\frac{1}{2}|u|^2}] \\
&\times [e^{\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}\sqrt{\sigma_0(\omega)}f(u)f(v)]dvdud\omega \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\iiint e^{|v|^2}(u^0)^2 g^{-2\beta}\sigma_0(\omega)(\partial_{v^1}f(v))^2e^{-|u|^2}dvdud\omega)^{\frac{1}{2}} \\
&\times (\iiint e^{|u|^2}e^{|v|^2}\sigma_0(\omega)f^2(u)f^2(v)dvdud\omega)^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-2}b^{-2}ds(\iiint e^{|v|^2}(u^0)^2\sigma_0(\omega)(\partial_{v^1}f(v))^2e^{-|u|^2}dvdud\omega)^{\frac{1}{2}} \\
&\times (\iiint e^{|u|^2}e^{|v|^2}\sigma_0(\omega)f^2(u)f^2(v)dvdud\omega)^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)g^{-2\beta}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times (\int_{\mathbb{R}^3}e^{|u|^2}f^2(u)du\int_{\mathbb{R}^3}e^{|v|^2}f^2(v)dv\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&+\leq C\int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times (\int_{\mathbb{R}^3}e^{|u|^2}f^2(u)du\int_{\mathbb{R}^3}e^{|v|^2}f^2(v)dv\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^1}f(s)\|_e\|f(s)\|_e^2)(\int_0^t a^{-2}b^{\beta-2}ds+\int_0^t a^{-2}b^{-2}ds). \qquad (3.29)
\end{aligned}
$$

For $k=2\,or\,3$, by (2.56) and as we have done above

$$
J_{2k}(t)\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e^2)(\int_0^t a^{-1}b^{\beta-3}ds+\int_0^t a^{-1}b^{-3}ds). \qquad (3.30)
$$

The result holds since $a^{-2}b^{\beta-3}$ is integrable over $\mathbb{R}_+$.

Estimate of $J_{3k}(t)$: For $k=1$, by (2.55), (2.18), (2.27), (2.25) and (3.12) as well as the Cauchy-

Schwartz inequality

$$
\begin{aligned}
J_{31}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}u^0(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^1}f(v)|f(v')f(u')d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint u^0\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&+C\int_0^t 2a^{-2}b^{-2}ds\iiint u^0\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint[u^0\sqrt{u^0}g^{-\beta}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)]\\
&\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0u^0}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}]d\omega dudv\\
&+C\int_0^t 2a^{-2}b^{-2}ds\iiint[u^0\sqrt{u^0}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)]\\
&\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0u^0}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}]d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\iiint(u^0)^3g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^1}f(v))^2d\omega dudv)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|v'|^2}(f^2(v'))^2e^{|u'|^2}(f^2(u'))^2d\omega dudv)^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-2}b^{-2}ds(\iiint(u^0)^3\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^1}f(v))^2d\omega dudv)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|v'|^2}(f^2(v'))^2e^{|u'|^2}(f^2(u'))^2d\omega dudv)^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{3}{2}}g^{-2\beta}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(f(u'))^2d\omega du'dv')^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{3}{2}}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(f(u'))^2d\omega du'dv')^{\frac{1}{2}}\\
&\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^1}f(s)\|_e\|f(s)\|_e^2)(\int_0^t a^{-2}b^{\beta-2}ds+\int_0^t a^{-2}b^{-2}ds).
\end{aligned}
\tag{3.31}
$$

For $k=2\,or\,3$ by (2.56) and as we have done before

$$
J_{3k}(t)\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e^2)(\int_0^t a^{-1}b^{\beta-3}ds+\int_0^t a^{-1}b^{-3}ds).
\tag{3.32}
$$

Since $a^{-1}b^{\beta-3}$ is integrable over $\mathbb{R}_+$, the result holds.

Estimate of $J_{4k}(t)$: Let us observe that with the relation (3.18)

$$\begin{aligned}\partial_{v^k[f(v')f(u')]} &= f(u')\partial_{v^k}(f(v')) + f(v')\partial_{v^k}(f(u'))\\ &= f(u')\sum_{j=1}^{3}\partial_{v^k}(v'^j)\partial_{v'^j}(f(v')) + f(v')\sum_{j=1}^{3}\partial_{v^k}(u'^j)\partial_{u'^j}(f(u'))\\ &= f(u')\sum_{j=1}^{3}\partial_{v^k}(v'^j)(\partial_{v^j}f)(v') + f(v')\sum_{j=1}^{3}\partial_{v^k}(u'^j)(\partial_{v^j}f)(u').\end{aligned}$$

We notice that $J_{4k}(t)$ is more difficult to handle due to the presence of derivatives of the post-collisional momenta which produce some singularities.

We fix a momentum v and note that since $a(t)$ and $b(t)$ are increasing functions with $a(0) = 1$, then it exists a finite time t_0 such that

$$t \geq t_0 \longleftrightarrow |v| \leq a(t).$$

For $k \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$ we break up the estimate of $\partial_{v^k}(v'^j)$ into a number of steps.

Step 1: $t \geq t_0$.

The relations $|v| \leq a(t)$ and $a(t) \leq b(t)$ allow the estimate of the derivatives of the post-collisional momenta with the first parametrization (1.56)-(1.57. From the relations (2.59), (2.60), (2.61) and (2.62), we have

$$\partial_{v^k}(v'^j) \leq C\sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2}(u^0)^4 \leq C\sqrt{1 + a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^4.$$

Step 2: $t < t_0$ and $|v| \leq 2|u|$.

In this case we recall (2.17). With the first parametrization (1.56)-(1.57 and the relations (2.59), (2.60), (2.61) and (2.62), the terms $|\partial_{v^k}(v'^j)|$ are controlled by $C(u^0)^5$.

Step 3: $t < t_0$ and $|v| \geq 2|u|$.

Here we are going to use the second parametrization (1.61)-(1.62). There are singularities in this region. To circumvent the difficulty, we remark that from the relation $|v| \geq 2|u|$, it follows that $|v - u| \geq \frac{1}{2}|v|$ and $|v + u| \geq \frac{1}{2}|v|$.

Then, from the estimates (2.67), (2.68), (2.69) and (2.70), using the assumption $a \leq b \leq \sqrt{2}a$, straightforward computations allow us to control all the terms $|\partial_{v^k}(v'^j)|$ by $C(u^0)^3$.

To summarize, since $u^0 \geq 1$ we can estimate all the terms $|\partial_{v^k}(v'^j)|$ and $|\partial_{v^k}(u'^j)|$ like this

$$|\partial_{v^k}(v'^j)| \leq C(u^0)^5 \qquad \textit{and} \qquad |\partial_{v^k}(u'^j)| \leq C(u^0)^5. \tag{3.33}$$

From the relation (3.18) we can deduce that

$$|\partial_{v^k[f(v')f(u')]}| \leq C(u^0)^5[f(u')\sum_{j=1}^{3}(\partial_{v^j}f)(v') + f(v')\sum_{j=1}^{3}(\partial_{v^j}f)(u')]. \tag{3.34}$$

Taking into account (3.34) and knowing that $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ and $\sqrt{a + b + c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$, we can estimate $J_{4k}(t)$ exactly as we have done the estimation of

$J_{3k}(t)$.
We have

$$J_{4k}(t) \leq Z_1(t) + Z_2(t) \tag{3.35}$$

where

$$Z_1(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^k}f(v)|f(u')\sum_{j=1}^{3}(\partial_{v^j}f)(v')d\omega dudv,$$

and

$$Z_2(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^k}f(v)|f(v')\sum_{j=1}^{3}(\partial_{v^j}f)(u')d\omega dudv.$$

Since $\vartheta_\phi \leq 4$ and following the steps of the estimation of $J_{3k}(t)$ we have:

$$Z_1(t) \leq C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}$$

$$\times\sum_{j=1}^{3}(\partial_{v^j}f)(v')e^{-\frac{1}{2}|u|^2}d\omega dudv$$

$$+C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}$$

$$\times\sum_{j=1}^{3}(\partial_{v^j}f)(v')e^{-\frac{1}{2}|u|^2}d\omega dudv$$

$$\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint [(u^0)^5\sqrt{u^0}g^{-\beta}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|]$$

$$\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0v^0}}e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}\sum_{j=1}^{3}(\partial_{v^j}f)(v')]d\omega dudv$$

$$+C\int_0^t 2a^{-1}b^{-2}ds \iiint [(u^0)^5\sqrt{u^0}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|]$$

$$\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0v^0}}e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}\sum_{j=1}^{3}(\partial_{v^j}f)(v')]d\omega dudv$$

$$\leq C\int_0^t 2a^{-1}b^{-2}ds(\iiint (u^0)^{11}g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^k}f(v))^2d\omega dudv)^{\frac{1}{2}}$$

$$\times(\iiint \frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2e^{|v'|^2}(\sum_{j=1}^{3}(\partial_{v^j}f)(v'))^2d\omega dudv)^{\frac{1}{2}}$$

$$+C\int_0^t 2a^{-1}b^{-2}ds(\iiint (u^0)^{11}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^k}f(v))^2d\omega dudv)^{\frac{1}{2}}$$

$$\times(\iiint \frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2e^{|v'|^2}(\sum_{j=1}^{3}(\partial_{v^j}f)(v'))^2d\omega dudv)^{\frac{1}{2}}$$

$$\leq C\int_0^t 2a^{-1}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^k}f(v))^2\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{11}{2}}g^{-2\beta}e^{-|u|^2}du)^{\frac{1}{2}}$$

$\times(\iiint \frac{v^0}{v'^0 u'^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2 e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2 d\omega du' dv')^{\frac{1}{2}}$

$+C\int_0^t 2a^{-1}b^{-2}ds(\int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k}f(v))^2\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{11}{2}}e^{-|u|^2}du)^{\frac{1}{2}}$

$\times(\iiint \frac{v^0}{v'^0 u'^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2 e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2 d\omega du' dv')^{\frac{1}{2}}$

$\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e ds$

$\times(\int_{\mathbb{R}^3} e^{|u'|^2}(f(u'))^2 du'\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3} e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$+C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e ds$

$\times(\int_{\mathbb{R}^3} e^{|u'|^2}(f(u'))^2 du'\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3} e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds$

$\times(\sum_{j=1}^3\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds$

$\times\sum_{j=1}^3(\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$+C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds$

$\times(\sum_{j=1}^3\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds$

$\times\sum_{j=1}^3(\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$+C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds$

$\times\sum_{j=1}^3(\int_{\mathbb{R}^3} e^{|v'|^2}((\partial_{v^j}f)(v'))^2 dv')^{\frac{1}{2}}$

$\leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f(s)\|_e)\int_0^t a^{-1}b^{\beta-2}ds$

$+C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f(s)\|_e)\int_0^t a^{-1}b^{\beta-2}ds$

$$\leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \textstyle\sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e)(\int_0^t a^{-1}b^{\beta-2}ds + \int_0^t a^{-1}b^{-2}ds).$$

Taking the same steps as we estimate $Z_1(t)$, we have

$$Z_2(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e)(\int_0^t a^{-1}b^{\beta-2}ds + \int_0^t a^{-1}b^{-2}ds).$$

By (3.35), it follows that

$$J_{4t} \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e)(\int_0^t a^{-1}b^{\beta-2}ds + \int_0^t a^{-1}b^{-2}ds). \tag{3.36}$$

By (3.28), (3.29), (3.30), (3.31), (3.32) and (3.36), we obtain

$$\begin{aligned}
\|\partial_{v^k} f(t)\|_e^2 &\leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e^2 \|f(s)\|_e)(\int_0^t a^{-1}b^{\beta-\frac{5}{2}}ds + \int_0^t a^{-1}b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2)(\int_0^t b^{\beta-2}ds + \int_0^t b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2)(\int_0^t b^{\beta-2}ds + \int_0^t b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e)(\int_0^t a^{-1}b^{\beta-2}ds + \int_0^t a^{-1}b^{-2}ds) \\
&\leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^3)(\int_0^\infty a^{-1}b^{\beta-\frac{5}{2}}ds + \int_0^\infty a^{-1}b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^3)(\int_0^\infty b^{\beta-2}ds + \int_0^\infty b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^3)(\int_0^\infty b^{\beta-2}ds + \int_0^\infty b^{-2}ds) \\
&+ C \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^3)(\int_0^\infty a^{-1}b^{\beta-2}ds + \int_0^\infty a^{-1}b^{-2}ds) \\
&\leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(\||f(s)\||_e^3).
\end{aligned}$$

□

3.2.3 L^2-energy estimates of the homogeneous equation for soft potentials

In this part we consider the additional assumption (2.50) on the scattering kernel.
We also consider the condition (3.22) on the metric tensor.

Lemma 3.5. Let f be a solution to the Cauchy problem (3.1)-(3.2). Then f satisfies

$$\|f(t)\|_e^2 \leq \|f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(\|f(s)\|_e^3) \tag{3.37}$$

for some constant C not depending on t.

Proof. We multiply the equation (3.1) by $2f(t,v)$ and integrate from 0 to t to obtain

$$f^2(t,v) = f^2(0,v) - 2\int_0^t f(s,v)Q_{gain}(f,f)(s,v)ds - 2\int_0^t f(s,v)Q_{loss}(f,f)(s,v)ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2}f^2(t,v) = e^{|v|^2}f^2(0,v) - 2\int_0^t e^{|v|^2}f(s,v)Q_{gain}(f,f)(s,v)ds - 2\int_0^t e^{|v|^2}f(s,v)Q_{loss}(f,f)(s,v)ds.$$

Integrating the above equation with respect to v yields

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2}f(v)\vartheta_\phi\sigma(g,\omega)f(v')f(u')d\omega dudv$$

$$-2\int_0^t a^{-1}b^{-2}ds\iiint f(v)\vartheta_\phi\sigma(g,\omega)f(v)f(u)d\omega dudv.$$

Since the function f is non-negative, we can ignore the loss term and have

$$\|f(t)\|_e^2 = \|f(0)\|_e^0 + 2\int_0^t a^{-1}b^{-2}A(s)ds \tag{3.38}$$

where

$$A(s) = \iiint e^{|v|^2}f(v)\vartheta_\phi\sigma(g,\omega)f(v')f(u')d\omega dudv.$$

By (2.43), (2.18), (2.7), (2.23), (3.12) and (2.27) as well as the Cauchy-Schwartz inequality we can state that

$$\begin{aligned}
A(s) &= \iiint e^{\frac{1}{2}|v|^2}f(v)\vartheta_\phi\sigma(g,\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\iiint e^{\frac{1}{2}|v|^2}f(v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}(|v|^2+|u|^2-|v'|^2-|u'|^2)}e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\iiint [g^{-\beta}\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}f(v)][\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')]d\omega dudv\\
&\leq C(\iiint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega dudv)^{\frac{1}{2}}\\
&\times(\iiint \vartheta_\phi\sigma_0(\omega)e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega dudv)^{\frac{1}{2}}\\
&\leq C(\iiint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}f^2(v)d\omega du)^{\frac{1}{2}}\\
&\times(\iint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}f^2(v')e^{|u'|^2}f^2(u')d\omega du'dv')^{\frac{1}{2}}\\
&\leq C(\int_{\mathbb{R}^3} g^{-2\beta}\vartheta_\phi e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}e^{|v|^2}f^2(v)dv)^{\frac{1}{2}}\\
&\times(\int_{\mathbb{R}^3}e^{|v'|^2}f^2(v')dv'\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}e^{|u'|^2}f^2(u')du')^{\frac{1}{2}}\\
&\leq C\|f(s)\|_e^3(\int_{\mathbb{R}^3}g^{-2\beta}\vartheta_\phi e^{-|u|^2}du)^{\frac{1}{2}}\\
&\leq Cb^{\beta-\frac{1}{2}}\|f(s)\|_e^3.
\end{aligned} \tag{3.39}$$

Inserting (3.39) into (3.38) yields

$$\|f(t)\|_e^2 \leq C \underset{s\in[0,t]}{Sup}(\|f(s)\|_e^3) \int_0^t a^{-1}(s)b^{\beta-\frac{5}{2}}(s)ds.$$

The desired result is obtained because $a^{-1}b^{\beta-\frac{5}{2}}$ is integrable over $\mathbb{R}_+$. □

Lemma 3.6. Let f be a solution to the Cauchy problem (3.1)-(3.2). Then for $k \in \{1, 2, , 3\}$, f satisfies the following estimate

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(\||f(s)\||_e^2) \tag{3.40}$$

for some constant C not depending on t.

Proof. We take the partial derivative of the Boltzmann equation (3.1) with respect to v^k

$$\partial_t\partial_{v^k} f(t,v) = \partial_{v^k} Q_{gain}(f,f)(t,v) - \partial_{v^k} Q_{loss}(f,f)(t,v).$$

We multiply the above equation by $2\partial_{v^k} f(t,v)$ and integrate from 0 to t to obtain

$$(\partial_{v^k} f)^2(t,v) = (\partial_{v^k} f)^2(0,v) - 2\int_0^t \partial_{v^k} f(s,v)\partial_{v^k} Q_{gain}(f,f)(s,v)ds - 2\int_0^t \partial_{v^k} f(s,v)\partial_{v^k} Q_{loss}(f,f)(s,v)ds.$$

We multiply this resulting equation by $e^{|v|^2}$ to obtain

$$e^{|v|^2}(\partial_{v^k} f)^2(t,v) = e^{|v|^2}(\partial_{v^k} f)^2(0,v) - 2\int_0^t e^{|v|^2}\partial_{v^k} f(s,v)\partial_{v^k} Q_{gain}(f,f)(s,v)ds$$

$$-2\int_0^t e^{|v|^2}\partial_{v^k} f(s,v)\partial_{v^k} Q_{loss}(f,f)(s,v)ds.$$

Integrating the above equation with respect to v yields

$$\|\partial_{v^k} f(t)\|_e^2 \leq \|\partial_{v^k} f(0)\|_e^k + J_{1k}(t) + J_{2k}(t) + J_{3k}(t) + J_{4t}(t) \tag{3.41}$$

where $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ are defined as follows:

$$J_{1k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\vartheta_\phi\sigma(g,\omega)(\partial_{v^k} f)^2(v)dvdud\omega,$$

$$J_{2k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)f(v)|\partial_{v^k} f(v)\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))|dvdud\omega,$$

$$J_{3k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))\partial_{v^k} f(v)|f(v')f(u')dvdud\omega,$$

$$J_{4k}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi\sigma(g,\omega)|\partial_{v^k} f(v)\partial_{v^k}[f(v')f(u')]|dvdud\omega.$$

Let's estimate $J_{1k}(t)$, $J_{2k}(t)$, $J_{3k}(t)$ and $J_{4k}(t)$ for $k \in \{1,2,3\}$.

Estimate of $J_{1k}(t)$: For any $k \in \{1,2,3\}$ and knowing that $\vartheta_\phi \leq 4$, we have

$$\begin{aligned}
J_{1k}(t) &\leq C \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} f(u)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(\partial_{v^k}f)^2(v)dvdud\omega \\
&\leq C \int_0^t 2a^{-1}b^{-2}ds \int_{\mathbb{R}^3} e^{|v|^2}(\partial_{v^k}f)^2(v)dv \iint g^{-\beta}\vartheta_\phi\sigma_0(\omega)f(u)d\omega du \\
&\leq C \int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e^2 ds \iint g^{-\beta}\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|u|^2}\sqrt{\vartheta_\phi\sigma_0(\omega)}f(u)d\omega du \\
&\leq C \int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e^2 ds (\iint g^{-2\beta}\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}d\omega du)^{\frac{1}{2}}(\iint \vartheta_\phi\sigma_0(\omega)(f(u))^2e^{|u|^2}d\omega du)^{\frac{1}{2}} \\
&\leq C \int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f(s)\|_e^2 ds (\int_{\mathbb{R}^3} g^{-2\beta}\vartheta_\phi e^{-|u|^2}du \int_{S_{ab}} \sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times (\int_{\mathbb{R}^3}\vartheta_\phi e^{|u|^2}(f(u))^2du \int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e^2\|f(s)\|_e)\int_0^t a^{-1}b^{\beta-\frac{5}{2}}ds. \qquad (3.42)
\end{aligned}$$

Estimate of $J_{2k}(t)$: For $k=1$, by (2.55)

$$\begin{aligned}
J_{21}(t) &\leq C \int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}u^0 g^{-\beta}\sigma_0(\omega)f(u)f(v)|\partial_{v^1}f(v)|dvdud\omega \\
&\leq C \int_0^t 2a^{-2}b^{-2}ds \iiint [e^{\frac{1}{2}|v|^2}u^0g^{-\beta}\sqrt{\sigma_0(\omega)}|\partial_{v^1}f(v)|e^{-\frac{1}{2}|u|^2}] \\
&\times [e^{\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}\sqrt{\sigma_0(\omega)}f(u)f(v)]dvdud\omega \\
&\leq C \int_0^t 2a^{-2}b^{-2}ds(\iiint e^{|v|^2}(u^0)^2g^{-2\beta}\sigma_0(\omega)(\partial_{v^1}f(v))^2e^{-|u|^2}dvdud\omega)^{\frac{1}{2}} \\
&\times (\iiint e^{|u|^2}e^{|v|^2}\sigma_0(\omega)f^2(u)f^2(v)dvdud\omega)^{\frac{1}{2}} \\
&\leq C \int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)g^{-2\beta}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\times (\int_{\mathbb{R}^3}e^{|u|^2}f^2(u)du\int_{\mathbb{R}^3}e^{|v|^2}f^2(v)dv\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^1}f(s)\|_e\|f(s)\|_e^2)\int_0^t a^{-2}b^{\beta-2}ds. \qquad (3.43)
\end{aligned}$$

For $k = 2\, or\, 3$, by (2.56) and as we have done above

$$J_{2k}(t) \leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e^2)\int_0^t a^{-1}b^{\beta-3}ds. \qquad (3.44)$$

The result holds since $a^{-2}b^{\beta-3}$ is integrable over $\mathbb{R}_+$.

Estimate of $J_{3k}(t)$: For $k=1$, by (2.55), (2.18), (2.27), (2.25) and (3.12), and as well as the Cauchy-

Schwartz inequality

$$\begin{aligned}
J_{31}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}u^0g^{-\beta}\sigma_0(\omega)|\partial_{v^1}f(v)|f(v')f(u')d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint u^0\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint[u^0\sqrt{u^0}g^{-\beta}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^1}f(v)]\\
&\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0u^0}}e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}f(u')e^{-\frac{1}{2}|u|^2}]d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\iiint(u^0)^3g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^1}f(v))^2d\omega dudv)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|v'|^2}(f^2(v'))^2e^{|u'|^2}(f^2(u'))^2d\omega dudv)^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^1}f(v))^2dv\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{3}{3}}g^{-2\beta}e^{-|u|^2}du\int_{S_{ab}}\sigma_0(\omega)d\omega)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(f(u'))^2d\omega du'dv')^{\frac{1}{2}}\\
&\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^1}f(s)\|_e\|f(s)\|_e^2)\int_0^t a^{-2}b^{\beta-2}ds. \qquad (3.45)
\end{aligned}$$

For $k=2\,or\,3$ by (2.56) and as we have done above

$$J_{3k}(t)\leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e^2)\int_0^t a^{-1}b^{\beta-3}ds. \qquad (3.46)$$

Since $a^{-1}b^{\beta-3}$ is integrable over $\mathbb{R}_+$ the results holds.
Estimate of $J_{4k}(t)$: Let us recall the relation (3.18)

$$\begin{aligned}
\partial_{v^k[f(v')f(u')]} &= f(u')\partial_{v^k}(f(v'))+f(v')\partial_{v^k}(f(u'))\\
&= f(u')\sum_{j=1}^3\partial_{v^k}(v'^j)\partial_{v'^j}(f(v'))+f(v')\sum_{j=1}^3\partial_{v^k}(u'^j)\partial_{u'^j}(f(u'))\\
&= f(u')\sum_{j=1}^3\partial_{v^k}(v'^j)(\partial_{v^j}f)(v')+f(v')\sum_{j=1}^3\partial_{v^k}(u'^j)(\partial_{v^j}f)(u').
\end{aligned}$$

We notice that $J_{4k}(t)$ is more difficult to handle due to the presence of derivatives of post-collisional momenta which produce singularities.
We fix a momentum v and note that since $a(t)$ and $b(t)$ are increasing functions with $a(0)=1$, then it exists a finite time t_0 such that

$$t\geq t_0\longleftrightarrow|v|\leq a(t).$$

For $k\in\{1,2,3\}$ and $j\in\{1,2,3\}$ we break up the estimate of $\partial_{v^k}(v'^j)$ into a number of steps.
Step 1: $t\geq t_0$.
The relation $|v|\leq a(t)$ and $a(t)\leq b(t)$ allow the estimate of the derivatives of the post-collisional

momenta with the first parametrization (1.56)-(1.57). From the relations (2.59), (2.60), (2.61) and (2.62), we have

$$\partial_{v^k}(v'^j) \leq C\sqrt{1+a^{-2}(v^1)^2+b^{-2}|\bar{v}|^2}(u^0)^4 \leq C\sqrt{1+a^{-2}|v|^2}(u^0)^4 \leq C(u^0)^4.$$

Step 2: $t < t_0$ and $|v| \leq 2|u|$.
In this case we recall (2.17). With the first parametrization (1.56)-(1.57) and the relations (2.59), (2.60), (2.61) and (2.62), the terms $|\partial_{v^k}(v'^j)|$ are controlled by $C(u^0)^5$.
Step 3: $t < t_0$ and $|v| \geq 2|u|$.
Here we are going to use the second parametrization (1.61)-(1.62). There are singularities in this region.To circumvent the difficulty, we remark that from the relation $|v| \geq 2|u|$, it follows that $|v - u| \geq \frac{1}{2}|v|$ and $|v+u| \geq \frac{1}{2}|v|$.
Then, from the estimates (2.67), (2.68), (2.69) and (2.70), using the assumption $a \leq b \leq \sqrt{2}a$, straightforward computations allow us to control all the terms $|\partial_{v^k}(v'^j)|$ by $C(u^0)^3$.
To summarize, since $u^0 \geq 1$ we can estimate all the terms $|\partial_{v^k}(v'^j)|$ and $|\partial_{v^k}(u'^j)|$ by recalling (3.33):

$$|\partial_{v^k}(v'^j)| \leq C(u^0)^5 \qquad and \qquad |\partial_{v^k}(u'^j)| \leq C(u^0)^5.$$

From the relation (3.18), we can recall (3.34):

$$|\partial_{v^k[f(v')f(u')]}| \leq C(u^0)^5[f(u')\sum_{j=1}^{3}(\partial_{v^j}f)(v') + f(v')\sum_{j=1}^{3}(\partial_{v^j}f)(u')].$$

Taking into account (3.34) and knowing that $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and $\sqrt{a+b+c} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$, we can estimate $J_{4k}(t)$ exactly as we have done the estimate of $J_{3k}(t)$.
We have

$$J_{4k}(t) \leq Z_1(t) + Z_2(t) \tag{3.47}$$

where

$$Z_1(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^k}f(v)|f(u')\sum_{j=1}^{3}(\partial_{v^j}f)(v')d\omega du dv,$$

and

$$Z_2(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^k}f(v)|f(v')\sum_{j=1}^{3}(\partial_{v^j}f)(u')d\omega du dv.$$

Since $\vartheta_\phi \leq 4$ and taking the same steps as we have done in the estimation of $J_{3k}(t)$,we obtain

$$\begin{aligned}
Z_1(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^5\frac{\sqrt{v^0u^0}}{\sqrt{v^0u^0}}g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}\\
&\times\sum_{j=1}^3(\partial_{v^j}f)(v')e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint [(u^0)^5\sqrt{u^0}g^{-\beta}\sqrt{\sigma_0(\omega)}e^{-\frac{1}{2}|u|^2}e^{\frac{1}{2}|v|^2}|\partial_{v^k}f(v)|]\\
&\times[\sqrt{\sigma_0(\omega)}\frac{\sqrt{v^0}}{\sqrt{v^0v^0}}e^{\frac{1}{2}|u'|^2}f(u')e^{\frac{1}{2}|v'|^2}\sum_{j=1}^3(\partial_{v^j}f)(v')]d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\iiint (u^0)^{11}g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}e^{|v|^2}(\partial_{v^k}f(v))^2d\omega dudv)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v^0u^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2d\omega dudv)^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds(\int_{\mathbb{R}^3}e^{|v|^2}(\partial_{v^k}f(v))^2\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}(1+|u|^2)^{\frac{11}{2}}g^{-2\beta}e^{-|u|^2}du)^{\frac{1}{2}}\\
&\times(\iiint\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f(u'))^2e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2d\omega du'dv')^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e ds\\
&\times(\int_{\mathbb{R}^3}e^{|u'|^2}(f(u'))^2du'\int_{S_{ab}}\sigma_0(\omega)d\omega\int_{\mathbb{R}^3}e^{|v'|^2}(\sum_{j=1}^3(\partial_{v^j}f)(v'))^2dv')^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds\\
&\times(\sum_{j=1}^3\int_{\mathbb{R}^3}e^{|v'|^2}((\partial_{v^j}f)(v'))^2dv')^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{\beta-2}\|\partial_{v^k}f(s)\|_e\|f(s)\|_e ds\\
&\times\sum_{j=1}^3(\int_{\mathbb{R}^3}e^{|v'|^2}((\partial_{v^j}f)(v'))^2dv')^{\frac{1}{2}}\\
&\leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f(s)\|_e)\int_0^t a^{-1}b^{\beta-2}ds.
\end{aligned}$$

Taking the same steps as the estimate of $Z_1(t)$, we have

$$Z_2(t) \leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f(s)\|_e)\int_0^t a^{-1}b^{\beta-2}ds.$$

By (3.47) it follows that

$$J_{4k}(t) \leq C\,\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f(s)\|_e\|f(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f(s)\|_e)\int_0^t a^{-1}b^{\beta-2}ds. \tag{3.48}$$

By (3.42), (3.43), (3.44), (3.45), (3.46) and (3.48) we obtain

$$\begin{aligned}
\|\partial_{v^k} f(t)\|_e^2 &\le \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e^2 \|f(s)\|_e) \int_0^t a^{-1} b^{\beta-\frac{5}{2}} ds \\
&+ C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2) \int_0^t b^{\beta-2} ds + C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e^2) \int_0^t b^{\beta-2} ds \\
&+ C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f(s)\|_e \|f(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f(s)\|_e) \int_0^t a^{-1} b^{\beta-2} ds \\
&\le \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(|||f(s)|||_e^3) \int_0^\infty a^{-1} b^{\beta-\frac{5}{2}} ds \\
&+ C \underset{s\in[0,t]}{Sup}(|||f(s)|||_e^3) \int_0^\infty b^{\beta-2} ds + C \underset{s\in[0,t]}{Sup}(|||f(s)|||_e^3) \int_0^\infty b^{\beta-2} ds \\
&+ C \underset{s\in[0,t]}{Sup}(|||f(s)|||_e^3) \int_0^\infty a^{-1} b^{\beta-2} ds \\
&\le \|\partial_{v^k} f(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}(|||f(s)|||_e^3).
\end{aligned}$$

□

3.3 L^2-global existence theorem for homogeneous equation

3.3.1 L^2-global existence theorem for Israel particles in the case of the homogeneous equation

In this section, we first construct a unique global-in-time solution to the Cauchy problem (3.1)-(3.2). The construction is based on a uniform energy estimate for a sequence of iterating approximative solutions.

Definition 3.1. Let f_0 be the initial data for the Cauchy problem (3.1)-(3.2). We define recursively the following sequence $(f^n)_{n\ge 0}$ by:

$$\partial_t f^{n+1} = Q_{gain}(f^n, f^n) - Q_{loss}(f^{n+1}, f^n), \tag{3.49}$$

$$f^{n+1}(0,v) = f_0(v), \qquad f^0(t,v) = f(0,v) = f_0(v). \tag{3.50}$$

We note that (3.49) is a linear partial differential equation in f^{n+1} for a given f^n.

Lemma 3.7. If f is a local-in-time solution of the Cauchy problem (3.1) with initial data f_0, then f is extended to a global-in-time solution, if initial data is given such that $|||f_0|||_e$ is sufficiently small.

Proof. Using the energy estimate (3.9) and (3.13), if f is a-local-in-time solution of (3.1) with initial data f_0, on a (short) time interval, we have

$$|||f(t)|||_e^2 \le |||f(0)|||_e^2 + C \underset{s\in[0,t]}{Sup} |||f(t)|||_e^3. \tag{3.51}$$

Since the norm $|||f|||_e$ contains first order derivatives with respect to v^i variables, (3.51) allows to bound all the derivatives of the local solution on each short time interval when the initial data is sufficiently small.
In fact, if $[0,T]$ is the maximal interval of the local solution, by (3.51), we have

$$\underset{s\in[0,T]}{Sup}\,|||f(t)|||_e^2 \leq |||f(0)|||_e^2 + C\,\underset{s\in[0,T]}{Sup}\,|||f(t)|||_e^3. \tag{3.52}$$

We are looking for a condition on $|||f(0)|||_e$ such that the following inequality holds:

$$C\theta^3 - \theta^2 + |||f(0)|||_e^2 \geq 0, \qquad for\;\; \theta \geq 0.$$

The relation (3.52) occurs if

$$1 - 4C|||f(0)|||_e^2 \geq 0.$$

The relation (3.52) holds if the initial data enjoy the smallness condition

$$|||f(0)|||_e \leq \frac{1}{2\sqrt{C}}.$$

This proves that the solution is extended to a global-in-time solution, if the initial data is given such that $|||f(0)|||_e$ is sufficiently small.

□

Now we turn to the construction of a unique local-in-time solution of the Cauchy problem.
In this section, we first construct a unique global-in-time solution to the Cauchy problem (3.1)-(3.2). The construction is based on uniform energy estimate for a sequence of iterating approximative solutions.

Proposition 3.1. The sequence $(f^n)_{n\geq 0}$ defined by (3.49) and (3.50) is locally well-defined and furthermore, if $\|f_0\|_e^2 \leq \frac{M_0}{2}$ and $\|f^n(t)\|_e^2 \leq M_0$ on the time interval $[0,T]$ with M_0 sufficiently small, then $\|f^{n+1}(t)\|_e^2 \leq M_0$ on $[0,T]$.

Proof. It is standard from the linear theory that if we know f^n, so we know f^{n+1}. The sequence is then locally well-defined. Our goal is to get uniform in n estimate for $\|f^n(t)\|_e^2$.
We multiply the first equation in (3.49) by $2e^{|v|^2}f^{n+1}(v)$ and then integrate from 0 to t to obtain

$$\begin{aligned} e^{|v|^2}(f^{n+1})^2(t,v) = e^{|v|^2}(f^{n+1})^2(0,v) &+ \int_0^t 2e^{|v|^2}f^{n+1}(v)Q_{gain}(f^n,f^n)(v)ds \\ &- \int_0^t 2e^{|v|^2}f^{n+1}(v)Q_{loss}(f^{n+1},f^n)(v)ds. \end{aligned} \tag{3.53}$$

Using the same argument as in Lemma 3.1 and integrating the above equation with respect to v we obtain

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0,v)\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f^{n+1}(v)f^n(v')f^n(u')d\omega du dv.$$

Then we have

$$\|f^{n+1}(t)\|_e^2 \le \|f^{n+1}(0)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint \left[\frac{\sigma_0(\omega)}{v^0u^0}e^{\frac{1}{2}|v|^2}f^{n+1}(v)e^{-\frac{1}{2}|u|^2}\right]$$
$$\times\left[\sqrt{\frac{\sigma_0(\omega)}{v^0u^0}}e^{\frac{1}{2}|v'|^2}f^n(v')e^{\frac{1}{2}|u'|^2}f^n(u')d\omega dudv\right]$$
$$\le \|f^{n+1}(0)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \sigma_0(\omega)e^{|v|^2}(f^{n+1})^2(v)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}$$
$$\times\left[\iiint \frac{1}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f^n)^2(v')e^{|u'|^2}(f^n)^2(u')d\omega du'dv'\right]^{\frac{1}{2}}$$
$$\le \|f^{n+1}(0)\|_e^2 + C\,\underset{s\in[0,t]}{Sup}\,(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^\infty a^{-1}(s)b^{-2}(s)ds.$$

Thus

$$\|f^{n+1}(t)\|_e^2 \le \|f^{n+1}(0)\|_e^2 + C\,\underset{s\in[0,t]}{Sup}\,(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2). \tag{3.54}$$

Next, we proceed to the estimation of the derivative of f^{n+1} with respect to the momenta variables. Let $k\in\{1,2,3\}$. We take the partial derivative ∂_{v^k} and multiply by $2e^{|v|^2}\partial_{v^k}f^{n+1}$ the equation (3.49) and obtain

$$2e^{|v|^2}\partial_{v^k}f^{n+1}\partial_t(\partial_{v^k}f^{n+1}) = 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{gain}(f^n,f^n)(v) - 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{loss}(f^{n+1},f^n)(v).$$

We take integration on $[0,t]$ to have

$$e^{|v|^2}(\partial_{v^k}f^{n+1})^2(t,v) = e^{|v|^2}(\partial_{v^k}f^{n+1})^2(0,v) + \int_0^t 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{gain}(f^n,f^n)(v)ds$$
$$-\int_0^t 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{loss}(f^{n+1},f^n)(v)ds. \tag{3.55}$$

Following the proof of the Lemma 3.2, we take integration of the above equation with respect to v

$$\|\partial_{v^k}f^{n+1}(t)\|_e^2 \le \|\partial_{v^k}f^{n+1}(0)\|_e^2 + J_{1k}^n(t) + J_{2k}^n(t) + J_{3k}^n(t) + J_{4k}^n(t)$$

where

$$J_{1k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}(\partial_{v^k}f^{n+1})^2(v)f^n(u)d\omega dudv,$$
$$J_{2k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^k}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||\partial_{v^k}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv,$$
$$J_{3k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^k}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||\partial_{v^k}f^{n+1}(v)|f^n(v')f^n(u')d\omega dudv,$$
$$J_{4k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^k}f^{n+1}(v)||\partial_{v^k}[f^n(v')f^n(u')]|d\omega dudv.$$

Following the same method as for $J_{1k}(t)$ we have

$$\begin{aligned}
J^n_{1k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\sigma_0(\omega)(\partial_{v^k}f^{n+1}(v))^2 f^n(u)d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f^{n+1}(s)\|_e^2 ds\left[\iint \sigma_0(\omega)e^{-|u|^2}d\omega du\right]^{\frac{1}{2}}\\
&\times\left[\iint \sigma_0(\omega)e^{|u|^2}(f^n(u))^2 d\omega du\right]^{\frac{1}{2}}\\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e^2\|f^n(s)\|_e)\int_0^t a^{-1}b^{-2}ds.
\end{aligned}$$

Doing the same as for $J_{2k}(t)$ we have

$$\begin{aligned}
J^n_{2k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}\sigma_0(\omega)e^{-|u|^2}(\partial_{v^k}f^{n+1}(v))^2 d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2 d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^{n+1}(s)\|_e\|f^n(s)\|_e)\int_0^t a^{-1}b^{-2}ds.
\end{aligned}$$

Doing the same as or $J_{3k}(t)$ we have

$$\begin{aligned}
J^n_{3k}(t) &\leq C\int_0^t a^{-1}b^{-2}ds\iiint e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^n(v')f^n(u')d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint u^0\sigma_0(\omega)e^{|v|^2}(\partial_{v^k}f^{n+1}(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2 d\omega du'dv'\right]^{\frac{1}{2}}\\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^t a^{-1}b^{-2}ds.
\end{aligned}$$

As for $J_{4k}(t)$ we now have

$$J^n_{4k}(t) \leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^k}f^{n+1}(v)||\partial_{v^k}(f^n(v')f^n(u'))|d\omega dudv.$$

Then

$$J^n_{4k}(t) \leq Z^n_1(t) + Z^n_2(t).$$

where

$$Z^n_1(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^3e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^n(u')\sum_{j=1}^{3}|(\partial_{v^j}f^n)(v')|d\omega dudv.$$

$$Z^n_2(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^3e^{|v|^2}\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^n(v')\sum_{j=1}^{3}|(\partial_{v^j}f^n)(u')|d\omega dudv.$$

Then we obtain

$$Z_1^n(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{-2}ds,$$

and

$$Z_2^n(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{-2}ds.$$

Then

$$J_{4k}^n(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^{3} \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{-2}ds.$$

At the end we have

$$\begin{aligned}
\|\partial_{v^k} f^{n+1}(t)\|_e^2 &\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + J_{1k}^n(t) + J_{2k}^n(t) + J_{3k}^n(t) + J_{4k}^n(t)\\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e)\\
&+ C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e) + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2)\\
&+ C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2)\\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e + |||f^{n+1}(s)|||_e |||f^n(s)|||_e^2).
\end{aligned} \tag{3.56}$$

Summing up (3.54) and (3.56), we obtain

$$\begin{aligned}
|||f^{n+1}(t)|||_e^2 &\leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e + |||f^{n+1}(s)|||_e |||f^n(s)|||_e^2)\\
&\leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup}\,|||f^n(s)|||_e^3 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e)
\end{aligned} \tag{3.57}$$

where we used the inequality $\lambda\mu^2 \leq \lambda^2\mu + \mu^3$ for non-negative λ and μ.
Using the fact that

$$|||f_0|||_e^2 \leq \frac{M_0}{2}, \qquad |||f^n(t)|||_e^2 \leq M_0$$

on the time interval $[0, T]$, we obtain

$$(1 - C\sqrt{M_0}) \underset{s\in[0,t]}{Sup}\,|||f^{n+1}(s)|||_e^2 \leq \frac{M_0}{2} + CM_0\sqrt{M_0}. \tag{3.58}$$

The desired result is obtained for small M_0; for example with M_0 such that $M_0 \leq \frac{1}{16C^2}$.

□

Theorem 3.1. Consider a Bianchi type I space-time where the metric tensor is such that $a = a(t)$ and $b = b(t)$ are given and satisfy assumptions (3.3)-(3.4). Let $f_0 = f(0, v)$ be the initial data of the Cauchy problem (3.1)-(3.2). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e^2 < M_0$, there exists a unique global solution to the Cauchy problem (3.1)-(3.2). Moreover

$$\underset{t\in[0,\infty[}{Sup}\,|||f(t)|||_e^2 \leq M_0. \tag{3.59}$$

Proof. **Existence:** Taking the limit in (3.53) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e^2 \leq M_0$ on the time interval $[0, T]$.
Next, we prove that the solution could be extended to $[0, \infty[$.
It suffices to bound the derivatives of the local solution with respect to the momentum variable on $[0, T]$. In order to do so, we combine the two energy inequalities (3.9)-(3.13) to obtain

$$|||f(t)|||_e^2 \leq |||f(0)|||_e^2 + C \underset{s\in[0,t]}{Sup} |||f(s)|||_e^3. \tag{3.60}$$

Using Lemma 3.7, this prove that the solution is extended to a global-in-time solution, if initial data is given such that $|||f(0)|||_e^2$ is sufficiently small.
Uniqueness: We now prove the uniqueness of the solution. We assume that there is another solution h to (3.1)-(3.2) such that $\underset{t\in[0,\infty[}{Sup} |||h(t)|||_e^2 \leq M_0$.
The difference $f - h$ satisfies

$$\partial_t(f - h) = Q(f - h, f) + Q(h, f - h). \tag{3.61}$$

Next we proceed as in the proof of the energy estimate. Since $f(0, v) = h(0, v) = f_0(v)$, we obtain

$$\begin{aligned} |||f(t) - h(t)|||_e^2 &\leq C \underset{s\in[0,t]}{Sup} \left[|||f(s)|||_e + |||h(s)|||_e\right] |||f(s) - h(s)|||_e^2 \\ &\leq 2C\sqrt{M_0} \underset{s\in[0,t]}{Sup} |||f(s) - h(s)|||_e^2. \end{aligned} \tag{3.62}$$

Since $M_0 \leq \frac{1}{16C^2}$, taking the supremum in (3.62) on the time interval $[0, T]$, we obtain

$$\underset{s\in[0,t]}{Sup} |||f(s) - h(s)|||_e^2 \leq \frac{1}{2} \underset{s\in[0,t]}{Sup} |||f(s) - h(s)|||_e^2.$$

Then

$$\underset{s\in[0,\infty]}{Sup} |||f(s) - h(s)|||_e^2 \leq \frac{1}{2} \underset{s\in[0,\infty]}{Sup} |||f(s) - h(s)|||_e^2.$$

Thus $f = h$ on $\mathbb{R}_+$. □

3.3.2 L^2-global existence theorem for hard potentials in the case of the homogeneous equation

In this section, we first construct a unique global-in-time solution to the Cauchy problem (3.1)-(3.2). The construction is based on a uniform energy estimate for a sequence of iterating approximative solutions.

Proposition 3.2. The sequence $(f^n)_{n\geq 0}$ defined by (3.49) and (3.50) is locally well-defined. Furthermore, if $\|f_0\|_e^2 \leq \frac{M_0}{2}$ and $\|f^n(t)\|_e^2 \leq M_0$ on the time interval $[0, T]$ with M_0 sufficiently small, then $\|f^{n+1}(t)\|_e^2 \leq M_0$ on $[0, T]$.

Proof. It is standard from the linear theory that if we know f^n, so we know f^{n+1}. The sequence is then locally well-defined. Our goal is to get uniform in n estimate for $\|f^n(t)\|_e^2$.
We multiply the first equation in (3.49) by $2e^{|v|^2}f^{n+1}(v)$ and then integrate from 0 to t to obtain

$$e^{|v|^2}(f^{n+1})^2(t,v) = e^{|v|^2}(f^{n+1})^2(0,v) + \int_0^t 2e^{|v|^2}f^{n+1}(v)Q_{gain}(f^n,f^n)(v)ds \\ - \int_0^t 2e^{|v|^2}f^{n+1}(v)Q_{loss}(f^{n+1},f^n)(v)ds. \quad (3.63)$$

Using the same argument as in Lemma 3.3 and integrating the above equation with respect to v we obtain

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0,v)\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})f^{n+1}(v)f^n(v')f^n(u')d\omega du dv.$$

Then we have

$$\begin{aligned}
\|f^{n+1}(t)\|_e^2 \leq{}& \|f^{n+1}(0,v)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint \left[\sqrt{\vartheta_\phi\sigma_0(\omega)}g^{-\beta}e^{\frac{1}{2}|v|^2}f^{n+1}(v)e^{-\frac{1}{2}|u|^2}\right] \\
&\times \left[\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f^n(v')e^{\frac{1}{2}|u'|^2}f^n(u')d\omega du dv\right] \\
&+ C\int_0^t 2a^{-1}b^{-2}ds \iiint \left[\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v|^2}f^{n+1}(v)e^{-\frac{1}{2}|u|^2}\right] \\
&\times \left[\sqrt{\vartheta_\phi\sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f^n(v')e^{\frac{1}{2}|u'|^2}f^n(u')d\omega du dv\right] \\
\leq{}& \|f^{n+1}(0,v)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi\sigma_0(\omega)g^{-2\beta}e^{|v|^2}(f^{n+1})^2(v)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f^n)^2(v')e^{|u'|^2}(f^n)^2(u')d\omega du'dv'\right]^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(f^{n+1})^2(v)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f^n)^2(v')e^{|u'|^2}(f^n)^2(u')d\omega du'dv'\right]^{\frac{1}{2}} \\
\leq{}& \|f^{n+1}(0,v)\|_e^2 + C\underset{s\in[0,t]}{Sup}(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^\infty (a^{-1}b^{\beta-\frac{5}{2}} + a^{-1}b^2)ds.
\end{aligned}$$

Thus

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0)\|_e^2 + C\underset{s\in[0,t]}{Sup}(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2). \quad (3.64)$$

Next, we proceed to the estimation of the derivative of f^{n+1} with respect to the momenta variables. Let $k\in\{1,2,3\}$. We take the partial derivative ∂_{v^k} and multiply by $2e^{|v|^2}\partial_{v^k}f^{n+1}$ the equation (3.49) and obtain

$$2e^{|v|^2}\partial_{v^k}f^{n+1}\partial_t(\partial_{v^k}f^{n+1}) = 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{gain}(f^n,f^n)(v) - 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{loss}(f^{n+1},f^n)(v).$$

We take integration on $[0,t]$ to have

$$e^{|v|^2}(\partial_{v^k} f^{n+1})^2(t,v) = e^{|v|^2}(\partial_{v^k} f^{n+1})^2(0,v) + \int_0^t 2e^{|v|^2}\partial_{v^k} f^{n+1}(v)\partial_{v^k} Q_{gain}(f^n, f^n)(v)ds$$
$$- \int_0^t 2e^{|v|^2}\partial_{v^k} f^{n+1}(v)\partial_{v^k} Q_{loss}(f^{n+1}, f^n)(v)ds. \quad (3.65)$$

Following the proof of the Lemma 3.4, we take integration of the above equation with respect to v

$$\|\partial_{v^k} f^{n+1}(t)\|_e^2 \leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + J_{1k}^n(t) + J_{2k}^n(t) + J_{3k}^n(t) + J_{4k}^n(t)$$

where

$$J_{1k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi \sigma(g,\omega)(\partial_{v^k} f^{n+1})^2(v) f^n(u) d\omega du dv,$$
$$J_{2k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi \sigma(g,\omega))||\partial_{v^k} f^{n+1}(v)| f^{n+1}(v) f^n(u) d\omega du dv,$$
$$J_{3k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi \sigma(g,\omega))||\partial_{v^k} f^{n+1}(v)| f^n(v') f^n(u') d\omega du dv,$$
$$J_{4k}^n(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi \sigma(g,\omega)|\partial_{v^k} f^{n+1}(v)||\partial_{v^k}[f^n(v') f^n(u')]| d\omega du dv.$$

Following the same idea as for $J_{1k}(t)$ we have

$$J_{1k}^n(t) \leq C \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi \sigma_0(\omega)(1+g^{-\beta})(\partial_{v^k} f^{n+1}(v))^2 f^n(u) d\omega du dv$$
$$\leq C \int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f^{n+1}(s)\|_e^2 ds \left[\iint \vartheta_\phi \sigma_0(\omega) g^{-2\beta} e^{-|u|^2} d\omega du\right]^{\frac{1}{2}}$$
$$\times \left[\iint \vartheta_\phi \sigma_0(\omega) e^{|u|^2}(f^n(u))^2 d\omega du\right]^{\frac{1}{2}}$$
$$+ C \int_0^t 2a^{-1}b^{-2}\|\partial_{v^k} f^{n+1}(s)\|_e^2 ds \left[\iint \vartheta_\phi \sigma_0(\omega) e^{-|u|^2} d\omega du\right]^{\frac{1}{2}}$$
$$\times \left[\iint \vartheta_\phi \sigma_0(\omega) e^{|u|^2}(f^n(u))^2 d\omega du\right]^{\frac{1}{2}}$$
$$\leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k} f^{n+1}(s)\|_e^2 \|f^n(s)\|_e) \int_0^t (a^{-1}b^{\beta-\frac{5}{2}} + a^{-1}b^{-2})ds.$$

Doing the same as for $J_{2k}(t)$ we have:

For $k = 1$

$$
\begin{aligned}
J_{21}^n(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}u^0(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^1}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint e^{|v|^2}(u^0)^2g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}(\partial_{v^1}f^{n+1}(v))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint e^{|v|^2}(u^0)^2\sigma_0(\omega)e^{-|u|^2}(\partial_{v^1}f^{n+1}(v))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^1}f^{n+1}(s)\|_e\|f^{n+1}(s)\|_e\|f^n(s)\|_e)\int_0^t(a^{-2}b^{\beta-2}+a^{-2}b^{-2})ds.
\end{aligned}
$$

For $k = 2, 3$

$$
\begin{aligned}
J_{22}^n(t) &\leq C\int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}u^0(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-3}ds\left[\iiint e^{|v|^2}(u^0)^2g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}(\partial_{v^k}f^{n+1}(v))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-1}b^{-3}ds\left[\iiint e^{|v|^2}(u^0)^2\sigma_0(\omega)e^{-|u|^2}(\partial_{v^k}f^{n+1}(v))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^{n+1}(s)\|_e\|f^n(s)\|_e)\int_0^t(a^{-2}b^{\beta-3}+a^{-2}b^{-3})ds.
\end{aligned}
$$

Following the same method as for $J_{3k}(t)$ we have:

For $k = 1$

$$
\begin{aligned}
J_{31}^n(t) &\leq C\int_0^t a^{-2}b^{-2}ds \iiint e^{|v|^2}u^0(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^1}f^{n+1}(v)|f^n(v')f^n(u')d\omega dudv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint (u^0)^3g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^1}f^{n+1}(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2d\omega du'dv'\right]^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint (u^0)^3\sigma_0(\omega)e^{|v|^2}(\partial_{v^1}f^{n+1}(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2d\omega du'dv'\right]^{\frac{1}{2}} \\
&\leq C\,\underset{s\in[0,t]}{Sup}\,(\|\partial_{v^1}f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^t(a^{-2}b^{\beta-2}+a^{-2}b^{-2})ds.
\end{aligned}
$$

For $k = 2, 3$

$$\begin{aligned}
J^n_{32}(t) &\leq C\int_0^t a^{-1}b^{-3}ds \iiint e^{|v|^2}u^0(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^n(v')f^n(u')d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-3}ds\left[\iiint (u^0)^3 g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^k}f^{n+1}(v))^2 e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2d\omega du'dv'\right]^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-3}ds\left[\iiint (u^0)^3\sigma_0(\omega)e^{|v|^2}(\partial_{v^k}f^{n+1}(v))^2 e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2d\omega du'dv'\right]^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^n(s)\|^2_e)\int_0^t (a^{-1}b^{\beta-3}+a^{-1}b^{-3})ds.
\end{aligned}$$

As for $J_{4k}(t)$, we now have

$$J^n_{4k}(t) \leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi\sigma_0(\omega)(1+g^{-\beta})|\partial_{v^k}f^{n+1}(v)||\partial_{v^k}(f^n(v')f^n(u'))|d\omega du dv.$$

Then

$$J^n_{4k}(t) \leq Z^n_1(t) + Z^n_2(t)$$

where

$$Z^n_1(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^5e^{|v|^2}\vartheta_\phi\sigma_0(\omega)(1+g^{-\beta})|\partial_{v^k}f^{n+1}(v)|f^n(u')\sum_{j=1}^3|(\partial_{v^j}f^n)(v')|d\omega du dv,$$

$$Z^n_2(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint (u^0)^5e^{|v|^2}\vartheta_\phi\sigma_0(\omega)(1+g^{-\beta})|\partial_{v^k}f^{n+1}(v)|f^n(v')\sum_{j=1}^3|(\partial_{v^j}f^n)(u')|d\omega du dv.$$

Then we obtain

$$Z^n_1(t) \leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^n(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f^n(s)\|_e)\int_0^t(a^{-1}b^{\beta-2}+a^{-1}b^{-2})ds,$$

and

$$Z^n_2(t) \leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^n(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f^n(s)\|_e)\int_0^t(a^{-1}b^{\beta-2}+a^{-1}b^{-2})ds.$$

Then

$$J^n_{4k}(t) \leq C\underset{s\in[0,t]}{Sup}(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^n(s)\|_e\sum_{j=1}^3\|\partial_{v^j}f^n(s)\|_e)\int_0^t(a^{-1}b^{\beta-2}+a^{-1}b^{-2})ds.$$

At the end we have

$$
\begin{aligned}
\|\partial_{v^k} f^{n+1}(t)\|_e^2 &\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + J_{1k}^n(t) + J_{2k}^n(t) + J_{3k}^n(t) + J_{4k}^n(t) \\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e) \\
&+ C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e) + C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2) \\
&+ C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2) \\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e + |||f^{n+1}(s)|||_e |||f^n(s)|||_e^2).
\end{aligned}
\tag{3.66}
$$

Summing up (3.64) and (3.66) we obtain

$$
\begin{aligned}
|||f^{n+1}(t)|||_e^2 &\leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e + |||f^{n+1}(s)|||_e |||f^n(s)|||_e^2) \\
&\leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup} |||f^n(s)|||_e^3 + C \underset{s\in[0,t]}{Sup} (|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e).
\end{aligned}
\tag{3.67}
$$

where we used the inequality $\lambda\mu^2 \leq \lambda^2\mu + \mu^3$ for non-negative λ and μ.
Using the fact that

$$
|||f_0|||_e^2 \leq \frac{M_0}{2}, \qquad |||f^n(t)|||_e^2 \leq M_0
$$

on the time interval $[0, T]$, we obtain

$$
(1 - C\sqrt{M_0}) \underset{s\in[0,t]}{Sup} |||f^{n+1}(s)|||_e^2 \leq \frac{M_0}{2} + CM_0\sqrt{M_0}. \tag{3.68}
$$

The desired result is obtained for small M_0; for example with M_0 such that $M_0 \leq \frac{1}{16C^2}$. □

Theorem 3.2. Consider a Bianchi type I space-time where the metric tensor is such that $a = a(t)$ and $b = b(t)$ are given and satisfy assumptions (3.3), (3.4) and (3.22). Let $f_0 = f(0, v)$ be the initial data of the Cauchy problem (3.1)-(3.2). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e^2 < M_0$, there exists a unique global solution to the Cauchy problem (3.1)-(3.2). Moreover

$$
\underset{t\in[0,\infty[}{Sup} |||f(t)|||_e^2 \leq M_0. \tag{3.69}
$$

Proof. **Existence:** Taking the limit in (3.63) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e^2 \leq M_0$ on the time interval $[0, T]$.
Next, we prove that the solution could be extended to $[0, \infty[$.
It suffices to bound the derivatives of the local solution with respect to the momentum variable on $[0, T]$. In order to do so, we combine the two energy inequalities (3.23)-(3.26) to obtain

$$
|||f(t)|||_e^2 \leq |||f(0)|||_e^2 + C \underset{s\in[0,t]}{Sup} |||f(s)|||_e^3. \tag{3.70}
$$

Using Lemma 3.7, this prove that the solution is extended to a global-in-time solution, if initial data is given such that $|||f(0)|||_e^2$ is sufficiently small.

Uniqueness: We now prove the uniqueness of the solution. We assume that there is another solution h to (3.1)-(3.2) such that $\underset{t\in[0,\infty[}{Sup}\, |||h(t)|||_e^2 \leq M_0$.
The difference $f-h$ satisfies

$$\partial_t(f-h) = Q(f-h,f) + Q(h,f-h). \tag{3.71}$$

Next we proceed as in the proof of the energy estimate. Since $f(0,v) = h(0,v) = f_0(v)$, we obtain

$$\begin{aligned} |||f(t)-h(t)|||_e^2 &\leq C \underset{s\in[0,t]}{Sup}\, [|||f(s)|||_e + |||h(s)|||_e]\, |||f(s)-h(s)|||_e^2 \\ &\leq 2C\sqrt{M_0}\, \underset{s\in[0,t]}{Sup}\, |||f(s)-h(s)|||_e^2. \end{aligned} \tag{3.72}$$

Since $M_0 \leq \frac{1}{16C^2}$, taking the supremum in (3.72) on the time interval $[0,T]$, we obtain

$$\underset{s\in[0,t]}{Sup}\, |||f(s)-h(s)|||_e^2 \leq \frac{1}{2} \underset{s\in[0,t]}{Sup}\, |||f(s)-h(s)|||_e^2.$$

Then

$$\underset{s\in[0,\infty]}{Sup}\, |||f(s)-h(s)|||_e^2 \leq \frac{1}{2} \underset{s\in[0,\infty]}{Sup}\, |||f(s)-h(s)|||_e^2.$$

Thus $f = h$ on $\mathbb{R}_+$. □

3.3.3 L^2-global existence theorem for soft potentials in the case of the homogeneous equation

In this section, we first construct a unique global-in-time solution to the Cauchy problem (3.1)-(3.2). The construction is based on uniform energy estimate for a sequence of iterating approximative solutions.

Proposition 3.3. The sequence $(f^n)_{n\geq 0}$ defined by (3.49) and (3.50) is locally well-defined. Furthermore, if $\|f_0\|_e^2 \leq \frac{M_0}{2}$ and $\|f^n(t)\|_e^2 \leq M_0$ on the time interval $[0,T]$ with M_0 sufficiently small, then $\|f^{n+1}(t)\|_e^2 \leq M_0$ on $[0,T]$.

Proof. It is standard from the linear theory that if we know f^n, so we know f^{n+1}. The sequence is then locally well-defined. Our goal is to get uniform in n estimate for $\|f^n(t)\|_e^2$.
We multiply the first equation in (3.49) by $2e^{|v|^2}f^{n+1}(v)$ and then integrate from 0 to t to obtain

$$\begin{aligned} e^{|v|^2}(f^{n+1})^2(t,v) &= e^{|v|^2}(f^{n+1})^2(0,v) + \int_0^t 2e^{|v|^2} f^{n+1}(v) Q_{gain}(f^n,f^n)(v)ds \\ &\quad - \int_0^t 2e^{|v|^2} f^{n+1}(v) Q_{loss}(f^{n+1},f^n)(v)ds. \end{aligned} \tag{3.73}$$

Using the same argument as in Lemma 3.5 and integrating the above equation with respect to v we obtain

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0,v)\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi \sigma(g,\omega) f^{n+1}(v) f^n(v') f^n(u') d\omega du dv.$$

Then we have

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0,v)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint \left[\sqrt{\vartheta_\phi \sigma_0(\omega)}g^{-\beta}e^{\frac{1}{2}|v|^2}f^{n+1}(v)e^{-\frac{1}{2}|u|^2}\right]$$
$$\times \left[\sqrt{\vartheta_\phi \sigma_0(\omega)}e^{\frac{1}{2}|v'|^2}f^n(v')e^{\frac{1}{2}|u'|^2}f^n(u')d\omega dudv\right]$$
$$\leq \|f^{n+1}(0,v)\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi\sigma_0(\omega)g^{-2\beta}e^{|v|^2}(f^{n+1})^2(v)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}$$
$$\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f^n)^2(v')e^{|u'|^2}(f^n)^2(u')d\omega du'dv'\right]^{\frac{1}{2}}$$
$$\leq \|f^{n+1}(0,v)\|_e^2 + C\underset{s\in[0,t]}{Sup}(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^\infty a^{-1}(s)b^{\beta-\frac{5}{2}}(s)ds.$$

Thus

$$\|f^{n+1}(t)\|_e^2 \leq \|f^{n+1}(0)\|_e^2 + C\underset{s\in[0,t]}{Sup}(\|f^{n+1}(s)\|_e\|f^n(s)\|_e^2). \tag{3.74}$$

Next, we proceed to the estimation of the derivative of f^{n+1} with respect to the momenta variables. Let $k\in\{1,2,3\}$. We take the partial derivative ∂_{v^k} and multiply by $2e^{|v|^2}\partial_{v^k}f^{n+1}$ the equation (3.49) and obtain

$$2e^{|v|^2}\partial_{v^k}f^{n+1}\partial_t(\partial_{v^k}f^{n+1}) = 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{gain}(f^n,f^n)(v) - 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_{loss}(f^{n+1},f^n)(v).$$

We take integration on $[0,t]$ to have

$$e^{|v|^2}(\partial_{v^k}f^{n+1})^2(t,v) = e^{|v|^2}(\partial_{v^k}f^{n+1})^2(0,v) + \int_0^t 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_g(f^n,f^n)(v)ds$$
$$-\int_0^t 2e^{|v|^2}\partial_{v^k}f^{n+1}(v)\partial_{v^k}Q_l(f^{n+1},f^n)(v)ds. \tag{3.75}$$

Following the proof of the Lemma 3.6, we take integration of the above equation with respect to v

$$\|\partial_{v^k}f^{n+1}(t)\|_e^2 \leq \|\partial_{v^k}f^{n+1}(0)\|_e^2 + J^n_{1k}(t) + J^n_{2k}(t) + J^n_{3k}(t) + J^n_{4k}(t)$$

where

$$J^n_{1k}(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi\sigma(g,\omega)(\partial_{v^k}f^{n+1})^2(v)f^n(u)d\omega dudv,$$
$$J^n_{2k}(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))||\partial_{v^k}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv,$$
$$J^n_{3k}(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^k}(\vartheta_\phi\sigma(g,\omega))||\partial_{v^k}f^{n+1}(v)|f^n(v')f^n(u')d\omega dudv,$$
$$J^n_{4k}(t) = \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi\sigma(g,\omega)|\partial_{v^k}f^{n+1}(v)||\partial_{v^k}[f^n(v')f^n(u')]|d\omega dudv.$$

Following the same idea as for $J_{1k}(t)$ we have

$$\begin{aligned} J^n_{1k}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi\sigma_0(\omega)g^{-\beta}(\partial_{v^k}f^{n+1}(v))^2 f^n(u)d\omega dudv \\ &\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^k}f^{n+1}(s)\|_e^2 ds\left[\iint \vartheta_\phi\sigma_0(\omega)g^{-2\beta}e^{-|u|^2}d\omega du\right]^{\frac{1}{2}} \\ &\times\left[\iint \vartheta_\phi\sigma_0(\omega)e^{|u|^2}(f^n(u))^2 d\omega du\right]^{\frac{1}{2}} \\ &\leq C\, \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e^2\|f^n(s)\|_e)\int_0^t a^{-1}b^{\beta-\frac{5}{2}}ds. \end{aligned}$$

Doing the same as for $J_{2k}(t)$ we have:
For $k=1$

$$\begin{aligned} J^n_{21}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}u^0g^{-\beta}\sigma_0(\omega)|\partial_{v^1}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv \\ &\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint e^{|v|^2}(u^0)^2g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}(\partial_{v^1}f^{n+1}(v))^2 d\omega dudv\right]^{\frac{1}{2}} \\ &\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2 d\omega dudv\right]^{\frac{1}{2}} \\ &\leq C\, \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^1}f^{n+1}(s)\|_e\|f^{n+1}(s)\|_e\|f^n(s)\|_e)\int_0^t a^{-2}b^{\beta-2}ds. \end{aligned}$$

For $k=2,3$

$$\begin{aligned} J^n_{22}(t) &\leq C\int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}u^0g^{-\beta}\sigma_0(\omega)|\partial_{v^k}f^{n+1}(v)|f^{n+1}(v)f^n(u)d\omega dudv \\ &\leq C\int_0^t 2a^{-1}b^{-3}ds\left[\iiint e^{|v|^2}(u^0)^2g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}(\partial_{v^k}f^{n+1}(v))^2 d\omega dudv\right]^{\frac{1}{2}} \\ &\times\left[\iiint e^{|v|^2}(f^{n+1}(v))^2e^{|u|^2}(f^n(u))^2 d\omega dudv\right]^{\frac{1}{2}} \\ &\leq C\, \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k}f^{n+1}(s)\|_e\|f^{n+1}(s)\|_e\|f^n(s)\|_e)\int_0^t a^{-2}b^{\beta-3}ds. \end{aligned}$$

Following the same method as for $J_{3k}(t)$ we have:
For $k=1$

$$\begin{aligned} J^n_{31}(t) &\leq C\int_0^t a^{-2}b^{-2}ds \iiint e^{|v|^2}u^0g^{-\beta}\sigma_0(\omega)|\partial_{v^1}f^{n+1}(v)|f^n(v')f^n(u')d\omega dudv \\ &\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint (u^0)^3g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^1}f^{n+1}(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\ &\times\left[\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2e^{|v'|^2}(f^n(v'))^2 d\omega du'dv'\right]^{\frac{1}{2}} \\ &\leq C\, \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^1}f^{n+1}(s)\|_e\|f^n(s)\|_e^2)\int_0^t a^{-2}b^{\beta-2}ds. \end{aligned}$$

For $k = 2, 3$

$$\begin{aligned}
J^n_{32}(t) &\leq C \int_0^t a^{-1}b^{-3}ds \iiint e^{|v|^2} u^0 g^{-\beta}\sigma_0(\omega)|\partial_{v^k} f^{n+1}(v)|f^n(v')f^n(u')d\omega du dv \\
&\leq C \int_0^t 2a^{-1}b^{-3}ds \left[\iiint (u^0)^3 g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^k} f^{n+1}(v))^2 e^{-|u|^2} d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\frac{v^0}{v'^0 u'^0}\sigma_0(\omega)e^{|u'|^2}(f^n(u'))^2 e^{|v'|^2}(f^n(v'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\
&\leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e^2) \int_0^t a^{-1}b^{\beta-3}ds.
\end{aligned}$$

As for $J_{4k}(t)$, we now have

$$J^n_{4k}(t) \leq C \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi \sigma_0(\omega)g^{-\beta}|\partial_{v^k} f^{n+1}(v)||\partial_{v^k}(f^n(v')f^n(u'))|d\omega du dv.$$

Then

$$J^n_{4k}(t) \leq Z^n_1(t) + Z^n_2(t)$$

where

$$Z^n_1(t) = C \int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi \sigma_0(\omega)g^{-\beta}|\partial_{v^k} f^{n+1}(v)|f^n(u') \sum_{j=1}^3 |(\partial_{v^j} f^n)(v')|d\omega du dv,$$

$$Z^n_2(t) = C \int_0^t 2a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi \sigma_0(\omega)g^{-\beta}|\partial_{v^k} f^{n+1}(v)|f^n(v') \sum_{j=1}^3 |(\partial_{v^j} f^n)(u')|d\omega du dv.$$

Then we obtain

$$Z^n_1(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^3 \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{\beta-2}ds,$$

and

$$Z^n_2(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^3 \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{\beta-2}ds.$$

Then

$$J^n_{4k}(t) \leq C \underset{s\in[0,t]}{Sup}\,(\|\partial_{v^k} f^{n+1}(s)\|_e \|f^n(s)\|_e \sum_{j=1}^3 \|\partial_{v^j} f^n(s)\|_e) \int_0^t a^{-1}b^{\beta-2}ds.$$

At the end, we have

$$\begin{aligned}
\|\partial_{v^k} f^{n+1}(t)\|_e^2 &\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + J^n_{1k}(t) + J^n_{2k}(t) + J^n_{3k}(t) + J^n_{4k}(t) \\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e) \\
&+ C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e) + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2) \\
&+ C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e |||f^n(s)|||_e^2) \\
&\leq \|\partial_{v^k} f^{n+1}(0)\|_e^2 + C \underset{s\in[0,t]}{Sup}\,(|||f^{n+1}(s)|||_e^2 |||f^n(s)|||_e + |||f^{n+1}(s)|||_e |||f^n(s)|||_e^2).
\end{aligned} \tag{3.76}$$

Summing up (3.74) and (3.76) we obtain

$$|||f^{n+1}(t)|||_e^2 \leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup}(|||f^{n+1}(s)|||_e^2|||f^n(s)|||_e + |||f^{n+1}(s)|||_e|||f^n(s)|||_e^2)$$
$$\leq |||f_0|||_e^2 + C \underset{s\in[0,t]}{Sup}|||f^n(s)|||_e^3 + C \underset{s\in[0,t]}{Sup}(|||f^{n+1}(s)|||_e^2|||f^n(s)|||_e). \tag{3.77}$$

where we used the inequality $\lambda\mu^2 \leq \lambda^2\mu + \mu^3$ for non-negative λ and μ.
Using the fact that

$$|||f_0|||_e^2 \leq \frac{M_0}{2}, \qquad |||f^n(t)|||_e^2 \leq M_0.$$

on the time interval $[0, T]$, we obtain

$$(1 - C\sqrt{M_0}) \underset{s\in[0,t]}{Sup}|||f^{n+1}(s)|||_e^2 \leq \frac{M_0}{2} + CM_0\sqrt{M_0}. \tag{3.78}$$

The desired result is obtained for small M_0; for example with M_0 such that $M_0 \leq \frac{1}{16C^2}$. □

Theorem 3.3. Consider a Bianchi type I space-time where the metric tensor is such that $a = a(t)$ and $b = b(t)$ are given and satisfy assumptions (3.3), (3.4) and (3.22). Let $f_0 = f(0, v)$ be the initial data of the Cauchy problem (3.1)-(3.2). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e^2 < M_0$, there exists a unique global solution to the Cauchy problem (3.1)-(3.2). Moreover

$$\underset{t\in[0,\infty[}{Sup} |||f(t)|||_e^2 \leq M_0. \tag{3.79}$$

Proof. **Existence:** Taking the limit in (3.73) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e^2 \leq M_0$ on the time interval $[0, T]$.
Next, we prove that the solution could be extended to $[0, \infty[$.
It suffices to bound the derivatives of the local solution with respect to the momentum variable on $[0, T]$. In order to do so, we combine the two energy inequalities (3.37)-(3.40) to obtain

$$|||f(t)|||_e^2 \leq |||f(0)|||_e^2 + C \underset{s\in[0,t]}{Sup}|||f(s)|||_e^3. \tag{3.80}$$

Using Lemma 3.7This prove that the solution is extended to a global-in-time solution, if initial data is given such that $|||f(0)|||_e^2$ is sufficiently small.
Uniqueness: We now prove the uniqueness of the solution. We assume that there is another solution h to (3.1)-(3.2) such that $\underset{t\in[0,\infty[}{Sup} |||h(t)|||_e^2 \leq M_0$.
The difference $f - h$ satisfies

$$\partial_t(f - h) = Q(f - h, f) + Q(h, f - h). \tag{3.81}$$

Next we proceed as in the proof of the energy estimate. Since $f(0, v) = h(0, v) = f_0(v)$, we obtain

$$|||f(t) - h(t)|||_e^2 \leq C \underset{s\in[0,t]}{Sup} [|||f(s)|||_e + |||h(s)|||_e] \, |||f(s) - h(s)|||_e^2$$
$$\leq 2C\sqrt{M_0} \underset{s\in[0,t]}{Sup}|||f(s) - h(s)|||_e^2. \tag{3.82}$$

Since $M_0 \leq \frac{1}{16C^2}$, taking the supremum in (3.82) on the time interval $[0, T]$, we obtain

$$\underset{s\in[0,t]}{Sup}|||f(s) - h(s)|||_e^2 \leq \frac{1}{2}\underset{s\in[0,t]}{Sup}|||f(s) - h(s)|||_e^2,$$

Then

$$\underset{s\in[0,\infty]}{Sup}|||f(s) - h(s)|||_e^2 \leq \frac{1}{2}\underset{s\in[0,\infty]}{Sup}|||f(s) - h(s)|||_e^2.$$

Thus $f = h$ on $\mathbb{R}_+$. □

3.4 L^2-stability for homogeneous solutions

In this section, we compare the difference between two solutions corresponding to different initial data.

3.4.1 L^2-stability for Israel particles in the case of homogeneous solutions

Theorem 3.4. Let the assumptions of Theorem 3.1 hold. Let $f_0(v)$ and $h_0(v)$ be two functions such that $max\{|||f_0|||_e^2, |||h_0|||_e^2\} \leq M_0$ for M_0 sufficiently small. If f and h are two solutions for the homogeneous relativistic Boltzmann equation (3.1) associated to the initial data f_0 and h_0, respectively, then

$$|||(f - h)(t)|||_e \leq C|||f_0 - h_0|||_e, \qquad \forall t \in [0, \infty[\tag{3.83}$$

where C is a constant which does not depend on t.

Proof. From the assumptions of the theorem, we have

$$\partial_t f = Q(f, f), \tag{3.84}$$

$$\partial_t h = Q(h, h), \tag{3.85}$$

$$f(0, v) = f_0(v), \qquad h(0, v) = h_0(v), \qquad \forall v \in \mathbb{R}^3. \tag{3.86}$$

Subtracting (3.85) from (3.84), we obtain

$$\partial_t(f - h) = Q(f - h, h) + Q(f, f - h). \tag{3.87}$$

Let us denote $\rho = f - h$.
Let us multiply (3.87) by $2e^{|v|^2}\rho(t, v)$, we have

$$2e^{|v|^2}\rho(t, v)\partial_t\rho(t, v) = 2e^{|v|^2}\rho(t, v)Q(\rho, h) + 2e^{|v|^2}\rho(t, v)Q(f, \rho).$$

We integrate the above equation on $[0, t]$ and obtain

$$e^{|v|^2}\rho^2(t, v) = e^{|v|^2}\rho^2(0, v) + 2e^{|v|^2}\rho(t, v)Q(\rho, h) + 2e^{|v|^2}\rho(t, v)Q(f, \rho).$$

We integrate the above equation with respect to v and obtain

$$\|\rho(t)\|_e^2 = \|f_0 - h_0\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2} \frac{4\sigma(\omega)}{v^0u^0\sqrt{s}}\rho(v) \\ \times [\rho(v')h(u') - \rho(v)h(u) + f(v')\rho(u') - f(v)\rho(u)]d\omega du dv. \quad (3.88)$$

Following the same idea as for Lemma 3.1, with the fact that $\|f(t)\|_e$ and $\|h(t)\|_e$ are both bounded, we have

$$\begin{aligned}\|\rho(t)\|_e^2 &= \|f_0 - h_0\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} \\ &\quad \times [|\rho(v')|h(u') + |\rho(v)|h(u) + f(v')|\rho(u')| + f(v)|\rho(u)|]d\omega du dv \\ &\leq \|f_0 - h_0\|_e^2 + A_1(t) + A_2(t) + A_3(t) + A_4(t)\end{aligned}$$

where

$$\begin{aligned}A_1(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)||\rho(v')|h(u')d\omega du dv, \\ A_2(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega du dv, \\ A_3(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega du dv, \\ A_4(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)|f(v)|\rho(u)|d\omega du dv.\end{aligned}$$

Then we have

$$\begin{aligned}A_1(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\rho(v)||\rho(v')|h(u')d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint \sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}|\rho(v')|e^{\frac{1}{2}|u'|^2}h(u')e^{-\frac{1}{2}|u|^2}d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint e^{|v|^2}(\rho(v))^2u^0\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\ &\times \left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2e^{|u'|^2}(h(u'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2\|h(s)\|_e ds \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2 ds.\end{aligned}$$

Next

$$\begin{aligned} A_2(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\ &\times \left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{|u|^2}(h(u))^2 d\omega du dv\right]^{\frac{1}{2}} \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2\|h(s)\|_e ds \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2 ds. \end{aligned}$$

Next

$$\begin{aligned} A_3(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint \sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}|\rho(u')|e^{-\frac{1}{2}|u|^2}d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint e^{|v|^2}(\rho(v))^2 u^0\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\ &\times \left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2 e^{|u'|^2}(\rho(u'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2\|f(s)\|_e ds \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2 ds. \end{aligned}$$

Next

$$\begin{aligned} A_4(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\sigma_0(\omega)|\rho(v)|f(v)|\rho(u)|d\omega du dv \\ &\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\ &\times \left[\iiint \sigma_0(\omega)e^{|v|^2}(f(v))^2 e^{|u|^2}(\rho(u))^2 d\omega du dv\right]^{\frac{1}{2}} \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2\|f(s)\|_e ds \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2\|f(s)\|_e ds \\ &\leq C\int_0^t a^{-1}b^{-2}\|\rho(s)\|_e^2 ds. \end{aligned}$$

we obtain the estimate

$$\|\rho(t)\|_e^2 \leq \|f_0 - h_0\|_e^2 + \int_0^t C\chi(s)\|\rho(s)\|_e^2 ds \tag{3.89}$$

where

$$\chi(s) = a^{-1}(s)b^{-2}(s).$$

Applying the Gronwall lemma to (3.89) leads to

$$\|\rho(t)\|_e^2 \leq C\|f_0 - h_0\|_e^2 exp(\int_0^t \chi(s)ds). \tag{3.90}$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$\|\rho(t)\|_e^2 \leq C\|f_0 - h_0\|_e^2. \tag{3.91}$$

Next we control the terms $\|\partial_{v^i}\rho(t)\|_e^2$ for $i = 1, 2, 3$. We apply ∂_{v^i} to (3.87), then we multiply the resulting equation by $2\partial_{v^i}\rho(t, v)$ and integrate from 0 to t. After this action, we multiply the resulting equation by $e^{|v|^2}$ and then integrate with respect to v to obtain

$$\|\partial_{v^i}\rho(t)\|_e^2 = \|\partial_{v^i}(f_0 - h_0)\|_e^2 + \int_0^t 2ds \int_{\mathbb{R}^3} e^{|v|^2}\partial_{v^i}\rho(s) \tag{3.92}$$

$$\times [\partial_{v^i}Q(\rho, h)(s, v) + \partial_{v^i}Q(f, \rho)(s, v)]dv. \tag{3.93}$$

Then we can state that

$$\|\partial_{v^i}\rho(t)\|_e^2 \leq \|\partial_{v^i}(f_0 - h_0)\|_e^2 + E_{1i}(t) + E_{2i}(t) + E_{3i}(t) + E_{4i}(t)$$
$$+ E_{5i}(t) + E_{6i}(t) + E_{7i}(t) + E_{8i}(t) \tag{3.94}$$

where

$$E_{1i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}f(v)||\rho(u)|d\omega dudv, \tag{3.95}$$

$$E_{2i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})|f(v)|\rho(u)|d\omega dudv, \tag{3.96}$$

$$E_{3i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}\rho(v)|h(u)d\omega dudv, \tag{3.97}$$

$$E_{4i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||\rho(v)|h(u)d\omega dudv, \tag{3.98}$$

$$E_{5i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})|f(v')|\rho(u')|d\omega dudv, \tag{3.99}$$

$$E_{6i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(f(v')\rho(u'))|d\omega dudv, \tag{3.100}$$

$$E_{7i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}})||\rho(v')|h(u')d\omega dudv, \tag{3.101}$$

$$E_{8i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(\rho(v')h(u'))|d\omega dudv. \tag{3.102}$$

Following the same method as for Lemma 3.2 with the fact that $\|f(t)\|_e$, $\|\partial_{v^i}f(t)\|_e$, $\|h(t)\|_e$, and $\|\partial_{v^i}h(t)\|_e$ are bounded, we control $E_{1i}(t)$, $E_{2i}(t)$, $E_{3i}(t)$, $E_{4i}(t)$, $E_{5i}(t)$, $E_{6i}(t)$, $E_{7i}(t)$ and $E_{8i}(t)$ as follows:

For (3.95), we have

$$\begin{aligned}
E_{1i}(t) \leq C \int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)|\partial_{v^i}f(v)||\rho(u)|d\omega dudv \\
C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}} \\
\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(\partial_{v^i}f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^i}\rho(s)\|_e\|\partial_{v^i}f(s)\|_e\|\rho(s)\|_e ds \\
\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.96), we have

$$\begin{aligned}
E_{2i}(t) \leq C \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)f(v)|\rho(u)|d\omega dudv \\
\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds \\
\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.97), we have

$$\begin{aligned}
E_{3i}(t) \leq C \int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)|\partial_{v^i}\rho(v)|h(u)d\omega dudv \\
C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}} \\
\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^i}\rho(s)\|_e^2\|h(s)\|_e ds \\
\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.98), we have

$$
\begin{aligned}
E_{4i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)|\rho(v)|h(u)d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \sigma_0(\omega)(u^0)^2e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{-2}\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.99),we have

$$
\begin{aligned}
E_{5i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)f(v')|\rho(u')|d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}(\partial_{v^i}\rho(v))^2u^0\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(\rho(u'))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.100), we have

$$
\begin{aligned}
E_{6i}(t) &= \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(f(v')\rho(u'))|d\omega dudv\\
&\leq C\int_0^t a^{-1}b^{-2}ds\iiint (u^0)^3e^{|v|^2}\sigma_0(\omega)f(v')\sum_{j=1}^{3}|(\partial_{v^j}\rho)(u')|d\omega dudv\\
&+\leq \int_0^t a^{-1}b^{-2}ds\iiint (u^0)^3e^{|v|^2}\sigma_0(\omega)|\rho(u')|\sum_{j=1}^{3}|(\partial_{v^j}f)(v')|d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}\left[\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\sum_{j=1}^{3}|\partial_{v^j}\rho(s)|+\|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=3}^{t}\|(\partial_{v^i}f)(s)\|_e\right]ds\\
&\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.101), we have

$$\begin{aligned}
E_{7i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\sigma_0(\omega)|\rho(v')h(u')d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}(\partial_{v^i}\rho(v))^2u^0\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint\frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2e^{|u'|^2}(h(u'))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-1}b^{\beta-2}ds\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2ds.
\end{aligned}$$

For (3.4.1), we have

$$\begin{aligned}
E_{8i}(t) &= \int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(\rho(v')h(u'))|d\omega dudv\\
&\leq C\int_0^t a^{-1}b^{-2}ds\iiint(u^0)^3e^{|v|^2}\sigma_0(\omega)|\rho(v')|\sum_{j=1}^{3}|(\partial_{v^j}h)(u')|d\omega dudv\\
&+\leq\int_0^t a^{-1}b^{-2}ds\iiint(u^0)^3e^{|v|^2}\sigma_0(\omega)h(u')\sum_{j=1}^{3}|(\partial_{v^j}f\rho)(v')|d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}\left[\|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=1}^{3}|\partial_{v^j}h(s)|+\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\sum_{j=3}^{t}\|(\partial_{v^i}\rho)(s)\|_e\right]ds\\
&\leq C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2ds.
\end{aligned}$$

Combining the above results with (3.94) we obtain

$$\begin{aligned}
\|\partial_{v^i}\rho(t)\|_e^2 &\leq \|\partial_{v^i}(f_0-h_0)\|_e^2+E_{1i}(t)+E_{2i}(t)+E_{3i}(t)+E_{4i}(t)\\
&+E_{5i}(t)+E_{6i}(t)+E_{7i}(t)+E_{8i}(t)\\
&\leq \|\partial_{v^i}(f_0-h_0)\|_e^2+C\int_0^t a^{-1}b^{-2}|||\rho(s)|||_e^2ds.
\end{aligned}\tag{3.103}$$

Summing up (3.89) and (3.103)and obtain

$$|||\rho(s)|||_e^2\leq|||f_0-h_0|||_e^2+\int_0^t C\chi(s)|||\rho(s)|||_e^2ds\tag{3.104}$$

where

$$\chi(s)=a^{-1}(s)b^{-2}(s).$$

Applying the Gronwall lemma to (3.104) leads to

$$|||\rho(s)|||_e^2\leq C|||f_0-h_0|||_e^2exp(\int_0^t\chi(s)ds).\tag{3.105}$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$|||\rho(s)|||_e^2\leq C|||f_0-h_0|||_e^2.$$

Thus we can state that

$$|||f(t) - h(t)|||_e^2 \leq C|||f_0 - h_0|||_e^2 \qquad \forall t \in [0, \infty[.$$

□

3.4.2 L^2-stability for hard potentials in the case of homogeneous solutions

Theorem 3.5. Let the assumptions of Theorem 3.2 hold. Let $f_0(v)$ and $h_0(v)$ be two functions such that $max\{|||f_0|||_e^2, |||h_0|||_e^2\} \leq M_0$ for M_0 sufficiently small. If f and h are two solutions for the homogeneous relativistic Boltzmann equation (3.1) associated to the initial data f_0 and h_0, respectively, then

$$|||(f - h)(t)|||_e \leq C|||f_0 - h_0|||_e, \qquad \forall t \in [0, \infty[. \tag{3.106}$$

where C is a constant independent on t.

Proof. From the assumptions of the theorem, we have

$$\partial_t f = Q(f, f), \tag{3.107}$$

$$\partial_t h = Q(h, h), \tag{3.108}$$

$$f(0, v) = f_0(v), \qquad h(0, v) = h_0(v), \qquad \forall v \in \mathbb{R}^3. \tag{3.109}$$

Subtracting (3.108) from (3.107), we obtain

$$\partial_t(f - h) = Q(f - h, h) + Q(f, f - h). \tag{3.110}$$

Let us denote $\rho = f - h$.
Let us multiply (3.110) by $2e^{|v|^2}\rho(t, v)$, we have

$$2e^{|v|^2}\rho(t, v)\partial_t\rho(t, v) = 2e^{|v|^2}\rho(t, v)Q(\rho, h) + 2e^{|v|^2}\rho(t, v)Q(f, \rho).$$

We integrate the above equation on $[0, t]$ and obtain

$$e^{|v|^2}\rho^2(t, v) = e^{|v|^2}\rho^2(0, v) + 2e^{|v|^2}\rho(t, v)Q(\rho, h) + 2e^{|v|^2}\rho(t, v)Q(f, \rho).$$

We integrate the above equation with respect to v and obtain

$$\begin{aligned} \|\rho(t)\|_e^2 = \|f_0 - h_0\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\rho(v)\vartheta_\phi\sigma(g, \omega)\rho(v) \\ \times [\rho(v')h(u') - \rho(v)h(u) + f(v')\rho(u') - f(v)\rho(u)]d\omega dudv. \end{aligned} \tag{3.111}$$

Following the same idea as for Lemma 3.3, with the fact that $\|f(t)\|_e$ and $\|h(t)\|_e$ are both bounded, we have

$$\begin{aligned} \|\rho(t)\|_e^2 &= \|f_0 - h_0\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\rho(v)|\vartheta_\phi\sigma(g, \omega) \\ &\times [|\rho(v')|h(u') + |\rho(v)|h(u) + f(v')|\rho(u')| + f(v)|\rho(u)|]d\omega dudv \\ &\leq \|f_0 - h_0\|_e^2 + A_1(t) + A_2(t) + A_3(t) + A_4(t) \end{aligned}$$

where

$$A_1(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)||\rho(v')|h(u')d\omega du dv,$$

$$A_2(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega du dv,$$

$$A_3(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega du dv,$$

$$A_4(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)|f(v)|\rho(u)|d\omega du dv.$$

Then we have

$$\begin{aligned}
A_1(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)||\rho(v')|h(u')d\omega du dv \\
&\le C\int_0^t 2a^{-1}b^{-2}ds \iiint \vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}|\rho(v')|e^{\frac{1}{2}|u'|^2}h(u')e^{-\frac{1}{2}|u|^2}d\omega du dv \\
&\le C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2e^{|u'|^2}(h(u'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2e^{|u'|^2}(h(u'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\
&\le C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}} + a^{-1}b^{-2})\|\rho(s)\|_e^2\|h(s)\|_e ds \\
&\le C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}} + a^{-1}b^{-2})\|\rho(s)\|_e^2 ds \\
&\le C\int_0^t (a^{-1}b^{\beta-2} + a^{-1}b^{-2})\|\rho(s)\|_e^2 ds.
\end{aligned}$$

Next

$$
\begin{aligned}
A_2(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2\|h(s)\|_e ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds.
\end{aligned}
$$

Next

$$
\begin{aligned}
A_3(t) &= C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint \vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}|\rho(u')|e^{-\frac{1}{2}|u|^2}d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(\rho(u'))^2d\omega du'dv'\right]^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(\rho(u'))^2d\omega du'dv'\right]^{\frac{1}{2}}\\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2\|f(s)\|_e ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds.
\end{aligned}
$$

Next

$$
\begin{aligned}
A_4(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v)||f(v)|\rho(u)|d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&+ C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2\|f(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\rho(s)\|_e^2 ds.
\end{aligned}
$$

we obtain the estimate

$$
\|\rho(t)\|_e^2 \leq \|f_0-h_0\|_e^2 + \int_0^t C\chi(s)\|\rho(s)\|_e^2 ds \tag{3.112}
$$

where

$$
\chi(s) = a^{-1}(s)b^{\beta-2}(s) + a^{-1}(s)b^{-2}(s).
$$

Applying the Gronwall lemma to (3.112) leads to

$$
\|\rho(t)\|_e^2 \leq C\|f_0-h_0\|_e^2 exp(\int_0^t \chi(s)ds). \tag{3.113}
$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$
\|\rho(t)\|_e^2 \leq C\|f_0-h_0\|_e^2. \tag{3.114}
$$

Next we control the terms $\|\partial_{v^i\rho(t)}\|_e^2$ for $i=1,2,3$. We apply ∂_{v^i} to (3.110), then we multiply the resulting equation by $2\partial_{v^i}\rho(t,v)$ and integrate from 0 to t. After this action, we multiply the resulting equation by $e^{|v|^2}$ and then integrate with respect to v to obtain

$$
\|\partial_{v^i}\rho(t)\|_e^2 = \|\partial_{v^i}(f_0-h_0)\|_e^2 + \int_0^t 2ds\int_{\mathbb{R}^3} e^{|v|^2}\partial_{v^i}\rho(s) \tag{3.115}
$$

$$
\times\left[\partial_{v^i}Q(\rho,h)(s,v) + \partial_{v^i}Q(f,\rho)(s,v)\right]dv. \tag{3.116}
$$

Then we can state that

$$
\begin{aligned}
\|\partial_{v^i}\rho(t)\|_e^2 \leq \|\partial_{v^i}(f_0-h_0)\|_e^2 + E_{1i}(t) + E_{2i}(t) + E_{3i}(t) + E_{4i}(t) \\
+ E_{5i}(t) + E_{6i}(t) + E_{7i}(t) + E_{8i}(t)
\end{aligned} \tag{3.117}
$$

where

$$E_{1i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}f(v)||\rho(u)|d\omega dudv, \tag{3.118}$$

$$E_{2i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))|f(v)|\rho(u)|d\omega dudv, \tag{3.119}$$

$$E_{3i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}\rho(v)|h(u)d\omega dudv, \tag{3.120}$$

$$E_{4i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))||\rho(v)|h(u)d\omega dudv, \tag{3.121}$$

$$E_{5i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))|f(v')|\rho(u')|d\omega dudv, \tag{3.122}$$

$$E_{6i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(f(v')\rho(u'))|d\omega dudv, \tag{3.123}$$

$$E_{7i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))||\rho(v')|h(u')d\omega dudv, \tag{3.124}$$

$$E_{8i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(\rho(v')h(u'))|d\omega dudv. \tag{3.125}$$

Following the same method as for Lemma 3.4 with the fact that $\|f(t)\|_e$, $\|\partial_{v^i}f(t)\|_e$, $\|h(t)\|_e$, and $\|\partial_{v^i}h(t)\|_e$ are bounded, we control $E_{1i}(t)$, $E_{2i}(t)$, $E_{3i}(t)$, $E_{4i}(t)$, $E_{5i}(t)$, $E_{6i}(t)$, $E_{7i}(t)$ and $E_{8i}(t)$ as follows:

For (3.118), we have

$$\begin{aligned}
E_{1i}(t) &\le C\int_0^t a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^i}f(v)||\rho(u)|d\omega dudv \\
&\le C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}} \\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}} \\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\le C\int_0^t 2(a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2})\|\partial_{v^i}\rho(s)\|_e\|\partial_{v^i}f(s)\|_e\|\rho(s)\|_e ds \\
&\le C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.119), we have
for $i=1$

$$
\begin{aligned}
E_{21}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^1}\rho(v)|u^0(1+g^{-\beta})\sigma_0(\omega)f(v)|\rho(u)|d\omega dudv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint g^{-2\beta}\sigma_0(\omega)(u^0)^2e^{|v|^2}(\partial_{v^1}\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint \sigma_0(\omega)(u^0)^2e^{|v|^2}(\partial_{v^1}\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2(a^{-2}b^{\beta-2}+a^{-1}b^{-2})\|\partial_{v^1}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\|\rho(s)\|\|_e^2ds,
\end{aligned}
$$

for $i=2,3$

$$
\begin{aligned}
E_{2i}(t) &\leq C\int_0^t 2a^{-1}b^{-3}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|u^0(1+g^{-\beta})\sigma_0(\omega)f(v)|\rho(u)|d\omega dudv\\
&\leq C\int_0^t 2(a^{-1}b^{\beta-3}+a^{-1}b^{-2})\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\|\rho(s)\|\|_e^2ds.
\end{aligned}
$$

For (3.120), we have

$$
\begin{aligned}
E_{3i}(t) &\leq C\int_0^t a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^i}\rho(v)|h(u)d\omega dudv\\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}}\\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&+C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}}\\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2(a^{-1}b^{\beta-\frac{5}{2}}+a^{1}b^{-2})\|\partial_{v^i}\rho(s)\|_e^2\|h(s)\|_e ds\\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})\|\|\rho(s)\|\|_e^2ds.
\end{aligned}
$$

For (3.121),we have

for $i = 1$

$$\begin{aligned}
E_{41}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|u^0(1+g^{-\beta})\sigma_0(\omega)|\rho(v)|h(u)d\omega du dv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint g^{-2\beta}\sigma_0(\omega)(u^0)^2 e^{|v|^2}(\partial_{v^1}\rho(v))^2 e^{-|u|^2} d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{|u|^2}(h(u))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&+ \leq C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint \sigma_0(\omega)(u^0)^2 e^{|v|^2}(\partial_{v^1}\rho(v))^2 e^{-|u|^2} d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{|u|^2}(h(u))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t 2(a^{-2}b^{\beta-2}+a^{-2}b^{-2})\|\partial_{v^1}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds,
\end{aligned}$$

for $i = 2, 3$

$$\begin{aligned}
E_{4i}(t) &\leq C\int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|u^0(1+g^{-\beta})\sigma_0(\omega)|\rho(v)|h(u)d\omega du dv \\
&\leq C\int_0^t 2(a^{-1}b^{\beta-3}+a^{-1}b^{-3})\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.122), we have

for $i = 1$

$$\begin{aligned}
E_{51}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|(1+g^{-\beta})u^0\sigma_0(\omega)f(v')|\rho(u')|d\omega du dv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2 g^{-2\beta}(u^0)^3\sigma_0(\omega)e^{-|u|^2} d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{v^0}{v'^0 u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2 e^{|u'|^2}(\rho(u'))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&+ \leq C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2 (u^0)^3\sigma_0(\omega)e^{-|u|^2} d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{v^0}{v'^0 u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2 e^{|u'|^2}(\rho(u'))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t 2(a^{-2}b^{\beta-2}+a^{-2}b^{-2})ds\|\partial_{v^1}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds,
\end{aligned}$$

for $i = 2, 3$

$$\begin{aligned}
E_{5i}(t) &\leq \int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|(1+g^{-\beta})u^0\sigma_0(\omega)f(v')|\rho(u')|d\omega du dv \\
&\leq C\int_0^t 2(a^{-1}b^{\beta-3}+a^{-1}b^{-3})ds\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.123), we have

$$\begin{aligned}
E_{6i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^i}(f(v')\rho(u'))|d\omega du dv \\
&\leq C\int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)f(v')\sum_{j=1}^{3}|(\partial_{v^j}\rho)(u')|d\omega du dv \\
&+\leq \int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(u')|\sum_{j=1}^{3}|(\partial_{v^j}f)(v')|d\omega du dv \\
&\leq C\int_0^t 2(a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2}) \\
&\left[\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\sum_{j=1}^{3}|\partial_{v^j}\rho(s)| + \|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=3}^{t}\|(\partial_{v^k}f)(s)\|_e\right]ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.124), we have

for $i = 1$

$$\begin{aligned}
E_{71}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|(1+g^{-\beta})u^0\sigma_0(\omega)|\rho(v')h(u')d\omega du dv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2 g^{-2\beta}(u^0)^3\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2 e^{|u'|^2}(h(u'))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&+C\int_0^t 2a^{-2}b^{-2}ds \left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2(u^0)^3\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2 e^{|u'|^2}(h(u'))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t 2(a^{-2}b^{\beta-2}+a^{-2}b^{-2})ds\|\partial_{v^1}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds,
\end{aligned}$$

for $i = 2, 3$

$$
\begin{aligned}
E_{7i}(t) &\leq \int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|(1+g^{-\beta})u^0\sigma_0(\omega)|\rho(v')|h(u')d\omega du dv \\
&\leq C\int_0^t 2(a^{-1}b^{\beta-3}+a^{-1}b^{-3})ds\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-3})|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.125), we have

$$
\begin{aligned}
E_{8i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\partial_{v^i}(\rho(v')h(u'))|d\omega du dv \\
&\leq C\int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)|\rho(v')|\sum_{j=1}^3|(\partial_{v^j}h)(u')|d\omega du dv \\
&+\leq \int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)h(u')\sum_{j=1}^3|(\partial_{v^j}f\rho)(v')|d\omega du dv \\
&\leq C\int_0^t 2(a^{-1}b^{\beta-\frac{5}{2}}+a^{-1}b^{-2}) \\
&\left[\|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=1}^3|\partial_{v^j}h(s)|+\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\sum_{j=3}^t\|(\partial_{v^i}\rho)(s)\|_e\right]ds \\
&\leq C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

Combining the above results with (3.117) we obtain

$$
\begin{aligned}
\|\partial_{v^i}\rho(t)\|_e^2 &\leq \|\partial_{v^i}(f_0-h_0)\|_e^2 + E_{1i}(t)+E_{2i}(t)+E_{3i}(t)+E_{4i}(t) \\
&+E_{5i}(t)+E_{6i}(t)+E_{7i}(t)+E_{8i}(t) \\
&\leq \|\partial_{v^i}(f_0-h_0)\|_e^2 + C\int_0^t (a^{-1}b^{\beta-2}+a^{-1}b^{-2})|||\rho(s)|||_e^2 ds. \qquad (3.126)
\end{aligned}
$$

Summing up (3.112) and (3.126) and obtain

$$
|||\rho(s)|||_e^2 \leq |||f_0-h_0|||_e^2 + \int_0^t C\chi(s)|||\rho(s)|||_e^2 ds \qquad (3.127)
$$

where

$$
\chi(s) = a^{-1}(s)b^{\beta-2}(s)+a^{-1}(s)b^{-2}(s).
$$

Applying the Gronwall lemma to (3.127) leads to

$$
|||\rho(s)|||_e^2 \leq C|||f_0-h_0|||_e^2 exp(\int_0^t \chi(s)ds). \qquad (3.128)
$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$
|||\rho(s)|||_e^2 \leq C|||f_0-h_0|||_e^2.
$$

Thus we can state that

$$
|||f(t)-h(t)|||_e^2 \leq C|||f_0-h_0|||_e^2, \qquad \forall t\in[0,\infty[.
$$

□

3.4.3 L^2-stability for soft potentials in the case of homogeneous solutions

Theorem 3.6. Let the assumptions of Theorem 3.3 hold. Let $f_0(v)$ and $h_0(v)$ be two functions such that $max\{|||f_0|||_e^2, |||h_0|||_e^2\} \leq M_0$ for M_0 sufficiently small. If f and h are two solutions for the homogeneous relativistic Boltzmann equation (3.1) associated to the initial data f_0 and h_0, respectively, then

$$|||(f-h)(t)|||_e \leq C|||f_0 - h_0|||_e, \qquad \forall t \in [0, \infty[\tag{3.129}$$

where C is a constant independent on t.

Proof. From the assumptions of the theorem, we have

$$\partial_t f = Q(f,f), \tag{3.130}$$

$$\partial_t h = Q(h,h), \tag{3.131}$$

$$f(0,v) = f_0(v), \qquad h(0,v) = h_0(v), \qquad \forall v \in \mathbb{R}^3. \tag{3.132}$$

Subtracting (3.131) from (3.130), we obtain

$$\partial_t(f-h) = Q(f-h,h) + Q(f, f-h). \tag{3.133}$$

Let us denote $\rho = f - h$.
Let us multiply (3.133) by $2e^{|v|^2}\rho(t,v)$, we have

$$2e^{|v|^2}\rho(t,v)\partial_t\rho(t,v) = 2e^{|v|^2}\rho(t,v)Q(\rho,h) + 2e^{|v|^2}\rho(t,v)Q(f,\rho).$$

We integrate the above equation on $[0,t]$ and obtain

$$e^{|v|^2}\rho^2(t,v) = e^{|v|^2}\rho^2(0,v) + 2e^{|v|^2}\rho(t,v)Q(\rho,h) + 2e^{|v|^2}\rho(t,v)Q(f,\rho).$$

We integrate the above equation with respect to v and obtain

$$\begin{aligned}\|\rho(t)\|_e^2 = \|f_0 - h_0\|_e^2 + \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\rho(v)\vartheta_\phi\sigma(g,\omega)\rho(v)\\ \times[\rho(v')h(u') - \rho(v)h(u) + f(v')\rho(u') - f(v)\rho(u)]d\omega du dv.\end{aligned} \tag{3.134}$$

Following the same idea as for Lemma 3.5, with the fact that $\|f(t)\|_e$ and $\|h(t)\|_e$ are both bounded, we have

$$\begin{aligned}\|\rho(t)\|_e^2 &= \|f_0 - h_0\|_e^2 + C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\rho(v)|\vartheta_\phi\sigma(g,\omega)\\ &\times[|\rho(v')|h(u') + |\rho(v)|h(u) + f(v')|\rho(u')| + f(v)|\rho(u)|]d\omega du dv\\ &\leq \|f_0 - h_0\|_e^2 + A_1(t) + A_2(t) + A_3(t) + A_4(t)\end{aligned}$$

where

$$A_1(t) = C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)||\rho(v')|h(u')d\omega du dv,$$

$$A_2(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega dudv,$$

$$A_3(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega dudv,$$

$$A_4(t) = C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)|f(v)|\rho(u)|d\omega dudv.$$

Then we have

$$\begin{aligned}
A_1(t) &= C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)||\rho(v')|h(u')d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint \vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}|\rho(v')|e^{\frac{1}{2}|u'|^2}h(u')e^{-\frac{1}{2}|u|^2}d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2e^{|u'|^2}(h(u'))^2d\omega du'dv'\right]^{\frac{1}{2}} \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2\|h(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2 ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}\|\rho(s)\|_e^2 ds.
\end{aligned}$$

Next

$$\begin{aligned}
A_2(t) &= C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)||\rho(v)|h(u)d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2\|h(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2 ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}\|\rho(s)\|_e^2 ds.
\end{aligned}$$

Next

$$\begin{aligned}
A_3(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)|f(v')|\rho(u')|d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint \vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{\frac{1}{2}|v|^2}|\rho(v)|e^{\frac{1}{2}|v'|^2}f(v')e^{\frac{1}{2}|u'|^2}|\rho(u')|e^{-\frac{1}{2}|u|^2}d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint e^{|v|^2}(\rho(v))^2\vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \frac{g\sqrt{s}}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2e^{|u'|^2}(\rho(u'))^2 d\omega du' dv'\right]^{\frac{1}{2}} \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2\|f(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2 ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}\|\rho(s)\|_e^2 ds.
\end{aligned}$$

Next

$$\begin{aligned}
A_4(t) &= C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v)|f(v)|\rho(u)|d\omega du dv \\
&\leq C\int_0^t 2a^{-1}b^{-2}ds \left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\rho(v))^2e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}} \\
&\times \left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2 d\omega du dv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t a^{-1}b^{\beta-\frac{5}{2}}\|\rho(s)\|_e^2\|f(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}\|\rho(s)\|_e^2\|f(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}\|\rho(s)\|_e^2 ds.
\end{aligned}$$

we obtain the estimate

$$\|\rho(t)\|_e^2 \leq \|f_0 - h_0\|_e^2 + \int_0^t C\chi(s)\|\rho(s)\|_e^2 ds \tag{3.135}$$

where

$$\chi(s) = a^{-1}(s)b^{\beta-2}(s).$$

Applying the Gronwall lemma to (3.135) leads to

$$\|\rho(t)\|_e^2 \leq C\|f_0 - h_0\|_e^2 exp(\int_0^t \chi(s)ds). \tag{3.136}$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$\|\rho(t)\|_e^2 \leq C\|f_0 - h_0\|_e^2. \tag{3.137}$$

Next we control the terms $\|\|_e^2$ for $i = 1, 2, 3$. We apply ∂_{v^i} to (3.133), then we multiply the resulting equation by $2\partial_{v^i}\rho(t,v)$ and integrate from 0 to t. After this action, we multiply the resulting equation by $e^{|v|^2}$ and then integrate with respect to v to obtain

$$\|\partial_{v^i}\rho(t)\|_e^2 = \|\partial_{v^i}(f_0 - h_0)\|_e^2 + \int_0^t 2ds \int_{\mathbb{R}^3} e^{|v|^2}\partial_{v^i}\rho(s) \tag{3.138}$$

$$\times [\partial_{v^i}Q(\rho,h)(s,v) + \partial_{v^i}Q(f,\rho)(s,v)]dv. \tag{3.139}$$

Then we can state that

$$\|\partial_{v^i}\rho(t)\|_e^2 \leq \|\partial_{v^i}(f_0 - h_0)\|_e^2 + E_{1i}(t) + E_{2i}(t) + E_{3i}(t) + E_{4i}(t) + E_{5i}(t) + E_{6i}(t) + E_{7i}(t) + E_{8i}(t) \tag{3.140}$$

where

$$E_{1i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}f(v)||\rho(u)|d\omega du dv, \tag{3.141}$$

$$E_{2i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))|f(v)|\rho(u)|d\omega du dv, \tag{3.142}$$

$$E_{3i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}\rho(v)|h(u)d\omega du dv, \tag{3.143}$$

$$E_{4i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))||\rho(v)|h(u)d\omega du dv, \tag{3.144}$$

$$E_{5i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))|f(v')|\rho(u')|d\omega du dv, \tag{3.145}$$

$$E_{6i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(f(v')\rho(u'))|d\omega du dv, \tag{3.146}$$

$$E_{7i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)||\partial_{v^i}(\vartheta_\phi\sigma(g,\omega))||\rho(v')|h(u')d\omega du dv, \tag{3.147}$$

$$E_{8i}(t) = \int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(\rho(v')h(u'))|d\omega du dv. \tag{3.148}$$

Following the same method as for Lemma 3.6 with the fact that $\|f(t)\|_e$, $\|\partial_{v^i}f(t)\|_e$, $\|h(t)\|_e$, and $\|\partial_{v^i}h(t)\|_e$ are bounded, we control $E_{1i}(t)$, $E_{2i}(t)$, $E_{3i}(t)$, $E_{4i}(t)$, $E_{5i}(t)$, $E_{6i}(t)$, $E_{7i}(t)$ and $E_{8i}(t)$ as follows:

For (3.141), we have

$$E_{1i}(t) \leq C\int_0^t a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^i}f(v)||\rho(u)|d\omega dudv$$

$$C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}}$$

$$\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}}$$

$$\leq C\int_0^t 2a^{-1}b^{\beta-\frac{5}{2}}\|\partial_{v^i}\rho(s)\|_e\|\partial_{v^i}f(s)\|_e\|\rho(s)\|_e ds$$

$$\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.$$

For (3.142), we have
for $i=1$

$$E_{21}(t) \leq C\int_0^t 2a^{-2}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^1}\rho(v)|u^0g^{-\beta}\sigma_0(\omega)f(v)|\rho(u)|d\omega dudv$$

$$\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint g^{-2\beta}\sigma_0(\omega)(u^0)^2e^{|v|^2}(\partial_{v^1}\rho(v))^2e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}}$$

$$\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(f(v))^2e^{|u|^2}(\rho(u))^2d\omega dudv\right]^{\frac{1}{2}}$$

$$\leq C\int_0^t 2a^{-2}b^{\beta-2}\|\partial_{v^1}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds$$

$$\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds,$$

for $i=2,3$

$$E_{2i}(t) \leq C\int_0^t 2a^{-1}b^{-3}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|u^0g^{-\beta}\sigma_0(\omega)f(v)|\rho(u)|d\omega dudv$$

$$\leq C\int_0^t 2a^{-2}b^{\beta-3}\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds$$

$$\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.$$

For (3.143), we have

$$E_{3i}(t) \leq C\int_0^t a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^i}\rho(v)|h(u)d\omega dudv$$

$$C\int_0^t 2a^{-1}b^{-2}ds\left[\iiint \vartheta_\phi g^{-2\beta}\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{-|u|^2}\right]^{\frac{1}{2}}$$

$$\times\left[\iiint \vartheta_\phi\sigma_0(\omega)e^{|v|^2}(\partial_{v^i}\rho(v))^2e^{|u|^2}(h(u))^2d\omega dudv\right]^{\frac{1}{2}}$$

$$\leq C\int_0^t 2a^{-1}b^{\beta-\frac{5}{2}}\|\partial_{v^i}\rho(s)\|_e^2\|h(s)\|_e ds$$

$$\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.$$

For (3.144), we have

for $i = 1$

$$
\begin{aligned}
E_{41}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|u^0 g^{-\beta}\sigma_0(\omega)|\rho(v)|h(u)d\omega dudv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint g^{-2\beta}\sigma_0(\omega)(u^0)^2 e^{|v|^2}(\partial_{v^1}\rho(v))^2 e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \sigma_0(\omega)e^{|v|^2}(\rho(v))^2 e^{|u|^2}(h(u))^2 d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-2}b^{\beta-2}\|\partial_{v^1}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds,
\end{aligned}
$$

for $i = 2, 3$

$$
\begin{aligned}
E_{4i}(t) &\leq C\int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|u^0 g^{-\beta}\sigma_0(\omega)|\rho(v)|h(u)d\omega dudv \\
&\leq C\int_0^t 2a^{-2}b^{\beta-3}\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.145), we have

for $i = 1$

$$
\begin{aligned}
E_{51}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|g^{-\beta}u^0\sigma_0(\omega)f(v')|\rho(u')|d\omega dudv \\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2 g^{-2\beta}(u^0)^3\sigma_0(\omega)e^{-|u|^2}d\omega dudv\right]^{\frac{1}{2}} \\
&\times\left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(f(v'))^2 e^{|u'|^2}(\rho(u'))^2 d\omega dudv\right]^{\frac{1}{2}} \\
&\leq C\int_0^t 2a^{-2}b^{\beta-2}ds\|\partial_{v^1}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds,
\end{aligned}
$$

for $i = 2, 3$

$$
\begin{aligned}
E_{5i}(t) &\leq \int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|g^{-\beta}u^0\sigma_0(\omega)f(v')|\rho(u')|d\omega dudv \\
&\leq C\int_0^t 2a^{-1}b^{\beta-3}ds\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\|\rho(s)\|_e ds \\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.
\end{aligned}
$$

For (3.146), we have

$$\begin{aligned}
E_{6i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^i}(f(v')\rho(u'))|d\omega du dv\\
&\leq C\int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)f(v')\sum_{j=1}^3|(\partial_{v^j}\rho)(u')|d\omega du dv\\
&+\leq \int_0^t a^{-1}b^{-2}ds \iiint (u^0)^5 e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(u')|\sum_{j=1}^3|(\partial_{v^j}f)(v')|d\omega du dv\\
&\leq C\int_0^t 2a^{-1}b^{\beta-\frac{5}{2}}\left[\|\partial_{v^i}\rho(s)\|_e\|f(s)\|_e\sum_{j=1}^3|\partial_{v^j}\rho(s)| + \|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=3}^t\|(\partial_{v^k}f)(s)\|_e\right]ds\\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.147), we have

for $i=1$

$$\begin{aligned}
E_{71}(t) &\leq C\int_0^t 2a^{-2}b^{-2}ds \iiint e^{|v|^2}|\partial_{v^1}\rho(v)|g^{-\beta}u^0\sigma_0(\omega)|\rho(v')h(u')d\omega du dv\\
&\leq C\int_0^t 2a^{-2}b^{-2}ds\left[\iiint e^{|v|^2}(\partial_{v^1}\rho(v))^2 g^{-2\beta}(u^0)^3\sigma_0(\omega)e^{-|u|^2}d\omega du dv\right]^{\frac{1}{2}}\\
&\times\left[\iiint \frac{v^0}{v'^0u'^0}\sigma_0(\omega)e^{|v'|^2}(\rho(v'))^2 e^{|u'|^2}(h(u'))^2 d\omega du dv\right]^{\frac{1}{2}}\\
&\leq C\int_0^t 2a^{-2}b^{\beta-2}ds\|\partial_{v^1}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds,
\end{aligned}$$

for $i=2,3$

$$\begin{aligned}
E_{7i}(t) &\leq \int_0^t 2a^{-1}b^{-3}ds \iiint e^{|v|^2}|\partial_{v^i}\rho(v)|g^{-\beta}u^0\sigma_0(\omega)|\rho(v')|h(u')d\omega du dv\\
&\leq C\int_0^t 2a^{-1}b^{\beta-3}ds\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\|\rho(s)\|_e ds\\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2 ds.
\end{aligned}$$

For (3.148), we have

$$
\begin{aligned}
E_{8i}(t) &\leq C\int_0^t 2a^{-1}b^{-2}ds\iiint e^{|v|^2}|\partial_{v^i}\rho(v)|\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\partial_{v^i}(\rho(v')h(u'))|d\omega du dv\\
&\leq C\int_0^t a^{-1}b^{-2}ds\iiint (u^0)^5e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)|\rho(v')|\sum_{j=1}^3|(\partial_{v^j}h)(u')|d\omega du dv\\
&+\leq \int_0^t a^{-1}b^{-2}ds\iiint (u^0)^5e^{|v|^2}\vartheta_\phi g^{-\beta}\sigma_0(\omega)h(u')\sum_{j=1}^3|(\partial_{v^j}f\rho)(v')|d\omega du dv\\
&\leq C\int_0^t 2a^{-1}b^{\beta-\frac{5}{2}}\left[\|\partial_{v^i}\rho(s)\|_e\|\rho(s)\|_e\sum_{j=1}^3|\partial_{v^j}h(s)|+\|\partial_{v^i}\rho(s)\|_e\|h(s)\|_e\sum_{j=3}^t\|(\partial_{v^i}\rho)(s)\|_e\right]ds\\
&\leq C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2ds.
\end{aligned}
$$

Combining the above results with (3.140) we obtain

$$
\begin{aligned}
\|\partial_{v^i}\rho(t)\|_e^2 &\leq \|\partial_{v^i}(f_0-h_0)\|_e^2+E_{1i}(t)+E_{2i}(t)+E_{3i}(t)+E_{4i}(t)\\
&+E_{5i}(t)+E_{6i}(t)+E_{7i}(t)+E_{8i}(t)\\
&\leq \|\partial_{v^i}(f_0-h_0)\|_e^2+C\int_0^t a^{-1}b^{\beta-2}|||\rho(s)|||_e^2ds. \qquad (3.149)
\end{aligned}
$$

Summing up (3.135) and (3.149)and obtain

$$
|||\rho(s)|||_e^2\leq |||f_0-h_0|||_e^2+\int_0^t C\chi(s)|||\rho(s)|||_e^2ds \qquad (3.150)
$$

where

$$
\chi(s)=a^{-1}(s)b^{\beta-2}(s).
$$

Applying the Gronwall lemma to (3.150) leads to

$$
|||\rho(s)|||_e^2\leq C|||f_0-h_0|||_e^2exp(\int_0^t\chi(s)ds). \qquad (3.151)
$$

Since χ is integrable over $\mathbb{R}_+$, we have

$$
|||\rho(s)|||_e^2\leq C|||f_0-h_0|||_e^2.
$$

Thus we can state that

$$
|||f(t)-h(t)|||_e^2\leq C|||f_0-h_0|||_e^2,\qquad \forall t\in[0,\infty[.
$$

□

CHAPTER FOUR

MILD SOLUTIONS OF THE INHOMOGENEOUS EQUATION

Contents

LEt's consider the set M that we will define in the sequel, the relativistic Boltzmann equation in f with initial data $f_0 \in M$ then reads in term of variables (t, x, v)

$$\frac{\partial f}{\partial t} + a^{-2}(t)\frac{v^1}{v^0}\frac{\partial f}{\partial x^1} + b^{-2}(t)\frac{v^2}{v^0}\frac{\partial f}{\partial x^2} + b^{-2}(t)\frac{v^3}{v^0}\frac{\partial f}{\partial x^3} = Q(f, f)(t, x, v). \tag{4.1}$$

We assume that the coefficients a and b of the Bianchi type I metric are given increasing functions of the time t and are such that:

$$a(0) \geq 1, \quad a \leq b \leq \sqrt{2}a, \quad \underset{t\longrightarrow\infty}{lim} a(t) = \infty, \tag{4.2}$$

$$\int_{\mathbb{R}_+} a^{-2}(\tau)d\tau < \infty. \tag{4.3}$$

In this chapter, we only use the third parametrization (1.65)-(1.66)-(1.67)-(1.68) of the post-collisional momenta, because we do not study the derivatives of the solutions.

4.1 Fundamental estimates

Notation 4.1. We suppose that at a position $x \in \mathbb{R}^3$, v and u stand for momenta of two particles before their collision, v' and u' for their momenta after the collision; we define the following vectors and scalars:

$$a_v = x \times v', \qquad b_v = v \times v', \qquad \nu_v = \frac{b_v}{|b_v|}, \qquad c_v = a_v.b_v, \tag{4.4}$$

$$a_u = x \times u', \qquad b_u = v \times u', \qquad \nu_u = \frac{b_u}{|b_u|}, \qquad c_u = a_u.b_u. \tag{4.5}$$

We also defined χ_1 and χ_2 by

$$\chi_1(\tau) = \int_0^\tau \frac{a^{-2}(s)ds}{\sqrt{1 + a^{-2}(s)(v^1)^2 + b^{-2}(s)|\bar{v}|^2}}, \tag{4.6}$$

$$\chi_2(\tau) = \int_0^\tau \frac{b^{-2}(s)ds}{\sqrt{1 + a^{-2}(s)(v^1)^2 + b^{-2}(s)|\bar{v}|^2}}. \tag{4.7}$$

Remark 4.1. By (4.2), we have

$$\frac{1}{2}\chi_1(\tau) \leq \chi_2(\tau) \leq \chi_1(\tau). \tag{4.8}$$

Lemma 4.1. For the functions χ_1 and χ_2, the following estimates hold:

$$|(\chi_1(\tau)v^1, \chi_2(\tau)\bar{v}) \times v'|^2 \geq (\chi(\tau))^2 |b_v|^2, \tag{4.9}$$

$$|(\chi_1(\tau)v^1, \chi_2(\tau)\bar{v}) \times u'|^2 \geq (\chi(\tau))^2 |b_u|^2, \tag{4.10}$$

where in function of the domain of $\mathbb{R}^3_v \times \mathbb{R}^3_x$, $\chi(\tau)$ is either $\chi_2(\tau)$ or $\frac{1}{2}\chi_1(\tau)$.

Proof. For the sake of simplicity, we note $\chi_1(\tau) = \chi_1$, $\chi_2(\tau) = \chi_2$. By a direct computation, we have

$$\begin{aligned}
|(\chi_1 v^1, \chi_2\bar{v}) \times v'|^2 &= \chi_2^2(v^2v'^3 - v^3v'^2)^2 + (\chi_1 v^1 v'^3 - \chi_2 v^3 v'^1)^2 + (\chi_1 v^1 v'^2 - \chi_2 v^2 v'^1)^2 \\
&= \chi_2^2(v^2v'^3 - v^3v'^2)^2 + \chi_1^2(v^1v'^3)^2 + \chi_2^2(v^3v'^1)^2 + \chi_1^2(v^1v'^2)^2 + \chi_2(v^2v'^1)^2 \\
&- 2\chi_1\chi_2 v^1 v'^1(v^2v'^2 + v^3v'^3)
\end{aligned}$$

and

$$\begin{aligned}|v\times v'|^2 &= (v^2v'^3 - v^3v'^2)^2 + (v^1v'^3 - v^3v'^1)^2 + (v^1v'^2 - v^2v'^1)^2\\ &= (v^2v'^3 - v^3v'^2)^2 + (v^1v'^3)^2 + (v^3v'^1)^2 + (v^1v'^2)^2 + (v^2v'^1)^2\\ &- 2v^1v'^1(v^2v'^2 + v^3v'^3).\end{aligned}$$

- If $v^1v'^1(v^2v'^2 + v^3v'^3) = v^1v'^1\bar{v}.\bar{v'} \geq 0$, we use (4.8) to obtain

$$|(\chi_1 v^1, \chi_2\bar{v})\times v'|^2 \geq \frac{1}{4}\chi_1^2|v\times v'|^2. \tag{4.11}$$

- If $v^1v'^1(v^2v'^2 + v^3v'^3) = v^1v'^1\bar{v}.\bar{v'} \leq 0$, we use (4.8) to obtain

$$|(\chi_1 v^1, \chi_2\bar{v})\times v'|^2 \geq \chi_2^2|v\times v'|^2. \tag{4.12}$$

For the second estimate (4.10), we have

$$\begin{aligned}|(\chi_1 v^1, \chi_2\bar{v})\times u'|^2 &= \chi_2^2(v^2u'^3 - v^3u'^2)^2 + (\chi_1 v^1u'^3 - \chi_2 v^3u'^1)^2 + (\chi_1 v^1u'^2 - \chi_2 v^2u'^1)^2\\ &= \chi_2^2(v^2u'^3 - v^3u'^2)^2 + \chi_1^2(v^1u'^3)^2 + \chi_2^2(v^3u'^1)^2 + \chi_1^2(v^1u'^2)^2 + \chi_2(v^2u'^1)^2\\ &- 2\chi_1\chi_2 v^1u'^1(v^2u'^2 + v^3u'^3)\end{aligned}$$

and

$$\begin{aligned}|v\times u'|^2 &= (v^2u'^3 - v^3u'^2)^2 + (v^1u'^3 - v^3u'^1)^2 + (v^1u'^2 - v^2u'^1)^2\\ &= (v^2u'^3 - v^3u'^2)^2 + (v^1u'^3)^2 + (v^3u'^1)^2 + (v^1u'^2)^2 + (v^2u'^1)^2\\ &- 2v^1u'^1(v^2u'^2 + v^3u'^3).\end{aligned}$$

- If $v^1u'^1(v^2u'^2 + v^3u'^3) = v^1u'^1\bar{v}.\bar{u'} \geq 0$, we use (4.8) to obtain

$$|(\chi_1 v^1, \chi_2\bar{v})\times u'|^2 \geq \frac{1}{4}\chi_1^2|v\times u'|^2. \tag{4.13}$$

- If $v^1u'^1(v^2u'^2 + v^3u'^3) = v^1u'^1\bar{v}.\bar{u'} \leq 0$, we use (4.8) to obtain

$$|(\chi_1 v^1, \chi_2\bar{v})\times u'|^2 \geq \chi_2^2|v\times u'|^2. \tag{4.14}$$

□

Lemma 4.2. For the functions χ_1 and χ_2, the following estimates hold:

$$(x\times v').((\chi_1 v^1, \chi_2\bar{v})\times v') \geq \chi(\tau)a_v.b_v, \tag{4.15}$$

$$(x\times u').((\chi_1 v^1, \chi_2\bar{v})\times u') \geq \chi(\tau)a_u.b_u, \tag{4.16}$$

where in function of the domain of $\mathbb{R}^3_v\times\mathbb{R}^3_x$, $\chi(\tau)$ is either $\chi_2(\tau)$ or $\frac{1}{2}\chi_1(\tau)$.

Proof. The proof is done using the same steps as that of Lemma 4.1.

$$\begin{aligned} a_v &= x \times v' \\ &(x^2v'^3 - x^3v'^2, x^3v'^1 - x^1v'^3, x^1v'^2 - x^2v'^1) \end{aligned}$$

and

$$\begin{aligned} a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times \bar{v'} &= (x^2v'^3 - x^3v'^2)\chi_2(v^2v'3 - v^3v'^2) \\ &+ (x^3v'^1 - x^1v'^3)(\chi_2 v^3v'^1 - \chi_1 v^1v'^3) \\ &+ (x^1v'^2 - x^2v'^1)(\chi_1 v^1v'^2 - \chi_2 v^2v'^1) \\ &= \chi_2(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) \\ &+ \chi_2\left[v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1)\right] \\ &+ \chi_1\left[v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3)\right] \end{aligned}$$

and

$$\begin{aligned} a_v.(v \times v') &= (x^2v'^3 - x^3v'^2)(v^2v'3 - v^3v'^2) \\ &+ (x^3v'^1 - x^1v'^3)(v^3v'^1 - v^1v'^3) \\ &+ (x^1v'^2 - x^2v'^1)(v^1v'^2 - v^2v'^1) \\ &= (x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) \\ &+ \left[v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1)\right] \\ &+ \left[v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3)\right]. \end{aligned}$$

• If

$$\begin{aligned} (x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\geq 0 \\ v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\geq 0 \\ v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\geq 0 \end{aligned}$$

$$thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \chi_2 a_v.v \times v'. \tag{4.17}$$

• If

$$\begin{aligned} (x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\geq 0 \\ v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\geq 0 \\ v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\leq 0 \end{aligned}$$

$$thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \frac{1}{2}\chi_1 a_v.v \times v'. \tag{4.18}$$

• If

$$
\begin{aligned}
(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\geq 0 \\
v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\leq 0 \\
v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\leq 0
\end{aligned}
$$

$$
thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \frac{1}{2}\chi_1 a_v.v \times v'. \tag{4.19}
$$

• If

$$
\begin{aligned}
(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\leq 0 \\
v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\leq 0 \\
v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\leq 0
\end{aligned}
$$

$$
thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \frac{1}{2}\chi_1 a_v.v \times v'. \tag{4.20}
$$

• If

$$
\begin{aligned}
(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\geq 0 \\
v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\leq 0 \\
v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\geq 0
\end{aligned}
$$

$$
thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \chi_2 a_v.v \times v'. \tag{4.21}
$$

• If

$$
\begin{aligned}
(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\leq 0 \\
v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\geq 0 \\
v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\geq 0
\end{aligned}
$$

$$
thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \chi_2 a_v.v \times v'. \tag{4.22}
$$

• If

$$
\begin{aligned}
(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\leq 0 \\
v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\geq 0 \\
v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\leq 0
\end{aligned}
$$

$$
thus \qquad a_v.(\chi_1 v^1, \chi_2 \bar{v}) \times v' \geq \frac{1}{2}\chi_1 a_v.v \times v'. \tag{4.23}
$$

• If

$$\begin{aligned}(x^2v'^3 - x^3v'^2)(v^2v'^3 - v^3v'^2) &\leq 0\\ v^3v'^1(x^3v'^1 - x^1v'^3) - v^2v'^1(x^1v'^2 - x^2v'^1) &\leq 0\\ v^1v'^2(x^1v'^2 - x^2v'^1) - v^1v'^3(x^3v'^1 - x^1v'^3) &\geq 0\end{aligned}$$

$$thus \qquad a_v.(\chi_1 v^1, \chi_2\bar{v}) \times v' \geq \chi_2 a_v.v \times v'. \tag{4.24}$$

We combine (4.17),(4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) to obtain the first inequality (4.15).

The second inequality (4.16) is done in the similar way to (4.15). We have

$$\begin{aligned}a_u &= x \times u'\\ &(x^2u'^3 - x^3u'^2, x^3u'^1 - x^1u'^3, x^1u'^2 - x^2u'^1)\end{aligned}$$

and

$$\begin{aligned}a_u.(\chi_1 v^1, \chi_2\bar{v}) \times \bar{u'} &= (x^2u'^3 - x^3u'^2)\chi_2(v^2u'3 - v^3u'^2)\\ &+ (x^3u'^1 - x^1u'^3)(\chi_2 v^3u'^1 - \chi_1 v^1u'^3)\\ &+ (x^1u'^2 - x^2u'^1)(\chi_1 v^1u'^2 - \chi_2 v^2u'^1)\\ &= \chi_2(x^2u'^3 - x^3u'^2)(v^2u'^3 - v^3u'^2)\\ &+ \chi_2\left[v^3u'^1(x^3u'^1 - x^1u'^3) - v^2u'^1(x^1u'^2 - x^2u'^1)\right]\\ &+ \chi_1\left[v^1u'^2(x^1u'^2 - x^2u'^1) - v^1u'^3(x^3u'^1 - x^1u'^3)\right]\end{aligned}$$

and

$$\begin{aligned}a_u.(v \times u') &= (x^2u'^3 - x^3u'^2)(v^2u'3 - v^3u'^2)\\ &+ (x^3u'^1 - x^1u'^3)(v^3u'^1 - v^1u'^3)\\ &+ (x^1u'^2 - x^2u'^1)(v^1u'^2 - v^2u'^1)\\ &= (x^2u'^3 - x^3u'^2)(v^2u'^3 - v^3u'^2)\\ &+ \left[v^3u'^1(x^3u'^1 - x^1u'^3) - v^2u'^1(x^1u'^2 - x^2u'^1)\right]\\ &+ \left[v^1u'^2(x^1u'^2 - x^2u'^1) - v^1u'^3(x^3u'^1 - x^1u'^3)\right].\end{aligned}$$

• If

$$\begin{aligned}(x^2u'^3 - x^3u'^2)(v^2u'^3 - v^3u'^2) &\geq 0\\ v^3u'^1(x^3u'^1 - x^1u'^3) - v^2u'^1(x^1u'^2 - x^2u'^1) &\geq 0\\ v^1u'^2(x^1u'^2 - x^2u'^1) - v^1u'^3(x^3u'^1 - x^1u'^3) &\geq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v}) \times u' \geq \chi_2 a_u.v \times u'. \tag{4.25}$$

• If

$$\begin{aligned}(x^2u'^3-x^3u'^2)(v^2u'^3-v^3u'^2) &\geq 0\\ v^3u'^1(x^3u'^1-x^1u'^3)-v^2u'^1(x^1u'^2-x^2u'^1) &\geq 0\\ v^1u'^2(x^1u'^2-x^2u'^1)-v^1u'^3(x^3u'^1-x^1u'^3) &\leq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v})\times u' \geq \frac{1}{2}\chi_1 a_u.v\times u'. \tag{4.26}$$

• If

$$\begin{aligned}(x^2u'^3-x^3u'^2)(v^2u'^3-v^3u'^2) &\geq 0\\ v^3u'^1(x^3u'^1-x^1u'^3)-v^2u'^1(x^1u'^2-x^2u'^1) &\leq 0\\ v^1u'^2(x^1u'^2-x^2u'^1)-v^1u'^3(x^3u'^1-x^1u'^3) &\leq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v})\times u' \geq \frac{1}{2}\chi_1 a_u.v\times u'. \tag{4.27}$$

• If

$$\begin{aligned}(x^2u'^3-x^3u'^2)(v^2u'^3-v^3u'^2) &\leq 0\\ v^3u'^1(x^3u'^1-x^1u'^3)-v^2u'^1(x^1u'^2-x^2u'^1) &\leq 0\\ v^1u'^2(x^1u'^2-x^2u'^1)-v^1u'^3(x^3u'^1-x^1u'^3) &\leq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v})\times u' \geq \frac{1}{2}\chi_1 a_u.v\times u'. \tag{4.28}$$

• If

$$\begin{aligned}(x^2u'^3-x^3u'^2)(v^2u'^3-v^3u'^2) &\geq 0\\ v^3u'^1(x^3u'^1-x^1u'^3)-v^2u'^1(x^1u'^2-x^2u'^1) &\leq 0\\ v^1u'^2(x^1u'^2-x^2u'^1)-v^1u'^3(x^3u'^1-x^1u'^3) &\geq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v})\times u' \geq \chi_2 a_u.v\times u'. \tag{4.29}$$

• If

$$\begin{aligned}(x^2u'^3-x^3u'^2)(v^2u'^3-v^3u'^2) &\leq 0\\ v^3u'^1(x^3u'^1-x^1u'^3)-v^2u'^1(x^1u'^2-x^2u'^1) &\geq 0\\ v^1u'^2(x^1u'^2-x^2u'^1)-v^1u'^3(x^3u'^1-x^1u'^3) &\geq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2\bar{v})\times u' \geq \chi_2 a_u.v\times u'. \tag{4.30}$$

• If

$$\begin{aligned}(x^2u'^3 - x^3u'^2)(v^2u'^3 - v^3u'^2) &\leq 0\\ v^3u'^1(x^3u'^1 - x^1u'^3) - v^2u'^1(x^1u'^2 - x^2u'^1) &\geq 0\\ v^1u'^2(x^1u'^2 - x^2u'^1) - v^1u'^3(x^3u'^1 - x^1u'^3) &\leq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2 \overline{v}) \times u' \geq \frac{1}{2}\chi_1 a_u.v \times u'. \tag{4.31}$$

• If

$$\begin{aligned}(x^2u'^3 - x^3u'^2)(v^2u'^3 - v^3u'^2) &\leq 0\\ v^3u'^1(x^3u'^1 - x^1u'^3) - v^2u'^1(x^1u'^2 - x^2u'^1) &\leq 0\\ v^1u'^2(x^1u'^2 - x^2u'^1) - v^1u'^3(x^3u'^1 - x^1u'^3) &\geq 0\end{aligned}$$

$$thus \qquad a_u.(\chi_1 v^1, \chi_2 \overline{v}) \times u' \geq \chi_2 a_u.v \times u'. \tag{4.32}$$

We combine (4.25),(4.26), (4.27), (4.28), (4.29), (4.30), (4.31) and (4.32) to obtain the second inequality (4.16).

□

Proposition 4.1. Let us define the scalar D by

$$D = |(x + (\chi_1(\tau)v^1, \chi_2(\tau)\overline{v})) \times v'|^2 + |(x + (\chi_1(\tau)v^1, \chi_2(\tau)\overline{v})) \times u'|^2. \tag{4.33}$$

We have

$$D \geq |\omega.(x \times v)|^2 \tag{4.34}$$

where ω denotes the parameter along the unit sphere, and which allows to parameterize the post-collisional momenta.

Proof. By elementary computation, we have

$$\begin{aligned}D = &|x \times v'|^2 + |x \times u'|^2\\ &+ 2(x \times v').((\chi_1 v^1, \chi_2\overline{v}) \times v') + 2(x \times u').((\chi_1 v^1, \chi_2\overline{v}) \times u')\\ &+ |(\chi_1 v^1, \chi_2\overline{v}) \times v'|^2 + |(\chi_1 v^1, \chi_2\overline{v}) \times u'|^2.\end{aligned} \tag{4.35}$$

Using the above Lemma 4.1 and Lemma 4.2 together with the notations (4.4) and (4.5), we obtain

$$D \geq (|b_v|^2 + |b_u|^2)\chi^2(\tau) + 2(c_v + c_u)\chi(\tau) + (|a_v|^2 + |a_u|^2) \tag{4.36}$$

where in function of the domain of $\mathbb{R}^3_v \times \mathbb{R}^3_x$, $\chi(\tau)$ is either $\chi_2(\tau)$ or $\frac{1}{2}\chi_1(\tau)$.
We denote by $\widetilde{D}$ the right hand side of (4.36). $\widetilde{D}$ is a polynomial of second order in $\chi(\tau)$. We are

going to prove that the opposite of its discriminant Δ is bounded from below.
We have

$$\begin{aligned}
-\Delta &= \left(|b_v|^2 + |b_u|^2\right)\left(|a_v|^2 + |a_u|^2\right) - (c_v + c_u)^2 \\
&= |b_v|^2|a_v|^2 + |b_v|^2|a_u|^2 + |b_u|^2|a_v|^2 + |b_u|^2|a_u|^2 - c_v^2 + 2c_vc_u - c_u^2 \\
&= |b_v|^2|a_v|^2 - (a_v.b_v)^2 + |b_u|^2|a_u|^2 - (a_u.b_u)^2 \\
&+ |b_v|^2|a_u|^2 + |b_u|^2|a_v|^2 - 2c_vc_u \\
&= |a_v \times b_v|^2 + |a_u \times b_u|^2 \\
&+ |b_v|^2\left[|a_u|^2|\nu_u|^2\right] + |b_u|^2\left[|a_v|^2|\nu_v|^2\right] - 2c_u|\frac{b_v}{b_u}|c_v|\frac{b_u}{b_v}| \\
&= |a_v \times b_v|^2 + |a_u \times b_u|^2 \\
&+ |b_v|^2\left[|a_u \times \nu_u|^2 + (a_u.\nu_u)^2\right] + |b_u|^2\left[|a_v \times \nu_v|^2 + (a_v.\nu_v)^2\right] - 2c_u|\frac{b_v}{b_u}|c_v|\frac{b_u}{b_v}| \\
&= |a_v \times b_v|^2 + |a_u \times b_u|^2 \\
&+ |b_v|^2\frac{(c_u)^2}{|b_u|^2} + |b_v|^2|a_u \times \nu_u|^2 + |b_u|^2\frac{(c_v)^2}{|b_v|^2} + |b_u|^2|a_v \times \nu_v|^2 - 2c_u|\frac{b_v}{b_u}|c_v|\frac{b_u}{b_v}| \\
&= |a_v \times b_v|^2 + |a_u \times b_u|^2 + |b_v|^2|a_u \times \nu_u|^2 + |b_u|^2|a_v \times \nu_v|^2 \\
&+ \left(\frac{|b_v|c_u}{|b_u|} - \frac{|b_u|c_v}{|b_v|}\right)^2 \\
&\geq |a_v \times b_v|^2 + |a_u \times b_u|^2 + |b_v|^2|a_u \times \nu_u|^2 + |b_u|^2|a_v \times \nu_v|^2. \qquad (4.37)
\end{aligned}$$

By (4.37), we have

$$\begin{aligned}
\widetilde{D} &= (|b_u|^2 + |b_v|^2)\left[\left(\chi(\tau) + \frac{c_v + c_u}{|b_v|^2 + |b_u|^2}\right)^2\right. \\
&\left.+\frac{(|b_v|^2 + |b_u|^2)(|a_v|^2 + |a_u|^2) - (c_v - c_u)^2}{(|b_u|^2 + |b_v|^2)^2}\right] \\
&\geq \frac{(|b_v|^2 + |b_u|^2)(|a_v|^2 + |a_u|^2) - (c_v - c_u)^2}{|b_u|^2 + |b_v|^2} \\
&\geq \frac{|a_v \times b_v|^2 + |a_u \times b_u|^2 + |b_v|^2|a_u \times \nu_u|^2 + |b_u|^2|a_v \times \nu_v|^2}{|b_u|^2 + |b_v|^2} \\
&= \frac{(|a_v \times \nu_v|^2 + |a_u \times \nu_u|^2)(|b_u|^2 + |b_v|^2)}{|b_u|^2 + |b_v|^2} \\
&= |a_v \times \nu_v|^2 + |a_u \times \nu_u|^2. \qquad (4.38)
\end{aligned}$$

We try to bound from below the terms $|a_v \times \nu_v|^2$ and $|a_u \times \nu_u|^2$.

We have

$$\begin{aligned}
|b_u &= |v \times u'| \\
&= |\omega||v \times u'| \\
&\geq |\omega.v \times u'| \\
&= |\omega.v \times (u + \widetilde{A}\omega)| \\
&= |\omega.p_u| \qquad (4.39)
\end{aligned}$$

where for a given x, p_x is defined by

$$p_x = v \times x \qquad (4.40)$$

and $\widetilde{A}$ is a parameter given by the third parametrization.
Let's recall the following vector identity for three vectors u, v and ω

$$u \times (v \times \omega) = (u.\omega)v - (u.v)\omega. \qquad (4.41)$$

Using (4.41), we have

$$\begin{aligned}
a_v \times b_v &= (x \times v') \times (v \times v') \\
&= ((u \times v').v')\, v - ((x \times v').v)\, v' \\
&= -\,(v.x \times v')\, v' \\
&= v'.(v \times x)v' \\
&= (v'.p_x)v'. \qquad (4.42)
\end{aligned}$$

The same arguments as above yields to

$$a_u \times b_u = (u'.p_x)u'. \qquad (4.43)$$

In the another hand, using the parametrization (1.65)-((1.66))

$$b_v = v \times v' = v \times (v - \widetilde{A}\omega) = \widetilde{A}p_\omega. \qquad (4.44)$$

This implies

$$|a_v \times \nu_v| = \frac{|a_v \times b_v|}{|b_v|} = \frac{|(v'.p_x)v'|}{|b_v|} = \frac{|v'.p_x||v'|}{|\widetilde{A}|}|p_\omega|.$$

In another hand, we have

$$|v'| \geq |\omega \times v'| = |\omega \times (v - \widetilde{A}\omega)| = |\omega \times v| = |p_\omega|,$$

$$|a_v \times \nu_v| \geq \frac{|v'.p_x|}{|\widetilde{A}|} = \frac{|(v - \widetilde{A}\omega).x \times v|}{|\widetilde{A}|} = |\omega.p_x|. \qquad (4.45)$$

Concerning $a_u \times \nu_u$, using the relation $a_u \times b_u = (u'.p_x)u'$, we have

$$|a_u \times \nu_u| \geq \frac{|u'.p_x|}{|v|}. \qquad (4.46)$$

(4.45) and (4.46) lead to

$$\widetilde{D} \geq |a_v \times \nu_v|^2 + |a_u \times \nu_u|^2 \geq |\omega.p_x|^2 + \frac{|u'.p_x|^2}{|v|^2} \geq |\omega.p_x|^2. \tag{4.47}$$

Since $D \geq \widetilde{D}$ we obtain the desired result. □

4.2 Differential characteristic system and functional space

Let's consider the inhomogeneous relativistic Boltzmann equation (4.1) which is a first order partial differential equation. For any fixed $(x, v) \in \mathbb{R}_x \times \mathbb{R}_v$, the characteristics $X^t(x, v)$ are defined by the following relations

$$\begin{cases} \frac{dX^{1t}}{dt}(x,v) &= a^{-2}(t)\frac{v^1}{v_0}, \\ \frac{dX^{2t}}{dt}(x,v) &= b^{-2}(t)\frac{v^2}{v^0}, \\ \frac{dX^{3t}}{dt}(x,v) &= b^{-2}(t)\frac{v^3}{v^0}, \end{cases} \tag{4.48}$$

$$X^t(x,v)|_{t=0} = x. \tag{4.49}$$

From the above expressions, we have

$$X^{1t}(x,v) = x^1 + (\int_0^t \frac{a^{-2}(s)ds}{\sqrt{1 + a^{-2}(s)(v^1)^2 + b^{-2}(s)(^{v^2})^2 + b^{-2}(s)(v^3)^2}})v^1, \tag{4.50}$$

$$X^{2t}(x,v) = x^2 + (\int_0^t \frac{b^{-2}(s)ds}{\sqrt{1 + a^{-2}(s)(v^1)^2 + b^{-2}(s)(^{v^2})^2 + b^{-2}(s)(v^3)^2}})v^2, \tag{4.51}$$

$$X^{3t}(x,v) = x^3 + (\int_0^t \frac{b^{-2}(s)ds}{\sqrt{1 + a^{-2}(s)(v^1)^2 + b^{-2}(s)(^{v^2})^2 + b^{-2}(s)(v^3)^2}})v^3. \tag{4.52}$$

Let's introduce the standard notation in the Boltzmann equation

$$f^{\#}(t,x,v) = f(t, X^t(x,v), v). \tag{4.53}$$

Using the above notation, we have

$$\begin{aligned} \frac{d}{dt}f^{\#}(t,x,v) &= \frac{\partial f}{\partial t} + \frac{\partial X^{it}}{\partial t}\frac{\partial f}{\partial x^i} \\ &= \frac{\partial f}{\partial t} + a^{-2}(t)\frac{v^1}{v^0}\frac{\partial f}{\partial x^1} + b^{-2}(t)\frac{v^2}{v^0}\frac{\partial f}{\partial x^2} + b^{-2}(t)\frac{v^3}{v^0}\frac{\partial f}{\partial x^3} \\ &= Q^{\#}(f,f)(t,x,v) \end{aligned} \tag{4.54}$$

where $Q^{\#}(f, f)$ is given by

$$Q^{\#}(f,f)(t,x,v) = Q(f,f)(t, X^t(x,v), v). \tag{4.55}$$

By (4.1) and (4.54), the inhomogeneous relativistic Boltzmann equation in terms of $f^{\#}$ reads

$$\frac{d}{dt}f^{\#}(t,x,v) = Q^{\#}(f,f)(t,x,v). \tag{4.56}$$

The Boltzmann equation in $f^{\#}$ with initial data $f^{\#}(0,x,v) = f(0,x,v) = f_0(x,v)$ leads to the following integral equation

$$f^{\#}(t,x,v) = f_0(x,v) + \int_0^t Q^{\#}(f,f)(s,x,v)ds. \tag{4.57}$$

Definition 4.1. (4.57) is called the mild form of the Boltzmann equation.

The solution of (4.57) is called the mild solution.

Lemma 4.3. For $s \in \mathbb{R}_+$, $v, u \in \mathbb{R}^3$, let us consider the vector function $\widetilde{b} = \widetilde{b}(s,v,u)$ defined by

$$\widetilde{b}^1(s,u,v) = (\int_0^s \frac{a^{-2}(\tau)}{v^0(\tau)}d\tau)v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u^0(\tau)}d\tau)u^1, \tag{4.58}$$

$$\widetilde{b}^2(s,u,v) = (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^2, \tag{4.59}$$

$$\widetilde{b}^3(s,u,v) = (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^3. \tag{4.60}$$

For $x \in \mathbb{R}^3$, if we let

$$y = x + \widetilde{b}(s,u,v) \tag{4.61}$$

the following relation holds

$$f(s, X^s(x,v), u) = f^{\#}(s,y,u). \tag{4.62}$$

Proof. By (4.53) we have

$$f^{\#}(s,y,u) = f(s, X^s(y,u), u).$$

Then the relation (4.62) holds if for all s, v and u

$$f(s, X^s(x,v), u) = f(s, X^s(y,u), u).$$

This is possible if $X^s(x,v) = X^s(y,u)$ for all s, v and u, that is to say

$$\begin{cases} X^{1s}(x,v) &= X^{1s}(y,u), \\ X^{2s}(x,v) &= X^{2s}(y,u), \\ X^{3s}(x,v) &= X^{3s}(y,u). \end{cases}$$

To get these equalities, we need that

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2}(\tau)}{v^0(\tau)}d\tau)v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u^0(\tau)}d\tau)u^1, \\ y^2 &= x^2 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^2, \\ y^3 &= x^3 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^3. \end{cases}$$

□

Remark 4.2. (4.62) is the link between f to $f^{\#}$.

Remark 4.3. In the remainder of this chapter, we are going to study the integro-equation (4.57) with an unknown function $f^{\#}$. Since the aim of this chapter is the study of mild solutions, we are looking for a continuous function $f^{\#}$ satisfying (4.57).

Remark 4.4. We are looking for a continuous bounded non-negative solution for the relativistic Boltzmann equation. Since the initial data is near vacuum, we allow f to decay exponentially in v and x variables. For this reason, we consider the weight function $\rho = \rho(x, v)$ defined by

$$\rho(x, v) = e^{(|v|^2+|x\times v|^2)}. \tag{4.63}$$

The function space in which we will seek the solution is defined as

$$M = \{f \in C^0([0, \infty] \times \mathbb{R}^3_x \times \mathbb{R}^3_v), \|f\| := \underset{(t,x,v)}{Sup}\,(\rho(x, v)|f(t, x, v)|) < \infty\}. \tag{4.64}$$

M is not an empty set. In fact $f(t, v) = e^{-2|v|^2}$ belong to M.

Remark 4.5. $(M, \| \|)$ is obviously a Banach space.

4.3 Global L^∞-existence theorem for mild solutions in the case of Israel particles

4.3.1 Estimates of the loss term

Lemma 4.4. There exists a constant C not depending on t, v and x such that :

$$\int_0^t |Q^{\#}_{loss}(f, f)(s, x, v)| \leq C\rho^{-1}(x, v)||f^{\#}||^2. \tag{4.65}$$

Proof. We recall that the loss term of the collision operator is expressed as follows

$$Q^{\#}_{loss}(f, f)(t, x, v) = a^{-1}(t)b^{-2}(t)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} f(t, X^t(x, v), v)f(t, X^t(x, v), u)du.$$

By (4.62) we have

$$\begin{aligned}|Q^{\#}_{loss}(f, f)(s, x, v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(s, X^s(x, v), v)||f(s, X^s(x, v), u)|du\\ &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f^{\#}(s, x, v)||f^{\#}(s, y, u)|du.\end{aligned}$$

From the relation (4.61), we will take $y = x + \widetilde{b}(s, u, v)$, that is

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2}(\tau)}{v^0(\tau)}d\tau)v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u^0(\tau)}d\tau)u^1,\\ y^2 &= x^2 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^2,\\ y^3 &= x^3 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^3.\end{cases}$$

It follows that

$$\begin{aligned}
|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f^{\#}(s,x,v)||f^{\#}(s,y,u)|du \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\rho^{-1}(y,u)du \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{du}{e^{|u|^2+|y\times u|^2}} \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{du}{e^{|u|^2}}.
\end{aligned}$$

Since $v^0 \geq 1$, $u^0 \geq 1$ and $2 \leq \sqrt{s}$, we have

$$\begin{aligned}
|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \sigma_0(\omega)e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{\mathbb{R}^3} e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2.
\end{aligned}$$

Then by an integration from 0 to t, we have

$$\begin{aligned}
\int_0^t |Q^{\#}_{loss}(f,f)(s,x,v)|ds &\leq C\rho^{-1}(x,v)||f^{\#}||^2 \int_0^t a^{-1}(s)b^{-2}(s)ds \\
&\leq C\rho^{-1}(x,v)||f^{\#}||^2.
\end{aligned}$$

□

4.3.2 Estimates of the gain term

As usual while working with the Boltzmann equation, the gain term is more difficult to handle.

Lemma 4.5. There exists a constant C not depending on t, v and x such that:

$$\int_0^t |Q^{\#}_{gain}(f,f)(s,x,v)|ds \leq C\rho^{-1}(x,v)\|f^{\#}\|^2. \tag{4.66}$$

Proof. We recall that the gain term of the collision operator is expressed as follows

$$Q^{\#}_{gain}(f,f)(t,x,v) = a^{-1}(t)b^{-2}(t)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} f(t,X^t(x,v),v')f(t,X^t(x,v),u')du.$$

By (4.62) we have

$$\begin{aligned}
|Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f(s,X^s(x,v),v')||f(s,X^s(x,v),u')|du \\
&\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|f^{\#}(s,y,v')||f^{\#}(s,z,u')|du.
\end{aligned}$$

We are looking for y and z such that

$$f^{\#}(s,y,v') = f(s,X^s(x,v),v') \qquad and \qquad f^{\#}(s,z,u') = f(s,X^s(x,v),u').$$

From the relation (4.61), we can choose y and z like this: $y = x + \widetilde{b}(s, v', v)$, meaning

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^1, \\ y^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^2, \\ y^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^3, \end{cases}$$

and $z = x + \widetilde{b}(s, u', v)$, that is

$$\begin{cases} z^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^1, \\ z^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^2, \\ z^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^3. \end{cases}$$

Then

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} \rho^{-1}(y,v')\rho^{-1}(z,u') du \\ &\leq a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} \frac{1}{e^{|v'|^2+|y\times v'|^2}} \frac{1}{e^{|u'|^2+|z\times u'|^2}} du \\ &\leq a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} \frac{1}{e^{|v'|^2+|u'|^2}} \frac{1}{e^{|y\times v'|^2+|z\times u'|^2}} du \\ &\leq Ca^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} \frac{1}{e^{|u|^2+|v|^2}} \frac{1}{e^{|y\times v'|^2+|z\times u'|^2}} \\ &\leq Ca^{-1}(s)b^{-2}(s)e^{-|v|^2}||f^{\#}||^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} e^{-|u|^2} \frac{du}{e^{|y\times v'|^2+|z\times u'|^2}}. \end{aligned}$$

Since $v^0 \geq 1$, $u^0 \geq 1$, $2 \leq \sqrt{s}$ and taking into account (2.27).
By (4.34) we obtain

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq Ce^{-|v|^2} a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \sigma_0(\omega) e^{-|u|^2} e^{-D} du \\ &\leq Ce^{-|v|^2} a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} \sigma_0(\omega) e^{-|\omega.p_x|^2} d\omega \int_{\mathbb{R}^3} e^{-|u|^2} du \\ &\leq Ce^{-|v|^2} a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma_0 (1+g^{-\beta}) e^{-|u|^2} e^{-|\omega.p_x|^2} du \\ &\leq Ce^{-|v|^2} a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{\mathbb{R}^3} e^{-|u|^2} du \\ &\leq C\rho^{-1}(x,v) a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2. \end{aligned}$$

We end the proof by this integration

$$\begin{aligned} \int_0^t |Q^{\#}_{gain}(f,f)(s,x,v)| ds &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 \int_0^t a^{-1}(s)b^{-2}(s) ds \\ &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 \end{aligned}$$

where C does not depend on t, x or v. □

4.3.3 L^∞-existence theorem for mild solutions

Theorem 4.1. Define the operator Υ on M by

$$\Upsilon f^\# = f_0(x,v) + \int_0^t Q^\#(f,f)(\tau,x,v)d\tau. \tag{4.67}$$

If we let $M_r = \{f \in M, \|f^\#\| \leq r\}$, under assumptions (1.69) on the scattering kernel and (4.2)-(4.3) on the coefficients of the metric tensor, there exists a constant r_0 such that if $\|f_0\|$ is sufficiently small, the integral equation $\Upsilon f^\# = f^\#$ has an unique solution $f^\# \in M_{r_0}$.

Proof. We remark that M_r is a closed subset of the Banach space $(M, \|\|)$.
Let us take the initial data f_0 such that $\|f_0\| \leq \frac{r_0}{2}$ for some r_0.
For $f \in M_{r_0}$, by (4.67)

$$\begin{aligned}|\Upsilon f^\#(t,x,v)| &\leq |f_0(x,v)| + \int_0^t |Q^\#(f,f)(\tau,x,v)|d\tau \\ &\leq \rho(x,v)^{-1}\|f_0\| + C\rho(x,v)^{-1}\|f^\#\|^2 \\ &\leq \rho(x,v)^{-1}[\frac{r_0}{2} + Cr_0^2].\end{aligned}$$

The second line is obtained by using the estimates of the loss and gain terms.
Thus

$$if \qquad \frac{r_0}{2} + Cr_0^2 < r_0 \qquad i.e \qquad r_0 < \frac{1}{2C}$$

after multiplying by $\rho(x,v)$ and taking the supremum with respect to t, x and v, we have

$$\|\Upsilon f^\#\| \leq r_0$$

Then Υ maps M_{r_0} into itself.
More over, if $\|f_0\| \leq \frac{r_0}{2}$ and $f, h \in M_{r_0}$, using the bilinearity of Q

$$\begin{aligned}Q(f,f) - Q(h,h) &= [Q_{gain}(f,f) - Q_{gain}(h,h)] + [Q_{loss}(h,h) - Q_{loss}(f,f)] \\ &= [Q_{gain}(f,f-h) - Q_{gain}(f-h,h)] + [Q_{loss}(h,h-f) - Q_{loss}(h-f,f)]\end{aligned} \tag{4.68}$$

we have

$$\begin{aligned}|\Upsilon f^\#(t,x,v) - \Upsilon h^\#(t,x,v)| &= |\int_0^t (Q^\#(f,f)(t,x,v) - Q^\#(h,h)(t,x,v))d\tau| \\ &\leq C\rho(x,v)^{-1}(\|f^\#\| + \|h^\#\|)\|f^\# - h^\#\| \\ &\leq 2Cr_0\rho(x,v)^{-1}\|f^\# - h^\#\|.\end{aligned}$$

So the desired result it obtained if $2Cr_0 < 1$.
In fact, if $r_0 < \frac{1}{2C}$, after multiplying the relation

$$|\Upsilon f^\#(t,x,v) - \Upsilon h^\#(t,x,v)| \leq 2Cr_0\rho(x,v)^{-1}\|f^\# - h^\#\|$$

by $\rho(x,v)$ and taking the supremum with respect to t, x and v, we obtain

$$\|\Upsilon f^{\#} - \Upsilon h^{\#}\| \leq 2Cr_0\|f^{\#} - h^{\#}\| < \|f^{\#} - h^{\#}\|.$$

So Υ is a contraction.
Using the fixed point theorem, we claim the desired result. □

4.4 Global L^∞-existence theorem for mild solutions in the case of hard potentials

In this part we take $\alpha = 0$ in (1.70).
We assume that the coefficient b of the metric tensor enjoys the condition

$$\int_{\mathbb{R}_+} b^{\beta-3}(\tau)d\tau < \infty. \tag{4.69}$$

4.4.1 Estimates of the loss term

Lemma 4.6. There exists a constant C not depending on t, v and x such that :

$$\int_0^t |Q^{\#}_{loss}(f,f)(s,x,v)| \leq C\rho^{-1}(x,v)||f^{\#}||^2. \tag{4.70}$$

Proof. The loss term of the collision operator is expressed as follows

$$Q^{\#}_{loss}(f,f)(t,x,v) = a^{-1}(t)b^{-2}(t)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) f(t, X^t(x,v), v) f(t, X^t(x,v), u) du.$$

By (4.62) we have

$$\begin{aligned}
|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)|f(s, X^s(x,v), v)||f(s, X^s(x,v), u)|du \\
&\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)|f^{\#}(s,x,v)||f^{\#}(s,y,u)|du.
\end{aligned}$$

From the relation (4.61), we will take $y = x + \widetilde{b}(s,u,v)$, and this is to say

$$\begin{cases}
y^1 &= x^1 + (\int_0^s \frac{a^{-2}(\tau)}{v^0(\tau)}d\tau)v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u^0(\tau)}d\tau)u^1, \\
y^2 &= x^2 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^2, \\
y^3 &= x^3 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^3.
\end{cases}$$

It follows that

$$
\begin{aligned}
|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)|f^{\#}(s,x,v)||f^{\#}(s,y,u)|du\\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)\rho^{-1}(y,u)du\\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)\frac{du}{e^{|u|^2+|y\times u|^2}}\\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)\frac{du}{e^{|u|^2}}\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2\int_{S^2}d\omega\int_{\mathbb{R}^3}(1+g^{-\beta})\sigma_0(\omega)\vartheta_\phi e^{-|u|^2}du\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2\int_{\mathbb{R}^3}(1+g^{-\beta})\vartheta_\phi e^{-|u|^2}du\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2(\int_{\mathbb{R}^3}\vartheta_\phi e^{-|u|^2}du+\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du)\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2(\int_{\mathbb{R}^3}e^{-|u|^2}du+\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du)\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2(1+\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du)\\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2(1+b^{\beta-1})\\
&\leq C\rho^{-1}(x,v)||f^{\#}||^2(a^{-1}(s)b^{-2}(s)+a^{-1}(s)b^{\beta-3}(s))
\end{aligned}
$$

where we use $\vartheta_\phi \leq 4$.

Integrating the above inequality from 0 to t we have

$$
\begin{aligned}
\int_0^t|Q^{\#}_{loss}(f,f)(s,x,v)|ds &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2\int_0^t[a^{-1}(s)b^{-2}(s)+a^{-1}(s)b^{\beta-3}(s)]ds\\
&\leq C\rho^{-1}(x,v)\|f^{\#}\|^2.
\end{aligned}
$$

□

4.4.2 Estimates of the gain term

Lemma 4.7. There exists a constant C not depending on t, v and x such that:

$$\int_0^t|Q^{\#}_{gain}(f,f)(s,x,v)|ds \leq C\rho^{-1}(x,v)||f^{\#}||^2. \tag{4.71}$$

Proof. We recall that the gain term of the collision operator is expressed as follows

$$Q^{\#}_{gain}(f,f)(t,x,v) = a^{-1}(t)b^{-2}(t)\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)f(t,X^t(x,v),v')f(t,X^t(x,v),u')du.$$

By (4.62) we have

$$
\begin{aligned}
|Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)|f(s,X^s(x,v),v')||f(s,X^s(x,v),u')|du\\
&\leq a^{-1}(s)b^{-2}(s)\int_{S^2}d\omega\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)|f^{\#}(s,y,v')||f^{\#}(s,z,u')|du.
\end{aligned}
$$

From the relation (4.61), we can choose y and z like this: $y = x + \widetilde{b}(s, v', v)$, meaning

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^1, \\ y^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^2, \\ y^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^3, \end{cases}$$

and $z = x + \widetilde{b}(s, u', v)$, that is

$$\begin{cases} z^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^1, \\ z^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^2, \\ z^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^3. \end{cases}$$

Then

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)\rho^{-1}(y,v')\rho^{-1}(z,u')du \\ &\leq a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|v'|^2+|y\times v'|^2}} \frac{1}{e^{|u'|^2+|z\times u'|^2}} du \\ &\leq a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|v'|^2+|u'|^2}} \frac{1}{e^{|y\times v'|^2+|z\times u'|^2}} du \\ &\leq Ca^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|u|^2+|v|^2}} \frac{1}{e^{|y\times v'|^2+|z\times u'|^2}} \\ &\leq Ca^{-1}(s)b^{-2}(s)e^{-|v|^2}\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta^\phi \sigma(g,\omega)e^{-|u|^2} \frac{du}{e^{|y\times v'|^2+|z\times u'|^2}} \end{aligned}$$

since (2.27) holds.
By (4.34) we have

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq Ce^{-|v|^2}a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)e^{-|u|^2}e^{-D}du \\ &\leq Ce^{-|v|^2}a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)e^{-|\omega.p_x|^2}e^{-|u|^2}du \\ &\leq Ce^{-|v|^2}a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma_0(\omega)(1+g^{-\beta})e^{-|u|^2}e^{-|\omega.p_x|^2}du \\ &\leq Ce^{-|v|^2}a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2 \int_{\mathbb{R}^3} e^{-|x\times v|^2}\vartheta_\phi(1+g^{-\beta})e^{-|u|^2}du \\ &\leq C\rho^{-1}(x,v)a^{-1}(s)b^{-2}(s)\|f^{\#}\|^2(1+b^{\beta-1}(s)) \\ &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2(a^{-1}(s)b^{-2}(s)+a^{-1}(s)b^{\beta-3}(s)). \end{aligned}$$

Then we end the proof by this integration

$$\begin{aligned} \int_0^t |Q^{\#}_{gain}(f,f)(s,x,v)|ds &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 \int_0^t [a^{-1}(s)b^{-2}(s)+a^{-1}(s)b^{\beta-3}(s)]ds \\ &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 \end{aligned}$$

where C does not depend on t, x or v. □

4.4.3 L^{∞}-existence theorem for mild solutions

Theorem 4.2. Define the operator Γ on M by

$$\Gamma f^{\#} = f_0(x,v) + \int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau. \tag{4.72}$$

If we let $M_r = \{f \in M, \|f^{\#}\| \leq r\}$, under assumptions (1.70) with $\alpha = 0$ and (2.50) on the scattering kernel and (4.2),(4.3) and (4.69) on the coefficients of the metric tensor, there exists a constant r_0 such that if $\|f_0\|$ is sufficiently small, the integral equation $\Gamma f^{\#} = f^{\#}$ has an unique solution $f^{\#} \in M_{r_0}$.

Proof. This proof is done exactly as in Theorem 4.1.

□

4.5 Global L^{∞}-existence theorem for mild solutions in the case of soft potentials

We assume that the coefficient b of the metric tensor enjoys the condition (4.69).

4.5.1 Estimates of the loss term

Lemma 4.8. There exists a constant C not depending on t, v and x such that :

$$\int_0^t |Q^{\#}_{loss}(f,f)(s,x,v)| \leq C\rho^{-1}(x,v)||f^{\#}||^2. \tag{4.73}$$

Proof. The loss term of the collision operator is expressed as follows

$$Q^{\#}_{loss}(f,f)(t,x,v) = a^{-1}(t)b^{-2}(t)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_{\phi}\sigma(g,\omega)f(t,X^t(x,v),v)f(t,X^t(x,v),u)du.$$

By (4.62) we have

$$\begin{aligned}|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_{\phi}\sigma(g,\omega)|f(s,X^s(x,v),v)||f(s,X^s(x,v),u)|du \\ &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_{\phi}\sigma(g,\omega)|f^{\#}(s,x,v)||f^{\#}(s,y,u)|du.\end{aligned}$$

From the relation (4.61), we will take $y = x + \widetilde{b}(s,u,v)$, and this is to say

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2}(\tau)}{v^0(\tau)}d\tau)v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u^0(\tau)}d\tau)u^1, \\ y^2 &= x^2 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^2, \\ y^3 &= x^3 + (\int_0^s \frac{b^{-2}(\tau)}{v^0(\tau)}d\tau)v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u^0(\tau)}d\tau)u^3. \end{cases}$$

It follows that

$$
\begin{aligned}
|Q^{\#}_{loss}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)|f^{\#}(s,x,v)||f^{\#}(s,y,u)|du \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)\rho^{-1}(y,u)du \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)\frac{du}{e^{|u|^2+|y\times u|^2}} \\
&\leq a^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)\frac{du}{e^{|u|^2}} \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} g^{-\beta}\sigma_0(\omega)\vartheta_\phi e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{\mathbb{R}^3} g^{-\beta}\vartheta_\phi e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 \int_{\mathbb{R}^3} \vartheta_\phi g^{-\beta} e^{-|u|^2}du \\
&\leq Ca^{-1}(s)b^{-2}(s)\rho^{-1}(x,v)\|f^{\#}\|^2 b^{\beta-1} \\
&\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 a^{-1}(s)b^{\beta-3}(s)
\end{aligned}
$$

where we use $\vartheta_\phi \leq 4$.

Integrating the above inequality from 0 to t we obtain

$$
\begin{aligned}
\int_0^t |Q^{\#}_{loss}(f,f)(s,x,v)|ds &\leq C\rho^{-1}(x,v)\|f^{\#}\|^2 \int_0^t a^{-1}(s)b^{\beta-3}(s)ds \\
&\leq C\rho^{-1}(x,v)\|f^{\#}\|^2.
\end{aligned}
$$

□

4.5.2 Estimates of the gain term

Lemma 4.9. There exists a constant C not depending on t, v and x such that:

$$
\int_0^t |Q^{\#}_{gain}(f,f)(s,x,v)|ds \leq C\rho^{-1}(x,v)||f^{\#}||^2. \tag{4.74}
$$

Proof. We recall that the gain term of the collision operator is expressed as follows

$$
Q^{\#}_{gain}(f,f)(t,x,v) = a^{-1}(t)b^{-2}(t)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) f(t,X^t(x,v),v')f(t,X^t(x,v),u')du.
$$

By (4.62) we have

$$
\begin{aligned}
|Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)|f(s,X^s(x,v),v')||f(s,X^s(x,v),u')|du \\
&\leq a^{-1}(s)b^{-2}(s)\int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)|f^{\#}(s,y,v')||f^{\#}(s,z,u')|du.
\end{aligned}
$$

From the relation (4.61), we can choose y and z like this: $y = x + \widetilde{b}(s, v', v)$, meaning

$$\begin{cases} y^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^1, \\ y^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^2, \\ y^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{v'^0(\tau)} d\tau) v'^3, \end{cases}$$

and $z = x + \widetilde{b}(s, u', v)$, that is

$$\begin{cases} z^1 &= x^1 + (\int_0^s \frac{a^{-2(\tau)}}{v^0(\tau)} d\tau) v^1 - (\int_0^s \frac{a^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^1, \\ z^2 &= x^2 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^2 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^2, \\ z^3 &= x^3 + (\int_0^s \frac{b^{-2(\tau)}}{v^0(\tau)} d\tau) v^3 - (\int_0^s \frac{b^{-2}(\tau)}{u'^0(\tau)} d\tau) u'^3. \end{cases}$$

Then

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \rho^{-1}(y,v') \rho^{-1}(z,u') du \\ &\leq a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|v'|^2 + |y \times v'|^2}} \frac{1}{e^{|u'|^2 + |z \times u'|^2}} du \\ &\leq a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|v'|^2 + |u'|^2}} \frac{1}{e^{|y \times v'|^2 + |z \times u'|^2}} du \\ &\leq C a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) \frac{1}{e^{|u|^2 + |v|^2}} \frac{1}{e^{|y \times v'|^2 + |z \times u'|^2}} \\ &\leq C a^{-1}(s) b^{-2}(s) e^{-|v|^2} \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta^\phi \sigma(g,\omega) e^{-|u|^2} \frac{du}{e^{|y \times v'|^2 + |z \times u'|^2}} \end{aligned}$$

since (2.27) holds.

By (4.34) we have

$$\begin{aligned} |Q^{\#}_{gain}(f,f)(s,x,v)| &\leq C e^{-|v|^2} a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{-|u|^2} e^{-D} du \\ &\leq C e^{-|v|^2} a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{-|\omega . p_x|^2} e^{-|u|^2} du \\ &\leq C e^{-|v|^2} a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{S^2} d\omega \int_{\mathbb{R}^3} \vartheta_\phi \sigma_0(\omega) g^{-\beta} e^{-|u|^2} e^{-|\omega . p_x|^2} du \\ &\leq C e^{-|v|^2} a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 \int_{\mathbb{R}^3} e^{-|x \times v|^2} \vartheta_\phi g^{-\beta} e^{-|u|^2} du \\ &\leq C \rho^{-1}(x,v) a^{-1}(s) b^{-2}(s) \|f^{\#}\|^2 b^{\beta - 1}(s) \\ &\leq C \rho^{-1}(x,v) \|f^{\#}\|^2 a^{-1}(s) b^{\beta - 3}(s). \end{aligned}$$

Then we end the proof by this integration

$$\begin{aligned} \int_0^t |Q^{\#}_{gain}(f,f)(s,x,v)| ds &\leq C \rho^{-1}(x,v) \|f^{\#}\|^2 \int_0^t a^{-1}(s) b^{\beta - 3}(s) ds \\ &\leq C \rho^{-1}(x,v) \|f^{\#}\|^2 \end{aligned}$$

where C does not depend on t, x or v. □

4.5.3 L^∞-existence theorem for mild solutions

Theorem 4.3. Define the operator Ψ on M by

$$\Psi f^\# = f_0(x,v) + \int_0^t Q^\#(f,f)(\tau,x,v)d\tau \tag{4.75}$$

If we let $M_r = \{f \in M, \|f^\#\| \leq r\}$, under assumptions (1.71)-(2.50) on the scattering kernel and (4.2),(4.3) and (4.69) on the coefficients of the metric tensor, there exists a constant r_0 such that if $\|f_0\|$ is sufficiently small, the integral equation $\Psi f^\# = f^\#$ has an unique solution $f^\# \in M_{r_0}$.

Proof. This proof is done exactly as in Theorem 4.1.

□

CHAPTER FIVE

L^∞-EXISTENCE THEOREM OF THE INHOMOGENEOUS RELATIVISTIC BOLTZMANN EQUATION IN THE BIANCHI TYPE I SPACE-TIME

Contents

THis part of our work provide the possibility to chose a suitable weighted framework to prove the global existence theorem.

Let's consider the set $\sum$ that we will define in the sequel; the inhomogeneous relativistic Boltzmann equation in f with initial data $f_0 \in \sum$ then reads in term of variables (t, x, v)

$$\frac{\partial f}{\partial t} + a^{-2}(t)\frac{v^1}{v^0}\frac{\partial f}{\partial x^1} + b^{-2}(t)\frac{v^2}{v^0}\frac{\partial f}{\partial x^2} + b^{-2}(t)\frac{v^3}{v^0}\frac{\partial f}{\partial x^3} = Q(f, f)(t, x, v). \tag{5.1}$$

We assume that the coefficients a and b of the Bianchi type I metric are given increasing functions of the time t and are such that:

$$a(0) \geq 1, \quad a \leq b \leq \sqrt{2}a, \quad \underset{t \longrightarrow \infty}{lim}\, a(t) = \infty, \tag{5.2}$$

$$\int_{\mathbb{R}_+} a^{-2}(\tau) d\tau < \infty. \tag{5.3}$$

5.1 Functional space

We are looking for a continuous bounded non-negative solution and we allow f to decay exponentially in v and x. For this reason we consider the weight function ρ defined by

$$\rho(x, v) = e^{(|v|^2 + |x \times v|^2)}.$$

We define the norms

$$\|g(t)\|_e = \underset{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3}{sup} (\rho(x, v)|g(t, x, v)|) \tag{5.4}$$

and

$$|||g(t)|||_e = \|g(t)\|_e \;+\; \sum_{k=1}^{3} (\|\partial_{v^k} g(t)\|_e \;+\; \|\partial_{x^k} g(t)\|_e). \tag{5.5}$$

The function space in which we seek the solution of the Boltzmann equation is

$$\Sigma = \{f \in C^1([0, \infty[\times \mathbb{R}^3 \times \mathbb{R}^3); |||f(t)|||_e < \infty; \quad \forall t \in \mathbb{R}_+\}. \tag{5.6}$$

Σ is not an empty set. In fact $f(t, v) = e^{-2|v|^2}$ belong to Σ.

5.2 Specific estimates on the derivatives of the collision kernel

5.2.1 Specific estimates for the case of Israel particles

Lemma 5.1. We have the following results:

$$|\partial_{v^1}(\frac{1}{v^0 u^0 \sqrt{s}})| \leq \frac{C_1}{a v^0 s \sqrt{s}}, \tag{5.7}$$

$$|\partial_{v^i}(\frac{1}{v^0 u^0 \sqrt{s}})| \leq \frac{C_2}{b v^0 s \sqrt{s}} \qquad for\ i = 2, 3 \tag{5.8}$$

where C_1 and C_2 do not depend on a or b.

Proof. A direct derivative leads to

$$\begin{aligned}
\partial_{v^1}(\frac{1}{v^0u^0\sqrt{s}}) &= -\frac{1}{(v^0)^2(u^0)^2s}[\partial_{v^1}(v^0u^0\sqrt{s})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\partial_{v^1}(v^0)u^0\sqrt{s} + v^0\partial_{v^1}(u^0)\sqrt{s} + v^0u^0\partial_{v^1}(\sqrt{s})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\frac{v^1}{a^2v^0}u^0\sqrt{s} + v^0u^0\frac{u^0}{a\sqrt{s}}(\frac{v^1}{av^0} - \frac{u^1}{au^0})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\frac{v^1}{a^2v^0}u^0\sqrt{s} + \frac{v^0(u^0)^2}{a\sqrt{s}}\frac{v^1}{av^0} - \frac{v^0(u^0)^2}{a\sqrt{s}}\frac{u^1}{au^0}] \\
&= \frac{1}{av^0\sqrt{s}}[-\frac{v^1}{av^0}\frac{1}{v^0u^0} - \frac{v^1}{av^0}\frac{1}{s} + \frac{u^1}{au^0}\frac{1}{s}].
\end{aligned}$$

Since $|\frac{v^1}{av^0}| < 1$, $|\frac{u^1}{au^0}| < 1$ and $\sqrt{s} \leq 2\sqrt{u^0v^0}$, then

$$\begin{aligned}
|\partial_{v^1}(\frac{1}{v^0u^0\sqrt{s}})| &\leq \frac{1}{av^0\sqrt{s}}(\frac{4}{s} + \frac{1}{s} + \frac{1}{s}) \\
&\leq \frac{6}{av^0s\sqrt{s}}.
\end{aligned}$$

In a similar way as above

$$\begin{aligned}
\partial_{v^i}(\frac{1}{v^0u^0\sqrt{s}}) &= -\frac{1}{(v^0)^2(u^0)^2s}[\partial_{v^i}(v^0u^0\sqrt{s})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\partial_{v^i}(v^0)u^0\sqrt{s} + v^0\partial_{v^i}(u^0)\sqrt{s} + v^0u^0\partial_{v^i}(\sqrt{s})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\frac{v^i}{b^2v^0}u^0\sqrt{s} + v^0u^0\frac{u^0}{b\sqrt{s}}(\frac{v^i}{bv^0} - \frac{u^i}{bu^0})] \\
&= -\frac{1}{(v^0)^2(u^0)^2s}[\frac{v^i}{b^2v^0}u^0\sqrt{s} + \frac{v^0(u^0)^2}{b\sqrt{s}}\frac{v^i}{bv^0} - \frac{v^0(u^0)^2}{b\sqrt{s}}\frac{u^i}{bu^0}] \\
&= \frac{1}{bv^0\sqrt{s}}[-\frac{v^i}{bv^0}\frac{1}{v^0u^0} - \frac{v^i}{bv^0}\frac{1}{s} + \frac{u^i}{bu^0}\frac{1}{s}].
\end{aligned}$$

Since $|\frac{v^i}{bv^0}| < 1$, $|\frac{u^i}{bu^0}| < 1$ and $\sqrt{s} \leq 2\sqrt{u^0v^0}$, then

$$\begin{aligned}
|\partial_{v^i}(\frac{1}{v^0u^0\sqrt{s}})| &\leq \frac{1}{bv^0\sqrt{s}}(\frac{4}{s} + \frac{1}{s} + \frac{1}{s}) \\
&\leq \frac{6}{bv^0s\sqrt{s}}.
\end{aligned}$$

□

Lemma 5.2. The following estimate hold:

$$|\partial_{v^i}(v'^0)| \leq \frac{C}{a}v^0(u^0)^4 \qquad for\ i = 1, 2, 3 \tag{5.9}$$

where C does not depend on a or b.

Proof. We recall that

$$v'^0 = \sqrt{1 + a^{-2}(v'^1)^2 + b^{-2}(v'^2)^2 + b^{-2}(v'^3)^2}.$$

Then

$$\begin{aligned}\partial_{v^i}(v'^0) &= \frac{1}{2v'^0}(2a^{-2}\partial_{v^i}(v'^1)v'^1 + 2b^{-2}\partial_{v^i}(v'^2)v'^2 + 2b^{-2}\partial_{v^i}(v'^3)v'^3)\\ &= \frac{v'^1}{a^2v'^0}\partial_{v^i}(v'^1) + \frac{v'^2}{b^2v'^0}\partial_{v^i}(v'^2) + \frac{v'^3}{b^2v'^0}\partial_{v^i}(v'^3).\end{aligned}$$

Hence

$$\begin{aligned}|\partial_{v^i}(v'^0)| &\leq \frac{1}{a}|\partial_{v^i}(v'^1)| + \frac{1}{b}|\partial_{v^i}(v'^2)| + \frac{1}{b}|\partial_{v^i}(v'^3)|\\ &\leq \frac{1}{a}(|\partial_{v^i}(v'^1)| + |\partial_{v^i}(v'^2)| + |\partial_{v^i}(v'^3)|)\\ &\leq \frac{C}{a}v^0(u^0)^4.\end{aligned}$$

□

Lemma 5.3. $\widetilde{b} = (\widetilde{b}^1, \widetilde{b}^2, \widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k = 1,\ 2,\ 3$ and $i = 1,\ 2,\ 3$:

$$|\partial_{v^i}(\widetilde{b}^k(t,u,v))| \leq C \tag{5.10}$$

where C is a constant which does not depend on t.

Proof. **Case $k = 1$:**

$$\widetilde{b}^1(t,u,v) = (\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1 - (\int_0^t \frac{1}{a^2(\tau)u^0(\tau)}d\tau)u^1,$$

$$\partial_{v^1}(\widetilde{b}^1(t,u,v)) = \int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1 - \frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u,v))| \leq 2\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i = 2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^1(t,u,v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^1(t,u,v))| \leq \int_0^\infty \frac{1}{a(\tau)b(\tau)}d\tau < \infty.$$

Case $k = 2\ or\ 3$:

$$\widetilde{b}^k(t,u,v) = (\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k - (\int_0^t \frac{1}{b^2(\tau)u^0(\tau)}d\tau)u^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,u,v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,u,v))| \leq \int_0^\infty \frac{1}{a(\tau)b(\tau)}d\tau < \infty.$$

For $i = 2$ or 3:

$$\partial_{v^i}(\widetilde{b}^k(t,u,v)) = \int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik} - \frac{v^i}{b(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,u,v))| \leq 2\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

□

Lemma 5.4. $\widetilde{b} = (\widetilde{b}^1, \widetilde{b}^2, \widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k = 1,\ 2,\ 3$ and $i = 1,\ 2,\ 3$:

$$|\partial_{v^i}(b^k(t,v',v))| \leq C + Cv^0(u^0)^4 \tag{5.11}$$

where C is a constant which does not depend on t.

Proof. **Case** $k = 1$**:**

$$\widetilde{b}^1(t,v',v) = (\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1 - (\int_0^t \frac{1}{a^2(\tau)v'^0(\tau)}d\tau)v'^1.$$

$$\partial_{v^1}(\widetilde{b}^1(t,v',v)) = \int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1 - \frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^1}(v'^1)}{a^2(\tau)v'^0} + \frac{v'^1\partial_{v^1}(v'^0)}{a^2(\tau)(v'^0)^2}]d\tau.$$

$$|\partial_{v^1}(\widetilde{b}^1(t,v',v))| \leq C + Cv^0(u^0)^4\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i = 2$ or 3:

$$\partial_{v^i}(\widetilde{b}^1(t,v',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^i}(v'^1)}{a^2(\tau)v'^0} + \frac{v'^1\partial_{v^i}(v'^0)}{a^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,v',v))| \leq C + Cv^0(u^0)^4\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

Case $k = 2$ or 3**:**

$$\widetilde{b}^k(t,v',v) = (\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k - (\int_0^t \frac{1}{b^2(\tau)v'^0(\tau)}d\tau)v'^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,v',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^1}(v'^k)}{b^2(\tau)v'^0(\tau)} + \frac{v'^k\partial_{v^1}(v'^0)}{b^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,v',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

For $i = 2 \ or \ 3$:

$$\partial_{v^i}(\widetilde{b}^k(t,v',v)) = \int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik} - \frac{v^i}{b(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^i}(v'^k)}{b^2(\tau)v'^0(\tau)} + \frac{v'^k\partial_{v^i}(v'^0)}{b^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,v',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

□

Lemma 5.5. $\widetilde{b} = (\widetilde{b}^1, \widetilde{b}^2, \widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k = 1,\ 2,\ 3$ and $i = 1,\ 2,\ 3$:

$$|\partial_{v^i}(\widetilde{b}^k(t,u',v))| \leq C + Cv^0(u^0)^4 \tag{5.12}$$

where C is a constant which does not depend on t.

Proof. **Case** $k = 1$**:**

$$\widetilde{b}^1(t,u',v) = (\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1 - (\int_0^t \frac{1}{a^2(\tau)u'^0(\tau)}d\tau)u'^1,$$

$$\partial_{v^1}(\widetilde{b}^1(t,u',v)) = \int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1 - \frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^1}(u'^1)}{a^2(\tau)u'^0} + \frac{u'^1\partial_{v^1}(u'^0)}{a^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i = 2 \ or \ 3$:

$$\partial_{v^i}(\widetilde{b}^1(t,u',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^i}(u'^1)}{a^2(\tau)u'^0} + \frac{u'^1\partial_{v^i}(u'^0)}{a^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

Case $k = 2 \ or \ 3$**:**

$$\widetilde{b}^k(t,u',v) = (\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k - (\int_0^t \frac{1}{b^2(\tau)u'^0(\tau)}d\tau)u'^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,u',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^1}(u'^k)}{b^2(\tau)u'^0(\tau)} + \frac{u'^k\partial_{v^1}(u'^0)}{b^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,u',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{b^2(\tau)} d\tau.$$

For $i = 2 \ or \ 3$:

$$\partial_{v^i}(\widetilde{b}^k(t,u',v)) = \int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik} - \frac{v^i}{b(\tau)v^0(\tau)} \frac{v^k}{b(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^i}(u'^k)}{b^2(\tau)u'^0(\tau)} + \frac{u'^k \partial_{v^i}(u'^0)}{b^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,u',v))| \leq C + Cv^0(u^0)^4 \int_0^\infty \frac{1}{b^2(\tau)} d\tau.$$

□

5.2.2 Specific estimates for the cases of hard and soft potentials

We split the integration domain into three integration domains:

$$A_0 = \{|v| \leq a\}, \qquad A_1 = \{|v| \geq a, \ |v| \leq 2|u|\} \qquad and \qquad A_3 = \{|v| \geq a, \ |v| \geq 2|u|\} \tag{5.13}$$

Lemma 5.6. For a fix finite time t, the derivatives of the post-collisional momenta are estimated as follows:
On the set A_0

$$|\partial_{v^i} v'^k| \lesssim (u^0)^4, \qquad for \ i = 1,2,3 \ and \ k = 1,2,3. \tag{5.14}$$

On the set A_1

$$|\partial_{v^i} v'^k| \lesssim (u^0)^5, \qquad for \ i = 1,2,3 \ and \ k = 1,2,3. \tag{5.15}$$

On the set A_2

$$|\partial_{v^i} v'^k| \lesssim (u^0)^3, \qquad for \ i = 1,2,3 \ and \ k = 1,2,3. \tag{5.16}$$

Proof. **On the set** A_0:
We have

$$\begin{aligned} v^0 &= \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}(v^2)^2 + b^{-2}(v^3)^2} \\ &\leq \sqrt{1 + a^{-2}|v|^2} \\ &\leq \sqrt{2}. \end{aligned}$$

Using the first parametrization, by (2.59)-(2.60)-(2.61)-(2.62)
we have: $|\partial_{v^i} v'^k| \leq Cv^0(u^0)^4$.
Then $|\partial_{v^i} v'^k| \lesssim (u^0)^4$.

On the set A_1:
By (2.17) we have: $v^0 \leq 2\sqrt{2}u^0$.
Using the first parametrization, by (2.59)-(2.60)-(2.61)-(2.62)

we have: $|\partial_{v^i} v'^k| \leq C v^0 (u^0)^4$.
Then $|\partial_{v^i} v'^k| \lesssim (u^0)^5$.

On the set A_2:
Using the second parametrization, by (2.67)-(2.68)-(2.69)-(2.70)
we have

$$|\partial_{v^i} v'^k| \leq C(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2})(u^0)^3 \qquad i = 1,2,3 \qquad and \qquad k = 1,2,3. \tag{5.17}$$

Let us observe that on A_2

$$|v| = |v-u+u| \leq |v-u| + |u| \leq |v-u| + \frac{1}{2}|v| \qquad \longrightarrow \qquad \frac{1}{2}|v| \leq |v-u|,$$
$$|v| = |v+u-u| \leq |v+u| + |u| \leq |v+u| + \frac{1}{2}|v| \qquad \longrightarrow \qquad \frac{1}{2}|v| \leq |v+u|.$$

By (5.17) we have

$$\begin{aligned} |\partial_{v^i} v'^k| &\lesssim (\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2})(u^0)^3 \\ &\lesssim (\frac{bv^0}{|v|} + \frac{bv^0}{|v|} + \frac{b^2(v^0)^2}{|v|^2})(u^0)^3 \\ &\lesssim (u^0)^3. \end{aligned}$$

□

Lemma 5.7. We have the following estimate:

$$|\partial_{v^i}(v'^0)| \leq \frac{C}{a}(u^0)^5, \qquad for\ i = 1,2,3 \tag{5.18}$$

where v'^0 is parameterized either by the first or the second parametrization

Proof. We recall that

$$v'^0 = \sqrt{1 + a^{-2}(v'^1)^2 + b^{-2}(v'^2)^2 + b^{-2} + (v'^3)^2}.$$

The derivative of v'^0 with respect to v^i leads to

$$\begin{aligned} \partial_{v^i}(v'^0) &= \frac{1}{2v'^0}(2a^{-2}\partial_{v^i}(v'^1)v'^1 + 2b^{-2}\partial_{v^i}(v'^2)v'^2 + 2b^{-2}\partial_{v^i}(v'^3)v'^3) \\ &= \frac{v'^1}{a^2 v'^0}\partial_{v^i}(v'^1) + \frac{v'^2}{b^2 v'^0}\partial_{v^i}(v'^2) + \frac{v'^3}{b^2 v'^0}\partial_{v^i}(v'^3). \end{aligned}$$

Hence

$$\begin{aligned} |\partial_{v^i}(v'^0)| &\leq \frac{1}{a}|\partial_{v^i}(v'^1)| + \frac{1}{b}|\partial_{v^i}(v'^2)| + \frac{1}{b}|\partial_{v^i}(v'^3)| \\ &\leq \frac{3}{a}(|\partial_{v^i}(v'^1)| + |\partial_{v^i}(v'^2)| + |\partial_{v^i}(v'^3)|) \\ &\leq \frac{C}{a}(u^0)^5. \end{aligned}$$

□

Lemma 5.8. $\widetilde{b} = (\widetilde{b}^1, \widetilde{b}^2, \widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k = 1,\ 2,\ 3$ and $i = 1,\ 2,\ 3$:

$$|\partial_{v^i}(\widetilde{b}^k(t,u,v))| \leq C \tag{5.19}$$

where C is a constant which does not depend on t.

Proof. **Case $k = 1$:**

$$\widetilde{b}^1(t,u,v) = (\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1 - (\int_0^t \frac{1}{a^2(\tau)u^0(\tau)}d\tau)u^1,$$

$$\partial_{v^1}(\widetilde{b}^1(t,u,v)) = \int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1 - \frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u,v))| \leq 2\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i = 2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^1(t,u,v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^1(t,u,v))| \leq \int_0^\infty \frac{1}{a(\tau)b(\tau)}d\tau < \infty.$$

Case $k = 2\ or\ 3$:

$$\widetilde{b}^k(t,u,v) = (\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k - (\int_0^t \frac{1}{b^2(\tau)u^0(\tau)}d\tau)u^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,u,v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,u,v))| \leq \int_0^\infty \frac{1}{a(\tau)b(\tau)}d\tau < \infty.$$

For $i = 2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^k(t,u,v)) = \int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik} - \frac{v^i}{b(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,u,v))| \leq 2\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

□

Lemma 5.9. $\widetilde{b} = (\widetilde{b}^1, \widetilde{b}^2, \widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k = 1,\ 2,\ 3$ and $i = 1,\ 2,\ 3$:

$$|\partial_{v^i}(\widetilde{b}^k(t,v',v))| \leq C + C(u^0)^5 \tag{5.20}$$

where C is a constant which does not depend on t.

Proof. **Case** $k=1$**:**

$$\widetilde{b}^1(t,v',v)=(\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1-(\int_0^t \frac{1}{a^2(\tau)v'^0(\tau)}d\tau)v'^1,$$

$$\partial_{v^1}(\widetilde{b}^1(t,v',v))=\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1-\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau-\int_0^t[\frac{\partial_{v^1}(v'^1)}{a^2(\tau)v'^0}+\frac{v'^1\partial_{v^1}(v'^0)}{a^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,v',v))|\leq C+C(u^0)^5\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i=2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^1(t,v',v))=-\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau-\int_0^t[\frac{\partial_{v^i}(v'^1)}{a^2(\tau)v'^0}+\frac{v'^1\partial_{v^i}(v'^0)}{a^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,v',v))|\leq C+C(u^0)^5\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

Case $k=2\ or\ 3$**:**

$$\widetilde{b}^k(t,v',v)=(\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k-(\int_0^t \frac{1}{b^2(\tau)v'^0(\tau)}d\tau)v'^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,v',v))=-\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau-\int_0^t[\frac{\partial_{v^1}(v'^k)}{b^2(\tau)v'^0(\tau)}+\frac{v'^k\partial_{v^1}(v'^0)}{b^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,v',v))|\leq C+C(u^0)^5\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

For $i=2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^k(t,v',v))=\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik}-\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}]d\tau-\int_0^t[\frac{\partial_{v^i}(v'^k)}{b^2(\tau)v'^0(\tau)}+\frac{v'^k\partial_{v^i}(v'^0)}{b^2(\tau)(v'^0)^2}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,v',v))|\leq C+C(u^0)^5\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

□

Lemma 5.10. $\widetilde{b}=(\widetilde{b}^1,\widetilde{b}^2,\widetilde{b}^3)$ defined by (4.58)-(4.59)-(4.60) satisfies for any $k=1,\ 2,\ 3$ and $i=1,\ 2,\ 3$:

$$|\partial_{v^i}(\widetilde{b}^k(t,u',v))|\leq C+C(u^0)^5 \tag{5.21}$$

where C is a constant which does not depend on t.

Proof. **Case $k=1$:**

$$\widetilde{b}^1(t,u',v) = (\int_0^t \frac{1}{a^2(\tau)v^0(\tau)}d\tau)v^1 - (\int_0^t \frac{1}{a^2(\tau)u'^0(\tau)}d\tau)u'^1,$$

$$\partial_{v^1}(\widetilde{b}^1(t,u',v)) = \int_0^t \frac{1}{a^2(\tau)v^0(\tau)}[1 - \frac{v^1}{a(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^1}(u'^1)}{a^2(\tau)u'^0} + \frac{u'^1\partial_{v^1}(u'^0)}{a^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u',v))| \leq C + C(u^0)^5\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

For $i = 2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^1(t,u',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^i}{b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^i}(u'^1)}{a^2(\tau)u'^0} + \frac{u'^1\partial_{v^i}(u'^0)}{a^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^1(t,u',v))| \leq C + C(u^0)^5\int_0^\infty \frac{1}{a^2(\tau)}d\tau.$$

Case $k = 2\ or\ 3$:

$$\widetilde{b}^k(t,u',v) = (\int_0^t \frac{1}{b^2(\tau)v^0(\tau)}d\tau)v^k - (\int_0^t \frac{1}{b^2(\tau)u'^0(\tau)}d\tau)u'^k,$$

$$\partial_{v^1}(\widetilde{b}^k(t,u',v)) = -\int_0^t \frac{1}{a(\tau)b(\tau)v^0(\tau)}\frac{v^1}{a(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}d\tau - \int_0^t [\frac{\partial_{v^1}(u'^k)}{b^2(\tau)u'^0(\tau)} + \frac{u'^k\partial_{v^1}(u'^0)}{b^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^1}(\widetilde{b}^k(t,u',v))| \leq C + C(u^0)^5\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

For $i = 2\ or\ 3$:

$$\partial_{v^i}(\widetilde{b}^k(t,u',v)) = \int_0^t \frac{1}{b^2(\tau)v^0(\tau)}[\delta^{ik} - \frac{v^i}{b(\tau)v^0(\tau)}\frac{v^k}{b(\tau)v^0(\tau)}]d\tau - \int_0^t [\frac{\partial_{v^i}(u'^k)}{b^2(\tau)u'^0(\tau)} + \frac{u'^k\partial_{v^i}(u'^0)}{b^2(\tau)(u'^0)^2}]d\tau,$$

$$|\partial_{v^i}(\widetilde{b}^k(t,u',v))| \leq C + C(u^0)^5\int_0^\infty \frac{1}{b^2(\tau)}d\tau.$$

□

5.3 L^∞-energy estimates

By (5.1) and (4.54), the Boltzmann equation in $f^\#$ with initial data $f^\#(0,x,v) = f(0,x,v) = f_0(x,v)$ reduces to the following integral equation

$$f^\#(t,x,v) = f_0(x,v) + \int_0^t Q^\#(f,f)(s,x,v)ds. \quad (5.22)$$

5.3.1 L^∞-energy estimates for Israel particles

Lemma 5.11. Let $f^\#$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|f^\#(t)\|_e \leq \|f_0\|_e + C \underset{\tau\in[0,t]}{sup}(\||f^\#(\tau)\||_e^2). \tag{5.23}$$

where C does not depend on t.

Proof. Let's consider the Boltzmann equation

$$\begin{aligned} f^\#(t,x,v) &= f_0(x,v) + \int_0^t Q^\#(f,f)(\tau,x,v)d\tau \\ &= f_0(x,v) + \int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned} \tag{5.24}$$

We multiply (5.24) by $\rho(x,v)$ to get

$$\begin{aligned} \rho(x,v)f^\#(t,x,v) = \rho(x,v)f_0(x,v) &+ \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau \\ &- \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned}$$

Now we are making our estimation like this

$$\rho(x,v)f^\#(t,x,v) \leq \|f(0)\|_e + S_1 + S_2 \tag{5.25}$$

where

$$S_1 = \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau \tag{5.26}$$

and

$$S_2 = \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.27}$$

For (5.26), we have

$$\begin{aligned} S_1 &= \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\omega du \\ &\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du \\ &\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\ &\leq \int_0^t a^{-1}b^{-2}A(\tau)d\tau \end{aligned} \tag{5.28}$$

where

$$A(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\tfrac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du.$$

We have

$$A(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\tfrac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\tfrac{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')$$

$$\times \frac{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du$$

$$\lesssim \|f^{\#}(\tau)\|_e^2 \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} e^{-(|u|^2+|v|^2)} e^{-(|(x+\widetilde{b}(\tau,v',v))\times v'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2)} d\omega du.$$

By (4.34), we know that $D \geq |\omega.(x\times v)|^2$.

By (2.27), we have

$$A(\tau) \lesssim \|f^{\#}(\tau)\|_e^2 \int_{\mathbb{R}^3} e^{|v|^2+|x\times v|^2} \frac{4}{v^0u^0\sqrt{s}} e^{-(|u|^2+|v|^2)} e^{-|x\times v|^2} du.$$

Since $v^0 \geq 1$, $u^0 \geq 1$ and $\sqrt{s} \geq 2$ we have

$$\begin{aligned} A(\tau) &\lesssim \|f^{\#}(\tau)\|_e^2 \int_{\mathbb{R}^3} e^{-|u|^2} du \\ &\lesssim \|f^{\#}(\tau)\|_e^2. \end{aligned}$$

So

$$\begin{aligned} S_1 &\lesssim \int_0^t a^{-1}(\tau)b^{-2}(\tau)\|f^{\#}(\tau)\|_e^2 d\tau \\ &\lesssim \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2) \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau. \end{aligned}$$

Since

$$\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \leq \int_0^\infty a^{-1}(\tau)b^{-2}(\tau)d\tau < \infty.$$

we can state that

$$S_1 \leq C \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2).$$

For (5.27), we have

$$\begin{aligned} S_2 &= \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau, X^\tau(x,v),v)d\tau \\ &\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du \\ &\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} f^{\#}(\tau,x,v)f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du \\ &\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)\|f^{\#}\|_e d\tau \iint_{S^2\times\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} \frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}} \\ &\times f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du \\ &\leq \int_0^t a^{-1}(\tau)b^{-2}(\tau)\|f^{\#}\|_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} e^{-|u|^2-|(x+\widetilde{b}(\tau,u,v))\times u|^2} d\omega du \\ &\leq \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2) \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int\int_{S^2\times\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}} e^{-|u|^2} d\omega du \\ &\leq C \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2) \int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{-|u|^2} du \\ &\leq C \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2) \int_0^\infty a^{-1}(\tau)b^{-2}(\tau)d\tau \int_{\mathbb{R}^3} e^{-|u|^2} du \\ &\leq C \underset{\tau\in[0,t]}{Sup}(\|f^{\#}(\tau)\|_e^2). \end{aligned}$$

By (5.25), we have

$$\rho(x,v)f^\#(t,x,v) \leq ||f(0)||_e + C\, \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

Taking the supremum with respect to x and v, we have

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup}(\rho(x,v)f^\#(t,x,v)) \leq ||f(0)||_e + C\, \underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2).$$

Then

$$\|f(t)\|_e \leq \|f(0)\|_e + C\, \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2.$$

□

Lemma 5.12. Let $f^\#$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|\partial_{v^i} f^\#(t)\|_e \leq \|\partial_{v^i} f_0\|_e + C\, \underset{\tau\in[0,t]}{sup}(\||f^\#(\tau)\||_e^2), \qquad for \qquad i=1,\,2,\,3 \tag{5.29}$$

where C does not depend on t.

Proof. The Boltzmannn equation is written as

$$\begin{aligned} f^\#(t,x,v) &= f(0,x,v) + \int_0^t Q^\#(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v) + \int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned}$$

We take ∂_{v^i} to this equation

$$\partial_{v^i} f^\#(t,x,v) = \partial_{v^i} f(0,x,v) + \int_0^t \partial_{v^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.30}$$

We multiply (5.30) by $\rho(x,v)$ and obtain
$\rho(x,v)\partial_{v^i} f^\#(t,x,v) = \rho(x,v)\partial_{v^i} f(0,x,v) + \int_0^t \rho(x,v)\partial_{v^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$
Then
$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t \rho(x,v)\partial_{v^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$
So
$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t \rho(x,v)\partial v_i$

$[a^{-1}b^{-2}\iint_{S^2\times\mathbb{R}^3} \frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}(f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u') - f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u))d\omega du]d\tau$

$\lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t a^{-1}b^{-2}d\tau[\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial_{v^i}[\frac{\sigma 0(\omega)}{v^0u^0\sqrt{s}}]f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du$

$+\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\partial_{v^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]d\omega du$

$-\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial_{v^i}[\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}]f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du$

$- \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\partial_{v^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)]d\omega du].$

We organize the previous expression like this:

$$\rho(x,v)\partial_{v^i}f^{\#}(t,x,v) \lesssim ||\partial_{v^i}f(0,x,v)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)[j_1(\tau)+j_2(\tau)+j_3(\tau)+j_4(\tau)]d\tau \quad (5.31)$$

where

$$j_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du, \quad (5.32)$$

$$j_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du, \quad (5.33)$$

$$j_3(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\partial_{v^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)]|d\omega du, \quad (5.34)$$

$$j_4(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\partial_{v^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]|d\omega du. \quad (5.35)$$

Now we control each of the four terms.

For (5.32), we have

$$\begin{aligned}
j_1(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du \\
&\lesssim \frac{1}{a}\int_{\mathbb{R}^3} \rho(x,v)\frac{1}{v^0s\sqrt{s}}f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)du \\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} \frac{1}{v^0s\sqrt{s}}f(\tau,X^\tau(x,v),u)du \\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} \frac{1}{v^0s\sqrt{s}}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}f(\tau,X^\tau(x,v),u)du \\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} \frac{1}{v^0s\sqrt{s}}e^{-|u|^2}du \\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} e^{-|u|^2}du \\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2.
\end{aligned}$$

For (5.33), by (4.34) we have

$$\begin{aligned}
j_2(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0s\sqrt{s}}f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0s\sqrt{s}}\frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v'v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}f(\tau,X^\tau(x,v),v')\\
&\times \frac{e^{|u'|^2+|(x+b(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+b(\tau,u',v))\times u'|^2}}f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0s\sqrt{s}}e^{-(|u|^2+|v|^2)}e^{-D}d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0s\sqrt{s}}e^{-(|u|^2+|v|^2)}e^{-|x\times v|^2}d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0s\sqrt{s}}e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2.
\end{aligned}$$

For (5.34), we notice that

$$\begin{aligned}
\partial_{v^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)] &= \partial_{v^i}\left[f^{\#}(\tau,x,v)f^{\sharp}(\tau,x+\widetilde{b}(\tau,u,v),u)\right]\\
&= \partial_{v^i}(f^{\#}(\tau,x,v))f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)\\
&+ f^{\#}(\tau,x,v)\sum_{k=1}^{3}\partial_{v^i}(\widetilde{b}^k(\tau,u,v))\partial_{x^i}\left(f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)\right).
\end{aligned} \tag{5.36}$$

With (5.36), we obtain

$$\begin{aligned}
j_3(t) &\lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}[|\partial_{v^i}(f^{\#}(\tau,x,v))|f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)\\
&+ f^{\#}(\tau,x,v)\sum_{k=1}^{3}|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du\\
&\lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{v^i}(f^{\#}(\tau,x,v)f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u))|d\omega du\\
&+ \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}f^{\#}(\tau,x,v)\sum_{k=1}^{3}|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du\\
&\lesssim ||\partial_{v^i}(f^{\#}(\tau))||_e\iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}|f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)|d\omega du\\
&+ ||f^{\#}(\tau)||_e\iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}\sum_{k=1}^{3}|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(x,u,v),u))|d\omega du.
\end{aligned}$$

Then we have

$$\begin{aligned}
j_3(\tau) &\lesssim ||\partial_{v^i} f^\#(\tau)||_e ||f^\#(\tau)||_e (\int_{\mathbb{R}^3} \frac{1}{v^0 u^0 \sqrt{s}} e^{-|u|^2} du) \\
&+ ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e (\int_{\mathbb{R}^3} \frac{1}{v^0 u^0 \sqrt{s}} e^{-|u|^2} du) \\
&\lesssim ||\partial_{v^i} f^\#(\tau)||^e ||f^\#(\tau)||_e \int_{\mathbb{R}^3} e^{-|u|^2} du \\
&+ ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e \int_{\mathbb{R}^3} e^{-|u|^2} du \\
&\lesssim \sum_{i=1}^{3} ||\partial_{v^i} f^\#(\tau)||_e ||f^\#(\tau)||_e + ||f^\#(\tau)||^e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} (||\partial_{v^i} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e).
\end{aligned}$$

For (5.35), we recall that

$$j_4(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\tfrac{\sigma_0(\omega)}{v^0 u^0 \sqrt{s}} \partial_{v^i}[f(\tau, X^\tau(x,v), v') f(\tau, X^\tau(x,v), u')]| d\omega du.$$

First remark:

$$\partial_{v^i}\left[f(\tau, X^\tau(x,v), v') f(\tau, X^\tau(x,v), u')\right] = \partial_{v^i}\left[f^\#(\tau, x+\widetilde{b}(\tau,v',v), v') f^\#(\tau, x+\widetilde{b}(\tau,u',v), u')\right]$$

$$= \partial_{v^i}\left[f^\#(\tau, x+\widetilde{b}(\tau,v',v), v')\right] f^\#(\tau, x+\widetilde{b}(\tau,u',v), u') + f^\#(\tau, x+\widetilde{b}(\tau,v',v), v')$$

$$\times\partial_{v^i}\left[f^\#(\tau, x+\widetilde{b}(\tau,u',v), u')\right]$$

$$= \left(\sum_{k=1}^3 \partial_{v^i}(\widetilde{b}^k(\tau,v',v))\partial_{x^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v')) + \sum_{k=1}^3 \partial_{v^i}(v'^k)\partial_{v'^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v'))\right)$$

$$\times f^\#(\tau, x+\widetilde{b}(\tau, x+\widetilde{b}(\tau,u',v), u'))$$

$$+ f^\#(\tau, x+\widetilde{b}(\tau,v',v), v')$$

$$\times\left(\sum_{k=1}^3 \partial_{v^i}(\widetilde{b}^k(\tau,u',v))\partial_{x^k}(f^\#(\tau, x+\widetilde{b}(\tau,u',v), u')) + \sum_{k=1}^3 \partial_{v^i}(u'^k)\partial_{u'^k}(f^\#(\tau, x+\widetilde{b}(\tau,u',v), u'))\right).$$

Then

$$|\partial_{v^i}[f(\tau, X^\tau(x,v), v') f(\tau, X^\tau(x,v), u')]| \lesssim v^0 (u^0)^4$$

$$\times \sum_{k=1}^3 \Big(|\partial_{x^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v'))| f^\#(\tau, x+\widetilde{b}(\tau,u',v), u')$$

$$+ |\partial_{v'^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v'))| f^\#(\tau, x+\widetilde{b}(\tau,u',v), u')$$

$$+f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))|$$

$$+f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))|\Big).$$

So we can recall that

$$j_4 =\lesssim H_1(\tau)+H_2(\tau)+H_3(\tau)+H_4(\tau) \tag{5.37}$$

where

$$H_1(\tau)=\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\tfrac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4\textstyle\sum_{k=1}^3|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du,$$

$$H_2(\tau)=\iint_{S^2\times\mathbb{R}^3}\tfrac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4\textstyle\sum_{k=1}^3|\partial_{v'^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du,$$

$$H_3(\tau)=\iint_{S^2\times\mathbb{R}^3}\tfrac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')\textstyle\sum_{k=1}^3|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))|d\omega du,$$

$$H_4(\tau)=\iint_{S^2\times\mathbb{R}^3}\tfrac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')\textstyle\sum_{k=1}^3|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))|d\omega du.$$

Let's control the term $H_1(\tau)$.
Since $v^0\geq 1$, $u^0\geq 1$ and $\sqrt{s}\geq 2$, we have

$$\begin{aligned}
H_1(\tau) &= \iint_{S^2\times\mathbb{R}^3}\rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4\sum_{k=1}^3\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3}\rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4\frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}\sum_{k=1}^3\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times\frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&\lesssim \sum_{k=1}^3||\partial_{x^k}f^\#(\tau)||_e||f^\#(\tau)||_e\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}v^0(u^0)^4e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \sum_{k=1}^3||\partial_{x^k}f^\#(\tau)||_e||f^\#(\tau)||_e\int_{\mathbb{R}^3}\frac{1}{\sqrt{s}}(u^0)^3e^{-|u|^2}du \\
&\lesssim \sum_{k=1}^3||\partial_{x^k}f^\#(\tau)||_e||f^\#(\tau)||_e\int_{\mathbb{R}^3}(1+|u|^2)^3e^{-|u|^2}du \\
&\lesssim \sum_{k=1}^3||\partial_{x^k}f^\#(\tau)||_e||f^\#(\tau)||_e.
\end{aligned}$$

Let's control the term $H_2(\tau)$.

Since $v^0 \geq 1$, $u^0 \geq 1$ and $\sqrt{s} \geq 2$, we have

$$\begin{aligned}
H_2(\tau) &= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 \sum_{k=1}^{3} \left|\partial_{v'^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} \sum_{k=1}^{3} \left|\partial_{v'^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&\lesssim \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \|f^{\#}(\tau)\|_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \|f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} \frac{1}{\sqrt{s}}(u^0)^3 e^{-|u|^2} du \\
&\lesssim \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e ||f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} (1+|u|^2)^3 e^{-|u|^2} du \\
&\lesssim \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \|f^{\#}(\tau)\|_e.
\end{aligned}$$

Let's control the term $H_3(\tau)$.

Since $v^0 \geq 1$, $u^0 \geq 1$ and $\sqrt{s} \geq 2$, we have

$$\begin{aligned}
H_3(\tau) &= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v') \sum_{k=1}^{3} \left|\partial_{x^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3} \left|\partial_{x^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{x^k} f^{\#}(\tau)\|_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{x^k} f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} \frac{1}{\sqrt{s}}(u^0)^3 e^{-|u|^2} du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{x^k} f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} (1+|u|^2)^3 e^{-|u|^2} du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{x^k} f^{\#}(\tau)\|_e.
\end{aligned}$$

Let's control the term $H_4(\tau)$.

Since $v^0 \geq 1$, $u^0 \geq 1$ and $\sqrt{s} \geq 2$, we have

$$\begin{aligned}
H_4(\tau) &= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0 (u^0)^4 f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v') \sum_{k=1}^{3} \left|\partial_{u'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0 (u^0)^4 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3} \left|\partial_{u'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}} v^0(u^0)^4 e^{-(|u|^2+|v|^2)} e^{-D} d\omega du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} \frac{1}{\sqrt{s}} (u^0)^3 e^{-|u|^2} du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e \int_{\mathbb{R}^3} (1+|u|^2)^3 e^{-|u|^2} du \\
&\lesssim \|f^{\#}(\tau)\|_e \sum_{k=1}^{3} \|\partial_{v^k} f^{\#}(\tau)\|_e.
\end{aligned}$$

By (5.37), we can assume that

$$j_4(\tau) \lesssim ||f^{\#}(\tau)||_e \sum_{k=1}^{3} (||\partial_{x^k} f^{\#}(\tau)||_e + ||\partial_{v^k} f^{\#}(\tau)||_e).$$

From the previous estimates, we obtain

$$\begin{aligned}
j_1(\tau)+j_2(\tau)+j_3(\tau)+j_4(\tau) &\lesssim \frac{1}{a}\|f^{\#}(\tau)\|_e^2 + \frac{1}{a}\|f^{\#}(\tau)\|_e^2 \\
&+ \|f^{\#}(\tau)\|_e \sum_{k=1}^{2} (\|\partial_{v^k} f^{\#}(\tau)\|_e + \|\partial_{x^k} f^{\#}(\tau)\|_e) \\
&+ \sum_{k=1}^{2} (||\partial_{v^k} f^{\#}(\tau)||_e + ||\partial_{x^k} f^{\#}(\tau)||_e) ||f^{\#}(\tau)||_e \\
&\lesssim \underset{\tau\in[0,t]}{Sup} [\|f^{\#}(\tau)\|_e (\|f^{\#}(\tau)\|_e + \sum_{k=1}^{3} (\|\partial_{v^k} f^{\#}(\tau)\|_e \|\partial_{x^k} f^{\#}(\tau)\|_e))].
\end{aligned}$$

By (5.31), we have

$$\rho(x,v)\partial_{v^i} f^{\#}(t,x,v) \lesssim \|\partial_{v^i} f(0)\|_e + \underset{\tau\in[0,t]}{Sup}(K(\tau)) \int_0^t a^{-1}(\tau) b^{-2}(\tau) d\tau$$

with

$$K(\tau) = \|f^{\#}(\tau)\|_e (\|f^{\#}(\tau)\|_e + \sum_{k=1}^{3} (\|\partial_{v^k} f^{\#}(\tau)\|_e + \|\partial_{x^k} f^{\#}(\tau)\|_e)).$$

Taking the supremum with respect to x and v, we have

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup} [\rho(x,v)\partial_{v^i} f^{\#}] \leq ||\partial_{v^i} f(0)||_e + C\, \underset{\tau\in[0,t]}{Sup}(K(\tau)).$$

We conclude at the end that

$$||\partial_{v^i} f^{\#}(t)||_e \leq ||\partial_{v^i} f(0)||_e + C \underset{\tau\in[0,t]}{Sup} (|||f^{\#}(\tau)|||_e^2).$$

□

Lemma 5.13. Let $f^{\#}$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|\partial_{x^i} f^{\#}(t)\|_e \leq \|\partial_{x^i} f_0\|_e + C \underset{t\in[0,t]}{sup} (\||f^{\#}(\tau)\||_e^2, \qquad for \qquad i = 1,\, 2,\, 3 \tag{5.38}$$

where C does not depend on t.

Proof. The Boltzmann equation is written as

$$\begin{aligned} f^{\#}(t,x,v) &= f(0,x,v) + \int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v) + \int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned} \tag{5.39}$$

We take ∂_{x^i} to (5.39) and get

$$\partial_{x^i} f^{\#}(t,x,v) = \partial_{x^i} f(0,x,v) + \int_0^t \partial_{x^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.40}$$

We multiply (5.40) by $\rho(x,v)$ and obtain

$\rho(x,v)\partial_{x^i} f^{\#}(t,x,v) = \rho(x,v)\partial_{x^i} f(0,x,v) + \int_0^t \rho(x,v)\partial_{x^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$

Then

$$\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)| \leq ||\partial_{x^i} f(0)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)\left[K_1(\tau)+K_2(\tau)\right]d\tau \tag{5.41}$$

where

$$K_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{x^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]|d\omega du$$

and

$$K_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}|\partial_{x^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)]|d\omega du.$$

We remark at first that:

$$\partial_{x^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')] = \partial_{x^i}[f(\tau,X^\tau(x,v),v')]f(\tau,X^\tau(x,v),u')$$

$$+f(\tau,X^\tau(x,v),v')\partial_{x^i}[f(\tau,X^\tau(x,v),u')]$$

and

$$\partial_{x^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)] = \partial_{x^i}[f(\tau,X^\tau(x,v),v)]f(\tau,X^\tau(x,v),u)$$

$$+f(\tau, X^\tau(x,v),v)\partial_{x^i}[f(\tau, X^\tau(x,v),u)].$$

Let's control the terms $K_1(\tau)$ and $K_2(\tau)$.

$$K_2(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}(|\partial_{x^i}[f(\tau, X^\tau(x,v),v)]|f(\tau, X^\tau(x,v),u)$$

$$+f(\tau, X^\tau(x,v),v)|\partial_{x^i}[f(\tau, X^\tau(x,v),u)]|)d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$+||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}} |\partial_{x^i}[f^{\#}(\tau, x+\widetilde{b}(\tau,u,v),u)]|d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e(\int_{\mathbb{R}^3} e^{-|u|^2}du)$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e.$$

We also have

$$K_1(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{4\sigma_0(\omega)}{v^0u^0\sqrt{s}}\,(|\partial_{x^i}[f(\tau, X^\tau(x,v),v')]|f(\tau, X^\tau(x,v),u')$$

$$+f(\tau, X^\tau(x,v),v')|\partial_{x^i}[f(\tau, X^\tau(x,v),u')]|)\,d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$+||f^{\#}(\tau)||_e||\partial_{x^i} f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\frac{\sigma_0(\omega)}{v^0u^0\sqrt{s}}e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e(\int_{\mathbb{R}^3} e^{-|u|^2})du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e.$$

Finally, by (5.41) we can state that

$$\begin{aligned}
\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)| &\leq ||\partial_{x^i} f(0)||_e + ||\partial_{x^i} f^{\#}(\tau)||_e||f^{\#}(\tau)||_e(\int_0^t a^{-1}(\tau)b^{-2}(\tau)d\tau)\\
&\leq ||\partial_{x^i} f(0)||_e + C||f^{\#}(\tau)||_e\sum_{i=1}^{3}||\partial_{x^i} f^{\#}(\tau)||_e\\
&\leq ||\partial_{x^i} f(0)||_e + C\,\underset{\tau\in[0,t]}{Sup}\,(||f^{\#}(\tau)||_e\sum_{i=1}^{3}||\partial_{x^i} f^{\#}(\tau)||_e)\\
&\leq ||\partial_{x^i} f(0)||_e + C\,\underset{\tau\in[0,t]}{Sup}\,(|||f^{\#}(\tau)|||_e^2).
\end{aligned}$$

Then

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup}(\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)|) \leq ||\partial_{x^i} f(0)||_e + C\,\underset{\tau\in[0,t]}{Sup}\,(|||f^{\#}(\tau)|||_e^2).$$

So

$$||\partial_{x^i} f^{\#}(t)||_e \leq ||\partial_{x^i} f(0)||_e + C \underset{\tau\in[0,t]}{Sup} (|||f^{\#}(\tau)|||_e^2).$$

□

5.3.2 L^∞-energy estimates for hard potentials

In this part we take $\alpha = 0$ in (1.70) and we have been working on the additional assumption (2.50). We also assume that the coefficient b of the metric tensor enjoys the condition

$$\int_{\mathbb{R}_+} b^{\beta-3}(\tau) d\tau < \infty. \tag{5.42}$$

Lemma 5.14. Let $f^{\#}$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|f^{\#}(t)\|_e \leq \|f_0\|_e + C \underset{\tau\in[0,t]}{sup} (\|f^{\#}(\tau)\|_e^2) \tag{5.43}$$

where C does not depend on t.

Proof. Let's consider the integral form of the Boltzmann equation

$$\begin{aligned} f^{\#}(t,x,v) &= f_0(x,v) + \int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau \\ &= f_0(x,v) + \int_0^t Q(f,f)(\tau, X^\tau(x,v),v)d\tau. \end{aligned} \tag{5.44}$$

We multiply (5.44) by $\rho(x,v)$ and get

$$\rho(x,v)f^{\#}(t,x,v) = \rho(x,v)f_0(x,v) + \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau \tag{5.45}$$

$$- \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.46}$$

So we are organizing our estimation like this:

$$\rho(x,v)f^{\#}(t,x,v) \leq ||f(0)||_e + S_1 + S_2 \tag{5.47}$$

where

$$S_1 = \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau \tag{5.48}$$

and

$$S_2 = \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.49}$$

Let's control the expression (5.48).
Since $\vartheta_\phi \leq 4$ and $\int_{\mathbb{R}^3} e^{-|u|^2} du < \infty$
we have

$$S_1 = \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau, X^{\tau}(x,v),v)d\tau$$

$$= \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)f(\tau,X^{\tau}(x,v),v')f(\tau,X^{\tau}(x,v),u')d\omega du$$

$$\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du$$

$$\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)\frac{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}$$

$$\times f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')\frac{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}}f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du$$

$$\leq C\int_0^t a^{-1}b^{-2}||f^{\#}(\tau)||_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}(1+g^{-\beta})\sigma_0(\omega)e^{-(|u'|^2+|v'|^2)}e^{-D}d\omega du$$

$$\leq C\int_0^t a^{-1}b^{-2}||f^{\#}(\tau)||_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}(1+g^{-\beta})\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$\leq C\underset{\tau\in[0,t]}{Sup}(||f^{\#}(\tau)||_e^2)\int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} e^{|\omega.(x\times v)|^2}\vartheta_{\phi}(1+g^{-\beta})\sigma_0(\omega)e^{-|u|^2}e^{-|\omega.(x\times v)|^2}d\omega du$$

$$\leq ||f(0)||_e + C\underset{\tau\in[0,t]}{Sup}(||f^{\#}(\tau)||_e^2)\int_0^t a^{-1}b^{-2}(\int_{\mathbb{R}^3}\vartheta_{\phi}e^{-|u|^2}du + \int_{\mathbb{R}^3}\vartheta_{\phi}g^{-\beta}e^{-|u|^2}du)d\tau$$

$$\leq C\underset{\tau\in[0,t]}{Sup}(||f^{\#}(\tau)||_e^2)\int_0^t[a^{-1}(\tau)b^{-2}(\tau)+a^{-1}(\tau)b^{\beta-3}(\tau)]d\tau$$

$$\leq C\underset{\tau\in[0,t]}{Sup}(||f^{\#}(\tau)||_e^2)\int_0^{\infty}[a^{-1}(\tau)b^{-2}(\tau)+a^{-1}(\tau)b^{\beta-3}(\tau)]d\tau.$$

Let's control the expression (5.49).

Since $\vartheta_{\phi} \leq 4$ and $\int_{\mathbb{R}^3} e^{-|u|^2}du < \infty$

we have

$$S_2 = \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau, X^{\tau}(x,v),v)d\tau$$

$$= \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)f(\tau,X^{\tau}(x,v),v)f(\tau,X^{\tau}(x,v),u)d\omega du$$

$$= \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)f^{\#}(\tau,x,v)f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$\leq \int_0^t a^{-1}b^{-2}\|f^{\#}(\tau)\|_e d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(x,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(x,u,v))\times u|^2}}$$

$$\times f^{\#}(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$\leq \int_0^t a^{-1}b^{-2}\|f^{\#}(\tau)\|_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}\sigma(g,\omega)e^{-|u|^2-|(x+\widetilde{b}(x,u,v))\times u|^2}d\omega du$$

$$\leq C\int_0^t a^{-1}b^{-2}\|f^{\#}(\tau)\|_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_{\phi}(1+g^{-\beta})\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2)\int_0^t a^{-1}b^{-2}\left(\int_{\mathbb{R}^3}\vartheta_\phi e^{-|u|^2}du+\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du\right)d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2)\int_0^t [a^{-1}(\tau)b^{-2}(\tau)+a^{-1}(\tau)b^{\beta-3}(\tau)]d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2)\int_0^\infty [a^{-1}(\tau)b^{-2}(\tau)+a^{-1}(\tau)b^{\beta-3}(\tau)]d\tau.$$

Then by (5.47), we obtain

$$\rho(x,v)f^\#(t,x,v)\leq \|f(0)\|_e + C\underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

Taking the supremum with respect to x and v, we have

$$\underset{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3}{Sup}(\rho(x,v)f^\#(t,x,v))\leq \|f(0)\|_e + C\underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

Finally

$$||f^\#(t)||_e\leq ||f(0)||_e + C\underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2).$$

□

Lemma 5.15. Let $f^\#$ be a solution of the inhomogeneous relativistic Boltzmann equation. The following estimate for $\partial_{v^i}f^\#$ holds for a fixed $i\in\{1,2,3\}$:

$$\|\partial_{v^i}f^\#(t)\|_e\leq \|\partial_{v^i}f_0\|_e + C\underset{\tau\in[0,t]}{sup}[\|f^\#(\tau)\|_e(\|f^\#(\tau)\|_e+\sum_{k=1}^{3}(\|\partial_{v^i}f^\#(\tau)\|^e+\|\partial_{x^k}f^\#(\tau)\|_e))] \quad (5.50)$$

for a constant C which does not depend on t.

Proof. The Boltzmannn equation is written as

$$\begin{aligned} f^\#(t,x,v) &= f(0,x,v)+\int_0^t Q^\#(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v)+\int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned}$$

We take ∂_{v^i} to this equation to obtain

$$\partial_{v^i}f^\#(t,x,v)=\partial_{v^i}f(0,x,v)+\int_0^t\partial_{v^i}Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$$

We multiply by $\rho(x,v)$ to have

$$\rho(x,v)\partial_{v^i}f^\#(t,x,v)=\rho(x,v)\partial_{v^i}f(0,x,v)+\int_0^t\rho(x,v)\partial_{v^i}Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$$

Then

$$\rho(x,v)\partial_{v^i}f^\#(t,x,v)\lesssim ||\partial_{v^i}f(0,x,v)||_e+\int_0^t\rho(x,v)\partial_{v^i}Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$$

So

$\rho(x,v)\partial_{v^i} f^{\#}(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e$

$+\int_0^t \rho(x,v)\partial_{v_i}[a^{-1}b^{-2}\iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)(f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')$

$-f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u))d\omega du d\tau$

$\lesssim ||\partial_{v^i} f(0,x,v)||_e+\int_0^t a^{-1}b^{-2}d\tau[\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial v^i\left[\vartheta_\phi\sigma(g,\omega)\right] f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du$

$+\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right] d\omega du$

$-\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right] f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du$

$-\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)\right] d\omega du].$

We can organize the previous expression a follows:

$$\rho(x,v)\partial_{v^i} f^{\#}(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)\left[j_1(\tau)+j_2(\tau)+j_3(\tau)+j_4(\tau)\right] d\tau \tag{5.51}$$

where

$$j_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du, \tag{5.52}$$

$$j_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du, \tag{5.53}$$

$$j_3(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)\right]|d\omega du, \tag{5.54}$$

$$j_4(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right]|d\omega du. \tag{5.55}$$

Now we are going to control each of the four terms.

For the expression (5.52) we have

$$\begin{aligned}
j_1(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\vartheta_\phi\sigma(g,\omega)]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0(1+g^{-\beta})\sigma_0(\omega)f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} u^0(1+g^{-\beta})f(\tau,X^\tau(x,v),u)du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} u^0(1+g^{-\beta})\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}f(\tau,X^\tau(x,v),u)du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} u^0(1+g^{-\beta})e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2(\int_{\mathbb{R}^3} u^0e^{-|u|^2}du+\int_{\mathbb{R}^3} u^0g^{-\beta}e^{-|u|^2}du)\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2(1+b^\beta)\\
&\lesssim (\frac{1}{b}+\frac{1}{b^{1-\beta}})||f^{\#}(\tau)||_e^2.
\end{aligned}$$

For the expression (5.53) we have

$$\begin{aligned}
j_2(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\vartheta_\phi\sigma(g,\omega)]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0(1+g^{-\beta})\sigma_0(\omega)f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0(1+g^{-\beta})\sigma_0(\omega)\frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v'v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}f(\tau,X^\tau(x,v),v')\\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0(1+g^{-\beta})\sigma_0(\omega)e^{-(|u'|^2+|v'|^2)}e^{-D}d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} \rho(x,v)u^0(1+g^{-\beta})e^{-(|u|^2+|v|^2)}e^{-|x\times v|^2}d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} u^0(1+g^{-\beta})e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2(\int_{\mathbb{R}^3} u^0e^{-|u|^2}du+\int_{\mathbb{R}^3} u^0g^{-\beta}e^{-|u|^2}du)\\
&\lesssim \frac{1}{a}(1+b^\beta)||f^{\#}(\tau)||_e^2\\
&\lesssim (\frac{1}{b}+\frac{1}{b^{1-\beta}})||f^{\#}(\tau)||_e^2.
\end{aligned}$$

For the expression (5.54), we recall (5.36) and we have

$$j_3(t) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)[|\partial_{v^i}(f^\#(\tau,x,v))|f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)$$
$$+ f^\#(\tau,x,v)\sum_{k=1}^{3}|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du$$
$$\lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(f^\#(\tau,x,v)f^\#(\tau,x+\widetilde{b}(\tau,u,v),u))|d\omega du$$
$$+ \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)f^\#(\tau,x,v)\sum_{k=1}^{3}|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du$$
$$\lesssim ||\partial_{v^i}(f^\#(\tau))||_e \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}|f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)|d\omega du$$
$$+ ||f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}\sum_{k=1}^{3}|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(x,u,v),u))|d\omega du.$$

Then we have

$$j_3(\tau) \lesssim ||\partial_{v^i}f^\#(\tau)||_e||f^\#(\tau)||_e(\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)e^{-|u|^2}du)$$
$$+ ||f^\#(\tau)||_e\sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e(\int_{\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)e^{-|u|^2}du)$$
$$\lesssim ||\partial_{v^i}f^\#(\tau)||_e||f^\#(\tau)||_e(b^{\beta-1})$$
$$+ ||f^\#(\tau)||_e\sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e(b^{\beta-1})$$
$$\lesssim \sum_{i=1}^{3}||\partial_{v^i}f^\#(\tau)||_e||f^\#(\tau)||_e + ||f^\#(\tau)||_e\sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e)b^{\beta-1}$$
$$\lesssim b^{\beta-1}||f^\#(\tau)||_e\sum_{k=1}^{3}(||\partial_{v^i}f^\#(\tau)||_e + ||\partial_{x^k}f^\#(\tau)||_e).$$

For the expression (5.55), we recall
$j_4(\tau) = \iint_{S^2\times\mathbb{R}^3}|\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]|d\omega du.$
We remark that

$$\partial_{v^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right] = \partial_{v^i}\left[f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')\right]$$

$$= \partial_{v^i}\left[f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')\right]f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')$$

$$+f^\#(\tau,x+\widetilde{b}(\tau,v',v),v')\partial_{v^i}\left[f^\#(\tau,x+\widetilde{b}(\tau,u',v),u')\right]$$

$$= \Big(\sum_{k=1}^{3}\partial_{v^i}(\widetilde{b}^k(\tau,v',v))\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))$$

$$+ \sum_{k=1}^{3}\partial_{v^i}(v'^k)\partial_{v'^k}(f^\#(\tau,x+\widetilde{b}(\tau,v',v),v'))\Big)f^\#(\tau,x+\widetilde{b}(\tau,x+b(\tau,u',v),u'))$$

$$+f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v')\left(\sum_{k=1}^{3}\partial_{v^i}(b^k(\tau,u',v))\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))\right.$$

$$\left.+\sum_{k=1}^{3}\partial_{v^i}(u'^k)\partial_{u'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))\right).$$

Then

$$|\partial_{v^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]| \lesssim$$

$$(u^0)^5\sum_{k=1}^{3}\left(|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')\right.$$

$$+|\partial_{v'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')$$

$$+f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))|$$

$$\left.+f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')|\partial_{u'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))|\right). \tag{5.56}$$

Inserting (5.56) in (5.55) we obtain

$$j_4 \lesssim H_1(\tau)+H_2(\tau)+H_3(\tau)+H_4(\tau)$$

where

$$H_1(\tau)=\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5\sum_{k=1}^{3}|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du,$$

$$H_2(\tau)=\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5\sum_{k=1}^{3}|\partial_{v'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))|f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du,$$

$$H_3(\tau)=\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')\sum_{k=1}^{3}|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))|d\omega du,$$

$$H_4(\tau)=\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')\sum_{k=1}^{3}|\partial_{u'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u'))|d\omega du.$$

Let's control the term $H_1(\tau)$.

Since $\vartheta_\phi \leq 4$ and by (2.20)we have

$$
\begin{aligned}
H_1(\tau) =& \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^{3} \left|\partial_{x^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} \sum_{k=1}^{3} \left|\partial_{x^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{x^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{x^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi(1+g^{-\beta})e^{-|u|^2}du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{x^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (\int_{\mathbb{R}^{\not\not 3}} (u^0)^5\vartheta_\phi e^{-|u|^2}du + \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du) \\
&\lesssim \sum_{k=1}^{3} ||\partial_{x^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (1+b^{\beta-1}).
\end{aligned}
$$

Let's control the term $H_2(\tau)$.

Since $\vartheta_\phi \leq 4$ and and by (2.20)we have

$$
\begin{aligned}
H_2(\tau) =& \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^{3} \left|\partial_{v'^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} \sum_{k=1}^{3} \left|\partial_{v'^k}(f^{\#}(\tau, x+\widetilde{b}(\tau,v',v),v'))\right| \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u',v),u')d\omega du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{v^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{v^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi(1+g^{-\beta})e^{-|u|^2}du \\
&\lesssim \sum_{k=1}^{3} ||\partial_{v^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (\int_{\mathbb{R}^{\not\not 3}} (u^0)^5\vartheta_\phi e^{-|u|^2}du + \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du) \\
&\lesssim \sum_{k=1}^{3} ||\partial_{v^k} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (1+b^{\beta-1}).
\end{aligned}
$$

Let's control the term $H_3(\tau)$.

Since $\vartheta_\phi \leq 4$ and and by (2.20)we have

$$\begin{aligned}
H_3(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \sum_{k=1}^{3}\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^\#(\tau,x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3}\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D} d\omega du \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi(1+g^{-\beta})e^{-|u|^2}du \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e (\int_{\mathbb{R}^{\not 3}} (u^0)^5\vartheta_\phi e^{-|u|^2}du + \int_{\mathbb{R}^3} (u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du) \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{x^k} f^\#(\tau)||_e (1+b^{\beta-1}).
\end{aligned}$$

Let's control the term $H_4(\tau)$.

Since $\vartheta_\phi \leq 4$ and and by (2.20), we have

$$\begin{aligned}
H_4(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \sum_{k=1}^{3}\left|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^\#(\tau,x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3}\left|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{v^k} f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D} d\omega du.
\end{aligned}$$

Then we have

$$H_4(\tau) \lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{v^k} f^\#(\tau)||_e \int_{\mathbb{R}^3} (u^0)^5 \vartheta_\phi (1+g^{-\beta}) e^{-|u|^2} du$$
$$\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{v^k} f^\#(\tau)||_e (\int_{\mathbb{R}^{\not 3}} (u^0)^5 \vartheta_\phi e^{-|u|^2} du + \int_{\mathbb{R}^3} (u^0)^5 \vartheta_\phi g^{-\beta} e^{-|u|^2} du)$$
$$\lesssim ||f^\#(\tau)||_e \sum_{k=1}^{3} ||\partial_{v^k} f^\#(\tau)||_e (1+b^{\beta-1}).$$

Summing up the above terms we obtain

$$j_4(\tau) \lesssim (1+b^{\beta-1}) ||f^\#(\tau)||_e \sum_{k=1}^{3} \left(||\partial_{x^k} f^\#(\tau)||_e + ||\partial_{v^k} f^\#(\tau)||_e \right).$$

So we can state that

$$j_1(\tau)+j_2(\tau)+j_3(\tau)+j_4(\tau) \lesssim \left(b^{-1}+b^{\beta-1}\right) ||f^\#(\tau)||_e^2 + \left(b^{-1}+b^{\beta-1}\right) ||f^\#(\tau)||_e^2$$

$$+b^{\beta-1} ||f^\#(\tau)||_e \textstyle\sum_{k=1}^{2} \left(||\partial_{v^k} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e \right)$$

$$+(1+b^{\beta-1}) \textstyle\sum_{k=1}^{2} \left(||\partial_{v^k} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e \right) ||f^\#(\tau)||_e$$

$$\lesssim (1+b^{\beta-1}) \underset{\tau\in[0,t]}{Sup} \left[||f^\#(\tau)||_e (||f^\#(\tau)||_e + \textstyle\sum_{k=1}^{3} (||\partial_{v^k} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e)) \right].$$

Then by (5.51) we have

$$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial_{v^i} f(0)||_e + \underset{\tau\in[0,t]}{Sup} (K(\tau)) \left[\int_0^t (a^{-1}(\tau) b^{-2}(\tau) + a^{-1}(\tau) b^{\beta-3}(\tau)) d\tau \right]$$

with

$$K(\tau) = ||f^\#(\tau)||_e \left[||f^\#(\tau)||_e + \sum_{k=1}^{3} \left(||\partial_{v^k} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e \right) \right].$$

Then

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup} [\rho(x,v)\partial_{v^i} f^\#(t,x,v)] \le ||\partial_{v^i} f(0)||_e + C \underset{\tau\in[0,t]}{Sup} (K(\tau)).$$

Finally we can conclude that

$$||\partial_{v^i} f^\#(t)||_e \le ||\partial_{v^i} f(0)||_e + C \underset{\tau\in[0,t]}{Sup} (K(\tau)).$$

□

Lemma 5.16. Let $f^\#$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|\partial_{x^i} f^\#(t)\|_e \le \|\partial_{x^i} f_0\|_e + C \underset{t\in[0,t]}{sup} (\|f^\#(\tau)\|_e \|\partial_{x^i} f^\#(\tau)\|_e), \qquad for \qquad i=1,\,2,\,3 \tag{5.57}$$

where C does not depend on t.

Proof. The Boltzmann equation is written as

$$\begin{aligned} f^{\#}(t,x,v) &= f(0,x,v) + \int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v) + \int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned} \quad (5.58)$$

We take ∂_{x^i} to (5.58) and get

$$\partial_{x^i} f^{\#}(t,x,v) = \partial_{x^i} f(0,x,v) + \int_0^t \partial_{x^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \quad (5.59)$$

We multiply (5.59) by $\rho(x,v)$ to obtain

$$\rho(x,v)\partial_{x^i} f^{\#}(t,x,v) = \rho(x,v)\partial_{x^i} f(0,x,v) + \int_0^t \rho(x,v)\partial_{x^i} Q(f,f)(\tau,X^\tau(x,v),v)d\tau.$$

Then

$$\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)| \le ||\partial_{x^i} f(0)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)\left[K_1(\tau)+K_2(\tau)\right]d\tau \quad (5.60)$$

where

$$K_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{x^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]|d\omega du \quad (5.61)$$

and

$$K_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{x^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)]|d\omega du. \quad (5.62)$$

We remark at first that

$$\partial_{x^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right] = \partial_{x^i}\left[f(\tau,X^\tau(x,v),v')\right]f(\tau,X^\tau(x,v),u')$$

$$+f(\tau,X^\tau(x,v),v')\partial_{x^i}\left[f(\tau,X^\tau(x,v),u')\right]$$

and

$$\partial_{x^i}\left[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)\right] = \partial_{x^i}\left[f(\tau,X^\tau(x,v),v)\right]f(\tau,X^\tau(x,v),u)$$

$$+f(\tau,X^\tau(x,v),v)\partial_{x^i}\left[f(\tau,X^\tau(x,v),u)\right].$$

Let's control the terms $K_1(\tau)$ and $K_2(\tau)$.

For the expression (5.62) we have

$$K_2(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\left(|\partial_{x^i}[f(\tau,X^\tau(x,v),v)]|f(\tau,X^\tau(x,v),u)\right.$$

$$\left.+f(\tau,X^\tau(x,v),v)|\partial_{x^i}[f(\tau,X^\tau(x,v),u)]|\right)d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}} f^{\#}(\tau, x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$+||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}} |\partial_{x^i}[f^{\#}(\tau, x+\widetilde{b}(\tau,u,v),u)]|d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (\textstyle\int_{\mathbb{R}^3} \vartheta_\phi(1+g^{-\beta})e^{-|u|^2}du)$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (1+b^{\beta-1}).$$

For the expression (5.61) we have

$$K_1(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\left(|\partial_{x^i}[f(\tau,X^\tau(x,v),v')]|f(\tau,X^\tau(x,v),u')\right.$$

$$\left.+f(\tau,X^\tau(x,v),v')|\partial_{x^i}[f(\tau,X^\tau(x,v),u')]|\right)d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$+||f^{\#}(\tau)||_e ||\partial_{x^i} f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi(1+g^{-\beta})\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (\textstyle\int_{\mathbb{R}^3} \vartheta_\phi(1+g^{-\beta})e^{-|u|^2})du$$

$$\lesssim ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e (1+b^{\beta-1}).$$

Finally by (5.60) we have

$$\begin{aligned}\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)| &\lesssim ||\partial_{x^i} f(0)||_e + ||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e \\ &\quad \times \left[\int_0^t (a^{-1}(\tau)b^{-2}(\tau)+a^{-1}(\tau)b^{\beta-3}(\tau))d\tau\right] \\ &\lesssim ||\partial_{x^i} f(0)||_e + \underset{\tau\in[0,t]}{Sup}\,(||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e).\end{aligned}$$

Then

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup}\left(\rho(x,v)|\partial_{x^i} f^{\#}(t,x,v)|\right) \leq ||\partial_{x^i} f(0)||_e + C\,\underset{\tau\in[0,t]}{Sup}\left(||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e\right).$$

So

$$||\partial_{x^i} f^{\#}(t)||_e \leq ||\partial_{x^i} f(0)||_e + C\,\underset{\tau\in[0,t]}{Sup}\left(||\partial_{x^i} f^{\#}(\tau)||_e ||f^{\#}(\tau)||_e\right).$$

□

5.3.3 L^∞-energy estimates for soft potentials

In this part we consider the additional assumption (2.50).

We also assume that the coefficient b of the metric tensor enjoys the condition (5.42).

Lemma 5.17. Let $f^{\#}$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$\|f^{\#}(t)\|_e \leq \|f_0\|_e + C \sup_{\tau\in[0,t]} (\|f^{\#}(\tau)\|_e^2) \tag{5.63}$$

where C does not depend on t.

Proof. Let's consider the Boltzmann equation

$$\begin{aligned} f^{\#}(t,x,v) &= f_0(x,v) + \int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau \\ &= f_0(x,v) + \int_0^t Q(f,f)(\tau,X^\tau(x,v),v)d\tau. \end{aligned} \tag{5.64}$$

We multiply (5.64) by $\rho(x,v)$ and get

$\rho(x,v)f^{\#}(t,x,v) = \rho(x,v)f_0(x,v) + \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau$

$-\int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau.$

So we organize our estimation like this:

$$\rho(x,v)f^{\#}(t,x,v) \leq \|f(0)\|_e + S_1 + S_2 \tag{5.65}$$

where

$$S_1 = \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau \tag{5.66}$$

and

$$S_2 = \int_0^t \rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau. \tag{5.67}$$

For the expression (5.66) we have

$S_1 = \int_0^t \rho(x,v)Q_{gain}(f,f)(\tau,X^\tau(x,v),v)d\tau$
$\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du$

$\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du$

$\leq \int_0^t a^{-1}b^{-2}d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\frac{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(x,v',v))\times v'|^2}}$

$\times f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v')\frac{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(x,u',v))\times u'|^2}}f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du$

$\leq C\int_0^t a^{-1}b^{-2}\|f^{\#}(\tau)\|_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-(|u'|^2+|v'|^2)}e^{-D}d\omega du$

$\leq C\int_0^t a^{-1}b^{-2}||f^{\#}(\tau)||_e^2 d\tau \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}b^{-2}d\tau\int_{S^2}\sigma_0(\omega)e^{-|\omega.(x\times v)|^2}d\omega\int_{\mathbb{R}^3}e^{|x\times v|^2}\vartheta_\phi g^{-\beta}e^{-|u|^2}du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}b^{-2}d\tau\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^\infty a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

For the expression (5.67) we have

$$S_2=\int_0^t\rho(x,v)Q_{loss}(f,f)(\tau,X^\tau(x,v),v)d\tau$$

$$\leq\int_0^t a^{-1}b^{-2}d\tau\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du$$

$$\leq\int_0^t a^{-1}b^{-2}d\tau\iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi\sigma(g,\omega)f^\#(\tau,x,v)f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$\leq\int_0^t a^{-1}b^{-2}\|f^\#(\tau)\|_e d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(x,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(x,u,v))\times u|^2}}f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$\leq C\int_0^t a^{-1}b^{-2}\|f^\#(\tau)\|_e^2 d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-|u|^2-|(x+\widetilde{b}(x,u,v))\times u|^2}d\omega du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}b^{-2}d\tau\iint_{S^2\times\mathbb{R}^3}\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-|u|^2}d\omega du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}b^{-2}d\tau\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2)\int_0^\infty a^{-1}(\tau)b^{\beta-3}(\tau)d\tau$$

$$\leq C \underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

Then by (5.65)

$$\rho(x,v)f^\#(t,x,v)\leq\|f(0)\|_e+C\underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

Taking the supremum with respect to x and v, we have

$$\underset{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3}{Sup}(\rho(x,v)f^\#(t,x,v))\leq||f(0)||_e+C\underset{\tau\in[0,t]}{Sup}(||f^\#(\tau)||_e^2).$$

So

$$\|f^\#(t)\|_e\leq\|f(0)\|_e+C\underset{\tau\in[0,t]}{Sup}(\|f^\#(\tau)\|_e^2).$$

□

Lemma 5.18. Let $f^\#$ be a solution of the inhomaogeneous relativistic Boltzmann equation (5.1). The following estimate for $\partial_{v^i} f^\#$ holds for a fixed $i \in \{1, 2, 3\}$:

$$\|\partial_{v^i} f^\#(t)\|_e \le \|\partial_{v^i} f_0\|_e + C \sup_{\tau\in[0,t]} [\|f^\#(\tau)\|_e(\|f^\#(\tau)\|_e + \sum_{k=1}^{3}(\|\partial_{v^i} f^\#(\tau)\|^e + \|\partial_{x^k} f^\#(\tau)\|_e))] \quad (5.68)$$

where C does not depend on t.

Proof. The Boltzmannn equation is written as

$$\begin{aligned} f^\#(t,x,v) &= f(0,x,v) + \int_0^t Q^\#(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v) + \int_0^t Q(f,f)(\tau, X^\tau(x,v), v)d\tau. \end{aligned}$$

We take ∂_{v^i} to this equation to obtain

$$\partial_{v^i} f^\#(t,x,v) = \partial v^i f(0,x,v) + \int_0^t \partial_{v^i} Q(f,f)(\tau, X^\tau(x,v), v)d\tau.$$

We multiply by $\rho(x,v)$ and get

$$\rho(x,v)\partial_{v^i} f^\#(t,x,v) = \rho(x,v)\partial v^i f(0,x,v) + \int_0^t \rho(x,v)\partial_{v^i} Q(f,f)(\tau, X^\tau(x,v), v)d\tau.$$

Then

$$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial v^i f(0,x,v)||_e + \int_0^t \rho(x,v)\partial_{v^i} Q(f,f)(\tau, X^\tau(x,v), v)d\tau.$$

So

$$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e$$

$$+ \int_0^t \rho(x,v)\partial_{v^i}[a^{-1}b^{-2} \iint_{S^2\times\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega)(f(\tau, X^\tau(x,v), v')f(\tau, X^\tau(x,v), u')$$

$$-f(\tau, X^\tau(x,v), v)f(\tau, X^\tau(x,v), u))d\omega du]d\tau$$

$$\lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t a^{-1}b^{-2}d\tau[\iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right] f(\tau, X^\tau(x,v), v')f(\tau, X^\tau(x,v), u')d\omega du$$

$$+ \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau, X^\tau(x,v), v')f(\tau, X^\tau(x,v), u')\right] d\omega du$$

$$- \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right] f(\tau, X^\tau(x,v), v)f(\tau, X^\tau(x,v), u)d\omega du$$

$$- \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau, X^\tau(x,v), v)f(\tau, X^\tau(x,v), u)\right] d\omega du].$$

We can organize the previous expression as follows:

$$\rho(x,v)\partial_{v^i} f^\#(t,x,v) \lesssim ||\partial_{v^i} f(0,x,v)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)[j_1(\tau) + j_2(\tau) + j_3(\tau) + j_4(\tau)]d\tau \quad (5.69)$$

where

$$j_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du, \tag{5.70}$$

$$j_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}\left[\vartheta_\phi\sigma(g,\omega)\right]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du, \tag{5.71}$$

$$j_3(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)\right]|d\omega du, \tag{5.72}$$

$$j_4(\tau) = \iint_{S^2\times\mathbb{R}^3} |\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right]|d\omega du. \tag{5.73}$$

Now we control each of the four terms.

For the expression (5.70), we have

$$\begin{aligned}
j_1(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\vartheta_\phi\sigma(g,\omega)]|f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0g^{-\beta}\sigma_0(\omega)f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)d\omega du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} u^0g^{-\beta}f(\tau,X^\tau(x,v),u)du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e\int_{\mathbb{R}^3} u^0g^{-\beta}\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}f(\tau,X^\tau(x,v),u)du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} u^0g^{-\beta}e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2\int_{\mathbb{R}^3} u^0g^{-\beta}e^{-|u|^2}du\\
&\lesssim \frac{1}{a}||f^{\#}(\tau)||_e^2 b^\beta\\
&\lesssim \frac{1}{b^{1-\beta}}||f^{\#}(\tau)||_e^2.
\end{aligned}$$

For the expression (5.71), we have

$$
\begin{aligned}
j_2(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)|\partial_{v^i}[\vartheta_\phi\sigma(g,\omega)]|f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0 g^{-\beta}\sigma_0(\omega)f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0 g^{-\beta}\sigma_0(\omega)\frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v'v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}f(\tau,X^\tau(x,v),v')\\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}f(\tau,X^\tau(x,v),u')d\omega du\\
&\lesssim \frac{1}{a}||f^\#(\tau)||_e^2\iint_{S^2\times\mathbb{R}^3} \rho(x,v)u^0(g^{-\beta}\sigma_0(\omega)e^{-(|u'|^2+|v'|^2)}e^{-D}d\omega du\\
&\lesssim \frac{1}{a}||f^\#(\tau)||_e^2\int_{\mathbb{R}^3} \rho(x,v)u^0 g^{-\beta}e^{-(|u|^2+|v|^2)}e^{-|x\times v|^2}d\omega du\\
&\lesssim \frac{1}{a}||f^\#(\tau)||_e^2\int_{\mathbb{R}^3} u^0 g^{-\beta}e^{-|u|^2}du\\
&\lesssim \frac{1}{a}b^\beta||f^\#(\tau)||_e^2\\
&\lesssim \frac{1}{b^{1-\beta}}||f^\#(\tau)||_e^2.
\end{aligned}
$$

For the expression (5.72), we recall (5.36) and obtain

$$
\begin{aligned}
j_3(t) &\lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)[|\partial_{v^i}(f^\#(\tau,x,v))|f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)\\
&+ f^\#(\tau,x,v)\sum_{k=1}^3|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du\\
&\lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{v^i}(f^\#(\tau,x,v)f^\#(\tau,x+\widetilde{b}(\tau,u,v),u))|d\omega du\\
&+ \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)f^\#(\tau,x,v)\sum_{k=1}^3|\partial_{v^i}(\widetilde{b}^k(\tau,u,v))||\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u,v)))|]d\omega du\\
&\lesssim ||\partial_{v^i}(f^\#(\tau))||_e\iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}|f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)|d\omega du\\
&+ ||f^\#(\tau)||_e\iint_{S^2\times\mathbb{R}^3} \vartheta_\phi\sigma(g,\omega)\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}\sum_{k=1}^3|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(x,u,v),u))|d\omega du.
\end{aligned}
$$

Then we have

$$\begin{aligned}
j_3(\tau) &\lesssim ||\partial_{v^i} f^\#(\tau)||_e ||f^\#(\tau)||_e (\int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{-|u|^2} du) \\
&+ ||f^\#(\tau)||_e \sum_{k=1}^3 ||\partial_{x^k} f^\#(\tau)||_e (\int_{\mathbb{R}^3} \vartheta_\phi \sigma(g,\omega) e^{-|u|^2} du) \\
&\lesssim ||\partial_{v^i} f^\#(\tau)||_e ||f^\#(\tau)||_e (b^{\beta-1}) \\
&+ ||f^\#(\tau)||_e \sum_{k=1}^3 ||\partial_{x^k} f^\#(\tau)||_e (b^{\beta-1}) \\
&\lesssim (\sum_{i=1}^3 ||\partial_{v^i} f^\#(\tau)||_e ||f^\#(\tau)||_e + ||f^\#(\tau)||_e \sum_{k=1}^3 ||\partial_{x^k} f^\#(\tau)||_e) b^{\beta-1} \\
&\lesssim b^{\beta-1} ||f^\#(\tau)||_e \sum_{k=1}^3 (||\partial_{v^i} f^\#(\tau)||_e + ||\partial_{x^k} f^\#(\tau)||_e).
\end{aligned}$$

For the expression (5.73), we recall that

$$j_4(\tau) = \int\int_{S^2\times\mathbb{R}^3} |\rho(x,v)\vartheta_\phi\sigma(g,\omega)\partial_{v^i}[f(\tau, X^\tau(x,v), v')f(\tau, X^\tau(x,v), u')]| d\omega du.$$

Inserting (5.56) in (5.73), we obtain

$$j_4 \lesssim H_1(\tau) + H_2(\tau) + H_3(\tau) + H_4(\tau) \tag{5.74}$$

where

$$H_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^3 |\partial_{x^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v'))| f^\#(\tau, x+\widetilde{b}(\tau,u',v), u') d\omega du,$$

$$H_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^3 |\partial_{v'^k}(f^\#(\tau, x+\widetilde{b}(\tau,v',v), v'))| f^\#(\tau, x+\widetilde{b}(\tau,u',v), u') d\omega du,$$

$$H_3(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v), v') \sum_{k=1}^3 |\partial_{x^k}(f^\#(\tau, x+\widetilde{b}(\tau,u',v), u'))| d\omega du,$$

$$H_4(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v), v') \sum_{k=1}^3 |\partial_{u'^k}(f^\#(\tau, x+\widetilde{b}(\tau,u',v), u'))| d\omega du.$$

Let's control the term $H_1(\tau)$.

$$\begin{aligned}
H_1(\tau) = & \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^{3}\left|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
& \times f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
& = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} \sum_{k=1}^{3}\left|\partial_{x^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
& \times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
& \lesssim \sum_{k=1}^{3}||\partial_{x^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(u^0)^5 \\
& \times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
& \lesssim \sum_{k=1}^{3}||\partial_{x^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e \int_{\mathbb{R}^3}(u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du \\
& \lesssim \sum_{k=1}^{3}||\partial_{x^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e b^{\beta-1}.
\end{aligned}$$

Let's control the term $H_2(\tau)$.

$$\begin{aligned}
H_2(\tau) = & \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \sum_{k=1}^{3}\left|\partial_{v'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
& \times f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
& = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} \sum_{k=1}^{3}\left|\partial_{v'^k}(f^{\#}(\tau,x+\widetilde{b}(\tau,v',v),v'))\right| \\
& \times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} f^{\#}(\tau,x+\widetilde{b}(\tau,u',v),u')d\omega du \\
& \lesssim \sum_{k=1}^{3}||\partial_{v'^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(u^0)^5 \\
& \times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
& \lesssim \sum_{k=1}^{3}||\partial_{v'^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e \int_{\mathbb{R}^3}(u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du \\
& \lesssim \sum_{k=1}^{3}||\partial_{v'^k}f^{\#}(\tau)||_e||f^{\#}(\tau)||_e b^{\beta-1}.
\end{aligned}$$

Let's control the term $H_3(\tau)$.

$$\begin{aligned}
H_3(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \sum_{k=1}^{3}\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3}\left|\partial_{x^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e \int_{\mathbb{R}^3}(u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{x^k}f^\#(\tau)||_e b^{\beta-1}.
\end{aligned}$$

Let's control the term $H_4(\tau)$.

$$\begin{aligned}
H_4(\tau) &= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \sum_{k=1}^{3}\left|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&= \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(u^0)^5 \frac{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}}{e^{|v'|^2+|(x+\widetilde{b}(\tau,v',v))\times v'|^2}} f^\#(\tau, x+\widetilde{b}(\tau,v',v),v') \\
&\times \frac{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}}{e^{|u'|^2+|(x+\widetilde{b}(\tau,u',v))\times u'|^2}} \sum_{k=1}^{3}\left|\partial_{u'^k}(f^\#(\tau,x+\widetilde{b}(\tau,u',v),u'))\right| d\omega du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{v^k}f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3}\rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)(u^0)^5 \\
&\times e^{-(|u|^2+|v|^2)}e^{-D}d\omega du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{v^k}f^\#(\tau)||_e \int_{\mathbb{R}^3}(u^0)^5\vartheta_\phi g^{-\beta}e^{-|u|^2}du \\
&\lesssim \|f^\#(\tau)\|_e \sum_{k=1}^{3}||\partial_{v^k}f^\#(\tau)||_e b^{\beta-1}.
\end{aligned}$$

By (5.74), we sum up the terms like this

$$j_4(\tau) \lesssim b^{\beta-1}||f^\#(\tau)||_e \sum_{k=1}^{3}\left(||\partial_{x^k}f^\#(\tau)||_e + ||\partial_{v^k}f^\#(\tau)||_e\right).$$

Now we can state that

$$j_1(\tau)+j_2(\tau)+j_3(\tau)+j_4(\tau) \lesssim b^{\beta-1}||f^{\#}(\tau)||_e^2$$

$$+b^{\beta-1}||f^{\#}(\tau)||_e^2$$

$$+b^{\beta-1}||f^{\#}(\tau)||_e \sum_{k=1}^{2}\left(||\partial_{v^k} f^{\#}(\tau)||_e+||\partial_{x^k} f^{\#}(\tau)||_e\right)$$

$$+b^{\beta-1}\sum_{k=1}^{2}\left(||\partial_{v^k} f^{\#}(\tau)||_e+||\partial_{x^k} f^{\#}(\tau)||_e\right)||f^{\#}(\tau)||_e$$

$$\lesssim b^{\beta-1} \underset{\tau\in[0,t]}{Sup}\left[||f^{\#}(\tau)||_e\left(||f^{\#}(\tau)||_e+\sum_{k=1}^{3}(||\partial_{v^k} f^{\#}(\tau)||_e+||\partial_{x^k} f^{\#}(\tau)||_e)\right)\right].$$

Now we state that

$$\rho(x,v)\partial_{v^i} f^{\#}(t,x,v) \lesssim ||\partial_{v^i} f(0)||_e + \underset{\tau\in[0,t]}{Sup}(K(\tau))\left[\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau\right]$$

with

$$K(\tau)=||f^{\#}(\tau)||_e[||f^{\#}(\tau)||_e+\sum_{k=1}^{3}\left(||\partial_{v^k} f^{\#}(\tau)||_e+||\partial_{x^k} f^{\#}(\tau)||_e\right)].$$

Then

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup}\left[\rho(x,v)\partial_{v^i} f^{\#}(t,x,v)\right] \leq ||\partial_{v^i} f(0)||_e + C\underset{\tau\in[0,t]}{Sup}(K(\tau)).$$

Finally we conclude that

$$||\partial_{v^i} f^{\#}(t)||_e \leq ||\partial_{v^i} f(0)||_e + C\underset{\tau\in[0,t]}{Sup}(K(\tau)).$$

□

Lemma 5.19. Let $f^{\#}$ be a solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0. Then

$$||\partial_{x^i} f^{\#}(t)||_e \leq ||\partial_{x^i} f_0||_e + C\underset{t\in[0,t]}{sup}(||f^{\#}(\tau)||_e||\partial_{x^i} f^{\#}(\tau)||_e), \qquad for \qquad i=1,\,2,\,3 \tag{5.75}$$

where C does not depend on t.

Proof. The Boltzmann equation is written as follows

$$\begin{aligned} f^{\#}(t,x,v) &= f(0,x,v)+\int_0^t Q^{\#}(f,f)(\tau,x,v)d\tau \\ &= f(0,x,v)+\int_0^t Q(f,f)(\tau,X^{\tau}(x,v),v)d\tau. \end{aligned} \tag{5.76}$$

We take ∂_{x^i} to (5.76) and get

$$\partial_{x^i} f^{\#}(t,x,v)=\partial_{x^i} f(0,x,v)+\int_0^t \partial_{x^i} Q(f,f)(\tau,X^{\tau}(x,v),v)d\tau. \tag{5.77}$$

We multiply (5.77) by $\rho(x,v)$ to obtain

$$\rho(x,v)\partial_{x^i} f^{\#}(t,x,v)=\rho(x,v)\partial_{x^i} f(0,x,v)+\int_0^t \rho(x,v)\partial_{x^i} Q(f,f)(\tau,X^{\tau}(x,v),v)d\tau.$$

Then

$$\rho(x,v)|\partial_{x^i}f^\#(t,x,v)| \le ||\partial_{x^i}f(0)||_e + \int_0^t a^{-1}(\tau)b^{-2}(\tau)[K_1(\tau)+K_2(\tau)]d\tau \tag{5.78}$$

where

$$K_1(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{x^i}[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')]|d\omega du \tag{5.79}$$

and

$$K_2(\tau) = \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)|\partial_{x^i}[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)]|d\omega du. \tag{5.80}$$

Let's remark that

$$\begin{aligned}\partial_{x^i}\left[f(\tau,X^\tau(x,v),v')f(\tau,X^\tau(x,v),u')\right] &= \partial_{x^i}\left[f(\tau,X^\tau(x,v),v')\right]f(\tau,X^\tau(x,v),u') \\ &+ f(\tau,X^\tau(x,v),v')\partial_{x^i}\left[f(\tau,X^\tau(x,v),u')\right]\end{aligned}$$

and also that

$$\begin{aligned}\partial_{x^i}\left[f(\tau,X^\tau(x,v),v)f(\tau,X^\tau(x,v),u)\right] &= \partial_{x^i}\left[f(\tau,X^\tau(x,v),v)\right]f(\tau,X^\tau(x,v),u) \\ &+ f(\tau,X^\tau(x,v),v)\partial_{x^i}\left[f(\tau,X^\tau(x,v),u)\right].\end{aligned}$$

Let's control the terms $K_1(\tau)$ and $K_2(\tau)$.
For the expression (5.80) we have

$$K_2(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(|\partial_{x^i}[f(\tau,X^\tau(x,v),v)]|f(\tau,X^\tau(x,v),u)$$

$$+f(\tau,X^\tau(x,v),v)|\partial_{x^i}[f(\tau,X^\tau(x,v),u)]|)d\omega du$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)$$

$$\times\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)d\omega du$$

$$+||f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)$$

$$\times\frac{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}{e^{|u|^2+|(x+\widetilde{b}(\tau,u,v))\times u|^2}}|\partial_{x^i}[f^\#(\tau,x+\widetilde{b}(\tau,u,v),u)]|d\omega du$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e(\textstyle\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2}du)$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e b^{\beta-1}.$$

For the expression (5.79) we have

$$K_1(\tau) \lesssim \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi\sigma(g,\omega)(|\partial_{x^i}[f(\tau,X^\tau(x,v),v')]|f(\tau,X^\tau(x,v),u')$$

$$+f(\tau,X^\tau(x,v),v')|\partial_{x^i}[f(\tau,X^\tau(x,v),u')]|)d\omega du$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$+||f^\#(\tau)||_e||\partial_{x^i}f^\#(\tau)||_e \iint_{S^2\times\mathbb{R}^3} \rho(x,v)\vartheta_\phi g^{-\beta}\sigma_0(\omega)e^{-(|u|^2+|v|^2)}e^{-D}d\omega du$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e(\int_{\mathbb{R}^3}\vartheta_\phi g^{-\beta}e^{-|u|^2})du$$

$$\lesssim ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e b^{\beta-1}.$$

By (5.78) we state that

$$\begin{aligned}\rho(x,v)|\partial_{x^i}f^\#(t,x,v)| &\lesssim ||\partial_{x^i}f(0)||_e + ||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e[\int_0^t a^{-1}(\tau)b^{\beta-3}(\tau)d\tau] \\ &\lesssim ||\partial_{x^i}f(0)||_e + \underset{\tau\in[0,t]}{Sup}(||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e).\end{aligned}$$

Then

$$\underset{(x,v)\in\mathbb{T}^3\times\mathbb{R}^3}{Sup}\left(\rho(x,v)|\partial_{x^i}f^\#(t,x,v)|\right) \leq ||\partial_{x^i}f(0)||_e + C\underset{\tau\in[0,t]}{Sup}\left(||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e\right).$$

Finally we obtain

$$||\partial_{x^i}f^\#(t)||_e \leq ||\partial_{x^i}f(0)||_e + C\underset{\tau\in[0,t]}{Sup}\left(||\partial_{x^i}f^\#(\tau)||_e||f^\#(\tau)||_e\right).$$

□

5.4 Global L^∞-existence theorem

5.4.1 Global L^∞-existence theorem for Israel particles

Lemma 5.20. If $f^\#$ is a local-in-time solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0, then $f^\#$ is extended to a global-in-time solution, if initial data is given such that $|||f(0)|||_e$ is sufficiently small.

Proof. Using the energy estimates (5.23), (5.29) and (5.38), if f is a local-in-time solution of (5.1) with initial data f_0, on a (short) time interval we have

$$|||f^\#(t)|||_e \leq |||f(0)|||_e + \underset{\tau\in[0,t]}{sup}(|||f^\#(\tau)|||_e^2. \tag{5.81}$$

Since the norm $|||f|||_e$ contains all first order derivatives with respect to x and v variables, (5.81) allows us to bound all the derivatives of the local solution on each short time interval when the initial data is sufficiently small. In fact if $[0,T]$ is the maximal interval of the local solution, by (5.81) we have

$$\underset{\tau\in[0,T]}{Sup}\,|||f^{\#}|||_e \leq |||f(0)|||_e + C\,\underset{\tau\in[0,T]}{Sup}\,|||f^{\#}|||_e^2. \tag{5.82}$$

The relation (5.82) occurs if $1-4C|||f(0)|||_e \geq 0$, that is with initial data which enjoy the littleness condition $|||f(0)|||_e \leq \frac{1}{4C}$. This proves that the solution is extended to a global-in-time solution, if initial data is given such that $|||f(0)|||_e$ is sufficiently small. □

Theorem 5.1. Consider a Bianchi type 1 space-time where the metric tensor is such that $a = a(t)\ and\ b = b(t)$ are given and satisfy (5.2) and (5.3). Let $f_0 = f(0,x,v)$ be the initial data of the Boltzmann equation (5.1), that is differentiable. Suppose that the scatteing kernel σ satisfies (1.69). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e < \frac{M_0}{2}$, there exists a unique global (in time) classical solution to the Boltzmann equation (5.1). Moreover

$$\underset{t\in[0,\infty[}{sup}\,(|||f^{\#}(\tau)|||_e \leq M_0. \tag{5.83}$$

Proof. Due to the Lemma 5.20, it suffices to prove a unique local existence theorem. The rest of the proof will be divided into two steps.

First step: Local existence theorem.

Let f_0 be the initial data for the Boltzmann equation (5.1). We recall the sequence $(f_n^{\#})_{n\geq 0}$ defined by

$$\partial_t f^{\#} = Q_{gain}(f_n^{\#}, f_n^{\#}) - Q_{loss}(f_{n+1}^{\#}, f_n^{\#}), \tag{5.84}$$

$$f_{n+1}^{\#}(0,x,v) = f(0,x,v), \tag{5.85}$$

$$f_0^{\#}(0,x,v) = f(0,x,v). \tag{5.86}$$

We note that for a given $f_n^{\#}$, (5.84) is a linear differential equation with $f_{n+1}^{\#}$ as unknown and f_0 as initial data. It is standard for the linear theory on the partial differential equation that (5.84) with initial data f_0 has an unique solution. So the sequence $(f_n^{\#})_{n\geq 0}$ is well defined.

Our main goal is to get an uniform in n estimate for $|||f_n^{\#}|||_e$. More precisely, we look for some small M_0 such that

$$\forall n \in \mathbb{N}, \qquad |||f_n^{\#}(t)|||_e \leq M_0 \tag{5.87}$$

on the local-in-time interval.

We are going to do it by induction.

We multiply (5.84) by $\rho(x,v)$ and integrate from 0 to t to obtain

$$\rho(x,v)f_{n+1}^{\#} = \rho(x,v)f_0 + \int_0^t \rho(x,v)Q_{gain}^{\#}(f_n,f_n)(\tau,x,v)d\tau$$
$$- \int_0^t \rho(x,v)Q_{loss}^{\#}(f_{n+1},f_n)(\tau,x,v)d\tau. \tag{5.88}$$

The same argument as in Lemma 5.11 allows us to obtain

$$\|f^{\#}_{n+1}\|_e \leq \|f_0(t)\|_e + C \underset{\tau\in[0,t]}{Sup}(|||f^{\#}_{n+1}(\tau)|||_e|||f^{\#}_n(\tau)|||_e + |||f^{\#}_n|||^2_e). \tag{5.89}$$

Next, we proceed to the estimate of the derivatives of $f^{\#}_{n+1}$ with respect to the momenta variables. Let $i \in 1, 2, 3$. We take ∂_{v^i}-derivatives to (5.84) and multiply it by $\rho(x, v)$. To the obtained equation, we integrate over $[0, t]$ to have

$$\begin{aligned}\rho(x, v)\partial_{v^i} f^{\#}_{n+1}(t, x, v) = \rho(x, v)(\partial_{v^i} f_0)(x, v) + \int_0^t \rho(x, v)\partial_{v^i} Q^{\#}_{gain}(f_n, f_n)(\tau, x, v)d\tau \\ - \int_0^t \rho(x, v)\partial_{v^i} Q^{\#}_{loss}(f_{n+1}, f_n)(\tau, x, v)d\tau.\end{aligned} \tag{5.90}$$

Following the proof of Lemma 5.12, we obtain the following estimate

$$\|\partial_{v^i} f^{\#}_{n+1}(t)\|_e \leq \|\partial_{v^i} f_0\|_e + C \underset{\tau\in[0,\tau]}{Sup}\left(|||f^{\#}_{n+1}|||_e|||f^{\#}_n|||_e + |||f^{\#}_n|||^2_e\right). \tag{5.91}$$

Next, we proceed to the estimate of the derivatives of $f^{\#}_{n+1}$ with respect to the x -variables. Let $i \in \{1, 2, 3\}$. We take ∂_{x^i} to (5.84) and we multiply by $\rho(x, v)$. To the obtained equation, we take integration on $[0, t]$ to have

$$\begin{aligned}\rho(x, v)\partial_{x^i} f^{\#}_{n+1}(t, x, v) = \rho(x, v)\partial_{x^i} f_0(x, v) + \int_0^t \rho(x, v)\partial_{x^i} Q^{\#}_{gain}(f_n, f_n)(\tau, x, v)d\tau \\ - \int_0^t \rho(x, v)\partial_{x^i} Q^{\#}_{loss}(f_{n+1}, f_n)(\tau, x, v)d\tau.\end{aligned} \tag{5.92}$$

Following the proof of Lemma5.13, we obtain the following estimate

$$\|\partial_{x^i} f^{\#}_{n+1}\|_e \leq \|\partial_{x^i} f_0\|_e + C \underset{\tau\in[0,t]}{Sup}\left(|||f^{\#}_{n+1}|||_e|||f^{\#}_n|||_e + |||f^{\#}_n|||^2_e\right). \tag{5.93}$$

Summing up (5.89), (5.91) and (5.93) we obtain

$$|||f^{\#}_{n+1}|||_e \leq |||f_0|||_e + C \underset{\tau\in[0,t]}{Sup}\left(|||f^{\#}_{n+1}(\tau)|||_e|||f^{\#}_n(\tau)|||_e + |||f^{\#}_n(\tau)|||^2_e\right). \tag{5.94}$$

Suppose now that there exists a positive M_0 such that $\|f_0\|_e \leq \frac{M_0}{2}$ and $\|f^{\#}_n\|_e \leq M_0$ on the local-in-time interval $[0, T]$, then we obtain the desired result, that is $|||f^{\#}_{n+1}(t)|||_e \leq M_0$ for $t \in [0, T]$, provided M_0 sufficiently small; for example with M_0 such that $M_0 \leq \frac{1}{4C}$
Finally, taking limit in (5.84) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e \leq M_0$ on the local-in-time interval $[0, T]$. Lemma 5.20 proves that if $|||f_0|||_e$ is sufficiently small, then the solution exists globally in time.

Second step: Uniqueness.

We now prove the uniqueness of the solution. We assume that there is another solution h to (5.1) with the same initial data f_0 such that $\underset{t\in[0,\infty[}{Sup} |||h^{\#}|||_e \leq M_0$. The difference $f - h$ satisfies

$$\partial_t(f^{\#} - h^{\#}) = Q^{\#}(f - h, f) + Q^{\#}(h, f - h). \tag{5.95}$$

We proceed as in the proof of the energy estimate. Since $f(0,x,v) = h(0,x,v)$

$$
\begin{aligned}
|||f^\#(t) - h^\#(t)|||_e &\leq C \underset{\tau\in[0,\infty[}{Sup} \left(|||f^\#(\tau)|||_e + |||h^\#(\tau)|||_e\right) |||f^\#(\tau) - h^\#(\tau)|||_e \\
&\leq 2CM_0 \underset{\tau\in[0,\infty[}{Sup} |||f^\#(\tau) - h^\#(\tau)|||_e. \qquad (5.96)
\end{aligned}
$$

Since $M_0 \leq \frac{1}{4C}$, taking the supremum in (5.96) on the interval $[0,\infty[$, we obtain

$$
\underset{t\in[0,\infty[}{Sup} |||f^\#(t) - h^\#(t)|||_e \leq \frac{1}{2} \underset{\tau\in[0,\infty[}{Sup} |||f^\#(\tau) - h^\#(\tau)|||_e.
$$

So $f^\# = h^\#$ on $\mathbb{R}_+$.

□

5.4.2 Global L^∞-existence theorem for hard potentials

Lemma 5.21. If $f^\#$ is a local-in-time solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0, then $f^\#$ is extended to a global-in-time solution,if initial data is given such that $|||f(0)|||_e$ is sufficiently small.

Proof. The proof is similar to that of Lemma 5.20. □

Theorem 5.2. Consider a Bianchi type 1 space-time where the metric tensor is such that $a = a(t)$ *and* $b = b(t)$ are given and satisfy (5.2), (5.3) and (5.42). Let $f_0 = f(0,x,v)$ be the initial data of the Boltzmann equation (5.1), that is differentiable. Suppose that the scattering kernel σ satisfies (1.70) with $\alpha = 0$ and (2.50). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e < \frac{M_0}{2}$, there exists a unique global (in time) classical solution to the Boltzmann equation (5.1). Moreover

$$
\underset{t\in[0,\infty[}{sup} (|||f^\#(\tau)|||_e \leq M_0. \qquad (5.97)
$$

Proof. Due to the Lemma 5.21, it suffices to prove a unique local existence theorem. The rest of the proof will be divided into two steps.

First step: Local existence theorem.

Let f_0 be the initial data for the Boltzmann equation (5.1). We recall the sequence $(f_n^\#)_{n\geq 0}$ defined by (5.84), (5.85) and (5.86).

Our main goal is to get an uniform in n estimate for $|||f_n^\#(t)|||_e$.

Precisely, we look for some small M_0 such that

$$
\forall n \in \mathbb{N}, \qquad |||f_n^\#(t)|||_e \leq M_0 \qquad (5.98)
$$

on the local-in-time interval.

We are going to do it by induction.

We multiply (5.84) by $\rho(x,v)$ and integrate from 0 to t to obtain

$$\rho(x,v)f^{\#}_{n+1} = \rho(x,v)f_0 + \int_0^t \rho(x,v)Q^{\#}_{gain}(f_n,f_n)(\tau,x,v)d\tau$$
$$- \int_0^t \rho(x,v)Q^{\#}_{loss}(f_{n+1},f_n)(\tau,x,v)d\tau. \quad (5.99)$$

The same argument as in Lemma 5.14 allows us to obtain

$$\|f^{\#}_{n+1}\|_e \leq \|f_0(t)\|_e + C \underset{\tau\in[0,t]}{Sup} (|||f^{\#}_{n+1}(\tau)|||_e|||f^{\#}_n(\tau)|||_e + |||f^{\#}_n|||^2_e). \quad (5.100)$$

Next, we proceed to the estimate of the derivatives of $f^{\#}_{n+1}$ with respect to the momenta variables. Let $i \in 1,2,3$. We take ∂_{v^i}-derivatives to (5.84) and multiply it by $\rho(x,v)$. To the obtained equation, we integrate over $[0,t]$ to have

$$\rho(x,v)\partial_{v^i}f^{\#}_{n+1}(t,x,v) = \rho(x,v)(\partial_{v^i}f_0)(x,v) + \int_0^t \rho(x,v)\partial_{v^i}Q^{\#}_{gain}(f_n,f_n)(\tau,x,v)d\tau$$
$$- \int_0^t \rho(x,v)\partial_{v^i}Q^{\#}_{loss}(f_{n+1},f_n)(\tau,x,v)d\tau. \quad (5.101)$$

Following the proof of Lemma 5.15, we obtain the following estimate

$$\|\partial_{v^i}f^{\#}_{n+1}(t)\|_e \leq \|\partial_{v^i}f_0\|_e + C \underset{\tau\in[0,\tau]}{Sup} \left(|||f^{\#}_{n+1}|||_e|||f^{\#}_n|||_e + |||f^{\#}_n|||^2_e\right). \quad (5.102)$$

Next, we proceed to the estimate of the derivatives of $f^{\#}_{n+1}$ with respect to the x -variables. Let $i \in \{1,2,3\}$. We take ∂_{x^i} to (5.84) and we multiply by $\rho(x,v)$. To the obtained equation, we take integration on $[0,t]$ to have

$$\rho(x,v)\partial_{x^i}f^{\#}_{n+1}(t,x,v) = \rho(x,v)\partial_{x^i}f_0(x,v) + \int_0^t \rho(x,v)\partial_{x^i}Q^{\#}_{gain}(f_n,f_n)(\tau,x,v)d\tau$$
$$- \int_0^t \rho(x,v)\partial_{x^i}Q^{\#}_{loss}(f_{n+1},f_n)(\tau,x,v)d\tau. \quad (5.103)$$

Following the proof of Lemma5.16, we obtain the following estimate

$$\|\partial_{x^i}f^{\#}_{n+1}\|_e \leq \|\partial_{x^i}f_0\|_e + C \underset{\tau\in[0,t]}{Sup} \left(|||f^{\#}_{n+1}|||_e|||f^{\#}_n|||_e + |||f^{\#}_n|||^2_e\right). \quad (5.104)$$

Summing up (5.100), (5.102) and (5.104) we obtain

$$|||f^{\#}_{n+1}|||_e \leq |||f_0|||_e + C \underset{\tau\in[0,t]}{Sup} \left(|||f^{\#}_{n+1}(\tau)|||_e|||f^{\#}_n(\tau)|||_e + |||f^{\#}_n(\tau)|||^2_e\right). \quad (5.105)$$

Suppose now that there exists a positive M_0 such that $\|f_0\|_e \leq \frac{M_0}{2}$ and $\|f^{\#}_n\|_e \leq M_0$ on the local-in-time interval $[0,T]$, then we obtain the desired result, that is $|||f^{\#}_{n+1}(t)|||_e \leq M_0$ for $t \in [0,T]$, provided M_0 sufficiently small; for example with M_0 such that $M_0 \leq \frac{1}{4C}$

Finally, taking limit in (5.84) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e \leq M_0$ on the local-in time-interval $[0,T]$. Lemma 5.21 proves that if $|||f_0|||_e$ is sufficiently small, then the solution exists globally in time.

Second step: Uniqueness.

We now prove the uniqueness of the solution. We assume that there is another solution h to (5.1) with the same initial data f_0 such that $\underset{t\in[0,\infty[}{Sup}\, |||h^{\#}|||_e \leq M_0$. The difference $f-h$ satisfies

$$\partial_t(f^{\#}-h^{\#}) = Q^{\#}(f-h,f) + Q^{\#}(h,f-h). \tag{5.106}$$

We proceed as in the proof of the energy estimate. Since $f(0,x,v) = h(0,x,v)$

$$\begin{aligned} |||f^{\#}(t)-h^{\#}(t)|||_e &\leq C \underset{\tau\in[0,\infty[}{Sup} \left(|||f^{\#}(\tau)|||_e + |||h^{\#}(\tau)|||_e\right) |||f^{\#}(\tau)-h^{\#}(\tau)|||_e \\ &\leq 2CM_0 \underset{\tau\in[0,\infty[}{Sup} |||f^{\#}(\tau)-h^{\#}(\tau)|||_e. \end{aligned} \tag{5.107}$$

Since $M_0 \leq \frac{1}{4C}$, taking the supremum in (5.107) on the interval $[0,\infty[$, we obtain

$$\underset{t\in[0,\infty[}{Sup} |||f^{\#}(t)-h^{\#}(t)|||_e \leq \frac{1}{2} \underset{\tau\in[0,\infty[}{Sup} |||f^{\#}(\tau)-h^{\#}(\tau)|||_e.$$

So $f^{\#} = h^{\#}$ on $\mathbb{R}_+$.

□

5.4.3 Global L^∞-existence theorem for soft potentials

Lemma 5.22. If $f^{\#}$ is a local-in-time solution of the inhomogeneous relativistic Boltzmann equation (5.1) with initial data f_0, then $f^{\#}$ is extended to a global-in-time solution,if initial data is given such that $|||f(0)|||_e$ is sufficiently small.

Proof. The proof is similar to that of Lemma 5.20. □

Theorem 5.3. Consider a Bianchi type 1 space-time where the metric tensor is such that $a = a(t)$ *and* $b = b(t)$ are given and satisfy (5.2), (5.3) and (5.42). Let $f_0 = f(0,x,v)$ be the initial data of the Boltzmann equation (5.1), that is differentiable. Suppose that the scattering kernel σ satisfies (1.71)-(2.50). Then there exists $M_0 > 0$ such that if $|||f(0)|||_e < \frac{M_0}{2}$, there exists a unique global (in time) classical solution to the Boltzmann equation (5.1). Moreover

$$\underset{t\in[0,\infty[}{sup} (|||f^{\#}(\tau)|||_e \leq M_0. \tag{5.108}$$

Proof. Due to the Lemma 5.22, it suffices to prove a unique local existence theorem. The rest of the proof will be divided into two steps.

First step: Local existence theorem.

Let f_0 be the initial data for the Boltzmann equation (5.1). We recall the sequence $(f_n^{\#})_{n\geq 0}$ defined by (5.84), (5.85) and (5.86).

Our main goal is to get an uniform in n estimate for $|||f_n^{\#}(t)|||_e$.
Precisely, we look for some small M_0 such that

$$\forall n \in \mathbb{N}, \qquad |||f_n^{\#}(t)|||_e \leq M_0 \tag{5.109}$$

on the local-in-time interval.
We are going to do it by induction.
We multiply (5.84) by $\rho(x, v)$ and integrate from 0 to t to obtain

$$\begin{aligned}\rho(x,v)f_{n+1}^{\#} = \rho(x,v)f_0 + \int_0^t \rho(x,v)Q_{gain}^{\#}(f_n, f_n)(\tau, x, v)d\tau \\ - \int_0^t \rho(x,v)Q_{loss}^{\#}(f_{n+1}, f_n)(\tau, x, v)d\tau.\end{aligned} \tag{5.110}$$

The same argument as in Lemma 5.17 allows us to obtain

$$\|f_{n+1}^{\#}\|_e \leq \|f_0(t)\|_e + C \underset{\tau\in[0,t]}{Sup} (|||f_{n+1}^{\#}(\tau)|||_e |||f_n^{\#}(\tau)|||_e + |||f_n^{\#}|||_e^2). \tag{5.111}$$

Next, we proceed to the estimate of the derivatives of $f_{n+1}^{\#}$ with respect to the momenta variables. Let $i \in 1, 2, 3$. We take ∂_{v^i}-derivatives to (5.84) and multiply it by $\rho(x, v)$. To the obtained equation, we integrate over $[0, t]$ to have

$$\begin{aligned}\rho(x,v)\partial_{v^i} f_{n+1}^{\#}(t,x,v) = \rho(x,v)(\partial_{v^i} f_0)(x,v) + \int_0^t \rho(x,v)\partial_{v^i} Q_{gain}^{\#}(f_n, f_n)(\tau, x, v)d\tau \\ - \int_0^t \rho(x,v)\partial_{v^i} Q_{loss}^{\#}(f_{n+1}, f_n)(\tau, x, v)d\tau.\end{aligned} \tag{5.112}$$

Following the proof of Lemma 5.18, we obtain the following estimate

$$\|\partial_{v^i} f_{n+1}^{\#}(t)\|_e \leq \|\partial_{v^i} f_0\|_e + C \underset{\tau\in[0,\tau]}{Sup} \left(|||f_{n+1}^{\#}|||_e |||f_n^{\#}|||_e + |||f_n^{\#}|||_e^2\right). \tag{5.113}$$

Next, we proceed to the estimate of the derivatives of $f_{n+1}^{\#}$ with respect to the x -variables. Let $i \in \{1, 2, 3\}$. We take ∂_{x^i} to (5.84) and we multiply by $\rho(x, v)$. To the obtained equation, we take integration on $[0, t]$ to have

$$\begin{aligned}\rho(x,v)\partial_{x^i} f_{n+1}^{\#}(t,x,v) = \rho(x,v)\partial_{x^i} f_0(x,v) + \int_0^t \rho(x,v)\partial_{x^i} Q_{gain}^{\#}(f_n, f_n)(\tau, x, v)d\tau \\ - \int_0^t \rho(x,v)\partial_{x^i} Q_{loss}^{\#}(f_{n+1}, f_n)(\tau, x, v)d\tau.\end{aligned} \tag{5.114}$$

Following the proof of Lemma5.19, we obtain the following estimate

$$\|\partial_{x^i} f_{n+1}^{\#}\|_e \leq \|\partial_{x^i} f_0\|_e + C \underset{\tau\in[0,t]}{Sup} \left(|||f_{n+1}^{\#}|||_e |||f_n^{\#}|||_e + |||f_n^{\#}|||_e^2\right). \tag{5.115}$$

Summing up (5.111), (5.113) and (5.115) we obtain

$$|||f_{n+1}^{\#}|||_e \leq |||f_0|||_e + C \underset{\tau\in[0,t]}{Sup} \left(|||f_{n+1}^{\#}(\tau)|||_e |||f_n^{\#}(\tau)|||_e + |||f_n^{\#}(\tau)|||_e^2\right). \tag{5.116}$$

Suppose now that there exists a positive M_0 such that $\|f_0\|_e \leq \frac{M_0}{2}$ and $\|f_n^\#\|_e \leq M_0$ on the local-in-time interval $[0,T]$, then we obtain the desired result, that is $|||f_{n+1}^\#(t)|||_e \leq M_0$ for $t \in [0,T]$, provided M_0 sufficiently small; for example with M_0 such that $M_0 \leq \frac{1}{4C}$

Finally, taking limit in (5.84) as n goes to infinity, we have a local-in-time solution such that $|||f(t)|||_e \leq M_0$ on the local-in-time interval $[0,T]$. Lemma 5.22 proves that if $|||f_0|||_e$ is sufficiently small, then the solution exists globally in time.

Second step: Uniqueness.

We now prove the uniqueness of the solution. We assume that there is another solution h to (5.1) with the same initial data f_0 such that $\underset{t\in[0,\infty[}{Sup} |||h^\#|||_e \leq M_0$. The difference $f-h$ satisfies

$$\partial_t(f^\# - h^\#) = Q^\#(f-h, f) + Q^\#(h, f-h). \tag{5.117}$$

We proceed as in the proof of the energy estimate. Since $f(0,x,v) = h(0,x,v)$

$$\begin{aligned} |||f^\#(t) - h^\#(t)|||_e &\leq C \underset{\tau\in[0,\infty[}{Sup} \left(|||f^\#(\tau)|||_e + |||h^\#(\tau)|||_e\right) |||f^\#(\tau) - h^\#(\tau)|||_e \\ &\leq 2CM_0 \underset{\tau\in[0,\infty[}{Sup} |||f^\#(\tau) - h^\#(\tau)|||_e. \end{aligned} \tag{5.118}$$

Since $M_0 \leq \frac{1}{4C}$, taking the supremum in (5.118) on the interval $[0,\infty[$, we obtain

$$\underset{t\in[0,\infty[}{Sup} |||f^\#(t) - h^\#(t)|||_e \leq \frac{1}{2} \underset{\tau\in[0,\infty[}{Sup} |||f^\#(\tau) - h^\#(\tau)|||_e.$$

So $f^\# = h^\#$ on $\mathbb{R}_+$.

□

Conclusion and Outlooks

We have studied the inhomogeneous relativistic Boltzmann equation in the spatially Bianchi type 1 space-time. We prove the global (with respect to the direction of time corresponding to the expansion of the universe) existence of classical solutions for small initial data in a suitable weighted space for some collisional kernels which fall separately the class of hard potentials, soft potentials or generated by the so-called Israel particles. Such kernels are closer to those which naturally arise in physical problems. Our result extends existing results such as the one of [35, 36] for the Minkowsky space-time, that of [25, 26] for the spatially homogeneous case in the Robertson-Walker space-time and also that of [42, 40, 41] for spatially inhomogeneous case in the Robertson-Walker space-time.

In this thesis, we discussed the existence and uniqueness of both the mild and classical solutions to relativistic Boltzmann equation with near vacuum initial data on a Bianchi type 1 space-time respectively for some hard potentials, soft potentials and potentials generated by Israel particles. One of the novelty here is the used of several parameterizations of post-collisional momenta. We follow the approach of [35, 40] and provide estimates for the loss and gain terms from which we derived our main result.

We used several methods to obtain our results. For the L^∞-existence theorem for classical solutions for the homogeneous equation, we use the fixed point theorem. The same method is used to prove the existence theorem for the mild solutions for the inhomogeneous equation. For the L^2-existence theorem for classical solutions for the homogeneous equation and for L^∞-existence theorem for classical solution for the inhomogeneous equation, we first proved energy estimates and then construct a suitable sequence which converges to the solution.

Further, it is a worthwhile problem to understand how the structure of the universe affects the asymptotic behavior of solutions. The main results of this paper allow us to claim that in the case where the space-time is for Bianchi type 1, when initial data are small in a suitable weighted framework, so does the global solution. As physical interpretation, this universe structure does not affect the asymptotic behavior of the solutions.

In our main results, we have obtained global existence of both mild solutions and classical solutions to the Boltzmann equation in transformed variable (t, x, v). A notable remark is that if f_0 is small, this implies f is small. So the solution and the initial data have the same size. We may compare this result with the Vlasov equation, which is obtained by simply ignoring the right hand side of the

Boltzmann equation; i.e. $L_X f = 0$ with the solution $f(t,x,v) = f_0(x,v)$. It is usual in the Vlasov case to assume that initial data has a compact support in impulsion variables; i.e $f_0 = 0$ for a large v. The smallness of the solutions allows to interpret that it converges in certain sense to the solution of the Vlasov equation for large v.

In this thesis, for the spatially homogeneous equation, we have proved global classical results in both the L^∞ and L^2 weighted framework. But for the spatially inhomogeneous situation, we have just obtained global classical results in a L^∞ framework. One of our next challenge will be to obtain such a result in a L^2-weighted framework.

As usual in the context of the Boltzmann equation, after establishing a global result, one of the main problem is to prove the non-negativity of the solution. We think that by using the arguments of M. Shinbrot we could resolve the problem.

The Boltzmann equation is usually coupled to the Einstein equations through the energy-momentum tensor, and the energy-momentum tensor of the Boltzmann equation has the same form with that of the Vlasov equation. Since existence results is known in the case of Einstein-Vlasov equation, one interesting open question is to know whether the result of this thesis can be extended to the Einstein-Boltzmann system when the distribution function is no longer spatially homogeneous. Another challenge could be the study of the properties of solutions.

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THE RELATIVISTIC BOLTZMANN EQUATION ON BIANCHI TYPE I SPACE TIME FOR HARD POTENTIALS

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In this paper, we consider the Cauchy problem for the spatially homogeneous relativistic Boltzmann equation with small initial data. The collision kernel considered here is for a hard potentials case. The background space-time in which the study is done is the Bianchi type I space-time. Under certain conditions made on the scattering kernel and on the metric, a uniqueness global (in time) solution is obtained in a suitable weighted functional space.

Keywords: relativistic Boltzmann equation, Bianchi type I space-time, hard potentials scattering kernel.

AMS subject classifications: 76P05, 35Q20.

1. Introduction

The expression “Boltzmann equation” is used in a more general sense and refers to any kinetic equation that describes the change of a macroscopic quantity in a thermodynamic system, such as energy, charge or particle number. The equation arises not by statistical analysis of all the individual positions and momenta of each particle in the fluid; rather by considering the probability that a number of particles occupy a very small region of space centered at the tip of the position vector, and have very nearly equal small changes in momenta from a momentum vector, at an instant of time. The Boltzmann equation can be used to determine how physical quantities, such as heat energy and momentum, change when a fluid is in transport.

Other characteristic properties to fluids such as viscosity, thermal conductivity, etc. can be derived.

Due to its importance in the kinetic theory, several authors have studied and proved local and global in time existence theorems for the Boltzmann equation, in both the nonrelativistic case, that considers particles with low velocities, and the full-relativistic case, which includes the case of fast moving particles with arbitrarily high velocities, such as, particles of ionized gas in some media at a very high temperature like: burning reactors, solar winds, nebular galaxies.

In the nonrelativistic case, the first original global result is due to T. Carleman in [4]; R. J. Diperna and P. L. Lions proved global existence and weak stability in [8]. R. Illner and M. Shinbrot proved a global result in [15], in the case of small initial data and without symmetry assumption; an analogous result is unknown in the full-relativistic case. For more details for the nonrelativistic Boltzmann equation we refer to [4, 15, 8] and references therein.

In the full-relativistic case, let $\Gamma^{\gamma}_{\mu\nu}$ denote the Christoffel symbols of the metric tensor ds^2 and $\tilde{Q}(f, f)$ denote the collisional operator; if we adopt the Einstein summation convention $a_\alpha b^\alpha = \sum a_\alpha b^\alpha$, the Boltzmann equation reads

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = \tilde{Q}(f, f). \tag{1.1}$$

Several authors proved local existence theorems, considering this equation alone, e.g. K. Bichteler in [3], D. Bancel in [1], or coupling it to other fields equations, e.g. D. Bancel and Y. Choquet-Bruhat, in [2]. The work [2] was done under an assumption of "$\mu - N$ regularity" on the collision operator (Section II [2]). With Minkowski space-time as background, R. T. Glassey and W. Strauss obtained a global result in [13], in the case of data near to that of an equilibrium solution with nonzero density. With Bianchi type I space-time as background and under assumption close to $\mu - N$ regularity, N. Noutchegueme, E. Takou and D. Dongo proved in [20] the existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data.

Unfortunately, the assumption of $\mu - N$ regularity on scattering kernels used in [20] is not physically well motivated. In fact, this does not allow a good interpretation of the type of collisions between particles. The scattering kernel is a quantity that determines the nature of collisions between particles, and in the nonrelativistic case, several different types of scattering kernel have been found to be of interest. For instance, the inverse power law gives the best known types of scattering kernel, and they are further classified into hard and soft potentials cases. In the relativistic setting, it is not very clear which types of the scattering kernel should be of interest, but a classification of (special) relativistic (hard and soft potentials) has been proposed in [9] by applying arguments similar to those used in the nonrelativistic case. This classification was recently reformulated to the full-relativistic case by R. Strain in [22]. As in the nonrelativistic case, the scattering kernels depend only on the relative momentum and scattering angle of two colliding particles. This will be specified in Section 2.

With the scattering kernel formulated as in [9, 22], H. Lee proved in [16] a global existence of solution to the relativistic Boltzmann equation in the Robertson–Walker space-time (FRW) with near vacuum initial data. Unlike FRW space-time which has the same scale factor for each of the three spatial directions, Bianchi type I space-time has a different scale factors in each direction, thereby introducing an anisotropy to the system. It is natural to try to see what happens in the relativistic Boltzmann equation when this metric is taken into account.

The purpose of this paper is to obtain analogous result of [16] in the Bianchi type I space-time in which the metric defined by (2.1) generalizes that of Robertson–Walker. One of the most important point to note here is the form of parametrization of the post-collisional momenta. The presence of the second factor in the metric imposes another formulations and proofs of several estimates used in [16].

The rest of the paper is organized as follows: In Section 2, we give a brief exposition of collision operator, we write the Boltzmann equation in the Bianchi type I space-time and we specify the kinds of parametrizations of post-collisional momenta used in this paper. We end this section by stating the main assumptions of the paper. In Section 3, we collect some preliminary results which will allow us to prove the existence and uniqueness theorem. In Section 4, we define the function space and we give some estimates of the derivatives of terms allowing to define the collision operator. The rest of Section 4 is devoted to the formulation and the proof of our main result in an appropriate functional framework.

2. The equation and main assumptions

2.1. Notations

Greek indices vary from 0 to 3 and Latin indices from 1 to 3; we adopt the Einstein summation convention $a_\alpha b^\alpha = \sum a_\alpha b^\alpha$. We consider as space-time, a Bianchi type I space-time denoted $(\mathbb{R}^4, ds^2)$, where for $x^\alpha = (x^0, x^i)$, $x^0 = t$ is the time and $x = (x^i)$ the space; ds^2 stands for the metric tensor with signature $(-, +, +, +)$ that can be written as

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)(dy^2 + dz^2). \tag{2.1}$$

In (2.1), $a = a(t) > 0$ and $b = b(t) > 0$ are given nonnegative regular, real-valued functions for which we will require certain conditions. The determinant of the metric tensor ds^2 is equal to a^2b^4.

In this work, we consider the collisional evolution of a kind of uncharged particles in the time-oriented curved space-time $(\mathbb{R}^4, ds^2)$. An essential tool to describe the dynamic of such particles is their distribution function that we denote by f, and that is a nonnegative real-valued function of both the position x^α, the 4-momentum $p^\alpha = (p^0, p) = (p^0, p^1, p^2, p^3)$ of the particles. More precisely, we have

$$f : T(\mathbb{R}^4) \cong \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}_+, \qquad (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha). \tag{2.2}$$

In this paper, we consider the usual inner product of $\mathbb{R}^3$ with the associated norm; i.e. for $p, q \in \mathbb{R}^3$, we let $p.q = p^1q^1 + p^2q^2 + p^3q^3$ and $|p| = \sqrt{p.p}$.

For any three-vector (d^1, d^2, d^3), due to the form of the metric and certain conveniences, we will sometimes let $\bar{d} = (d^2, d^3)$.

In this work, we consider massive particles with the same rest mass that can be rescaled to $m = 1$. The particles are then required to move on the future sheet of the mass-shell whose equation is $-(p^0)^2 + a^2(p^1)^2 + b^2|\bar{p}|^2 = -1$, or equivalently

$$p^0 = \sqrt{1 + a^2(p^1)^2 + b^2|\bar{p}|^2}. \tag{2.3}$$

We consider the homogeneous case for which f depends only on the time t and the impulsion p.

2.2. The collision operator

In the instantaneous, binary and elastic scheme due to A. Lichnerowicz [19], we consider that at a given point (t, x), only two particles collide instantaneously without destroying each other. The collision affects only the momenta of the two particles that change after the collision; only the sum of the two momenta is preserved.

Let us suppose p^α and q^α stand for the momenta of the two particles before their collision, p'^α and q'^α stand for their momenta after the collision. By the energy-momentum conservation principle, we have

$$p^\alpha + q^\alpha = p'^\alpha + q'^\alpha. \tag{2.4}$$

The expressions of p'^α and q'^α as functions of p^α and q^α will be specified soon. In such case, the collision operator Q that acts only on the momentum variable, is defined as follows: regardless for the time t, and where f and h are two functions on $\mathbb{R} \times \mathbb{R}^3$,

$$Q(f, h) = Q_{\mathrm{g}}(f, h) - Q_{\mathrm{l}}(f, h). \tag{2.5}$$

$Q_{\mathrm{g}}(f, h)$ and $Q_{\mathrm{l}}(f, h)$ represent respectively the gain term and the lost term. Taking into account the fact that the space-time is defined by (2.1), $Q_{\mathrm{g}}(f, h)$ and $Q_{\mathrm{l}}(f, h)$ are given by the following relations:

$$Q_{\mathrm{g}}(f, h)(t, p) = ab^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{g\sqrt{s}}{p^0 q^0} \sigma(g, \omega) f(p') h(q') d\omega dq, \tag{2.6}$$

$$Q_{\mathrm{l}}(f, h)(t, p) = ab^2 \int_{\mathbb{R}^3} \int_{S^2} \frac{g\sqrt{s}}{p^0 q^0} \sigma(g, \omega) f(p) h(q) d\omega dq. \tag{2.7}$$

In (2.6) and (2.7):

- $f(p)$, $f(q)$, $f(p')$ and $f(q')$ represent respectively abbreviations of $f(t, p)$, $f(t, q)$, $f(t, p')$ and $f(t, q')$;
- $\sigma(g, \omega)$ is called the scattering kernel. It measures interactions between particles and determines their natures;

– the quantities g and s are respectively called the relative momentum and energy in the center of momentum system. They are defined by

$$s = s(p^\alpha, q^\alpha) = -(p^\alpha + q^\alpha)(p_\alpha + q_\alpha), \tag{2.8}$$

$$g = g(p^\alpha, q^\alpha) = \sqrt{(p^\alpha - q^\alpha)(p_\alpha - q_\alpha)}. \tag{2.9}$$

– the quantity

$$v_\phi = \frac{g\sqrt{s}}{p^0 q^0}$$

is called the Mϕller velocity.

Now, we are going to introduce a change of variables so that the Boltzmann equation in the Bianchi type I space-time is written in a simple form. In our context (where we consider the Bianchi type I space-time), the Boltzmann equation is written in a simple form if we use covariant variables. So, the distribution function f will be considered as a function of t and $v = (v^1, v^2, v^3) = (v^1, \bar{v})$ where

$$v^1 = g_{1i} p^i = a^2 p^1, \qquad v^2 = g_{2i} p^i = b^2 p^2, \qquad v^3 = g_{3i} p^i = b^2 p^3. \tag{2.10}$$

It is easy to see that $dv = a^2 b^4 dp$. Let us observe that if we set $v^0 := \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2}$, then $v^0 = p^0$.

Using these new variables and setting $\tilde{f}(t, v) = f(t, p)$, we can express the collision operator in term of new variables as follows: if we let $v = (a^2 p^1, b^2 \bar{p})$, $u = (a^2 q^1, b^2 \bar{q})$, $v' = (v'^1, \bar{v}') = (a^2 p'^1, b^2 \bar{p}')$ and $u' = (u'^1, \bar{u}') = (a^2 q'^1, b^2 \bar{q}')$,

$$\begin{aligned} Q(\tilde{f}, \tilde{f})(t, v) &= a^{-1} b^{-2} \int_{S^2} d\omega \int_{\mathbb{R}^3} du \frac{g\sqrt{s}}{v^0 u^0} \sigma(g, \omega)[\tilde{f}(t, v')\tilde{f}(t, u') - \tilde{f}(t, v)\tilde{f}(t, u)] \\ &:= Q_{\mathrm{g}}(\tilde{f}, \tilde{f})(t, v) - Q_{\mathrm{l}}(\tilde{f}, \tilde{f})(t, v). \end{aligned} \tag{2.11}$$

2.3. The equation

After computing all the Christoffel symbols and denoting by "dot" the derivative with respect to t, if we let $Q = \frac{1}{p^0}\tilde{Q}$, (1.1) reduces to

$$\partial_t f - 2\frac{\dot{a}}{a} p^1 \partial_{p^1} f - 2\frac{\dot{b}}{b} p^2 \partial_{p^2} f - 2\frac{\dot{b}}{b} p^3 \partial_{p^3} f = Q(f, f). \tag{2.12}$$

Using the expression $\tilde{f}(t, v) = f(t, p)$, it follows directly that the left-hand side of (2.12) is equal to $\partial_t \tilde{f}(t, v)$.

For simplicity of notation, it will cause no confusion if we use the same letter f to designate $\tilde{f}$ in the remainder of the paper. Thus the Boltzmann equation in f with initial data f_0 becomes

$$\begin{cases} \partial_t f(t, v) = Q_{\mathrm{g}}(f, f)(t, v) - Q_{\mathrm{l}}(f, f)(t, v), \\ f(0, v) = f_0(v). \end{cases}$$

So, f is the solution of the Boltzmann equation with initial data f_0 if and only if f is the solution of the following integral equation,

$$f(t, v) = f_0(v) + \int_0^t Q(f, f)(\tau, v)d\tau. \tag{2.13}$$

In the remainder of this paper, the term Boltzmann equation refers to (2.13).

2.4. The post-collisional momenta

One of the main terms allowing to describe the Boltzmann equation (2.13) is the collision operator. This operator is expressed by using the post-collisional momenta. This section is devoted to express the post collisional momenta as functions of the pre-collisional momenta. In the present work, we consider two kinds of parametrization.

2.4.1. First parametrization

We consider a parametrization of post-collisional momenta introduced in [17]. Suppose that p^α and q^α are given, and consider the following four-vectors,

$$n^\alpha = p^\alpha + q^\alpha, \qquad t^\alpha = (n_i\omega^i, n^0\omega), \qquad \omega \in S^2. \tag{2.14}$$

p'^α and q'^α can be parametrized by

$$p'^\alpha = \frac{p^\alpha + q^\alpha}{2} + \frac{g}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \tag{2.15}$$

$$q'^\alpha = \frac{p^\alpha + q^\alpha}{2} - \frac{g}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}}. \tag{2.16}$$

This parametrization has an advantage that it looks like the usual parametrization in the classical Boltzmann equation.

From (2.14) and (2.15), we express easily p'^0 and q'^0 as functions of p^0 and q^0,

$$\begin{cases} p'^0 = \dfrac{p^0 + q^0}{2} + \dfrac{g}{2}\dfrac{a^2n^1\omega^1 + b^2n^2\omega^2 + b^2n^3\omega^3}{\sqrt{t_\beta t^\beta}}, \\ q'^0 = \dfrac{p^0 + q^0}{2} - \dfrac{g}{2}\dfrac{a^2n^1\omega^1 + b^2n^2\omega^2 + b^2n^3\omega^3}{\sqrt{t_\beta t^\beta}}. \end{cases} \tag{2.17}$$

If we let $\tilde{n} = v + u$ and $\tilde{n}^0 = n^0$, p'^0, p'^1 and p'^k $(k = 2, 3)$ express as functions of v^1, v^2 and v^3 as follows

$$p'^0 = \frac{\tilde{n}^0}{2} + \frac{\frac{g}{2}(\tilde{n}^1\omega^1 + \tilde{n}^2\omega^2 + \tilde{n}^3\omega^3)}{\sqrt{-(\tilde{n}^1\omega^1 + \tilde{n}^2\omega^2 + \tilde{n}^3\omega^3)^2 + (\tilde{n^0})^2(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)}}, \tag{2.18}$$

$$p'^1 = \frac{\tilde{n}^1}{2a^2} + \frac{\frac{g}{2}\tilde{n}^0\omega^1}{\sqrt{-(\tilde{n}^1\omega^1 + \tilde{n}^2\omega^2 + \tilde{n}^3\omega^3)^2 + (\tilde{n}^0)^2(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)}}, \tag{2.19}$$

$$p'^k = \frac{\tilde{n}^k}{2b^2} + \frac{\frac{g}{2}\tilde{n}^0\omega^k}{\sqrt{-(\tilde{n}^1\omega^1 + \tilde{n}^2\omega^2 + \tilde{n}^3\omega^3)^2 + (\tilde{n}^0)^2(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)}}. \tag{2.20}$$

REMARK 2.1. In the sequel, by abuse of notation to avoid any confusion, we write $n = v + u$ instead of $\tilde{n} = v + u$.

Using the relations $v'^1 = a^2 p'^1$, $v'^2 = b^2 p'^2$, $v'^3 = b^2 p'^3$; we have v'^1, v'^2 and v'^3 expressed as functions of v^1, v^2 and v^3 as follows:

$$v'^0 = \frac{n^0}{2} + \frac{g}{2}\frac{n.w}{\sqrt{-(n.w)^2 + (n^0)^2(a^2(\omega^1)^2 + b^2|\bar{w}|^2)}}, \tag{2.21}$$

$$v'^1 = \frac{v^1 + u^1}{2} + \frac{a^2 g}{2}\frac{n^0\omega^1}{\sqrt{-(n.w)^2 + (n^0)^2(a^2(\omega^1)^2 + b^2|\bar{w}|^2)}}, \tag{2.22}$$

$$v'^k = \frac{v^k + u^k}{2} + \frac{b^2 g}{2}\frac{n^0\omega^k}{\sqrt{-(n.w)^2 + (n^0)^2(a^2(\omega^1)^2 + b^2|\bar{w}|^2)}}. \tag{2.23}$$

2.4.2. Second parametrization

By using the Minkowski space-time, R. Strain has found in [22] the following parametrization of post-collisional momenta

$$\begin{cases} p' = \dfrac{p+q}{2} + \dfrac{g}{2}\left(\omega + (\gamma - 1)\dfrac{(p+q).\omega}{|p+q|^2}\right), \\ q' = \dfrac{p+q}{2} - \dfrac{g}{2}\left(\omega + (\gamma - 1)\dfrac{(p+q).\omega}{|p+q|^2}\right), \qquad \omega \in S^2, \end{cases} \tag{2.24}$$

where $\gamma = (p^0 + q^0)/\sqrt{s}$. In this work, we generalise this parametrisation to the Bianchi type I space-time. So, in term of new variables, after some calculations, we have for the parameter $\omega \in S^2$,

$$v'^1 = \frac{n^1}{2} + \frac{ag}{2}\left[\left(w^1 - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^1\bar{n})|^2}a^{-1}n^1\right) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^1\bar{n})|^2}a^{-1}n^1\right], \tag{2.25}$$

$$v'^k = \frac{n^k}{2} + \frac{bg}{2}\left[\left(w^k - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^1\bar{n})|^2}b^{-1}n^k\right) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^1\bar{n})|^2}b^{-1}n^k\right], \quad k = 2, 3. \tag{2.26}$$

Let us observe that this second parametrization provides singularities when $v+u = 0$. So we will avoid to use it in such region.

2.5. Assumptions of the paper

Henceforth we let C, and sometimes c denote generic positive inessential constants whose values may change from line to line. The notation $A \lesssim B$ will imply that a positive constant C exists such that $A \leq CB$ holds uniformly over the range of parameters which are present in the inequality and moreover that the precise magnitude of the constant is not important.

2.5.1. Assumption on the scattering kernel

In order to get one global existence theorem, it is necessary to put some restrictions on the scattering kernel σ. We shall make two standing assumptions on the scattering kernel under consideration.

We first recall a brief and usual description of the scattering kernel in the nonrelativistic Boltzmann equation. In the classical Boltzmann equation, the inverse power law gives the best-known types of scattering kernel. The scattering kernel is then classified into soft and hard potentials. This classification was first adapted in the general relativity case by M. Dudyński and M. Ekiel-Jeżewska in [9] and recently reformulated by R. Strain in [22]. In this work, we suppose that the scattering kernel falls into hard potentials; these allow to model strong shocks. In such situations, one assumes that there exist $\gamma > -2$, $0 \leq \alpha \leq \gamma + 2$ and $0 < \beta < \min\{4, 4 + \gamma\}$ such that the scattering kernel $\sigma(g, \omega)$ satisfies the following growth/decay estimates,

$$\frac{g}{\sqrt{s}} g^{\beta} \sigma_0(\omega) \lesssim \sigma(g, \omega) \lesssim (g^{\alpha} + g^{-\beta}) \sigma_0(\omega). \tag{2.27}$$

In (2.27) $\sigma_0(\omega)$ is such that $\sigma_0(\omega) \lesssim \sin^{\gamma} \theta$ where θ stands for the scattering angle. Note that under (2.4), the scattering angle θ is well defined in [10] (see Lemma 3.15.3]) by relation

$$\cos \theta = \frac{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)}{g^2}.$$

In this work, by choosing $\alpha = 0$, we work under the additional assumption

$$|\partial_g \sigma(g, \omega)| \lesssim g^{-1-\beta} \sigma_0(\omega). \tag{2.28}$$

2.5.2. Assumption on the metric tensor

On the coefficients of the metric tensor (2.1), the following assumptions will be needed throughout the paper. We assume that the coefficients a and b of the Bianchi's type I metric are given as increasing functions of the time t and are such that

$$a(0) = 1, \qquad a \leq b \leq \sqrt{2} a, \tag{2.29}$$

$$\int_0^{+\infty} (a^{-1} b^{-2} + a^{-1} b^{\beta - 3})(t) dt < +\infty, \tag{2.30}$$

β is the same as in (2.28).

Before studying our main result, we are going to collect some fundamental estimates.

3. Preliminary results

The relative momentum g and the energy s in the center of momentum are two of the most important quantities in the definition of the collision operator. We are going to collect some fundamental estimates on them.

LEMMA 3.1. *The relative momentum and the energy in the center of momentum of the system fulfill the following estimates:*

$$s = 4 + g^2, \qquad 2 \le \sqrt{s}, \qquad g \le \sqrt{s}, \tag{3.1}$$

$$g \le \sqrt{s} \le 2\sqrt{v^0 u^0}. \tag{3.2}$$

Proof: Our proof starts with the observation that

$$s = 2 - 2p_\alpha q^\alpha \qquad \text{and} \qquad g^2 = -2 - 2p_\alpha q^\alpha. \tag{3.3}$$

Then we have $s = 4 + g^2$ and this implies $\sqrt{s} \ge 2$ and $\sqrt{s} \ge g$. Since $v^0 = p^0$ and $u^0 = q^0$, we obtain

$$\begin{aligned} s &= 2 - 2[-p^0q^0 + a^2p^1q^1 + b^2p^2q^2 + b^2p^3q^3] \\ &= 2p^0q^0 + 2[1 - a^2p^1q^1 - b^2p^2q^2 - b^2p^3q^3] \\ &\le 2p^0q^0 + 2[1 + a^2|p^1||q^1| + b^2|p^2||q^2| + b^2|p^3||q^3|] \\ &= 2p^0q^0 + 2(1, a|p^1|, b|p^2|, b|p^3|).(1, a|q^1|, b|q^2|, b|q^3|) \\ &\le 2p^0q^0 + 2\sqrt{1 + a^2(p^1)^2 + b^2|\bar{p}|^2}\sqrt{1 + a^2(q^1)^2 + b2|\bar{q}|^2} \\ &= 4p^0q^0. \end{aligned}$$

□

LEMMA 3.2. *The relative momentum fulfills the estimates:*

$$\frac{|v-u|}{\sqrt{v^0u^0}} \le bg, \qquad ag \le |v-u|. \tag{3.4}$$

Proof: For the first inequality, by direct computation we have

$$\begin{aligned} g^2 &= 2p^0q^0 - 2[1 + a^2p^1q^1 + b^2p^2q^2 + b^2p^3q^3] \\ &= 2p^0q^0 - 2[1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})] \\ &= 2\frac{(p^0q^0)^0 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2}{p^0q^0 + [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]}. \end{aligned}$$

It is obvious to see that

$$(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2 \ge |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2$$

and we notice that if we set

$$\Delta = (p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2,$$

we have

$$\begin{aligned} \Delta &= 1 + (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2)(a^2(q^1)^2 \\ &\quad + b^2|\bar{q}|^2) - \Delta_1 \\ &= (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) - 2(ap^1, b\bar{p}).(aq^1, b\bar{q}) + \Delta_2 \end{aligned}$$

$$\geq (a^2(q^1)^2 + b^2|\bar{q}|^2) + (a^2(p^1)^2 + b^2|\bar{p}|^2) - 2(ap^1, b\bar{p}).(aq^1, b\bar{q})$$
$$= [(ap^1, b\bar{p}) - (aq^1, b\bar{q})].[(ap^1, b\bar{p}) - (aq^1, b\bar{q})]$$
$$= |(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2,$$

where Δ_1 and Δ_1 are defined by

$$\Delta_1 = 1 + 2(ap^1, b\bar{p})(aq^1, b\bar{q}) + [(ap^1, b\bar{p})(aq^1, b\bar{q})]^2,$$
$$\Delta_2 = [(a^2(p^1)^2 + b^2|\bar{p}|^2)(a^2(q^1)^2 + b^2|\bar{q}|^2) - [(ap^1, b\bar{p}).(aq^1, b\bar{q})]^2].$$

By the relation

$$(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2 \geq 0,$$

we have

$$2p^0q^0 \geq 2|1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})| \geq p^0q^0 + 1 + (ap^1, b\bar{p}).(aq^1, b\bar{q}).$$

$$g^2 = 2\frac{(p^0q^0)^2 - [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]^2}{p^0q^0 + [1 + (ap^1, b\bar{p}).(aq^1, b\bar{q})]} \geq 2\frac{|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2}{2p^0q^0}.$$

Let us observe $|v - u|^2 = a^4(p^1 - q^1)^2 + b^4|\bar{p} - \bar{p}|^2$. Since $a \leq b$, we have

$$|v - u|^2 \leq b^2[a^2(p^1 - q^1)^2 + b^2|\bar{p} - \bar{q}|^2] \leq b^2|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2.$$

This leads to

$$g^2 \geq \frac{|(ap^1, b\bar{p}) - (aq^1, b\bar{q})|^2}{p^0q^0} \geq \frac{|v - u|^2}{b^2p^0q^0}$$

and then

$$bg \geq \frac{|v - u|}{\sqrt{v^0u^0}}.$$

For the second inequality in (3.4), after computation we have

$$(v^0)^2 - (u^0)^2 = a^{-2}(v^1 - u^1)n^1 + b^{-2}(\bar{v} - \bar{u}).\bar{n}$$
$$= (a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u})).(a^{-1}n^1, b^{-1}\bar{n}).$$

Let us denote by θ_0 the angle between $(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))$ and $(a^{-1}n^1, b^{-1}\bar{n})$, then

$$g^2 = a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 - (v^0 - u^0)^2$$
$$= a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 - \left[\frac{a^{-2}(v^1 - u^1)n^1 + b^{-2}(\bar{v} - \bar{u}).\bar{n}}{n^0}\right]^2$$
$$= |(a^{-1}(v^1 - u^1), b^{-1}(\bar{v} - \bar{u}))|^2\left[1 - \left(\frac{|(a^{-1}n^1, b^{-1}\bar{n})| \cos\theta_0}{n^0}\right)^2\right].$$

Thus

$$g^2 \leq a^{-2}(v^1 - u^1)^2 + b^{-2}|\bar{v} - \bar{u}|^2 \leq a^{-2}|v - u|^2. \qquad \square$$

LEMMA 3.3. *For* $0 \leq \beta < 4$, *we have the following estimate*

$$\int_{\mathbb{R}^3} v_\phi g^{-\beta} e^{-|u|^2} du \leq \begin{cases} C & \text{for } 0 \leq \beta \leq 1, \\ C b^{\beta-1} & \text{for } 1 \leq \beta \leq 4, \end{cases} \tag{3.5}$$

where C *is a positive constant depending on* β.

Proof: The proof of this lemma is similar to that of [16]. □

LEMMA 3.4. *For the increasing functions* $a = a(t)$ *and* $b = b(t)$ *such that* $a(0) = 1$ *and* $a(t) \leq b(t)$, *the following identities hold*

$$|v| \leq b v^0, \qquad v^0 \leq \sqrt{1 + |v|^2}. \tag{3.6}$$

Proof: The proof is obvious. □

LEMMA 3.5. *For the unit vector* $\omega \in S^2$, *setting* $\bar{\omega} = (\omega^2, \omega^3)$, *if we set*

$$r = \sqrt{t_\alpha t^\alpha} = \sqrt{-(n.\omega)^2 + (n^0)^2 (a^2(\omega^1)^2 + b^2 |\bar{\omega}|^2)},$$

the quadratic vector $t^\alpha = (n_i \omega^i, n^0 \omega)$ *fulfills the estimate*

$$\sqrt{t_\beta t^\beta} \geq \sqrt{s}[a^2(\omega^1)^2 + b^2|\bar{\omega}|^2]^{\frac{1}{2}}. \tag{3.7}$$

Proof: By using elementary algebra we have

$$\begin{aligned} t_\beta t^\beta &= -(t^0)^2 + a^2(t^1)^2 + b^2|\bar{t}|^2 \\ &= -(a^2 n^1 \omega^1 + b^2 \bar{n}.\bar{w})^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2] \\ &= -[(an^1, b\bar{n}).(a\omega^1, b\bar{w})]^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2] \\ &\geq -|(an^1, b\bar{n})|^2 |(a\omega^1, b\bar{w})|^2 + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2] \\ &\geq -[a^2(n^1)^2 + b^2|\bar{n}|^2][a^2(\omega^1)^2 + b^2|\bar{w}|^2] + (n^0)^2[a^2(\omega^1)^2 + b^2|\bar{w}|^2] \\ &\geq [a^2(\omega^1)^2 + b^2|\bar{w}|^2][(n^0)^2 - a^2(n^1)^2 - b^2|\bar{n}|^2] \\ &\geq [a^2(\omega^1)^2 + b^2|\bar{w}|^2]s, \end{aligned}$$

which is the desired result. □

LEMMA 3.6. *The energy* s *enjoys the estimates*

$$\sqrt{s} \geq \max\left(\sqrt{\frac{v^0}{u^0}}, \sqrt{\frac{u^0}{v^0}}\right) \qquad \text{and} \qquad r \geq a\sqrt{s} \geq a \max\left(\sqrt{\frac{v^0}{u^0}}, \sqrt{\frac{u^0}{v^0}}\right). \tag{3.8}$$

Proof: From the definition of s, we have

$$\begin{aligned} s &= (v^0)^2 + 2v^0u^0 + (u^0)^2 - a^{-2}(v^1)^2 - a^{-2}(u^1)^2 - 2a^{-2}v^1u^1 - b^{-2}|\bar{v}|^2 - b^{-2}|u|^2 - 2b^{-2}\bar{v}.\bar{u} \\ &= 2 + 2\sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2}\sqrt{1 + a^{-2}(u^1)^2 + b^{-2}|\bar{u}|^2} - 2(a^{-1}v^1, b^{-1}\bar{v}).(a^{-1}u^1, b^{-1}\bar{u}) \end{aligned}$$

$$\geq 2+2\sqrt{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1+|(a^{-1}u^1, b^{-1}\bar{u})|^2}-2|(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})|$$

$$\geq 2+2\frac{(1+|(a^{-1}v^1, b^{-1}\bar{v})|^2)(1+|(a^{-1}u^1, b^{-1}\bar{u})|^2)-|(a^{-1}v^1, b^{-1}\bar{v})|^2|(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1+|(a^{-1}u^1, b^{-1}\bar{u})|^2}+|(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})|}$$

$$\geq 2+2\frac{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2+|(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1+|(a^{-1}u^1, b^{-1}\bar{u})|^2}+|(a^{-1}v^1, b^{-1}\bar{v})||(a^{-1}u^1, b^{-1}\bar{u})|}$$

$$\geq 2+\frac{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2+|(a^{-1}u^1, b^{-1}\bar{u})|^2}{\sqrt{1+|(a^{-1}v^1, b^{-1}\bar{v})|^2}\sqrt{1+|(a^{-1}u^1, b^{-1}\bar{u})|^2}}$$

$$\geq 2+\frac{(v^0)^2+(u^0)^2-1}{v^0u^0}=\frac{(v^0)^2+(u^0)^2+2v^0u^0-1}{v^0u^0}\geq\frac{(v^0)^2+(u^0)^2}{v^0u^0}\geq\frac{v^0}{u^0}+\frac{u^0}{v^0}. \qquad \square$$

4. The global existence theorem

4.1. Functional spaces

Our aim in this work is to study the relativistic Boltzmann equation in the Bianchi type I space-time in the case of hard potentials situation near vacuum initial data. Our goal is to establish new results concerning the existence theorem. Let us introduce the functional framework we will work with. We choose the weight function as $e^{|v|^2}$. For $f : [0, +\infty[\times\mathbb{R}^3 \to \mathbb{R}_+$, we let

$$\|f(t)\| := \sup\{|e^{|v|^2}|\partial^j_{v^k} f(t, v)|;\ v \in \mathbb{R}^3,\ j = 0, 1;\ k = 1, 2, 3\}, \tag{4.1}$$

$$\Lambda = \{f \in C^0([0, +\infty[\times\mathbb{R}^3), \|f(t)\| < +\infty\ \forall t \in [0, +\infty[\}, \tag{4.2}$$

where Λ is the function space in which we will seek the solution. Endowed with the norm $|||f||| := \sup_{t\in\mathbb{R}_+} \|f(t)\|$, Λ is a Banach space.

Let C be a positive real number, we set

$$S_{ab} = \left\{w \in S^2, \frac{|\omega^1||\bar{v}-\bar{u}|}{|\bar{\omega}||v^1-u^1|} \leq 1, |\frac{1}{2}\frac{|v-u|}{s}\frac{|n\times w|^2+3|n|^2}{a^2(\omega^1)^2+b^2|\bar{w}|^2} \leq C\right\}. \tag{4.3}$$

REMARK 4.1. In the remainder of this paper, we will use a cutoff S_{ab} on the angular part of the scattering kernel. This cutoff depends on t and on pre-collisional momenta v and u. So, henceforth, unless otherwise specified, the parameter ω will always belong to S_{ab}.

LEMMA 4.1. *Let v and u be given. Suppose that v' and u' are post-collisional momenta with a parameter $\omega \in S_{ab}$. If $a^2 \leq b^2 \leq 2a^2$, we have*

$$|v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq C. \tag{4.4}$$

Proof: We let $A = |v|^2 + |u|^2 - |v'|^2 - |u'|^2$ and we recall that $r = \sqrt{t_\alpha t^\alpha}$. Using the parametrization (2.22)–(2.23), a straightforward computation leads to

$$A = \frac{1}{2}|v-u|^2 - \frac{1}{2}\frac{g^2(n^0)^2}{r^2}|(a^2\omega^1, b^2\bar{\omega})|^2, \tag{4.5}$$

$$(v^0-u^0)^2 = \frac{1}{(n^0)^2}|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2\cos^2\theta_0,$$

and

$$\begin{aligned}(n^0)^2g^2 &= -|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2\cos^2\theta_0 \\ &\quad + (n^0)^2|(a^{-1}(v^1-u^1), b^{-1}(\bar{v}-\bar{u}))|^2.\end{aligned}$$

Let us set

$$\begin{aligned}A_1 &= r^2|v-u|^2 - (a^4(\omega^1)^2 + b^4|\bar{\omega}|^2)(n^0)^2|(a^{-1}(v^1-u^1)), b^{-1}(\bar{v}-\bar{u}))|^2 \\ &= [-(n.\omega)^2 + (n^0)^2(a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)]|v-u|^2 \\ &\quad - (a^4(\omega^1)^2 + b^4|\bar{\omega}|^2)(n^0)^2|(a^{-1}(v^1-u^1)), b^{-1}(\bar{v}-\bar{u}))|^2 \\ &= -(n.\omega)^2 + (n^0)^2A_2,\end{aligned}$$

where

$$\begin{aligned}A_2 &= (a^2(\omega^1)^2 + b^2|\bar{\omega}|^2)(|v^1-u^1|^2 + |\bar{v}-\bar{u}|^2) \\ &\quad - (a^4(\omega^1)^2 + b^4|\bar{\omega}|^2)(a^{-2}|v^1-u^1|^2 + b^{-2}|\bar{v}-\bar{u}|^2) \\ &= a^2\left(1-\left(\frac{a}{b}\right)^2\right)(\omega^1)^2|\bar{v}-\bar{u}|^2 + b^2\left(1-\left(\frac{b}{a}\right)^2\right)|\bar{\omega}|^2|v^1-u^1|^2.\end{aligned}$$

Using expressions above of $(v^0-u^0)^2$ and $(n^0)^2g^2$, we obtain

$$A = \frac{1}{2r^2}\left[A_1 + |(a^{-1}(v^1-u^1)), b^{-1}(\bar{v}-\bar{u})|^2|(a^2\omega^1, b^2\bar{\omega})|^2|(a^{-1}n^1, b^{-1}\bar{n})|^2\cos^2\theta_0\right].$$

If we let $t = (\frac{a}{b})^2$, then $t \in]0,1[$. Since the parameter ω enjoys (4.3), one has

$$\begin{aligned}A_2 &= t(1-t)(\omega^1)^2|\bar{v}-\bar{u}|^2 + \left(1-\frac{1}{t}\right)|\bar{\omega}|^2|v^1-u^1|^2 \\ &= \frac{1-t}{t}[t^2(\omega^1)^2|\bar{v}-\bar{u}|^2 - |\bar{\omega}|^2|v^1-u^1|^2] \\ &\leq \frac{1-t}{t}[(\omega^1)^2|\bar{v}-\bar{u}|^2 - |\bar{\omega}|^2|v^1-u^1|^2] \leq 0.\end{aligned}$$

Since $A_2 \leq 0$, we have $A_1 \leq -(n.\omega)^2$. Thus

$$\begin{aligned}A &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[-(n.w)^2 + a^{-4}|n|^2b^4\cos^2\theta_0] \\ &\leq \frac{1}{2}\frac{|v-u|^2}{r^2}\left[-(n.w)^2 + \left(\frac{b}{a}\right)^4|n|^2|w|^2\right]\end{aligned}$$

$$\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[-(n.w)^2 + 4|n|^2|w|^2]$$
$$\leq \frac{1}{2}\frac{|v-u|^2}{r^2}[|n \times w|^2 + 3|n|^2] \leq C. \qquad \square$$

REMARK 4.2. With the parametrization (2.25)–(2.26), we have the same estimate by following the same method.

4.2. Estimates of derivatives of g and $\sqrt{s}$

LEMMA 4.2. *The derivatives of v^0 with respect to v^i fulfill the following estimates:*

$$|\partial_{v^1} v^0| \leq \frac{1}{a} \quad \text{and} \quad |\partial_{v^i} v^0| \leq \frac{1}{b} \quad \text{for } i = 2, 3. \tag{4.6}$$

Proof: Since

$$v^0 = \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2},$$

we have

$$(v^1)^2 \leq \left(\frac{v^0}{a}\right)^2, \qquad (v^i)^2 \leq \left(\frac{v^0}{b}\right)^2, \qquad \partial_{v^1} v^o = \frac{v^1}{a^2 v^0}$$

and

$$\partial_{v^i} v^0 = \frac{v^i}{b^2 v^0} \qquad \text{for } i = 2, 3.$$

It follows that

$$\left|\frac{v^1}{a v^0}\right| \leq 1, \qquad \left|\frac{v^i}{b v^0}\right| \leq 1, \qquad i = 2, 3.$$

Thus

$$|\partial_{v^1} v^0| \leq \frac{1}{a}, \qquad |\partial_{v^i} v^0| \leq \frac{1}{b}, \qquad i = 2, 3. \qquad \square$$

LEMMA 4.3. *The derivatives of g and $\sqrt{s}$ with respect to v^i enjoy the following estimates:*

$$|\partial_{v^1} g| \leq \frac{2u^0}{ag} \quad \text{and} \quad |\partial_{v^i} g| \leq \frac{2u^0}{bg} \quad \text{for } i = 2, 3, \tag{4.7}$$

$$|\partial_{v^1} \sqrt{s}| \leq \frac{2u^0}{a\sqrt{s}} \quad \text{and} \quad |\partial_{v^i} \sqrt{s}| \leq \frac{2u^0}{b\sqrt{s}} \quad \text{for } i = 2, 3. \tag{4.8}$$

Proof: From the relation

$$g^2 = -2 + 2v^0u^0 - 2[a^{-2}v^1u^1 + b^{-2}v^2u^2 + b^{-2}v^3u^3],$$

we have

$$\partial_{v^1} g = \frac{u^0}{ag}\left[\frac{v^1}{av^0} - \frac{u^1}{au^0}\right], \qquad \text{then} \qquad |\partial_{v^1} g| \leq \frac{u^0}{ag}\left|\frac{a^{-1}v^1}{v^0} + \frac{a^{-1}u^1}{u^0}\right| \leq \frac{2u^0}{ag}.$$

$$\partial_{v^i} g = \frac{u^0}{bg}\left[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}\right], \qquad \text{then} \qquad |\partial_{v^i} g| \leq \frac{2u^0}{bg}, \qquad i = 2, 3.$$

On the other hand

$$s = 2 + 2v^0u^0 - 2[a^{-2}v^1u^1 + b^{-2}v^2u^2 + b^{-2}v^3u^3].$$

So

$$\partial_{v^1}\sqrt{s} = \frac{u^0}{a\sqrt{s}}\left[\frac{v^1}{av^0} - \frac{u^1}{au^0}\right], \qquad \text{then} \qquad |\partial_{v^1}\sqrt{s}| \leq \frac{2u^0}{a\sqrt{s}},$$

$$\partial_{v^i}\sqrt{s} = \frac{u^0}{b\sqrt{s}}\left[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}\right], \qquad \text{then} \qquad |\partial_{v^i}\sqrt{s}| \leq \frac{2u^0}{b\sqrt{s}}, \qquad \text{for } i = 2, 3. \qquad \square$$

LEMMA 4.4. *If we let* $G := G(\omega, a, b) = a^2(\omega^1)^2 + b^2|\bar{\omega}|^2$, *then*

$$|\partial_{v^1} r| \leq \frac{(\frac{b^2}{a} + b)(n^0)}{\sqrt{(n^0)^2G - (n.w)^2}} \qquad \textit{and} \qquad |\partial_{v^i} r| \leq \frac{2b(n^0)}{\sqrt{(n^0)^2G - (n.w)^2}}, \qquad i = 2, 3. \tag{4.9}$$

Proof: After expanding r, we have

$$\partial_{v^1} r = \frac{u^0}{a\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}\left(\frac{v^1}{av^0}G(\omega, a, b) - \frac{(u.w)\omega^1}{u^0}a\right)$$
$$+ \frac{v^0}{a\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}\left(\frac{v^1}{av^0}G(\omega, a, b) - \frac{(v.w)\omega^1}{v^0}a\right),$$

$$\partial_{v^i} r = \frac{u^0}{b\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}\left(\frac{v^i}{bv^0}G(\omega, a, b) - \frac{(u.w)\omega^i}{u^0}b\right)$$
$$+ \frac{v^0}{b\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}\left(\frac{v^i}{bv^0}G(\omega, a, b) - \frac{(v.w)\omega^i}{v^0}b\right), \qquad i = 2, 3.$$

It is easy to see that $a^2 \leq G(\omega, a, b) \leq b^2$. From the equalities above, we have

$$|\partial_{v^1} r| \leq \left[\frac{u^0}{a\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}} + \frac{v^0}{a\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}\right](b^2 + ba)$$
$$\leq \left(\frac{b^2}{a} + b\right)\frac{u^0 + v^0}{\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}.$$

Similarly, we have

$$|\partial_{v^i} r| \leq 2b\frac{u^0 + v^0}{\sqrt{(n^0)^2G(\omega, a, b) - (n.w)^2}}, \qquad i = 2, 3. \qquad \square$$

LEMMA 4.5. *We have the following two estimates:*

$$\left|\frac{v^1}{av^0}-\frac{u^1}{au^0}\right|\leq\frac{1}{a}\left(1+\frac{b^2}{a^2}\right)|v-u|, \tag{4.10}$$

$$\left|\frac{v^i}{bv^0}-\frac{u^i}{bu^0}\right|\leq\frac{1}{b}\left(1+\frac{b^2}{a^2}\right)|v-u|, \qquad i=2,3. \tag{4.11}$$

Proof: For the first inequality, we can write

$$\left|\frac{v^1}{v^0}-\frac{u^1}{u^0}\right|=\frac{1}{v^0u^0}|u^0(v^1-u^1)+u^1(u^0-v^0)|\leq\frac{1}{v^0u^0}[|v-u|u^0+|u||u^0-v^0|].$$

We now try to control $|v^0-u^0|$. One has

$$\begin{aligned}|(v^0)^2-(u^0)^2|&=|(a^{-1}(u^1-v^1),b^{-1}(\bar{u}-\bar{v})).(a^{-1}n^1,b^{-1}\bar{n})|\\&\leq a^{-2}|u-v||u+v|.\end{aligned}$$

On the other hand, we have $v^0+u^0\geq b^{-1}(|v|+|u|)$. Thus

$$|u^0-v^0|=\frac{|(u^0)^2-(v^0)^2|}{n^0}\leq\frac{a^{-2}|v-u||v+u|}{b^{-1}(|v|+|u|)}\leq\frac{b}{a^2}|v-u|,$$

$$\left|\frac{v^1}{v^0}-\frac{u^1}{u^0}\right|\leq|v-u|\left[\frac{u^0}{v^0u^0}+\frac{b}{a^2}\frac{|u|}{v^0u^0}\right].$$

Since $u^0\geq b^{-1}|u|$ and $(v^0\geq 1$, these estimates lead to

$$\left|\frac{v^1}{v^0}-\frac{u^1}{u^0}\right|\leq|v-u|\left[\frac{u^0}{v^0u^0}+\frac{b^2}{a^2}\frac{u^0}{v^0u^0}\right]\leq\left(1+\frac{b^2}{a^2}\right)|v-u|.$$

Using the same method, we obtain the relation (4.11). □

LEMMA 4.6. *The partial derivatives of g and $\sqrt{s}$ with respect to v^i enjoy the following estimates:*

$$|\partial_{v^1}g|\leq\frac{b}{a^2}\left(1+\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \tag{4.12}$$

$$|\partial_{v^i}g|\leq\frac{1}{b}\left(1+\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \qquad i=2,3, \tag{4.13}$$

$$|\partial_{v^1}\sqrt{s}|\leq\frac{b}{a^2}\left(1+\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \tag{4.14}$$

$$|\partial_{v^i}\sqrt{s}|\leq\frac{1}{b}\left(1+\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \qquad i=2,3. \tag{4.15}$$

Proof: Using (3.4) and (4.10)–(4.11), we can deduce:

$$\partial_{v^1}g=\frac{u^0}{ag}\left[\frac{v^1}{av^0}-\frac{u^1}{au^0}\right], \qquad |\partial_{v^1}g|\leq\frac{b}{a^2}\left(1+\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0},$$

$$\partial_{v^i} g = \frac{u^0}{bg}\left[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}\right], \qquad |\partial_{v^i} g| \le \frac{1}{b}\left(1 + \frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \qquad i = 2, 3,$$

$$\partial_{v^1}\sqrt{s} = \frac{u^0}{a\sqrt{s}}\left[\frac{v^1}{av^0} - \frac{u^1}{au^0}\right], \qquad |\partial_{v^1}\sqrt{s}| \le \frac{b}{a^2}\left(1 + \frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0},$$

$$\partial_{v^i}\sqrt{s} = \frac{u^0}{b\sqrt{s}}\left[\frac{v^i}{bv^0} - \frac{u^i}{bu^0}\right], \qquad |\partial_{v^i}\sqrt{s}| \le \frac{1}{b}\left(1 + \frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}, \qquad i = 2, 3. \quad \square$$

LEMMA 4.7. *Under assumptions* (2.27)–(2.28), *we have the following estimates*

$$|\partial_{v^1}[v_\phi\sigma(g,\omega)]| \le ca^{-1}u^0(1 + g^{-\beta})\sigma_0(\omega), \tag{4.16}$$

$$|\partial_{v^i}[v_\phi\sigma(g,\omega)]| \le cb^{-1}u^0(1 + g^{-\beta})\sigma_0(\omega), \qquad \text{for } i = 1, 2. \tag{4.17}$$

Proof: By direct computation we have

$$\partial_{v^i}[v_\phi\sigma(g,\omega)] = \left[(\partial_{v^i} g)\frac{\sqrt{s}}{v^0u^0} + (\partial_{v^i}\sqrt{s})\frac{g}{v^0u^0} - (\partial_{v^i} v^0)\frac{g\sqrt{s}}{(v^0)^2u^0}\right]\sigma(g,\omega) + \frac{g\sqrt{s}}{v^0u^0}(\partial_{v^i} g)(\partial_g\sigma(g,\omega)).$$

Using the estimate(4.12)–(4.15) of derivatives of g and $\sqrt{s}$, we have

$$\begin{aligned}
&|\partial_{v^1}[v_\phi\sigma(g,\omega)]| \\
&\le \frac{b}{a^2}\left(1 + \frac{b^2}{a^2}\right)\left[\frac{u^0}{\sqrt{v^0u^0}}(\sqrt{s} + g)\sigma(g,\omega) + |\partial_g\sigma(g,\omega)|\frac{u^0g\sqrt{s}}{\sqrt{v^0u^0}}\right] + \frac{1}{a}\frac{g\sqrt{s}}{(v^0)^2u^0}\sigma(g,\omega) \\
&\le \frac{cu^0}{a}(\sigma(g,\omega) + g|\partial_g\sigma(g,\omega)|) \le ca^{-1}u^0(1 + g^{-\beta})\sigma_0(\omega),
\end{aligned}$$

$$\begin{aligned}
|\partial_{v^i}[v_\phi\sigma(g,\omega)]| &\le \frac{1}{b}\left(1 + \frac{b^2}{a^2}\right)\left[u^0\sqrt{v^0u^0}\frac{\sqrt{s}}{v^0u^0}\sigma(g,\omega) + u^0\sqrt{v^0u^0}\frac{g}{v^0u^0}\sigma(g,\omega)\right] \\
&\quad + \frac{1}{b}\frac{g\sqrt{s}}{(v^0)^2u^0}\sigma(g,\omega) + \frac{1}{b}\left(1 + \frac{b^2}{a^2}\right)\frac{g\sqrt{s}}{v^0u^0}|\partial_g\sigma(g,\omega)|u^0\sqrt{v^0u^0} \\
&\le \frac{cu^0}{b}(\sigma(g,\omega) + g|\partial_g\sigma(g,\omega)|) \le cb^{-1}u^0(1 + g^{-\beta})\sigma_0(\omega), \qquad i = 2, 3.
\end{aligned}$$

$\square$

4.3. Estimates of derivatives of the post-collisional momenta

LEMMA 4.8. *Consider the representation for v' in* (2.22)–(2.23). *We have the following estimates*

$$|\partial_{v^i} v'^k| \le Cv^0(u^0)^4, \qquad k = 1, 2, 3. \tag{4.18}$$

where C does not depend on a or b.

Proof: Let us recall that

$$r \geq \sqrt{s}(G(\omega, a, b))^{\frac{1}{2}}, \qquad \sqrt{s} \geq \max\left(\sqrt{\frac{v^0}{u^0}}, \sqrt{\frac{u^0}{v^0}}\right).$$

Straightforward computations lead to the following relations:

$$\partial_{v^i} v'^1 = \frac{\delta^{i1}}{2} + \frac{a^2(\partial_{v^i} g)}{2}\frac{n^0\omega^1}{r} + \frac{a^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^1}{r} - \frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i} r), \tag{4.19}$$

$$\left|\frac{a^2 g}{2}\frac{(\partial_{v^1} v^0)\omega^1}{r}\right| \leq \frac{a\sqrt{v^0u^0}}{r}, \tag{4.20}$$

$$\left|\frac{a^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^1}{r}\right| \leq \frac{a^2\sqrt{v^0u^0}}{br}, \tag{4.21}$$

$$\left|\frac{a^2(\partial_{v^1} g)}{2}\frac{n^0\omega^1}{r}\right| \leq \frac{b}{2r}\left(1 + 3\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}(n^0), \tag{4.22}$$

$$\left|\frac{a^2(\partial_{v^i} g)}{2}\frac{n^0\omega^1}{r}\right| \leq \frac{a^2}{2br}\left(1 + 3\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}(n^0), \tag{4.23}$$

$$\left|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^1} r)\right| \leq \frac{a^2}{r^3}\left(\frac{b^2}{a} + b\right)\sqrt{v^0u^0}(n^0)^2, \tag{4.24}$$

$$\left|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i} r)\right| \leq \frac{2ba^2}{r^3}\sqrt{v^0u^0}(n^0)^2, \tag{4.25}$$

$$\partial_{v^i} v'^k = \frac{\delta^{ik}}{2} + \frac{b^2(\partial_{v^i} g)}{2}\frac{n^0\omega^k}{r} + \frac{b^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^k}{r} - \frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i} r), \qquad k = 2, 3, \tag{4.26}$$

$$\left|\frac{b^2(\partial_{v^1} g)}{2}\frac{n^0\omega^k}{r}\right| \leq \frac{b^3}{2a^2 r}\left(1 + 3\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}(n^0), \tag{4.27}$$

$$\left|\frac{b^2(\partial_{v^i} g)}{2}\frac{n^0\omega^k}{r}\right| \leq \frac{b}{2r}\left(1 + 3\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}(n^0), \tag{4.28}$$

$$\left|\frac{b^2 g}{2}\frac{(\partial_{v^1} v^0)\omega^k}{r}\right| \leq \frac{b^2}{ar}\sqrt{v^0u^0}, \tag{4.29}$$

$$\left|\frac{b^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^k}{r}\right| \leq \frac{b}{r}\sqrt{v^0u^0}, \tag{4.30}$$

$$\left|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^1} r)\right| \leq \frac{b^2}{r^3}\left(\frac{b^2}{a} + b\right)\sqrt{v^0u^0}(u^0 + v^0)^2, \tag{4.31}$$

$$\left|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i} r)\right| \leq \frac{2b^3}{r^3}\sqrt{v^0u^0}(n^0)^2. \tag{4.32}$$

For the reader convenience, we consider the four cases for the rest of the proof.

Case 1: Estimation of $\partial_{v^1} v'^1$,

$$\left|\frac{a^2(\partial_{v^1} g)}{2}\frac{n^0\omega^1}{r}\right| \le \frac{b}{a^2}\left(1+3\frac{b^2}{a^2}\right)u^0\sqrt{v^0u^0}(n^0)\frac{1}{r} \le \frac{b}{2a}(1+3\frac{b^2}{a^2})(u^0)^2(n^0),$$

$$\left|\frac{a^2 g}{2}\frac{(\partial_{v^1} v^0)\omega^1}{r}\right| \le \frac{a}{r}\sqrt{v^0u^0} \le \frac{a}{\sqrt{G(\omega,a,b)}}u^0 \le u^0,$$

$$\left|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^1} r)\right| \le a^2\sqrt{v^0u^0}(n^0)\left(\frac{b^2}{a}+b\right)(n^0)\frac{1}{r^3} \le \left(\frac{b^2}{a^2}\right)\frac{(u^0)^2}{v^0}(n^0)^2.$$

We combine the above estimates to obtain the desired result.

Case 2: Estimation of $\partial_{v^i} v'^1 \;\; i = 2, 3$.

The following estimate holds; thanks to (4.12)–(4.13),

$$\left|\frac{a^2(\partial_{v^i} g)}{2}\frac{n^0\omega^1}{r}\right| \le \frac{a^2}{2b}\left(1+3\frac{b^2}{a^2}\right)u^0\sqrt{u^0v^0}(u^0+v^0)\frac{1}{r} \le \frac{a}{2b}(u^0)^2n^0,$$

$$\left|\frac{a^2 g}{2}\frac{(\partial_{v^i} v^0)\omega^1}{r}\right| \le \frac{a^2}{br}\sqrt{v^0u^0} \le \frac{a}{b}u^0 \le u^0,$$

$$\left|\frac{a^2 g}{2}\frac{n^0\omega^1}{r^2}(\partial_{v^i} r)\right| \le \frac{2a^2b}{r^3}\sqrt{v^0u^0}(n^0)^2 \le \frac{2b}{a}\frac{(u^0)^2}{v^0}(n^0)^2.$$

We combine the above estimates to obtain the desired result.

Case 3: Estimation of $\partial_{v^1} v'^k$,

$$\left|\frac{b^2(\partial_{v^1} g)}{2}\frac{n^0\omega^1}{r}\right| \le \frac{b^3}{2a^2}(1+3\frac{b^2}{a^2})u^0\sqrt{u^0v^0}(n^0) \le \frac{b^3(u^0)^2n^0}{a\sqrt{G(\omega,a,b)}} \le \frac{b^3}{2a^3}(u^0)^2n^0,$$

$$\left|\frac{b^2 g}{2}\frac{(\partial_{v^1} v^0)\omega^k}{r}\right| \le \frac{b^2}{ar}\sqrt{v^0u^0} \le \frac{b^2}{a\sqrt{G(\omega,a,b)}}u^0 \le \frac{b^2}{a^2}u^0 \le u^0,$$

$$\left|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^1} r)\right| \le \frac{b^2}{r^3}\sqrt{v^0u^0}(n^0)^2\left(\frac{b^2}{a}+b\right) \le \left(\frac{b^4}{a^4}+\frac{b^3}{a^3}\right)\frac{(u^0)^2}{v^0}(n^0)^2.$$

We combine the above estimates to obtain the desired result.

Case 4: Estimation of $\partial_{v^i} v'^k$, $i = 2, 3$.

Taking into account (4.28), we have

$$\left|\frac{b^2(\partial_{v^i} g)}{2}\frac{n^0\omega^k}{r}\right| \le \frac{b}{2r}\left(1+3\frac{b^2}{a^2}\right)u^0\sqrt{u^0v^0}(u^0+v^0) \le \frac{b}{2a}\left(1+3\frac{b^2}{a^2}\right)(u^0)^2(n^0),$$

$$\left|\frac{b^2 g}{2}\frac{(\partial_{v^i} n^0)\omega^k}{r}\right| \le \frac{b}{r}\sqrt{v^0u^0} \le \frac{b}{\sqrt{G(\omega,a,b)}}u^0 \le \frac{b}{a}u^0$$

$$\left|\frac{b^2 g}{2}\frac{n^0\omega^k}{r^2}(\partial_{v^i} r)\right| \le \frac{2b^3}{r^3}\sqrt{v^0u^0}(n^0)^2 \le \frac{2b^3}{a^3}\frac{(u^0)^2}{v^0}(u^0+v^0)^2.$$

We combine the above estimates to obtain the desired result. □

LEMMA 4.9. *With the parametrization* (2.25)–(2.26) *of* v', *we have the following estimates:*

$$|\partial_{v^1} v'^k| \leq C_1\left(\frac{av^0}{|v-u|} + \frac{av^0}{|v+u|} + \frac{a^2(v^0)^2}{|v-u|^2}\right)(u^0)^3, \qquad k = 1, 2, 3, \tag{4.33}$$

$$|\partial_{v^i} v'^k| \leq C_2\left(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2}\right)(u^0)^3, \qquad i = 2, 3, \qquad k = 1, 2, 3. \tag{4.34}$$

where the constants C_j, $j = 1, 2$, *do not depend on* a *nor* b.

Proof: Let us compute $\partial_{v^1} v'^1$, $\partial_{v^i} v'^1$ $(i = 2, 3)$ and $\partial_{v^1} v'^k$,$\partial_{v^i} v'^k$ $(i = 2, 3)$,

$$\begin{aligned}
\partial_{v^1} v'^1 = {} & \frac{1}{2} + \frac{a}{2}(\partial_{v^1} g)\left[\left(\omega^1 - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1\right) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1\right] \\
& + \frac{ag}{2}\left[-a^{-1}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{a^{-2}n^1\omega^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-3}(n^1)^2\right. \\
& + \frac{\partial_{v^1} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1 - \frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1 \\
& + \frac{n^0}{\sqrt{s}}\frac{a^{-2}n^1\omega^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-3}(n^1)^2 \\
& \left. + a^{-1}\frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}\right],
\end{aligned}$$

$$\begin{aligned}
\partial_{v^i} v'^1 = {} & \frac{a}{2}(\partial_{v^i} g)\left[\left(\omega^1 - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1\right) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1\right] \\
& + \frac{ag}{2}\left[-\frac{a^{-1}b^{-1}n^1\omega^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-1}b^{-2}n^1n^i\right. \\
& + \frac{\partial_{v^i} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1 - \frac{n^0}{s}(\partial_{v^i}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}a^{-1}n^1 \\
& \left. + \frac{n^0}{\sqrt{s}}\frac{a^{-1}b^{-1}n^1\omega^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-1}b^{-2}n^1n^i\right],
\end{aligned}$$

$$\begin{aligned}
\partial_{v^1} v'^k = {} & \frac{b}{2}(\partial_{v^1} g)\left[\left(\omega^k - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}b^{-1}n^k\right) + \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}b^{-1}n^k\right] \\
& + \frac{bg}{2}\left[-\frac{a^{-1}b^{-1}n^k\omega^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-2}b^{-1}n^1n^k\right. \\
& + \frac{\partial_{v^1} v^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}b^{-1}n^k - \frac{n^0}{s}(\partial_{v^1}\sqrt{s})\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}b^{-1}n^k \\
& \left. + \frac{n^0}{\sqrt{s}}\frac{a^{-1}b^{-1}n^k\omega^1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{n^0}{\sqrt{s}}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4}2a^{-2}b^{-1}n^1n^k\right],
\end{aligned}$$

$$\begin{aligned}\partial_{v^i} v'^k = & \frac{\delta^{ik}}{2} + \frac{b}{2}(\partial_{v^i} g)\Bigg[\left(\omega^k - \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} b^{-1}n^k\right) + \frac{n^0}{\sqrt{s}} \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} b^{-1}n^k\Bigg] \\ & + \frac{bg}{2}\Bigg[-\delta^{ik} b^{-1}\frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{b^{-2}n^k\omega^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} + \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4} 2b^{-3}n^k n^i \\ & + \frac{\partial_{v^i} v^0}{\sqrt{s}} \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} b^{-1}n^k - \frac{n^0}{s}(\partial_{v^i}\sqrt{s}) \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} b^{-1}n^k \\ & + \frac{n^0}{\sqrt{s}} \frac{b^{-2}n^k\omega^i}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} - \frac{n^0}{\sqrt{s}} \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^4} 2b^{-3}n^k n^i \\ & + \delta^{ik} b^{-1} \frac{n^0}{\sqrt{s}} \frac{(a^{-1}n^1, b^{-1}\bar{n}).w}{|(a^{-1}n^1, b^{-1}\bar{n})|^2}\Bigg].\end{aligned}$$

Let us bound these quantities. In order to do so, we will use estimates of derivatives of g, $\sqrt{s}$ and v^0 done in Lemma 4.3 and the fact that $|v^j - u^j| \leq |v - u|$.

Estimate of $\partial_{v^1} v'^1$,

$$\begin{aligned}|\partial_{v^1} v'^1| \leq & \frac{1}{2} + \frac{bu^0\sqrt{v^0u^0}}{|v-u|} + \frac{bu^0\sqrt{v^0u^0}}{|v-u|} \frac{a^{-1}|n|a^{-1}|n|}{b^{-2}|n|^2} \\ & + \frac{bu^0\sqrt{v^0u^0}}{|v-u|}(n^0)\frac{b\sqrt{v^0u^0}}{|v-u|}\frac{a^{-2}|n|^2}{b^{-2}|n|^2} + \sqrt{v^0u^0}\frac{a^{-1}|n|}{b^{-2}|n|^2} + \sqrt{v^0u^0}\frac{a^{-1}|n|}{b^{-2}|n|^2} \\ & + \sqrt{v^0u^0}\frac{|n|2a^{-3}|n|^2}{b^{-4}|n|^4} + \sqrt{v^0u^0}\frac{b\sqrt{v^0u^0}}{|v-u|}\frac{a^{-2}|n|^2}{b^{-2}|n|^2} + \sqrt{v^0u^0}n^0\frac{b}{2}\frac{b\sqrt{v^0u^0}}{|v-u|}\frac{a^{-2}|n|^2}{b^{-2}|n|^2} \\ & + \sqrt{v^0u^0}n^0\frac{1}{2}\frac{a^{-1}|n|}{b^{-2}n|^2} + \sqrt{v^0u^0}(v^0+u^0)\frac{1}{2}\frac{|n|2a^{-3}|n|^2}{b^{-4}|n|^4} + \sqrt{v^0u^0}n^0\frac{1}{2}\frac{a^{-1}|n|}{b^{-2}|n|^2}.\end{aligned}$$

In virtue of the above estimate, after some rearrangements, the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \text{and} \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result.

Estimate of $\partial_{v^1} v'^i$, $i = 2, 3$,

$$\begin{aligned}|\partial_{v^i} v'1| \leq & \frac{au^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^2}{a}\frac{u^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^3}{a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} + b\frac{\sqrt{v^0u^0}}{|v+u|} \\ & + \frac{2b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} + \frac{b^2}{a}\frac{v^0u^0}{|v-u|} + \frac{b^2}{2a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|} \\ & + \frac{b}{2}\frac{\sqrt{v^0u^0}(v^0+u^0)}{|v+u|} + \frac{b^2}{a}\frac{\sqrt{v^0u^0}(v^0+u^0)}{|v+u|}.\end{aligned}$$

In virtue of the above estimate, after some rearrangements, the use of the fact that

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2$$

leads to the desired result.

Estimates of $\partial_{v^1} v'^k$, $k = 2, 3$.

First of all, we remark that the following inequalities hold:

$$\left|\frac{\partial_{v^1} v^0}{\sqrt{s}}\right| \leq \frac{b}{a}\frac{\sqrt{v^0 u^0}}{|v-u|}, \qquad \left|\frac{\partial_{v^1}\sqrt{s}}{s}\right| \leq \frac{b}{2a}\frac{u^0\sqrt{v^0u^0}}{|v-u|},$$

$$|(a^{-1}n^1, b^{-1}\bar{n}).\omega| \leq a^{-1}|v+u|, \qquad \frac{1}{|(a^{-1}n^1, b^{-1}\bar{n})|^2} \leq \frac{1}{b^{-2}|v+u|^2}.$$

We then have

$$\begin{aligned}|\partial_{v^1} v'^k| \leq{}& \frac{b^2}{a}\frac{u^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^3}{a^2}\frac{u^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^4}{a^2}\frac{v^0(u^0)^2n^0}{|v-u|^2} + \frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|}\\ &+ \frac{b^4}{a^3}\frac{\sqrt{v^0u^0}}{|v+u|} + \frac{b^3}{a^2}\frac{v^0u^0}{|v-u|} + \frac{b^3}{2a^2}\frac{v^0(u^0)^2n^0}{|v-u|}\\ &+ \frac{b^2}{a}\frac{\sqrt{v^0u^0}n^0}{|v+u|} + \frac{b^4}{a^3}\frac{\sqrt{v^0u^0}n^0}{|v+u|}.\end{aligned}$$

Since

$$v^0 \geq 1, \qquad u^0 \geq 1, \qquad \frac{b^2}{a^2} \leq 2,$$

we have the desired result.

Estimate of $\partial_{v^i} v'^k$, $i = 2, 3$, $k = 2, 3$.

Using the same arguments, we have

$$\begin{aligned}|\partial_{v^i} v'k| \leq{}& \frac{1}{2} + \frac{bu^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^2}{a}\frac{u^0\sqrt{v^0u^0}}{|v-u|} + \frac{b^3}{a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|^2} + \frac{b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|}\\ &+ b\frac{\sqrt{v^0u^0}}{|v+u|} + \frac{2b^2}{a}\frac{\sqrt{v^0u^0}}{|v+u|} + \frac{b^2}{a}\frac{v^0u^0}{|v-u|} + \frac{b^2}{2a}\frac{v^0(u^0)^2(v^0+u^0)}{|v-u|}\\ &+ \frac{b}{2}\frac{\sqrt{v^0u^0}(v^0+u^0)}{|v+u|} + \frac{b^2}{a}\frac{\sqrt{v^0u^0}(v^0+u^0)}{|v+u|} + \frac{b^2}{2a}\frac{\sqrt{v^0u^0}(v^0+u^0)}{|v+u|}.\end{aligned}$$

In the same way as we did earlier, we have the desired estimate. □

REMARK 4.3. Let us observe that since $a \leq b$, we can summarise all the previous estimates to the following relations

$$|\partial_{v^i} v'^k| \leq C\left(\frac{bv^0}{|v-u|} + \frac{bv^0}{|v+u|} + \frac{b^2(v^0)^2}{|v-u|^2}\right)(u^0)^3, \qquad i = 1, 2, 3, \qquad k = 1, 2, 3.$$

4.4. Estimates of the lost term and the gain term

PROPOSITION 4.1. *Under the hypothesis* (2.27) *on the collisional cross section* $\sigma(g, \omega)$ *and the assumptions* (2.29)–(2.30) *on* a *and* b, *for any* $t \geq 0$ *and* $f \in M$, *there is a constant* c *independent on* t, x, v, *for which*

$$\int_0^t |Q_1(f, f)(\tau, v)| d\tau \leq c e^{-|v|^2} |||f|||^2, \tag{4.35}$$

$$\partial_{v^k} \left(\int_0^t |Q_1(f, f)(\tau, v)| d\tau \right) \leq c e^{-|v|^2} |||f|||^2, \qquad k = 1, 2, 3. \tag{4.36}$$

Proof: For the inequality (4.35), using (3.5) and (2.30), we have

$$\begin{aligned}
e^{|v|^2} \int_0^t & |Q_1(f, f)(\tau, v)| d\tau \\
&= \int_0^t d\tau a^{-1} b^{-2} \iint_{S_{ab} \times \mathbb{R}^3} v_\phi \sigma(g, w)(e^{|v|^2} f(v))(e^{|u|^2} f(u)) e^{-|u|^2} d\omega du \\
&\leq |||f|||^2 \int_0^t d\tau a^{-1} b^{-2} \iint_{S_{ab} \times \mathbb{R}^3} v_\phi \sigma(g, w) e^{-|u|^2} d\omega du \\
&\leq c |||f|||^2 \int_0^t d\tau a^{-1} b^{-2} \left(\int_{\mathbb{R}^3} v_\phi e^{-|u|^2} du + \int_{\mathbb{R}^3} v_\phi g^{-\beta} e^{-|u|^2} du \right) \\
&\leq c |||f|||^2 \int_0^t d\tau (a^{-1} b^{-2} + a^{-1} b^{\beta - 3}) \leq c |||f|||^2.
\end{aligned}$$

In the rest of the proof and for the next lemma, $\iint$ means $\iint_{S_{ab} \times \mathbb{R}^3}$. As for the inequality (4.36), we have

$$e^{|v|^2} \partial_{v^i} \left(\int_0^t |Q_1(f, f)(\tau, x, v)| d\tau \right) = I_1 + I_2,$$

where

$$\begin{cases} I_1 = \int_0^t a^{-1} b^{-2} \iint \partial_{v^i} [v_\phi \sigma(g, \omega)] e^{|v|^2} f(v) f(u) d\omega du d\tau, \\ I_2 = \int_0^t a^{-1} b^{-2} \iint v_\phi \sigma(g, \omega) e^{|v|^2} (\partial_{v^i} f)(v) f(u) d\omega du d\tau. \end{cases}$$

The estimate of I_2 is obvious. It gives

$$I_2 \leq c |||f|||^2 \int_0^t d\tau (a^{-1} b^{-2} + a^{-1} b^{\beta - 3}) \leq c |||f|||^2. \tag{4.37}$$

For the estimate of I_1, we separate it into two cases and we use the same reasoning as in the estimate (4.16)–(4.17).

The case $i = 1$: From the estimate (4.16) of $\partial_{v^1} [v_\phi \sigma(g, \omega)]$, we have for $i = 1$,

$$\begin{aligned}
I_1 &\leq \int_0^t d\tau a^{-2} b^{-2} \iint u^0 (1 + g^{-\beta}) \sigma_0(\omega) e^{|v|^2} f(v) f(u) d\omega du \\
&\leq c |||f|||^2 \int_0^t d\tau a^{-2} b^{-2} \iint (1 + g^{-\beta}) \sigma_0(\omega) \sqrt{1 + |u|^2} e^{-|u|^2} d\omega du \\
&\leq c |||f|||^2 \int_0^t (a^{-2} b^{-2} + a^{-2} b^{\beta - 3}) d\tau \\
&\leq c |||f|||^2 \int_0^t (a^{-1} b^{-2} + a^{-1} b^{\beta - 3}) d\tau \leq c |||f|||^2.
\end{aligned}$$

The case $i = 1, 2$: From the estimate (4.17) of $\partial_{v^k}[v_\phi \sigma(g, \omega)]$,

$$I_1 \leq \int_0^t d\tau a^{-1} b^{-3} \iint u^0 (1 + g^{-\beta}) \sigma_0(\omega) e^{|v|^2} f(v) f(u) d\omega du$$

$$\leq c |||f|||^2 \int_0^t d\tau a^{-1} b^{-3} \iint (1 + g^{-\beta}) \sigma_0(\omega) \sqrt{1 + |u|^2} e^{-|u|^2} d\omega du$$

$$\leq c |||f|||^2 \int_0^t (a^{-1} b^{-3} + a^{-1} b^{\beta - 4}) d\tau \leq c |||f|||^2. \qquad \square$$

PROPOSITION 4.2. *Under the hypothesis* (2.27) *on the collisional cross section* $\sigma(g, \omega)$ *and the assumptions* (2.29)–(2.30) *on* a *and* b, *for any* $t \geq 0$ *and* $f \in M$, *there is a constant* c *independent on* t, x, v, *for which*

$$\int_0^t |Q_g(f, f)(\tau, v)| d\tau \leq c e^{-|v|^2} |||f|||^2, \tag{4.38}$$

$$\partial_{v^i} \left(\int_0^t |Q_g(f, f)(\tau, v)| d\tau \right) \leq c e^{-|v|^2} |||f|||^2, \qquad i = 1, 2, 3. \tag{4.39}$$

Proof: As for the inequality (4.38), let us remind that

$$\int_{\mathbb{R}^3} v_\phi g^{-\beta} e^{-|u|^2} du \leq C b^{\beta - 1}.$$

So, since

$$|v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq C$$

where C is a positive constant, if we let

$$I_g = e^{|v|^2} \int_0^t |Q_g(f, f)(\tau, x, v)| d\tau,$$

by direct computation, we have

$$I_g \leq \int_0^t d\tau a^{-1} b^{-2} |||f|||^2 \iint v_\phi \sigma(g, w) e^{|v|^2 + |u|^2 - |v'|^2 - |u'|^2} e^{-|u|^2} d\omega du$$

$$\leq \int_0^t d\tau a^{-1} b^{-2} |||f|||^2 \iint v_\phi \sigma(g, w) e^{-|u|^2} d\omega du$$

$$\leq c \int_0^t d\tau a^{-1} b^{-2} |||f|||^2 \left(\int_{\mathbb{R}^3} v_\phi e^{-|u|^2} du + \int_{\mathbb{R}^3} v_\phi g^{-\beta} e^{-|u|^2} du \right)$$

$$\leq c |||f|||^2 \int_0^t (a^{-1} b^{-2} + a^{-1} b^{\beta - 3}) d\tau \leq c |||f|||^2. \tag{4.40}$$

As expected, the derivatives of gain term is much more difficult to handle. First, we have

$$e^{|v|^2} \partial_{v^i} \left(\int_0^t |Q_g(f, f)(\tau, v)| d\tau \right) = J_1 + J_2, \tag{4.41}$$

where J_1 and J_2 are defined as:

$$\begin{cases} J_1 = \int_0^t d\tau a^{-1} b^{-2} \iint \partial_{v^i}[v_\phi \sigma(g,\omega)] e^{|v|^2} f(v') f(u') d\omega du, \\ J_2 = \int_0^t d\tau a^{-1} b^{-2} \iint v_\phi \sigma(g,\omega) e^{|v|^2} \partial_{v^i}[f(v') f(u')] d\omega du. \end{cases}$$

After bounding $\partial_{v^i}[v_\phi \sigma(g,\omega)]$, the estimate of J_1 is done easily by following the estimate of $\int_0^t |Q_{\mathrm{g}}(f,f)(\tau,x,v)| d\tau$.

As for the estimate of J_2, let us observe that

$$\partial_{v^i}[f(v')f(u')] = f(u') \sum_{k=1}^{3} (\partial_{v^i} v'^k)(\partial_{v^k} f)(v') + f(v') \sum_{k=1}^{3} (\partial_{v^i} u'^k)(\partial_{v^k} f)(u'). \quad (4.42)$$

We let

$$j_2(t) = a^{-1} b^{-2} \iint v_\phi \sigma(g,\omega) e^{|v|^2} \partial_{v^i}[f(v')f(u')] d\omega du$$

and we fix a momentum v. We notice that $a(t)$ and $b(t)$ are increasing functions with $a(0) = 1$. Then it exists a finite time t_0 such that: $t \geq t_0$ if and only if $|v| \leq a(t)$. We break up the estimate of $j_2(t)$ into a number of steps.

Step 1: $t \geq t_0$. From the relations $|v| \leq a(t)$ and (4.18) allowing the estimate of derivatives of the post-collisional momenta (2.22)–(2.23), we have:

$$|\partial_{v^i} v'^k| \leq c\sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2}(u^0)^4 \leq c\sqrt{1 + a^{-2}|v|^2} \leq c(u^0)^4.$$

In this case, to control $j_2(t)$ we use the same reasoning which allowed us to control I_g. This leads to

$$|j_2(t)| \leq c\|f(t)\|^2 (a^{-1}b^{-2} + a^{-1}b^{\beta-3}). \quad (4.43)$$

Step 2: $t < t_0$ and $|v| \leq 2|u|$. In this case we have

$$v^0 = \sqrt{1 + a^{-2}(v^1)^2 + b^{-2}|\bar{v}|^2} \leq \sqrt{1 + a^{-2}|v|^2} \leq 2u^0. \quad (4.44)$$

From (2.22)–(2.23), all the term $|\partial_{v^i} v'^k|$ are controlled by $c(u^0)^5$ and $|j_2(t)|$ is exactly controlled as in the first step.

Step 3: $t < t_0$ and $|v| \geq 2|u|$. In this case, instead of the parametrization (2.22)–(2.23), we use (2.25)–(2.26). From the relation $|v| \geq 2|u|$, it follows that

$$|v - u| \geq \frac{1}{2}|v|, \qquad |v + u| \geq \frac{1}{2}|v|. \quad (4.45)$$

From the estimates (4.33)–(4.34), using the assumption $a(t) \leq b(t) \leq \sqrt{2}a(t)$, a straightforward computation allows us to control all the term $|\partial_{v^i} v'^k|$ by $c(u^0)^3$. Finally, $|j_2(t)|$ is exactly controlled as in the first step.

Finally, we integrate $j_2(\tau)$ over $[0,t]$. This leads to the estimate of J_2. $\square$

4.5. The main result

Our main result is stated as follows.

THEOREM 4.1. *Consider the relativistic Boltzmann equation in the Bianchi type I space-time in the form of* (2.1). *Suppose that the scattering kernel satisfies* (2.27)–(2.28), *and let the coefficients a and b be given and satisfy* (2.29)–(2.30). *Let* f_0 *be an initial data such that it is differentiable and satisfies* $\|f_0\| \le r_0$ *for some positive constant* r_0. *If* r_0 *is sufficiently small, then there exists a unique nonnegative classical solution of the Boltzmann equation* (2.13) *such that* $\sup_{t\in\mathbb{R}_+} \|f(t)\| \le C_{r_0}$ *where* C_{r_0} *is some positive constant depending on* r_0.

Proof: Proving the main theorem is equivalent to proving the existence and uniqueness solution of the integral equation (2.13). In order to do so, we are going to use the fixed point theorem. We define the map Υ from Λ by

$$\Upsilon(f)(t, v) = f_0(v) + \int_0^t Q(f, f)(\tau, v)d\tau. \tag{4.46}$$

If we let $\Lambda_{r_0} = \{f \in \Lambda, |||f||| \le r_0\}$, suppose that $\|f_0\| \le r_0/2$ and $f \in \Lambda_{r_0}$, from (4.46) and the relation

$$\partial_{v^i}\Upsilon(f)(t, v) = \partial_{v^i} f_0 + \partial_{v^i} \int_0^t Q(f, f)(\tau, v)d\tau,$$

we have the following two inequalities for any (t, v):

$$|\Upsilon(f)(t, v)| \le e^{-|v|^2}\|f_0\| + ce^{-|v|^2}|||f|||^2 \le e^{-|v|^2}\left[\frac{r_0}{2} + cr_0^2\right], \tag{4.47}$$

$$|\partial_{v^i}\Upsilon(f)(t, v)| \le e^{-|v|^2}\|f_0\| + ce^{-|v|^2}|||f|||^2 \le e^{-|v|^2}\left[\frac{r_0}{2} + cr_0^2\right]. \tag{4.48}$$

Thus, if

$$\frac{r_0}{2} + cr_0^2 \le r_0, \quad \text{i.e } r_0 \le \frac{1}{2c},$$

after multiplying (4.47) and (4.48) by $e^{-|v|^2}$ and taking the upper bounds with respect to t and v, it follows that Υ is a map from Λ_{r_0} to itself.

On the other hand, using the bilinearity of Q, we prove in such situation that Υ is a contraction. In fact, if $\|f_0\| \le r_0/2$ and $f, g \in \Lambda_{r_0}$, then

$$|\Upsilon f(t, v) - \Upsilon g(t, v)| \le ce^{-|v|^2}(\|f\| + \|g\|)\|f - g\| \le 2cr_0e^{-|v|^2}\|f - g\|, \tag{4.49}$$

$$|\partial_{v^i}\Upsilon f(t, v) - \partial_{v^i}\Upsilon g(t, v)| \le ce^{-|v|^2}(\|f\| + \|g\|)\|f - g\| \le 2cr_0e^{-|v|^2}\|f - g\|. \tag{4.50}$$

The desired result is obtained if $2cr_0 < 1$. In fact, if $r_0 < \frac{1}{2c}$, after multiplying (4.49) and (4.50) by $e^{-|v|^2}$ and taking the upper bounds with respect to t and v, it follows that Υ is a contraction. So, using the fixed point theorem, we claim that the desired result is proved. □

Summary

We have studied the relativistic Boltzmann equation in a spatially homogeneous Bianchi type I space-time. We have proved the global existence of solutions in a suitable weighted space.

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