REPUBLIQUE DU CAMEROUN
Paix – Travail – Patrie
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UNIVERSITE DE YAOUNDE I
Faculté des Sciences
DEPARTEMENT DE MATHEMATIQUES
****
de Mathématiques Appliquées

REPUBLIC OF CAMEROUN
Peace – Work – Fatherland
****
UNIVERSITY OF YAOUNDE I
Faculty of Science
DEPARTMENT OF MATHEMATICS
****
of Applied Mathematics

A jump-diffusion model for pricing electricity under price cap regulation and parameters estimation

THESIS
submitted to the Postgraduate School of Science, Technology and Geoscience, University of Yaoundé I, Cameroon,
in fulfilment of the requirements for the degree of PhD in Mathematics

Par : **Kegnenlezom Martin**

Sous la direction de
Mr. Emvudu Wono Yves:
Professor, University of Yaoundé I
Mr. Andjiga Nicolas Gabriel:
Professor, University of Yaoundé I

Année Académique :

RÉPUBLIQUE DU CAMEROUN
PAIX-TRAVAIL-PATRIE
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REPUBLIC OF CAMEROON
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POSTGRADUATE SCHOOL OF SCIENCE, TECHNOLOGY AND GEOSCIENCES
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DÉPARTEMENT DE MATHÉMATIQUES
DEPARTMENT OF MATHEMATICS

ATTESTATION DE CORRECTION DE LA THÈSE DE DOCTORAT / Ph.D

Nous soussignés, **CHAMENI Célestin, Pr., UYII, Président du jury** ; **EMVUDU Yves, Pr., UYI, Rapporteur** ; **FOTSO Simeon, MC, UYI , Examinateur,** membres du jury de la thèse de Doctorat / Ph.D présenté par **M. KEGNENLEZOM Martin, Matricule 04T177**, intitulé: « **A jump-diffusion model for pricing electricity under price-cap regulation and parameters estimation** » et soutenue en vue de l'obtention du diplôme de Doctorat / Ph.D **en mathématiques**, Spécialité **: Mathématiques et Applications Fondamentales,** Option : **Probabilités-Statistiques-Finance**, attestons que toutes les corrections demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

En foi de quoi la présente attestation lui est délivrée pour servir et valoir ce que de droit.

Président	**Rapporteurs**	**Examinateurs**
CHAMENI Célestin, Pr., UYII	EMVUDU Yves, Pr., UYI	FOTSO Siméon, MC, UYI

Université de Yaoundé I
Centre de Recherche et de Formation Doctorale en Science, Technologies et Geosciences
Unité de Recherche et de Formation Doctorale en Mathématiques Informatiques, Bioinformatiques et Applications

University of Yaoundé I
Postgraduate School of Science, Technology and Geoscience
Research and Training Unit for Doctorate in Mathematics Computer Sciences, Bioinformatics and Applications

DEPARTMENT OF MATHEMATICS
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Laboratory of Applied Mathematics
Laboratoire de Mathématiques Appliquées

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THESIS

submitted to the Postgraduate School of Science, Technology and Geoscience,
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Option: Probability-Statistics-Finance
Option: Probabilités-Statistiques-Finance

by:

Kegnenlezom Martin
Registration number: 04T177

Directors:
Mr. Emvudu Wono Yves: Professor, University of Yaoundé I
Mr. Andjiga Nicolas Gabriel: Professor, University of Yaoundé I

Year: 2021

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LISTE DES ENSEIGNANTS PERMANENTS		**LIST OF PERMANENT TEACHING STAFF**

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(Par Département et par Grade)
DATE D'ACTUALISATION 12 Juin 2020

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32	MANANGA Marlyse Joséphine	Chargée de Cours	En poste
33	MBONG ANGIE M. Mary Anne	Chargée de Cours	En poste
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37	OWONA AYISSI Vincent Brice	Assistant	En poste
38	WILFRIED ANGIE Abia	Assistante	En poste

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9	NJIOKOU Flobert	Professeur	En Poste
10	NOLA Moïse	Professeur	En poste
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21	NGUEGUIM TSOFACK Florence	Maître de Conférences	En poste
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26	ATSAMO Albert Donatien	Chargé de Cours	En poste
27	BELLET EDIMO Oscar Roger	Chargé de Cours	En poste
28	DONFACK Mireille	Chargée de Cours	En poste
29	ETEME ENAMA Serge	Chargé de Cours	En poste
30	GOUNOUE KAMKUMO Raceline	Chargée de Cours	En poste
31	KANDEDA KAVAYE Antoine	Chargé de Cours	En poste

32	LEKEUFACK FOLEFACK Guy B.	Chargé de Cours	En poste
33	MAHOB Raymond Joseph	Chargé de Cours	En poste
34	MBENOUN MASSE Paul Serge	Chargé de Cours	En poste
35	MOUNGANG Luciane Marlyse	Chargée de Cours	En poste
36	MVEYO NDANKEU Yves Patrick	Chargé de Cours	En poste
37	NGOUATEU KENFACK Omer Bébé	Chargé de Cours	En poste
38	NGUEMBOK	Chargé de Cours	En poste
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40	NOAH EWOTI Olive Vivien	Chargée de Cours	En poste
41	TADU Zephyrin	Chargé de Cours	En poste
42	TAMSA ARFAO Antoine	Chargé de Cours	En poste
43	YEDE	Chargé de Cours	En poste

44	BASSOCK BAYIHA Etienne Didier	Assistant	En poste
45	ESSAMA MBIDA Désirée Sandrine	Assistante	En poste
46	KOGA MANG DOBARA	Assistant	En poste
47	LEME BANOCK Lucie	Assistante	En poste
48	YOUNOUSSA LAME	Assistant	En poste

3- DÉPARTEMENT DE BIOLOGIE ET PHYSIOLOGIE VÉGÉTALES (BPV) (33)

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3	DJOCGOUE Pierre François	Professeur	En poste
4	MOSSEBO Dominique Claude	Professeur	En poste
5	YOUMBI Emmanuel	Professeur	Chef de Département
6	ZAPFACK Louis	Professeur	En poste

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10	MALA Armand William	Maître de Conférences	En poste
11	MBARGA BINDZI Marie Alain	Maître de Conférences	CT/ MINESUP
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16	TSOATA Esaïe	Maître de Conférences	En poste
17	TONFACK Libert Brice	Maître de Conférences	En poste

18	DJEUANI Astride Carole	Chargé de Cours	En poste
19	GOMANDJE Christelle	Chargée de Cours	En poste
20	MAFFO MAFFO Nicole Liliane	Chargé de Cours	En poste
21	MAHBOU SOMO TOUKAM. Gabriel	Chargé de Cours	En poste
22	NGALLE Hermine BILLE	Chargée de Cours	En poste
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24	NNANGA MEBENGA Ruth Laure	Chargé de Cours	En poste
25	NOUKEU KOUAKAM Armelle	Chargé de Cours	En poste
26	ONANA JEAN MICHEL	Chargé de Cours	En poste
27	GODSWILL NTSOMBAH NTSEFONG	Assistant	En poste
28	KABELONG BANAHO Louis-Paul-Roger	Assistant	En poste
29	KONO Léon Dieudonné	Assistant	En poste
30	LIBALAH Moses BAKONCK	Assistant	En poste
31	LIKENG-LI-NGUE Benoit C	Assistant	En poste
32	TAEDOUNG Evariste Hermann	Assistant	En poste
33	TEMEGNE NONO Carine	Assistant	En poste

4- DÉPARTEMENT DE CHIMIE INORGANIQUE (CI) (34)

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2	ELIMBI Antoine	Professeur	En poste
3	Florence UFI CHINJE épouse MELO	Professeur	*Recteur Univ.Ngaoundere*
4	GHOGOMU Paul MINGO	Professeur	*Ministre Chargé deMiss.PR*
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6	NDIFON Peter TEKE	Professeur	*CT MINRESI*
7	NGOMO Horace MANGA	Professeur	*Vice Chancelor/UB*
8	NDIKONTAR Maurice KOR	Professeur	*Vice-Doyen Univ. Bamenda*
9	NENWA Justin	Professeur	En poste
10	NGAMENI Emmanuel	Professeur	*DOYEN FS UDs*

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12	DJOUFAC WOUMFO Emmanuel	Maître de Conférences	En poste
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30	PABOUDAM GBAMBIE A.	Chargée de Cours	En poste
31	TCHAKOUTE KOUAMO Hervé	Chargé de Cours	En poste
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33	PATOUOSSA ISSOFA	Assistant	En poste
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15	MBAZOA née DJAMA Céline	Maître de Conférences	En poste
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17	NOTE LOUGBOT Olivier Placide	Maître de Conférences	Chef Service/MINESUP
18	NGO MBING Joséphine	Maître de Conférences	Sous/Direct. MINERESI
19	NGONO BIKOBO Dominique Serge	Maître de Conférences	En poste
20	NOUNGOUE TCHAMO Diderot	Maître de Conférences	En poste
21	TABOPDA KUATE Turibio	Maître de Conférences	En poste
22	TCHOUANKEU Jean-Claude	Maître de Conférences	*Doyen /FS/ UYI*
23	TIH née NGO BILONG E. Anastasie	Maître de Conférences	En poste
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26	KAMTO Eutrophe Le Doux	Chargé de Cours	En poste
27	MVOT AKAK CARINE	Chargé de Cours	En poste
28	NGNINTEDO Dominique	Chargé de Cours	En poste
29	NGOMO Orléans	Chargée de Cours	En poste
30	OUAHOUO WACHE Blandine M.	Chargée de Cours	En poste
31	SIELINOU TEDJON Valérie	Chargé de Cours	En poste
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33	ZONDENDEGOUMBA Ernestine	Chargée de Cours	En poste

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22	MAKEMBE. S . Oswald	Assistant	En poste
23	MESSI NGUELE Thomas	Assistant	En poste
24	MEYEMDOU Nadège Sylvianne	Assistante	En poste
25	NKONDOCK. MI. BAHANACK.N.	Assistant	En poste

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4	MBANG Joseph	Maître de Conférences	En poste
5	MBELE BIDIMA Martin Ledoux	Maître de Conférences	En poste
6	NKUIMI JUGNIA Célestin	Maître de Conférences	En poste
7	NOUNDJEU Pierre	Maître de Conférences	*Chef service des programmes & Diplômes*
8	MBEHOU Mohamed	Maître de Conférences	En poste
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10	TCHOUNDJA Edgar Landry	Maître de Conférences	En poste

11	AGHOUKENG JIOFACK Jean Gérard	Chargé de Cours	Chef Cellule MINPLAMAT
12	CHENDJOU Gilbert	Chargé de Cours	En poste
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14	DOUANLA YONTA Herman	Chargé de Cours	En poste
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16	KIKI Maxime Armand	Chargé de Cours	En poste
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20	NIMPA PEFOUKEU Romain	Chargée de Cours	En poste
21	POLA DOUNDOU Emmanuel	Chargé de Cours	En poste
22	TAKAM SOH Patrice	Chargé de Cours	En poste
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25	TIAYA TSAGUE N. Anne-Marie	Chargée de Cours	En poste
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28	MBATAKOU Salomon Joseph	Assistant	En poste
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12	TCHIKOUA Roger	Chargé de Cours	En poste
13	ESSONO Damien Marie	Assistant	En poste
14	LAMYE Glory MOH	Assistant	En poste
15	MEYIN A EBONG Solange	Assistante	En poste
16	NKOUDOU ZE Nardis	Assistant	En poste
17	SAKE NGANE Carole Stéphanie	Assistante	En poste
18	TOBOLBAÏ Richard	Assistant	En poste

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4	KOFANE Timoléon Crépin	Professeur	En poste

5	NANA ENGO Serge Guy	Professeur	En poste
6	NDJAKA Jean Marie Bienvenu	Professeur	Chef de Département
7	NOUAYOU Robert	Professeur	En poste
8	NJANDJOCK NOUCK Philippe	Professeur	*Sous Directeur/ MINRESI*
9	PEMHA Elkana	Professeur	En poste
10	TABOD Charles TABOD	Professeur	Doyen Univ/Bda
11	TCHAWOUA Clément	Professeur	En poste
12	WOAFO Paul	Professeur	En poste

13	BIYA MOTTO Frédéric	Maître de Conférences	DG/HYDRO Mekin
14	BODO Bertrand	Maître de Conférences	En poste
15	DJUIDJE KENMOE épouse ALOYEM	Maître de Conférences	En poste
16	EYEBE FOUDA Jean sire	Maître de Conférences	En poste
17	FEWO Serge Ibraïd	Maître de Conférences	En poste
18	HONA Jacques	Maître de Conférences	En poste
19	MBANE BIOUELE César	Maître de Conférences	En poste
20	NANA NBENDJO Blaise	Maître de Conférences	En poste
21	NDOP Joseph	Maître de Conférences	En poste
22	SAIDOU	Maître de Conférences	MINERESI
23	SIEWE SIEWE Martin	Maître de Conférences	En poste
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25	VONDOU Derbetini Appolinaire	Maître de Conférences	En poste
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32	MBINACK Clément	Chargé de Cours	En poste
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Répartition chiffrée des Enseignants de la Faculté des Sciences de l'Université de Yaoundé I

NOMBRE D’ENSEIGNANTS					
DÉPARTEMENT	**Professeurs**	**Maîtres de Conférences**	**Chargés de Cours**	**Assistants**	**Total**
BCH	9 (1)	13 (09)	14 (06)	3 (2)	**39 (18)**
BPA	13 (1)	09 (06)	19 (05)	05 (2)	**46 (14)**
BPV	06 (0)	11 (02)	9 (06)	07 (01)	**33 (9)**
CI	10 (1)	9 (02)	12 (02)	03 (0)	**34 (5)**
CO	7 (0)	17 (04)	09 (03)	02 (0)	**35(7)**
IN	2 (0)	1 (0)	13 (01)	09 (01)	**25 (2)**
MAT	1 (0)	5 (0)	19 (01)	05 (02)	**30 (3)**
MIB	1 (0)	5 (02)	06 (01)	06 (02)	**18 (5)**
PHY	12 (0)	15 (02)	10 (03)	03 (0)	**40 (5)**
ST	8 (1)	14 (01)	19 (05)	02 (0)	**43(7)**
Total	**69 (4)**	**99 (28)**	**130 (33)**	**45 (10)**	**343 (75)**

Soit un total de **344 (**75**)** dont :

- Professeurs **68 (4)**
- Maîtres de Conférences **99 (**28**)**
- Chargés de Cours **130 (33)**
- Assistants **46** (10**)**

( ) = Nombre de Femmes **75**

Dedication

To my parents and my family for all sacrifices they made in order to ensure my schooling.

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I am highly grateful to the Almighty God for making it possible for me to complete this thesis.

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I will not end without thanking my wife Mrs Kengne Nzonde Christelle, my children Herman and Christagie, and the rest of the family for morale, spiritual and financial support all along this thesis.

Abstract

One of the major developments in the electricity sector is the functional separation of the industry into three main phases, namely: the generation, transmission and distribution phases. This reform has leat to the openness of the electricity industry to competition, with the progressive replacement of state monopoly in favour of independent producers, and recently to the opening of deregulated or free electricity markets. Markets regulated by economic principles such as "price cap" or "revenue cap" to cap the fluctuation of the electricity prices were also introduced . In this thesis, we propose a new model for pricing electricity based on the price cap principle and derive some financial derivatives. To achieve the goal we divided the work into three parts.

In the first part, we modelling the dynamics of spot electricity prices under price cap regulated market.The particularity of the model is that the asset price is an exponential functional of a jump Lévy process. This model can capture both mean reversion and jumps which are observed in electricity market.

In the second part, we derive the forward contract and the European option using two approaches. The first is based on the use of Fourier transforms, while the other uses the price of the option as a solution of an integro-differential equation (PIDE). It is shown that the value of an European option of this asset is the unique viscosity solution of a partial integro-differential equation. A numerical approximation of this solution by the finite differences method is provided. The consistency, stability and convergence results of the scheme are given. Numerical simulations are performed under a smooth initial condition.

In the last part we propose a maximum likelihood approach for estimating the parameters of the model via estimating the transition density by the saddlepoint method.

Keywords: Mean reverting, jump-diffusion, option pricing, price-cap, integro-differential equation, viscosity solution, parameters estimation, saddlepoint

Résumé

L'une des principales évolutions dans le secteur de l'électricité est la séparation fonctionnelle de l'industrie en trois phases principales, à savoir : la production, la transmission et la distribution. Cette réforme a conduit à l'ouverture du secteur de l'électricité à la concurrence, avec le remplacement progressif du monopole d'État en faveur des producteurs indépendants, et récemment à l'ouverture de marchés de l'électricité déréglementés ou libres. Des marchés régulés par des principes économiques tels que le "price cap" ou le "revenue cap" pour limiter la fluctuation des prix de l'électricité ont également été introduits. Dans cette thèse, nous proposons un nouveau modèle de tarification de l'électricité basé sur le principe du price cap et en déduisons quelques dérivés financiers. Pour atteindre cet objectif, nous avons divisé le travail en trois parties.

Dans la première partie, nous modélisons la dynamique des prix spot de l'électricité dans le cadre d'un marché réglementé par plafonnement des prix. La particularité du modèle est que le prix de l'actif est une fonction exponentielle d'un processus de L'evy à saut. Ce modèle peut aussi capturer à la fois la propriété de moyenne renversante et les sauts qui sont observés sur le marché de l'électricité.

Dans la deuxième partie, nous dérivons le contrat à terme et l'option européenne en utilisant deux approches. La première est basée sur l'utilisation des transformées de Fourier, tandis que l'autre utilise le prix de l'option comme solution d'une équation intégro-différentielle (PIDE). On montre que la valeur d'une option européenne de cet actif est la solution unique de viscosité d'une équation intégro-différentielle partielle. Une approximation numérique de cette solution par la méthode des différences finies est fournie. Les résultats de cohérence, de stabilité et de convergence du schéma sont donnés. Des simulations numériques sont effectuées avec une condition initiale lisse.

Dans la dernière partie, nous proposons une approche de maximum de vraisemblance pour estimer les paramètres du modèle via l'estimation de la densité de transition par la méthode du point de selle.

Mots clés: Retour à la moyenne, saut de diffusion, prix des options, prix plafond, équation

intégro-différentielle, solution de viscosité, estimation des paramètres, point de selle

Contents

List of Figures

List of Tables

General introduction

Electricity pricing problem

Electricity is generated at power plants and moves through a complex system (sometimes called the grid of electricity substations), involving transformers and power lines that connect electricity producers and consumers. Most local grids are interconnected for reliability and commercial purposes, forming larger and more dependable networks that enhance the coordination and planning of electricity supply.

Cameroon is second in terms of hydro-electric potential in Sub-Saharan Africa, with an estimated 20 GW electricity power it can produce per year. Despite the institutional reforms of 1998, electricity costs in Cameroon is increasing more and more. In Cameroon, the law and the regulatory framework provide that it is Electricity Regulatory Agency that sets electricity tariffs on the proposal of the operator.

Pricing remains a central issue in the energy production system because it is an important determinant of justice, equity and economic viability. The demand for electricity, which is constantly changing, is confronted by tariff measures which, depending on the country, are a concern for consumer organizations. According to Piebalgs (2014), no economic development is possible without the availability of electricity. Hence, the introduction of competitive prices through the opening of the electricity market is one of the solution proposed. Electricity market has not only brought new opportunities in electricity industries, but also, challenges in electricity pricing for companies which are now exposed to price risks characterized by volatility, jumps and peaks. In addition, electricity spot prices in emerging power markets are volatile, a consequence of the unstorable nature of electricity. Uncontrolled exposure to market price risks can lead to devastating consequences for market participants in the restructured electricity industry. Lessons learned from the financial market suggest that financial derivatives, when well understood and properly utilized, are beneficial to the sharing and controlling of undesired risks through hedging strategies. This pricing problem requires a new solution when we are in the presence of a new spot price model, which is the case in this thesis.

Evolution of pricing reforms

For a long time, electricity, unlike other commodities, was characterized by a flat rate without obeying the demand/supply rule. This pricing structure clearly distorts reality. In effect, electricity production may vary considerably, depending on the time of day, the week of the month and even the seasons of the year. Thereafter, electricity pricing was initially approached essentially from an economic point of view, with one of the main objectives being to improve the conditions of access to electricity (coverage and distribution of costs, investment incentives, etc.). The economic models used for this purpose have evolved over time to include mathematical models. Two main mechanisms have governed the evolution of electricity pricing, namely, *regulation* and *deregulation*, which lead to several pricing modes such as tranche pricing, real time pricing, increasing-block pricing, price cap pricing and dynamic pricing.

Objective of the thesis

The objective of our thesis is to propose a mathematical model for pricing electricity derivatives where the underlying asset spot price is regulated by price cap regulation.

Mathematical option pricing models in electricity: A review

Modelling electricity price is the most crucial component in pricing electricity derivatives. Due to the unique physical and operational characteristics of electricity production and transmission processes, electricity price exhibits a behaviour different from other financial prices. The development of electricity financial and physical instruments evolve together with electricity spot price modelling. We start by reviewing some existing electricity pricing models.

Diffusion model

Pilipovic model (Pilipovic (2007))

Merton (1976) was among the first to develop a diffusion model, but used it in the context of stock markets. Pilipovic (2007) adapted the model to electricity market. His mathematical model took into account the mean reversion property, an important electricity characteristic. He obtained the following stochastic differential system of equations:

$$\begin{aligned} \mathrm{d}S_t &= \alpha(L_t - S_t)\mathrm{d}t + S_t\sigma\mathrm{d}z_t \\ \mathrm{d}L_t &= \mu L_t\mathrm{d}t + \xi L_t\mathrm{d}w_t, \end{aligned}$$

where

- S_t represents spot price at time t,
- L_t is the equilibrium price,
- μ is rate of return or drift rate
- α represents the Mean reversion rate,
- σ is the volatility,
- ξ represents the volatility of equilibrium value a long run,
- w and z are independent Brownian motions.

Lucia and Schwartz model Lucía and Schwartz (2002)

Another classical model along these lines is the exponential Ornstein-Uhlenbeck process suggested by Lucía and Schwartz (2002) in the electricity market, with inspiration from Schwartz (1997). They considered spot price as a stochastic process with two components represented by

$$P_t = f(t) + X_t \ \ t \in [0, \infty), \tag{1}$$

where f is a deterministic differentiable function, and X_t is the stochastic component satisfying

$$dX_t = -\alpha X_t dt + \sigma dW_t, \tag{2}$$

with $\alpha > 0$ representing the speed of mean-reversion, $X(0) = x_0$ is the initial condition and W_t is a standard Brownian motion. Applying Itô's formula on (2), they obtained the following Vasicek form for spot price:

$$dP_t = \alpha(a(t) - P_t)dt + \sigma dW_t, \tag{3}$$

where

$$a(t) = \frac{1}{\alpha}f'(t) + f(t).$$

In the same paper, they also considered $\log$ of spot prices i.e. $\ln P_t = f(t) + Y_t$ where Y_t follows process (2). In this case they obtained

$$dP_t = \alpha(b(t) - \ln P_t)P_t dt + \sigma dW_t, \tag{4}$$

where $\ln P_t = f(t) + Y_t$,

$$b(t) = \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + f'(t)\right) + f(t).$$

This model captures the mean-reverting feature which is one of the main characteristics of electricity, but does not take into account spikes which can occur in electricity markets.

Carlos Blanco-David Soronow model (Blanco and Soronow (2001))

Blanco and Soronow (2001), in their first model, took into account the mean reverting property by modelling the spot price as follow:

$$\underbrace{S_{t+1} - S_t}_{\text{price change}} = \underbrace{\alpha S_t \Delta t + S_t \sigma \varepsilon_{1t} \sqrt{\Delta t}}_{geometric\ Brownian\ motion},$$

where

- S^* represents equilibrium length price,
- S_t is the spot price at time t,
- ε_{1t} is a normally distributed random variable,
- σ represents price volatility,

- α is the fluctuation rate.

This first model fails to take into account sharp increase in prices during periods of stress, translated into electricity models by jumps or spikes. To account for that, they developed a *jump diffusion model* (also called Lévy model) given by

$$\underbrace{S_{t+1} - S_t}_{\text{price change}} = \underbrace{\alpha(S^* - S_t)\Delta t}_{\text{drift } component} + \underbrace{S_t \sigma \varepsilon_{1t}\sqrt{\Delta t}}_{\text{diffusion } component} + \underbrace{\eta[S_t(\kappa + \delta\varepsilon_{2t})]}_{\text{jump } component},$$

where ε_{2t} this is a normally distributed random jump. In the same line Clewlow et al. (2001) used different technic to take into account spikes and jumps, by proposing the following jump diffusion model:

$$\mathrm{d}S_t = \alpha(b - \ln S_t)\mathrm{d}t + \sigma S_t \mathrm{d}z_t + kS_t \mathrm{d}\hat{q},$$

where $\hat{q}$ is a Poisson process and k is random variable such that $\log(1+k)$ standard normally distributed. This model captures the mean reverting property and jumps in electricity prices, but fails to take into account seasonality.

Jump diffusion model

Cartea-Figueroa jump diffusion model Cartea and Figueroa (2007)

Cartea and Figueroa (2007) in the context of deregulation markets extended Schwartz (1997) model by adding a jump term, giving a mean-reverting and jump-diffusion model. They also took into account seasonality from $\ln S_t = g(t) + Y_t$, where g is a seasonal deterministic function and that Y_t is a stochastic process given by

$$dY_t = -\alpha Y_t dt + \sigma dW_t + \ln J dq_t.$$

Their complete model then takes the form:

$$dS_t = \alpha(\rho(t) - \ln S_t)S_t dt + \sigma(t) S_t dW_t + S_t(J-1)dq_t, \tag{5}$$

where J is the proportional size of jump, q_t is a Poisson process and

$$\rho(t) = \frac{1}{\alpha}\left(g'(t) + \frac{1}{2}\sigma^2(t)\right) + g(t).$$

Barlow model (Barlow (2002))

From demand (or supply) model for electricity, Barlow (2002) obtained a jump diffusion model for spot price that can exhibit price spikes. The demand model is

$$\begin{aligned} D_t &= \overline{D_t} + X_t \\ \mathrm{d}X_t &= (\mu + \lambda X_t)\mathrm{d}t + \sigma \mathrm{d}w_t, \end{aligned}$$

where $\overline{D_t}$ is the seasonal demand component, X_t is the stationary stochastic process. Then Barlow model takes the form

$$P_t = \begin{cases} \left(\frac{a_0 - D_t}{b_0}\right)^{1/\alpha}, & D_t < a_0 - b_0\varepsilon \\ \varepsilon^{1/\alpha}, & D_t \leq a_0 - b_0\varepsilon, \end{cases}$$

where $g(x) = a_0 - b_0 x^\alpha$ and $P_t = g^{-1}(D_t)$.

Multi-factor jump-diffusion model

Multi-factor model have recently been introduced in deregulated electricity spot markets. Benth et al. (2011) was among the first.

Benth, Kiesel and Nazarova model (Benth et al. (2011))

Benth et al. (2011) used three different classes of models for electricity pricing in Germany. To start with, they proposed the following model:

$$\begin{aligned} S(t) &= e^{\mu(t)} X(t) \\ \mathrm{d}\ln(X(t)) &= -\alpha \ln X(t) dt + \sigma(t) dW(t) + \ln J \mathrm{d}q(t), \end{aligned}$$

where

- $X(t)$ models random fluctuation of price,
- α is the fluctuate parameter around average price,
- $\sigma(t)$ is the time depending volatility,
- J is the proportional random jump size, $\ln J \sim \mathcal{N}(\mu_J, \sigma_J^2)$,

- dq represents the increments of Poisson process define by

$$\mathrm{d}q(t) = \begin{cases} 1, \text{with probability} & Idt \\ 0, \text{with probability} & 1 - Idt \end{cases}$$

- I represents intensity of peaks.

This proposed model captures the three most important electricity characteristics. But to take into account the direction of jumps and spikes of the electricity spot price, they extended the preceding model to:

$$\begin{aligned} S(t) &= e^{\mu(t)} X(t) \\ \mathrm{d} \ln(X(t) &= -\theta_1 \ln X(t) \mathrm{d}t + \sigma(t) \mathrm{d}W(t) + h(\ln X(t^-)) \mathrm{d}J(t) \end{aligned}$$

where J is a time-inhomogeneous compound Poisson process:

$$J = \sum_{i=1}^{N(t)} J_i$$

where h take two values that represents the direction of jump, . $N(t)$ is a Poisson process with time-dependent jump intensity and counts the spikes up to time t and J_1,J_2,... model the magnitudes of the spikes and are assumed to be independent and identically distributed random variables.

After this precedent model to ensure positivity of the prices. they developed the following model

$$S(t) = e^{\mu(t)} X(t)$$

where

- $X(t)$ is a stochastic process defined by equation $\sum_{i=1}^{n} w_i Y_i(t)$, where
- w_i are weighted functions; λ_i are mean-reversion coefficients; Y_i is a non-Gaussian Ornstein Uhlenbeck process define by $\mathrm{d}Y_i = -\lambda_i Y_i(t)\mathrm{d}t + \mathrm{d}L_i(t)$, where
- L_i is a additive process independent of cádlág pure jump.

This last model succeeded to capture well both the stylized facts of electricity spot prices and is analytically tractable for derivatives pricing and risk management.

Meyer and Tankov (Meyer-Brandis and Tankov (2008))

In order to model the intra-day hourly spot price series they built the following model:

$$X_t^h = Y_t f(t,h) + \varepsilon_t^h,$$

where

- X_t^h, $h \in \{1, , 24\}$ represents price for day t and hour h,
- Y_t is the common factor in the daily price,
- $f(t,h)$ is a slowly varying daily pattern,
- ε_t^h is a white noise.

Their aim was to study the most salient statistical features of electricity prices with a particular attention to the European energy ex-changes.

Based on this review, one can see that mathematical models for electricity pricing have evolved in recent years. From diffusion models, we arrived at the sum of Orstein-Ulhenbeck processes also called multi-factor jump-diffusion model, passing through the mean reversion jump diffusion model and polynomial process introduce in the power market by (Ware, 2019; Kleisinger-Yu et al., 2020). The review thus suggests mean reversion, jumps, spikes and seasonality as the main characteristics for electricity pricing models. In the following section, we review pricing option models in electricity.

Option pricing in electricity

Option pricing uses variables (e.g. stock price, exercise price, volatility, interest rate, time to expiration) to theoretically value an option. Essentially, it provides an estimation of an option's fair value which traders incorporate into their strategies to maximize profits. A large variety of electricity derivatives are traded among market participants, including forward contracts, swaps, plain vanilla options (European style and American style), exotic (i.e. non-standard) options like spark spread options, and swing options. Pricing or valuating option is a mathematical problem that consists in performing conditional expectation under risk neutral probability of the discounted payoff, translated by the following equation:

Definition 0.1. (Option value, see (Etheridge, 2002, p. 116))

$$C(S_T, T, t, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\phi(S_T)|\mathcal{F}_t\right] \tag{6}$$

where

- $\mathbb{Q}$ represents risk neutral probability,
- $\mathcal{F}_t$ represents the history of prices up to t,
- r represents the risk-free interest rate of the investment of the portfolio items,
- S_T is the spot price at maturity,
- $\phi(S_T)$ represents the payoff or payoff function,
- K represents strikes.

If $\phi(S_T) = \max(S_T - K, 0) = (S_T - K)^+$, option is called *call option*, that is, an option contract giving the owner the right (but not the obligation) to buy a specified amount of the underlying at a specified price within a specified time. If $\phi(S_T) = \max(K - S_T, 0) = (K - S_T)^+$, option is called *put option* that give owners the right (but not the obligation) to sell a specified amount of the underlying at a specified price within a specified time. Electricity call and put options offer their purchasers the right (but not the obligation) to buy or sell a fixed amount of electricity at a pre-specified strike price up to the option expiration time. They have similar payoff structures as those of regular call and put options in financial securities. Generally, the put option is determined by the relation $C(S_T, t, T, K) - P(S_T, t, T, K) = S - Ke^{-r(T-t)}$, called the *put-call parity*. A close form formula of option price differs from one model to another because of the expression of changes in the expression of the spot price from one model to another. In what follows, we present our view of the state of arts of forward and European options pricing models.

Forward option pricing model

Black and Scholes (1973) were the first to use diffusion models for pricing option in finance. From the mathematical definition of option price, they obtained an explicit analytic closed form formula of the European option defined as follows

Black and Scholes pricing option model (Black and Scholes (1973))

The price at time t of the call option derived by Black and Scholes is given by the equation

$$C(X_T, t, T, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\phi(S_T)|\mathcal{F}_t\right] = X_t\phi(d_+) - Ke^{-r(T-t)}\phi(d_-), \tag{7}$$

where ϕ is the cumulative distribution function from a normal distribution, and

$$d_{\pm} = \frac{\ln(X_t/Ke^{-r(T-t)}) \pm \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}.$$

The dynamic of the underlying asset is given by the Black and Scholes model

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

It is important to note that this explicit form is computationally feasible, thanks to the fact that the distribution of X is known. This pricing option assumes constant volatility, and does not take into account the mean reverting property and price jumps. Lucía and Schwartz (2002) were the first to propose an option pricing model for energy commodities.

Lucia and Schwartz forward pricing model Lucía and Schwartz (2002)

Electricity forward contracts represent the obligation to buy or sell a fixed amount of electricity at a pre-specified contract price (known as the forward price) at a given time in the future (called maturity or expiration time). In other words, electricity forwards are custom-tailored supply contracts between a buyer and a seller, where the buyer is compelled to buy and the seller is compelled to supply. The value of the forward contract is obtained by replacing $\phi(S_T) = S_T$ in equation (6). The forward price derived from the Black and Scholes (1973) model is given by the relation

$$F(t, T, X_t) = X_t e^{-r(T-t)},$$

since it is martingale under neutral risk probability. This forward contract formula is less suitable in energy commodities, particularly because it lacks the mean reverting property. Lucía and Schwartz (2002) derived from equation (6), the following forward price model which incorporates that property:

$$F(t, T) = f(T) + X_t e^{-\alpha(T-t)} - \sigma\int_t^T e^{-\alpha(T-s)}\lambda_s ds, \tag{8}$$

where λ_t is the market price for diffusion risk. However, the model does not account for non-constant volatility and the price jumps in electricity, situations handled in Cartea and Figueroa (2007).

Cartea and Figueroa forward pricing model Cartea and Figueroa (2007)

Cartea and Figueroa (2007) proposed the following extension of the Lucía and Schwartz (2002) forward price model to account for non-constant volatility and price jumps:

$$F(t,T) = G(T)\left(\frac{S(t)}{G(t)}\right)^{e^{-\alpha(T-t)}} e^{\int_t^T \frac{1}{2}\sigma^2(s)e^{-2\alpha(T-s)} - \lambda\sigma(s)e^{-\alpha(T-s)}ds + \int_t^T \xi(\sigma_J,\alpha,T,s)\ell ds - \ell(T-t)}, \quad (9)$$

where the parameters are described in their spot price model above. Their pricing model has captures all the essential properties of the electricity as the underlying asset.

European option pricing model

In the literature the European option has been derived using two different approaches. The first approach, which uses partial differential equations, is advantageous in that no knowledge of the distribution of the underlying process is required, while the second uses Fourier transform methods.

PDE for pricing European option

From Itô formula and some financial properties, Black and Scholes (1973) obtained the following partial differential equation for pricing option:

$$\frac{\partial f}{\partial t} + rX\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} = rf. \quad (10)$$

This PDE has many solutions, corresponding to all the different derivatives that can be defined with X as the underlying variable. The particular derivative obtained when the equation is solved, depends on the underlying boundary conditions. Equation (10) specifies the values of the derivative at the boundaries of possible values of X at time t. In the case of a European call option, the key boundary condition is $f = \max(X - K, 0)$, when $t = T$.

PIDE for pricing European option

Jump-diffusion models are a recent class of spot price models in electricity, involving PIDE for pricing European option. One reason for the development of this class of model is their popularity in electricity pricing. Cont and Ekaterina (2005b) derived the PIDE model associated to Lévy process in the case of infinite activity that generalize this class of models. The value of an option with terminal payoff H_T is obtained as a discounted conditional expectation under the (risk-neutral) pricing measure $\mathbb{Q}$ and is given by: $c(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\phi(S_T)|\mathcal{F}_t.\right]$ From Markov property, they obtained

$$c(t, S) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\phi(S_T)|S = S_t\right].$$

They also introduced the following change of variable $\tau = T - t$, $x = \ln\left(\frac{S}{S_0}\right)$, and defined $h(x) = H(S_0e^x)$, so that the latter equation becomes:

$$u(\tau, x) = \mathbb{E}^{\mathbb{Q}}\left[h(x + Y_\tau)\right].$$

They then showed that u is a solution to the Cauchy problem

$$\frac{\partial u}{\partial \tau} = L^Y u, \quad \text{on} \quad (0, T] \times \mathbb{R}_+, \quad \text{with} \quad u(0, x) = h(x) \quad \forall x \in \mathbb{R},$$

where $Y_t = rt + X_t$, X_t is a Lévy process under $\mathbb{Q}$, and L^Y is the infinitesimal generator defined by:

$$L^Y f = L^X f + r\frac{\partial f}{\partial x},$$

with

$$L^X f(x) = \frac{\sigma^2}{2}\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right] + \int_{-\infty}^{+\infty} \nu(dy)\left[f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}\right],$$

This finite difference method used to solve the PIDE obtained necessitates several approximations, which do not facilitate de search for solutions in this approach. Fourier methods are generally faster than finite difference ones.

Fourier Methods in Option Pricing

The Fourier transform has become a popular tool in option pricing when the coefficients of the equations are deterministic and constant. This popularity is partially due to the contributions of Carr and Madan (1999). Consider the problem of valuing a European call with maturity T written on the terminal spot price S_T of some underlying asset. The characteristic function of $s_T = \ln(S_T)$ is defined by

$$\Phi_T(u) = \mathbb{E}\left[\exp(ius_T)\right].$$

Let k denote de log of the strike price K, and let $C_T(k)$ be the desired value of call option with strike $\exp(k)$. Let the risk-neutral density of the log price s_T be q_T. Then characteristic function of this density is given by

$$\Phi_T(u) = \int_{-\infty}^{+\infty} e^{ius} q_T(s)ds. \tag{11}$$

The initial call value $C_T(k)$ is related to the risk-neutral density q_T by:

$$C_T(k) = \int_{-\infty}^{+\infty} e^{-rT}(e^s - s^k)q_T(s)ds. \tag{12}$$

Note that $C_T(k)$ tends to S_0 as k tends to $-\infty$, and hence the call pricing function is not square integrable. To obtain a square integrable function, we consider the modified call price $c_T(k)$ defined by

$$c_T(k) = \exp(\alpha k)C_T(k). \tag{13}$$

For $\alpha > 0$, we expect that $c_T(k)$ be square integrable in k, over the entire real line. We comment later on the choice of α. Consider now the Fourier transform of $c_T(k)$ defined by

$$\psi_T(v) = \int_{-\infty}^{+\infty} e^{ivk} c_T(k)dk. \tag{14}$$

Carr and Madan (1999) first develop an analytical expression for $\psi_T(v)$ in terms of ϕ_T and then obtained call prices numerically, using the inverse transform:

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{+\infty} e^{ivk}\psi_T(v)dv = \frac{\exp(-\alpha k)}{\pi} \int_{0}^{+\infty} e^{ivk}\psi_T(v)dv. \tag{15}$$

The second equality holds because $C_T(k)$ is real, implying that the function $\psi_T(v)$ is odd in its imaginary part and even in its real part. The expression for $\psi_T(v)$ is then written as follows

$$\psi_T = \frac{e^{-rt}\phi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v} \tag{16}$$

Call values are determined by substituting (16) into (15) and performing the required integration.

In summary, the difficulty in pricing option lies in the fact that, for more complex models, the distribution is unknown contrary to the case where the distribution is available and therefore the option can be expressed using exact closed form. In the case where the distribution of the underlying asset price is unavailable, two approaches are usually employed to valuing option, namely PDE or PIDE and valuing option using Fourier transform. One important step in pricing consist to calibrating the model with historical market data.

Model calibration

In absence of jumps, calibrating a model consist in finding the log likelihood if there is an analytical expression, or to approximate it with some numerical schemes. The log likelihood is then maximized to obtained the model parameters. Calibrating the SDE in the absent of jumps when the transition function is available has been explored by **?** and Black and Scholes (1973). In cases where the transition function is unknown, it is obtained by calculating the conditional expectation of the discretized process as in Jensen and Poulsen (2002) and Carr and Madan (1999). Non parametric calibration based on option pricing is another approach for calibrating in jump diffusion models. We remark that in general, non parametric calibration leads to an inverse problem which is solved via non-linear least squares. Such problems are ill-posed because of non uniqueness of solution. Regularization method based on relative entropy has therefore been proposed as one solution of this problem. One major contribution in this field has recently been made by Cont and Tankov (2004). They used observed option prices to calibrate parameters of one large class of finance models called exponential Lévy models, characterized by the triplet (σ, ν, γ) (see, Sato (1999) for better explanation of Lévy processes). We now review their approach. For that, the calibration problem for type of model leads, in general, to solving the following equation:

$$(\sigma^*, \nu^*) = \arg\inf_{\sigma,\nu} \sum_{i=1}^{N} \omega_i \mid C^{\sigma,\nu}(t=0, S_0, T_i, K_i) - C_0^*(T_i, K_i) \mid^2, \tag{17}$$

where $C_0^*(T_i, K_i)$ is the market price of a call option observed at $t = 0$ and $C^{\sigma,\nu}(t = 0, S_0, T_i, K_i)$ is the price of this option computed in an exponential Lévy model with volatility σ and Lévy measure ν. To obtain a unique solution in a stable manner, they introduced a regularization method by adding to the least squares criterion (17), a convex penalization term:

$$(\sigma^*, \nu^*) = \arg\inf_{\sigma,\nu} \sum_{i=1}^{N} \omega_i \mid C^{\sigma,\nu}(t=0, S_0, T_i, K_i) - C_0^*(T_i, K_i) \mid^2 + \alpha F(\sigma, \nu), \tag{18}$$

where F, a measure of closeness of the model $\mathbb{Q}$ to a prior model $\mathbb{Q}_0$, is chosen such that the problem (18) becomes well-posed. The weights ω_i, which are positive and sum to one, reflect the pricing error tolerance for the option i. The choice of weights and prior model $\mathbb{Q}_0$ are addressed in more detail in their work. Another recent approach called saddlepoint method was initiated by Daniels (1954), and latter developed in Rogers and Zane (1999), Jensen (1995) and Aït-Sahalia and Yu (2006). This approach consists in approximating the transition density by the saddlepoint. We first recall that density function and characteristic functions are linked by the Fourier inversion formula

$$
\begin{aligned}
p(\Delta, y \mid x) &= (2\pi)^{-m} \int_{-\infty}^{+\infty} \exp(-iu.y)\, \phi(\Delta, iu \mid x)\, du \\
&= (2\pi i)^{-m} \int_{\hat{u}-i\infty}^{\hat{u}+i\infty} \exp(K(\Delta, u \mid x) - u.y)\, du,
\end{aligned} \tag{19}
$$

where $\phi(\Delta, . \mid x)$ represents the characteristic function, while K is the cumulant generating function of the Markov process X. From Aït-Sahalia and Yu (2006), the key to the saddlepoint method consists in choosing u in (19) such that Taylor expand of the integrand will be easily computed using normal distributed density which integrates to 1. This amounts to solving the equation:

$$
\frac{\partial K(\Delta, u \mid x)}{\partial u} = y. \tag{20}
$$

Structure of the Thesis

In this dissertation, We will develop new mathematical models for pricing European option underlying electricity market regulated by price cap principle. To achieve that, in chapter 1, we gain inspiration from the modeling approach introduced by Merton (1976) to first of all built a new electricity spot price model as simple stochastic differential equation using price cap formula. To complete our model, we take into account spikes and jump of prices using the same approach as in Cartea and Figueroa (2007). This first chapter constitutes the main step in deriving option price.

In chapter 2 we derive European option price using two approaches. The first approach consists in performing European option based on Fourier methods as in Carr and Madan (1999). From the Fayman-Kac representation theorem we develop a new Partial Integro-Differential Equation (PIDE) as in Cont and Ekaterina (2005b), we analyse the numerical scheme used to approximate the viscosity solution of the PIDE obtained, and end the chapter with some numerical results.

In chapter 3 we calibrate our spot price model by estimating the model parameters using maximum likelihood and simulated data (as real data are rare in this this new market). The exact form of the transition density of the obtained model is not available due to the sum of two processes in the model (the Poisson process and Brownian motion) with two different distributions. Following Aït-Sahalia (2010) we approximate the transition density using saddlepoint method.

CHAPTER ONE

A JUMP-DIFFUSION MODEL FOR PRICING ELECTRICITY UNDER PRICE CAP REGULATION

Contents

In this chapter, we derive a new jump-diffusion model for electricity spot price from the price cap principle using stochastic calculus tools. Next, we show that the model has a non classical mean-reverting linear drift and solve the latter model. Finally, we provide some simulations from the model.

1.1 Introduction

The limitations and shortcomings of the experimental method have given rise to the modelling and simulation activities necessary for humans to understand the phenomena surrounding them and to solve the problems they face. Geraud et al. (1974) define modelling as the operation by which the model of a complex system is established, in order to more conveniently study and measure the effects of variation on that system brought by its component elements.

The wind of liberalization and reforms in the electricity sector blowing across several countries has left behind a complex field in the electricity industry in which an analysis of the interactions and relationships between the components has given rise to several areas of research. One aspect of electricity that remains at the heart of debates is pricing, and is therefore a focal point for researchers.

Several pricing methods have been developed, such as, unit-based pricing, real-time pricing (RTP), increasing-Block Pricing, Profit cap constraints, dynamic pricing model, rate of return pricing and price cap, each with specific economic objectives. In order to transfer this objectives to the electricity market which is now open to competition, regulatory activity has been introduced for this purpose. Price cap is the one of interest here.

Electricity prices generally reflect the costs associated with the construction, financing, maintenance, management and operation of power plants and the electricity grid as a whole, i.e. the complex system of transmission and distribution of power on the one hand and the operating and administrative costs of the utilities that provide electricity to consumers on the other. However, there are a few key factors that can cause the latter to fluctuate (Administration (2011)). In effect, fuel in the form of coal is relatively less expensive compared to natural gas. Also, the construction and maintenance costs are higher for some types of power plants than others. The transmission and distribution system also contributes to electricity cost. Un favourable Weather periods, like over heating can increase electricity demand for cooling and regulation, there by provoking an increase in price. For these reasons some prices are fully regulated by public service commissions, while in others, there is a combination of regulated prices (for generators) and regulated prices (for transmission and distribution).

In addition to that, electricity is not a classical financial asset and presents some particularities. Firstly, it is non storable and can not be traded as any financial asset. Secondly, prices have a delivery period, that is, the electricity is not delivered instantly but continuously during a period of 1 hour for instance. All these particularities lead to the following features of electricity spot prices time series:

Seasonality: they exhibit daily, weekly and yearly seasonalities which are highly related to electricity consumption due to its non-storability.

Spikes: prices jump upward or downward to high values before reverting quickly to their original level. positive can appear when demand is abnormally high or temperature abnormally low or high. In case of high temperature, air conditioning produces these spikes and in case of low temperature, heating is responsible. Negative spikes may be a consequence of non-storability. If the production is higher than expected, the cost of stopping a production plant may be high and the producer may prefer to pay for consuming electricity. In Germany, unexpected production is caused by the penetration of the renewable energies in the system. For instance, high unexpected wind production may cause negative spikes.

Mean reversion: mean reversion is present even in the presence of spike.

In the rest of this chapter, we, in the light of the existing electricity price models largely reviewed in Bodily and De Buono (2002), propose a new electricity pricing model when the market is regulated by the Price Cap principle, and which takes into account the factors and characteristics of the electricity price described above.

1.2 Model Derivation

Our model is inspired from the electricity price-cap regulation proposed by Littlechild (1983) and adopted in several countries today.

1.2.1 Price Cap Market Regulation

Price cap regulation was implemented for the first time in the UK in the early 1980s to regulate the newly privatized telecommunication market that emerged after the privatization of British Telecoms. Steven Littlechild, who recommended this measure to the British government (Littlechild (1983)), envisaged the price cap as a transitory measure that would eventually wither away as the market became more competitive. The point of the price cap was to hold the fort until competition arrived. However, it has taken longer than expected for utility industries to become competitive, and price cap regulation has gained popularity in many countries as a permanent form of regulation in the utilities industry as electricity.

In the electricity market the price cap regulation is an economical principle which aims to establish an incentive scheme for the regulated market. A key objective is to enable companies to maximize the well-being while seeking to maximize their own interests, see Acton and Vogelsang (1989). Its principle is to cap the market price. The main components of the price cap include: the efficiency factor (G), for transferring the gains to consumers through the reduction of costs; the inflation rate (I), which drives the prices changes; the exogenous factors such as Customer portion of earnings sharing (E), Service quality penalties (H) and Flow through and uncontrollable costs, if any (F). ENMAX (2009) proposed price cap formula:

$$\frac{P_i - P_{i-1}}{P_{i-1}} = I_i - G_i + \left(\frac{-E_i - H_i + F_i}{P_{i-1}} \right), \tag{1.1}$$

where P_i represents the current year's price and P_{i-1} preceding year price. Later we would be inspired by the economic formula (1.1) to model the drift of the model.

1.2.2 Some Definitions of elements of stochastic calculus

The underlying set-up for stochastic processes consists 0f a complete probability space

Definition 1.1. (Filtered Probability Space)
The triple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ consisting of the sample space Ω, the $\sigma-$algebra $\mathcal{F}$ of subset of Ω, probability measure $\mathbb{P}$ defined on $\mathcal{F}$ and the non-decreasing family $(\mathcal{F}_t)_{0\leq t\leq T}$ of sub-$\sigma-$ fields of $\mathcal{F}$ is called filtered probability space.

Here, $\mathcal{F}_t$ represents the information available at time t, and the filtration $(\mathcal{F}_t)_{0\leq t\leq \infty}$ represents the information flow evolving over time and accruing to all agents in the economy.

Definition 1.2. (Stochastic Process)
A scalar (n-vector) stochastic process denoted by $\{X_t\}_{0\leq t\leq \infty}$, is a family of random variables (n-vectors) indexed by the parameter set $[0, \infty[$, where the parameter t will refer to time in our application. The process is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq \infty}, \mathbb{P})$ with value in $\mathbb{R}^n$. We say the process is adapted if $X_t \in \mathcal{F}_t$ for each time t, i.e. X_t is $\mathcal{F}_t-$measurable; thus, X_t is known when $\mathcal{F}_t$ is known. Further, if a filtration is generated by a stochastic process, i.e.

$\mathcal{F}_t = \sigma(X_s; 0 \leq s \leq t)$, we call $\mathcal{F}_t$ the natural filtration of the process $\{X_t\}_{0\leq t\leq\infty}$. Thus a process is always adapted to its natural filtration.

Definition 1.3. (Standard Brownian Motion)

A continuous parameter process $\{W_t\}_{t\geq 0}$ is a Brownian motion under the probability measure $\mathbb{P}$ if

(i) every increment $W_{t+\tau} - W_t$ is normally distributed with $\mathcal{N}(\mu\tau, \sigma^2\tau)$ under $\mathbb{P}$, and

(ii) for every pair of disjoint time intervals $[t_1, t_2]$, $[t_3, t_4]$ (with $t_1 < t_2 \leq t_3 < t_4$) the increments $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent random variables, and

(iii) $\mathbb{P}(W_0 = 0) = 1$ and W_t is continuous at t=0.

For $\{W_t\}_{t\geq 0}$ to be a standard Brownian motion (or Wiener process), we need to fix the parameter at $\mu = 0$ and $\sigma = 1$.

Definition 1.4. The process $(N_t)_{t\in\mathbb{R}_+}$ define by

$$N_t = \sum_{k\geq 1} \mathbb{1}_{[T_k,\infty)}, \quad t \in \mathbb{R}_+,$$

where

$$\mathbb{1}_{[T_k,\infty)}(t) = \begin{cases} 1 & \text{if } t \geq T_k, \\ 0 & \text{if } 0 \leq t < T_k, \end{cases}$$

$k \geq 1$, $(T_k)_{k\geq 1}$ is the increasing family of jump times of $(N_t)_{t\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} T_k = +\infty$$

. is called standard Poisson process.

In addition, the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ is assumed to satisfy the following conditions:

1 Independence of increments: for all $0 \leq t_0 < t_1 < ... < t_n$ and $n \geq 1$ the increments

$$N_{t_1} - N_{t_0}, ..., N_{t_n} - N_{t_{n-1}},$$

are mutually independent random variables.

2 Stationarity of increments: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leq s \leq t$.

Let $(Z_k)_{k\geq 1}$ denote an i.i.d. sequence of square-integrable random variables distributed as the common random variable Z with probability distribution $\nu(dy)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

Definition 1.5. The process $(Y_t)_{t\in\mathbb{R}_+}$ given by the random sum $Y_t := Z_1 + Z_2 + ... + Z_{N_t}$ is called a compound Poisson process

Theorem 1.1. (Etheridge (2002))(Itô's formula with jumps)

Suppose

$$dY_t = \mu_t dt + \sigma_t dW_t + \nu_t dN_t$$

where, under $\mathbb{P}$, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ. If f is a twice continuously differentiable function on $\mathbb{R}$ then

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_s^-)dY_s + \int_0^t f''(Y_s^-)\sigma^2 \mathrm{d}s - \sum_{k=1}^{N_t} f'(Y_{\tau_k^-})(Y_{\tau_k}) - Y_{\tau_k^-})) + \sum_{k=1}^{N_t}(f'(Y_{\tau_k}) - f'(Y_{\tau_k^-})),$$

where $\{\tau_k\}$ are the times of the jumps of the Poisson process.

1.2.3 Spot Price Modeling

The daily (resp. weekly and monthly) change in price is the difference between today's price (resp. this week's price and this month's price) and yesterday's price (resp. last week's price and last month's price). In general, one denotes a change over a given time period $\mathrm{d}t$ by $\mathrm{d}S_t$. For a daily change, we therefore have $\mathrm{d}t = \frac{1}{365}$, $\mathrm{d}t = \frac{1}{52}$ for weekly change and $dt = \frac{1}{12}$ for a monthly change. The change in price $\mathrm{d}S_t$ over a given time period $\mathrm{d}t$ is given by de following theorem.

Before stating the following theorem let recall that a *càdlàg* stochastic process is the right continuous with left limits stochastic process.

Theorem 1.2. Suppose that the spot prices S_t is a *càdlàg* process in a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $(\mathcal{F}_t)_t$ is a natural filtration of S_t (we assume this to get closer to practice since the jump process has irregularities). Assume the following conditions:

i) the stock prices jumps from the previous value S_{t^-} to a next value JS_{t^-} where J is the proportional size of the random jump assumed log-normally distributed i.e. $\ln J \sim \mathcal{N}(m_J, \sigma_J^2)$ with $\mathbb{E}[J] = 1$,

ii) the change before and after the jumps, is driven by increments $\mathrm{d}q_t$ of a Poisson process q_t defined by

$$dq_t = \begin{cases} 1, & \text{with probability } \ell \mathrm{d}t \\ 0, & \text{with probability } 1 - \ell \mathrm{d}t, \end{cases}$$

where ℓ is the intensity or frequency of the process.

Then the Price-cap principle (1.1) yields the Stochastic Differential Equation (SDE) also called a *jump diffusion model* below

$$\mathrm{d}S_t = -\alpha(t)(\gamma(t) - S_t)\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t + (J-1)S_t\mathrm{d}q_t, \tag{1.2}$$

where W_t is the standard Brownian motion, and the coefficients involved are deterministic functions of time denoted as such: $\sigma(t)$ is the volatility, $\beta(t) := E(t) + H(t) - F(t)$ defines the exogenous factors, $\alpha(t) := I(t) - G(t)$, $\gamma(t) := \beta(t)/\alpha(t)$.

Proof. We will first model the prices dynamic without taking into account the jumps and then later take into account the jumps.

Let S(t) denote electricity price and assume that the electricity market regulated by price cap verify the 9 assumptions made in (Merton, 1976, pp.377-380.) such that market structure is hold. For each quantity of energy supply in the opportunity set at each point in time t, the expected rate of return per unit time, is defined by

$$r \equiv \frac{1}{h}\mathbb{E}_t\left\{\frac{S(t+h) - S(t)}{S(t)}\right\}, \tag{1.3}$$

which represents the right member of (1.1) and the variance of the return per unit time, is defined by

$$\sigma^2 \equiv \frac{1}{h}\mathbb{E}_t\left\{\left(\frac{S(t+h) - S(t)}{S(t)} - rh\right)^2\right\}, \tag{1.4}$$

where $\mathbb{E}_t$ denote conditional expectation respect to $\mathcal{F}_t$.

From (1.3) and (1.4) rate of return dynamic is give by

$$\frac{S(t+h) - S(t)}{S(t)} = rh + \sigma y(t)\sqrt{h} \tag{1.5}$$

where: $y(t)$ is purely the random process where, by construction

$E_t(y) = 0, E_t(y^2) = 1$, that is, $y(t)$ and $y(t+s)$, for $s > 0$, are identically distributed and mutually independent see Samulson and Merton (1974) for further discussion. Further, assumed that y(t) is Gaussian distributed, it is defined the stochastic process, z(t), by z(t + h) = z(t) + y(t)$\sqrt{h}$, then for h tends to zero z(t + h)-z(t) describes a Wiener process or Brownian motion. In the formalism of stochastic differential equations, we have

$$\mathrm{d}z \equiv y(t)\sqrt{\mathrm{d}t} \tag{1.6}$$

from (1.6), (1.5) is a stochastic differential equation, for the instantaneous return is giving by

$$\frac{\mathrm{d}S(t)}{S(t)} = r\mathrm{d}t + \sigma \mathrm{d}z \tag{1.7}$$

Hence, we obtain the following the SDE

$$\mathrm{d}S_t = \left[S_t\left(I(t) - G(t)\right) - \beta(t)\right] dt + \sigma(t) S_t dW_t. \tag{1.8}$$

Next, to capture the market shocks we add the jump term in (1.8) using Cartea and Figueroa (2007) idea as follows. We suppose that the stock prices jumps from the previous value S_{t^-} to a next value JS_{t^-} where J is the proportional size of the random jump assumed log-normally distributed such that $\mathbb{E}(J) = 1$ this assumption is motivated by the fact that under regulation we want that the risk of the market shocks fluctuate around unit. Next, from assumption (i)

$$S_t = JS_{t^-}$$

by removing S_t^- in the two side of equality we obtain

$$S_t - S_{t^-} = JS_{t^-} - S_{t^-} = S_{t^-}(J-1),$$

which represents the price change after the jump. Then from assumption (ii) the equation (1.8), setting $\alpha(t) := I(t) - G(t)$, $\gamma(t) := \beta(t)/\alpha(t)$, we finally obtain the SDE (1.2). □

1.3 Mean-Reversion Condition

A means-reverting process has a drift term that brings the variable being pulled back to some equilibrium. This feature is captured by one stochastic differential equation if the following definition is verified.

Definition 1.6. (Condition (A_3) of Mbele and Miklós (2014)).

Consider a Jump diffusion process Y_t with a differentiable drift function $\mu(.)$.

If

$$\limsup_{|Y_t|\to\infty} \frac{| Y_t + \mu(Y_t) |}{| Y_t |} < 1,$$

then Y_t is mean-reverting.

From this definition we have the following Proposition

Proposition 1.1. The jump-diffusion model (1.2) is mean-reverting.

Proof. It is straightforward and is based on the economic fact that $\beta(t)$ is bounded on $[0, T]$ and we have $| 1 + \alpha(t) |< 1$ for all $t \in [0, T]$. □

Proposition 1.2. The solution of equation (1.2) is the process $(S_t, 0 \le t \le T)$ define by

$$S_t = Z_t \left(S_0 - \int_0^t \beta(s) Z_s^{-1} \mathrm{d}s \right),$$

where $Z_t = e^{\left(\int_0^t (\alpha(s) - \frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_0^t \sigma(s) dW_s + \int_0^t \ln J dq_s\right)}$.

Proof. To solve equation (1.2) we consider a process Z, solution of the following equation

$$\mathrm{d}Z_t = Z_t \left(\alpha(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t + (J-1)dq_t \right) \text{ and } Z_0 = 1.$$

Applying Itô formula with jumps stated in 7.10 Etheridge (2002), we obtain
$Z_t = Z_0 e^{\left(\int_0^t (\alpha(s) - \frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W_s + \int_0^t \ln J \mathrm{d}q_s\right)}$.

Now, let us set $f(S_t, Z_t) = \frac{S_t}{Z_t}$. By applying Itô formula with jumps one more, we obtain

$$\begin{aligned} \frac{S_t}{Z_t} &= \frac{S_0}{Z_0} + \int_0^t \frac{1}{Z_s} \left[\alpha(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_s + (J-1)dq_s)S_s - \beta(s)\mathrm{d}s \right] \\ &\quad - \int_0^t \frac{S_s}{(Z_s)^2} Z_s \left(\alpha(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_s + (J-1)\mathrm{d}q_s \right) \\ &\quad + \frac{1}{2} \left(\int_0^t 2 \frac{S_s}{(Z_s)^3} (\sigma(s) Z_s)^2 \mathrm{d}s - \frac{2}{(Z_s)^2} \sigma(s)^2 S_s Z_s \mathrm{d}s \right). \end{aligned} \tag{1.9}$$

The development of (1.9) leads to

$$\begin{aligned} \frac{S_t}{Z_t} &= \frac{S_0}{Z_0} + \int_0^t \frac{S_s}{Z_s} \alpha(s)\mathrm{d}s + \sigma(s)dW_s + (J-1)dq_s) - \int_0^t \frac{1}{Z_s} \beta(s)\mathrm{d}s \\ &\quad - \int_0^t \frac{S_s}{Z_s} \alpha(s)\mathrm{d}s + \sigma(s)\mathrm{d}W_s + (J-1)dq_s) \\ &\quad + \int_0^t \frac{S_s}{Z_s} \sigma(s)^2 \mathrm{d}s - \frac{S_s}{Z_s} \sigma(s)^2 \mathrm{d}s. \end{aligned} \tag{1.10}$$

By simplifying the second and the fourth terms, the last two terms together in (1.10) and since $Z_0 = 1$ we obtain

$$\frac{S_t}{Z_t} = S_0 - \int_0^t Z_s^{-1}\beta(s)\mathrm{d}s.$$

Finally, we obtain

$$S_t = Z_t S_0 - Z_t \int_0^t Z_s^{-1}\beta(s)\mathrm{d}s.$$

This end the proof. □

Furthermore the solution of (1.2) at T starting at t is given by

$$S_T = \frac{Z_T}{Z_t} S_t - Z_T \int_t^T \beta(s) Z_s^{-1}\mathrm{d}s, \tag{1.11}$$

where $Z_T = Z_t e^{\left(\int_t^T \alpha(s-\frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_t^T \sigma(s)dW_s + \int_t^T \ln J dq_s\right)}$. Thus, the solution of (1.2) is a functional of Lévy process.

1.4 Some Illustrative Curves of Spot Price in Regulated Electricity Market

This section deals with the numerical simulations of the Spot price in order to illustrate some meaningful behaviors of the model and in comparison with the model develop in Cartea and Figueroa (2007). The proposed simulations also aim at highlighting the fundamental role of some particular parameters in the outcomes of the prices. For the numerical computation, we approximated the integrals using the trapezium and the Stratonovich integration methods. The parameters used in the simulations are plausable relative to those used in the literature.

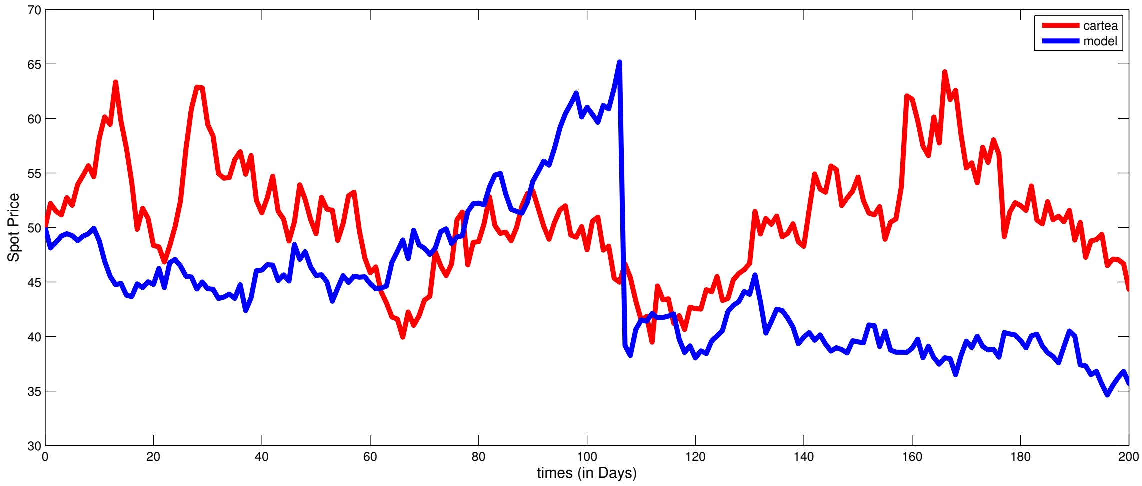

Figure 1.1: These curves compare spot price propose in this work in color blue with the spot price introduce by Cartea et al. in Cartea and Figueroa (2007) with the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; ℓ= 2.85; $\sigma = 0.75$; $\sigma J = 0.67$; S(0)=50.

Figure 1.2 shows a simulated spot price of our propose model when increase the jumps intensity parameter ℓ end one can observe that the number of the jumps increase too. Figure 1.1 shows a simulated spot price of our propose model in color blue compared to the spot price develop by Cartea and Figueroa (2007) without the seasonal part in color red. One can observe that the proposed model captures some characteristics discussed in the regulated market such as mean-reversion, a property also observed in figures 1.2 confirming the theoretical results. It is further relevant to discuss that in our model, the frequency of jumps is less than in the deregulated market. In a nut shell, these illustrations show that our model with the mean-reversion property captures the main objective of regulation principle, which is to cap prices within a given range.

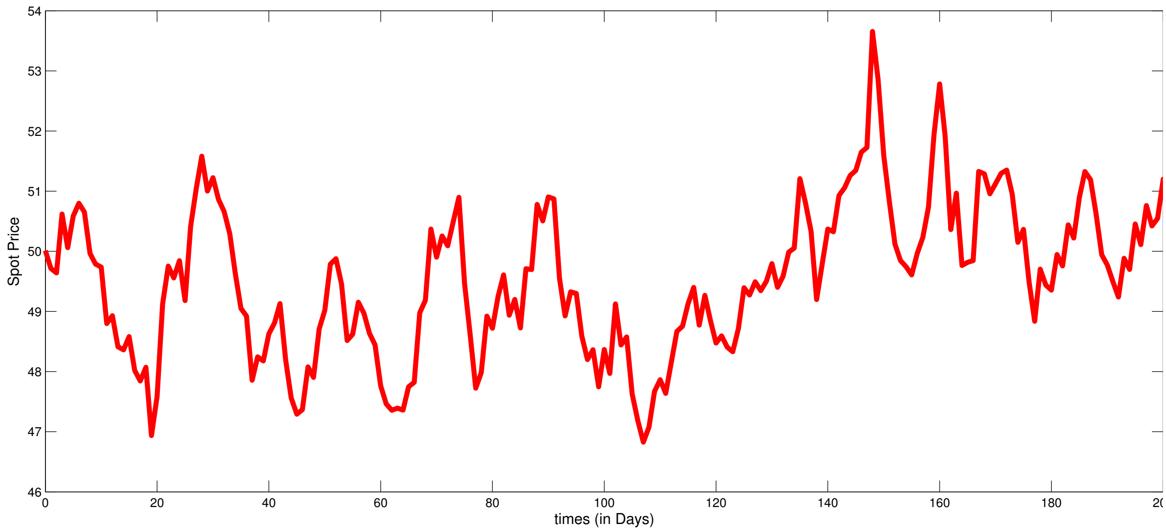

Figure 1.2: Spot prices for the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\ell = 8.85$; $\sigma = 0.75$; $\sigma J = 0.67$; S(0)=50.

1.5 Condition for the implementation of the theory developed in a specific case: the case of Cameroon

To implement the theory developed, it is essential to have data on the evolution of prices over time. Indeed, this data will allow two things:

- Justify the randomness of the model (Brownian aspect and possible jumps);
- The calibration of the model using observed data to estimate model parameters.

Conclusion

In this chapter we have proposed a new model of spot price in the regulated electricity market. The proposed model leads to non classical Ornstein-Uhlenbeck process due to the non constant speed of the mean-reversion. This model is useful to better capture the dynamics of the electricity prices and its behaviour in regulated electricity market. It also gives a better quantification of financial tools that help to hedge against financial risks.

CHAPTER TWO

OPTION PRICING FROM JUMP DIFFUSION MODEL OF ELECTRICITY PRICES UNDER PRICE CAP

Contents

This chapter concerns the derivation of the closed-form formula of forward contract and European call option, which are two financial tools used to hedge risk in the electricity market. To price European call option, two approaches were discussed: the approach based on Fourier transform and the one based on Partial Integro-Differential Equations (PIDE). Some simulations were conducted after analysing the stability of the explicit-implicit scheme used.

2.1 Introduction

Electricity spot prices in the emerging power markets are volatile, a consequence of the unique physical attributes of electricity production and distribution. Uncontrolled exposure to market price risks can lead to devastating consequences for market participants in the restructured electricity industry. Lessons learned from the financial markets suggest that financial derivatives are a major tool in risk management. Options pricing is recent in electricity and constitutes an attractive new research pole in financial mathematics.

Electricity forward contracts are the primary instruments used in electricity price risk management. Other power marketers usually use forwards to hedge their positions in electricity options and other complex electricity derivatives. Two standard approaches have often been used to derive it in the literature. The first consists in modelling directly the forward curve dynamics and deduce forward contract (Aid et al., 2013; Clewlow and Strickland, 2000). The second approach starts from a spot price model to derive forward contract price as the conditional expectation of the spot price under a risk-neutral probability (Lucía and Schwartz, 2002; Cartea and Figueroa, 2007).

In finance, options are tools that help to guard against risks. However, it is difficult to know the value of an option before the maturity date since, one has to estimate the value of the underlying in the future. In the early 1970's, Black and Scholes (1973) brought a major contribution in the evaluation of options. In the case where the underlying is a share that does not pay dividends, they construct a risk-neutral portofolio that replicates the winning profile of an option, which allows to perform the theoretical value of a European option under a closed formula. In the case of the Black and Scholes model, this formula is derived from some classical results in discounting, statistics, stochastic and differential calculus. On the contrary, the valuation of options remains an open topic in the case of jump-diffusion processes due to the additional jump term that complicates calculation of option prices. This has been investigated by several authors. One may distinguish two main approaches used by these authors. The first one, which relies on Fourier transform-based methods, has been introduced by Carr and Madan (1999) to price and analyses European option prices. Other authors like Chiarella et al. (2009), Lewis (2001) and **?** followed the same idea to evaluate option price. This approach relies on its high computational efficiency when the characteristic function is available. The idea here is to apply the direct discounted expectation method to evaluate the integral of the discounted payoff and risk neutral density function of the underlying process. Continuous and discrete Fourier transforms were successfully applied to price options of three models: the classical Black-Scoles model, the Merton jump diffusion model (an exponential Lévy model with finite arrival rate of jumps), and the variance Gamma model (an exponential Lévy model with infinite arrival rate of jumps). However, for many other jump diffusion processes, it is not possible even by applying the Lévy-Khinchin representation to determine an analytical expression of the characteristic function. This may be due to the complexity of diffusion processes or to the fact that two or more processes are combined in the same model. Therefore one expects to evaluate option prices with rather an approximation of the characteristic function. The debate is then on the choice of a good approximation. Indeed, when the characteristic function can not be expressed explicitly, the Fourier transform method requires two levels of approximation which increases the error level in option valuation. In order to reduce approximation errors, a second approach based on the Feynman-Kac formula was introduced by some authors like Alvarez and Tourin (1996), Bales et al. (1991) and Cont and Ekaterina (2005b). This approach relates the risk-neutral valuation formula to either the solution of a partial differential equation (PDE), when the model is a continuous exponential Lévy one, or to the solution of a partial integro-differential equation (PIDE) in the case of continuous Lévy process. The PDE, respectively PIDE obtained may be complex and its theoretical analysis requires new mathematical tools. However, after a numerical approximation step, the solution leads directly to the value of the option. So, we have one level of approximation. Nevertheless, theoretical analysis of the approximation (consistency, stability and convergence of the scheme) remains a challenge. Several finite difference schemes are used in the literature (Alvarez and Tourin, 1996; Cont and Ekaterina, 2005a; Rama and Tankov, 2003). Most of these works focus on pricing option when the underlying is driven by either a Lévy process or an exponential Lévy process. Some of the main difficulties are related to the local integral term due to the fact that, on the one hand, the approximation of the risk-neutral density can often involve an infinite summation, and on the other hand, a local integral term requires a specific treatment at both theoretical and numerical levels. We may have other difficulties as the smoothness of option prices and even the degeneracy of the diffusion coefficient. To overcome such difficulties, the notion of viscosity solution was introduced by Grandall et al. (1992) for PDEs and, more generally, for PIDEs by Alvarez and Tourin (1996) and Bales et al. (1991). Precisely, one can split the integro-differential operator into a non local and a local part, and then treat the non local term using an implicit step, and the local term using an explicit step. This idea was applied by Cont and Ekaterina (2005a) to obtain a better approximation of option prices than the previous ones.

In the rest of this chapter, and from the spot price model derived in the first chapter and the second approach mentioned above we obtain a closed form formula of the forward contract. We also calculated the European option using the Fourier transform and the Feynman-Kac approaches. Indeed, in our case, the characteristic function of the jump diffusion process is unavailable, hence was approximated. For the second approach, we took the advantage of the Markov property in the electricity spot price to prove that the European option price solves a PIDE. We then apply an explicit-implicit scheme to compute numerically the viscosity solution of the PIDE founded. We also studied the consistency, the stability and the convergence of this scheme using a technique similar to that in Cont and Ekaterina (2005a), with the exception that (i) the market price here is the sum of the exponential Lévy process and Lévy process, (ii) the presence of this additional term (see equation 2.2) has created an additional difficulty at all levels of the analysis since it still depends on the process, (iii) the parameters used in numerical simulations are derived from the Cameroonian context, and (iv) we worked in the case of time-dependent parameters. Finally, some numerical simulations are provided under a smooth initial condition.

2.2 Some Mathematical and Financial Definitions and Results

Definition 2.1. (Characteristic function)

The characteristic function of random variable X is function

$$\psi_X : \mathbb{R} \longrightarrow \mathbb{C}$$

define by

$$\psi_X(t) = \mathbb{E}\left[e^{itX}\right], \ t \in \mathbb{R}.$$

Definition 2.2. A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if under $\mathbb{P}^*$ on average the expected yield of the risky asset equals the risk-free interest rate obtained by investing on the savings account with interest rate

Also known as the risk-neutral measure, $\mathbb{P}^*$-measure is a way of measuring probability such that the current value of a financial asset is the sum of the expected future payoffs discounted at the risk-free rate. The risk-free rate is the return on investment on a riskless asset. Q-measure is used in the pricing of financial derivatives under the assumption that the market is free of arbitrage.

Definition 2.3. (Equivalent Probability measure)

A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is said to be equivalent to another probability measure $\mathbb{P}^*$ when

$$\mathbb{P}^*(A) = 0 \ \ if\, and\, only\, if \ \ \mathbb{P}(A) = 0, \ \ for\, all\; A \in \mathcal{F}$$

Theorem 2.1. (Delbaen and Schachermayer (1994))(Fundamental Theorem of Asset Pricing)

The no-arbitrage condition and the existence of an equivalent martingale measure are fundamentally connected as follows:

(i) the market is arbitrage free, if and only if there exists at least one equivalent martingale measure $\mathbb{Q}$, and

(ii) the market is complete, if and only if there exists only one equivalent martingale measure $\mathbb{Q}$

for a modeled capital market.

Definition 2.4. A (European) put option is a contract that gives its holder the right (but not the obligation) to sell a quantity of assets at a predefined price K called the strike price (or exercise price) and at a predefined date T called the maturity.

Definition 2.5. A (European) call option is a contract that gives its holder the right (but not the obligation) to buy a quantity of assets at a predefined price K called the strike and at a predefined date T called the maturity.

We consider in this chapter that the underlying asset is electricity which the spot price dynamic is given by a jump diffusion equation

$$\mathrm{d}S_t = (\alpha(t)S_t - \beta(t))\,\mathrm{d}t + \sigma_t S_t \mathrm{d}W_t + (J-1)S_t \mathrm{d}q_t, \tag{2.1}$$

where S_t represents the electricity spot prices in regulated market and the rest of parameters remain as define in chapter 1. In addition to the assumptions considered in chapter 1 we add the following

(iii) The random jump size J, $\mathrm{d}q_t$ and $\mathrm{d}W_t$ are independent.

We will further assume that the electricity parameters $\alpha(.)$, $\beta(.)$ and $\sigma(.)$ are bounded. We recall that the exact solution to (2.1) is given, using Itô formula for jump diffusion process, by

$$S_t = S_0 e^{X_t} - \int_0^t \beta(s) e^{X_t - X_s} ds, \tag{2.2}$$

where $X_t = \int_0^t \alpha(s) - \dfrac{1}{2}\sigma(s)^2 ds + \int_0^t \sigma(s) dW_s + \int_0^t \ln J dq_s$ is a Lévy process.

2.2.1 Computation of Regulated Electricity Forward Price

The price at time t of the forward expiring at time T (i.e. $\mathbf{F}(t,T)$) is obtained as the expected value of the spot price under an equivalent $\mathbb{Q}$-martingale measure, conditional on the information set available up to time t, precisely

$$\mathbf{F}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right].$$

Where $\mathbb{E}_t^{\mathbb{Q}}$ represents the conditional expectation knowing a natural filtration of S_t under the risk-neutral probability Q. To incorporate the non opportunity of arbitrage in the model, we use the same approach as in Lucía and Schwartz (2002) and Cartea and Figueroa (2007), which consists of incorporating a market price of risk in the drift, to obtain

$$\widehat{\gamma}(t) = \gamma(t) - \lambda \frac{\sigma(t)}{\alpha(t)} S_t, \tag{2.3}$$

where λ denotes the market price of risk per unit risk linked to the state variable S_t. This market price of risk to be calibrated from market information, pins down the choice of one particular martingale measure. Recall that when a market subjected to that measure, the opportunity of arbitrage is theoretically excluded in this market. Hence, under this equivalent martingale measure SDE (2.4) becomes

$$dS_t = -\alpha(t)\left(\widehat{\gamma}(t) - S_t\right)dt + \sigma(t)S_t d\widehat{W}_t + (J-1)S_t dq_t, \tag{2.4}$$

substituting (2.3) in (2.4), we obtain

$$dS_t = -\alpha(t)\left(\gamma(t) - \left(1 + \lambda\frac{\sigma(t)}{\alpha(t)}\right)S_t\right)dt + \sigma(t)S_t d\widehat{W}_t + (J-1)S_t dq_t, \tag{2.5}$$

where $d\widehat{W}$ is the increment of a Brownian motion in the $\mathbb{Q}$-martingale measure specified by the choice of λ.

The next addresses the forward price computations.

Proposition 2.1. Assume that J, the increments of q_t and W_t are independent. Under the risk-neutral or martingale measure $\mathbb{Q}$ and Novikov hypothesis i.e. $\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \sigma(s)^2 ds}\right] < \infty$, electricity forward price under regulated market is given by

$$\mathbf{F}(t,T) = S_t e^{\int_t^T (\alpha(s)+\lambda\sigma(s))ds} - \int_t^T \beta(s) e^{\int_s^T (\alpha(u)+\lambda\sigma(u))du} ds. \tag{2.6}$$

Before proving the Proposition 2.1 let us first prove the following lemmas.

Lemma 2.1. If J is a log-normal distributed process with $\mathbb{E}[J] = 1$ and q a Poisson process, then $\mathbb{E}_t^{\mathbb{Q}}[e^{\int_t^T \ln J_s dq_s}] = 1$.

Proof. Firstly, we use differentiation method to compute $\mathbb{E}_t^{\mathbb{Q}}[e^{\int_0^t \ln J_s dq_s}]$.
Let us define L_t such that

$$\begin{aligned} L_t &\equiv e^{\int_0^t \ln J_s dq_s}, \\ &\equiv e^{m_t} \end{aligned} \tag{2.7}$$

where m_t is

$$m_t = \int_0^t \ln J_s dq_s,$$

or equivalently

$$dm_t = \ln J_t dq_t. \tag{2.8}$$

In order to write the dynamic followed by L_t for process define in (2.8) we use the generalization form of Itô's lemma Etheridge (2002). The SDE verified by L_t is

$$\begin{aligned} L_t &= L_0 + \int_0^t L_s \ln J_s dq_s - \int_0^t L_s \ln J_s dq_s + \int_0^t L_s(e^{\ln J_s} - 1)dq_s \\ &= 1 + \int_0^t L_s(e^{\ln J_s} - 1)dq_s. \end{aligned} \tag{2.9}$$

Then, from (2.9) we obtain

$$\begin{aligned} \mathbb{E}[L_t] &= 1 + \int_0^t \mathbb{E}[L_s](\mathbb{E}[e^{\ln J_s}] - 1)\ell ds \\ &= 1. \end{aligned}$$

Alternatively we can remark that $\int_0^t \ln J_s dq_s = \sum_{i=0}^{q_t} \ln J_s$ which is the particular case of Lévy process with the moment generating function. Using Lévy-Khintchine representation, we have

$$\begin{aligned} \mathbb{E}\left[e^{iu\int_0^t \ln J_s dq_s}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{iu\int_0^t \ln J dq_s}|q_t\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\varphi(u)^{q_t}\right]\right] \\ &= e^{t\ell(\varphi(u)-1)}. \end{aligned} \tag{2.10}$$

where φ is the moment generating function of the jump $\ln J$. Evaluating (2.10) at $u = -i$ leads to desired result. □

Proof. of Proposition 2.1 Before stating let recall that forward price formula is given by $\mathbf{F}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right]$. *By substituting* S_T *with* (1.11) *we obtain*

$$\begin{aligned} \mathbf{F}(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right] \\ &= S_t\mathbb{E}_t^{\mathbb{Q}}\left[e^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)ds+\int_t^T\sigma(s)d\widehat{W}_s+\int_t^T\ln J_s dq_s\right)}\right] \\ &\quad -\mathbb{E}_t^{\mathbb{Q}}\left[Z_t e^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)ds+\int_t^T\sigma(s)d\widehat{W}_s+\int_t^T\ln J_s dq_s\right)}\right. \\ &\qquad \left.\times\int_t^T \alpha(s)\mu(s)Z_s^{-1}ds\right]. \end{aligned} \tag{2.11}$$

We first compute $\mathbb{E}_t^{\mathbb{Q}}\left[e^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)ds+\int_t^T\sigma(s)d\widehat{W}_s+\int_t^T\ln J_s dq_s\right)}\right] \equiv A.$

From independence between J, dq_t *and* dW_t *we obtain*

$$\begin{aligned} A &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)ds+\int_t^T\sigma(s)d\widehat{W}_s\right)}\right]\mathbb{E}_t^{\mathbb{Q}}[e^{\int_t^T\ln J dq_s}] \\ &= e^{\int_t^T(\alpha(s)+\lambda\sigma(s))ds}\mathbb{E}_t^{\mathbb{Q}}[e^{\int_t^T\ln J dq_s}]. \end{aligned} \tag{2.12}$$

We now compute
$\mathbb{E}_t^{\mathbb{Q}}\left[Z_t e^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)ds+\int_t^T\sigma(s)d\widehat{W}_s+\int_t^T \ln J dq_s\right)}\int_t^T\alpha(s)\gamma(s)Z_s^{-1}ds\right]\equiv A_1.$
Replacing Z_s^{-1} by its expression, using independence between J, dq_t and dW_t and Fubini theorem Veraar (2011) we obtain

$$\begin{aligned} A_1 &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \alpha(s)\gamma(s)e^{\left(\int_s^T(\lambda\sigma(u)+\alpha(u)-\frac{1}{2}\sigma(u)^2)du+\int_s^T\ln J dq_u+\int_s^T\sigma(u)d\widehat{W}_u\right)}ds\right] \\ &= \int_t^T \mathbb{E}_t^{\mathbb{Q}}[e^{\int_s^T\ln J dq_u}]\alpha(s)\gamma(s)\mathbb{E}_t^{\mathbb{Q}}\left[e^{\left(\int_s^T(\lambda\sigma(u)+\alpha(u)-\frac{1}{2}\sigma(u)^2)du+\int_s^T\sigma(u)d\widehat{W}_u\right)}\right]ds \\ &= \int_t^T \alpha(s)\gamma(s)e^{\int_s^T(\alpha(u)+\lambda\sigma(u))du}ds. \end{aligned} \tag{2.13}$$

By replacing finally (2.12) *and* (2.13) *in* (2.11) *we obtain the forward price*

$$\mathbf{F}(t,T) = S_t e^{\int_t^T(\alpha(s)+\lambda\sigma(s))ds} - \int_t^T \beta(s)e^{\int_s^T(\alpha(u)+\lambda\sigma(u))du}ds.$$

This completes the proof. □

2.2.2 Analytical Comparison with Forward Price in Cartea et al. (2005)

Recall that forward price obtained in Cartea and Figueroa (2007) is given by

$$F(t,T) = G(T)\left(\frac{S(t)}{G(t)}\right)^{e^{-\alpha(T-t)}} e^{\int_t^T \frac{1}{2}\sigma^2(s)e^{-2\alpha(T-s)}-\lambda\sigma(s)e^{-\alpha(T-s)}ds+\int_t^T\xi(\sigma_J,\alpha,T,s)\ell ds-\ell(T-t)}$$

The forward price formula (2.6) derived in this work is an affine function of the spot price S_t, Unlike in the works of Cartea and Figueroa (2007), where they have obtained a power function of the spot price. This is what justifies the presence of fewer jumps in the forward prices. This is in line with the fact that we are in a regulating context where prices are likely to undergo less variations.

2.3 Some Illustrative Curves of Forward in Regulated Electricity Market

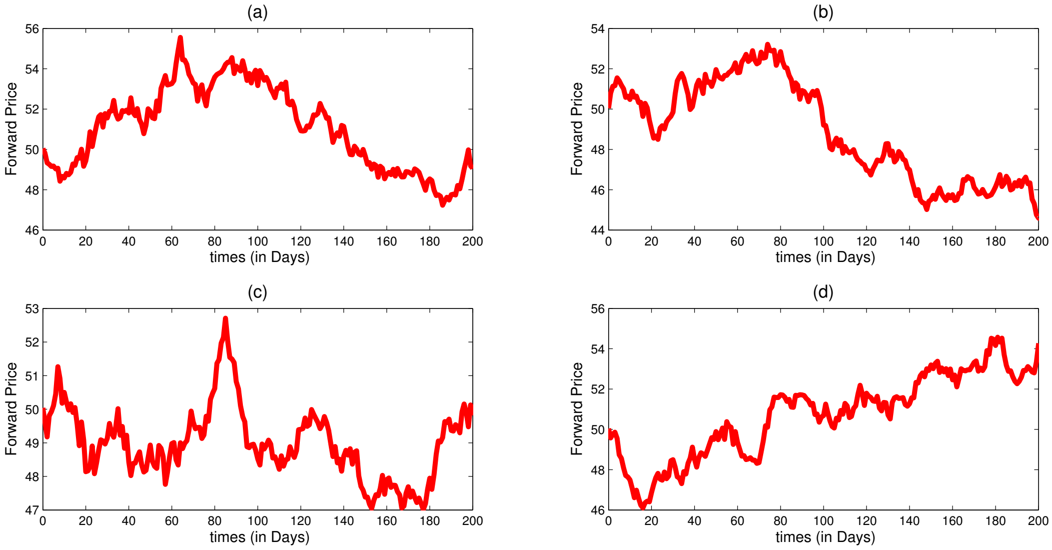

Figure 2.1: forward price for the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\sigma = 0.75$; $\sigma J = 0.67$; S(0)=50.

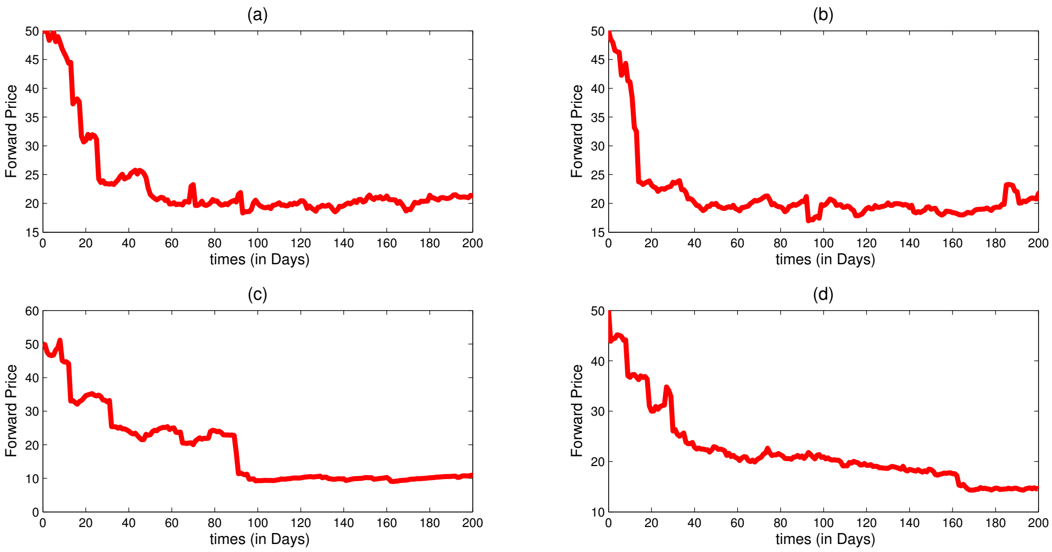

Figure 2.2: forward prices for the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\ell = 8.5$; $\sigma = 0.75$; $\sigma J = 0.67$; S(0)=50.

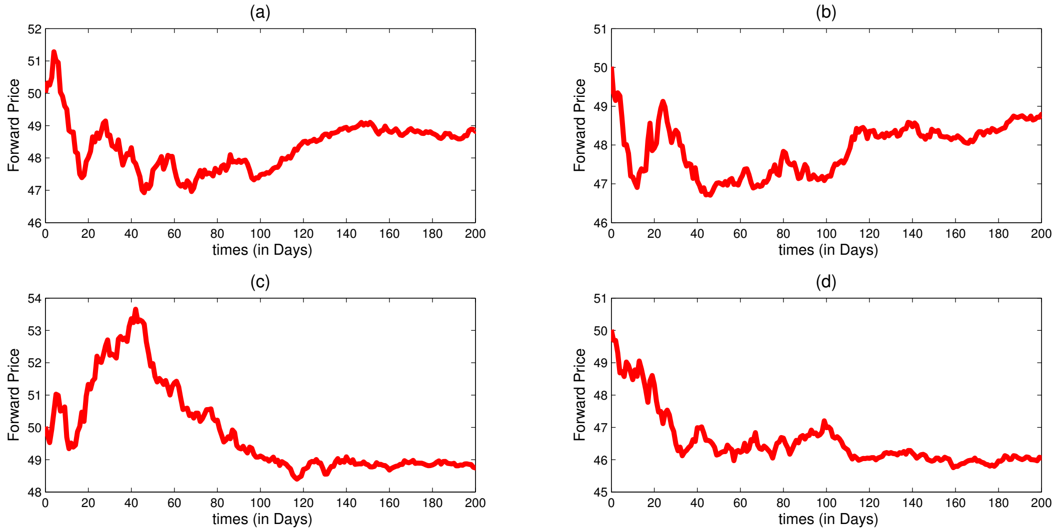

Figure 2.3: forward prices for the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\sigma = \exp(-0.01t)$; $\sigma J = 0.67$;S(0)=50.

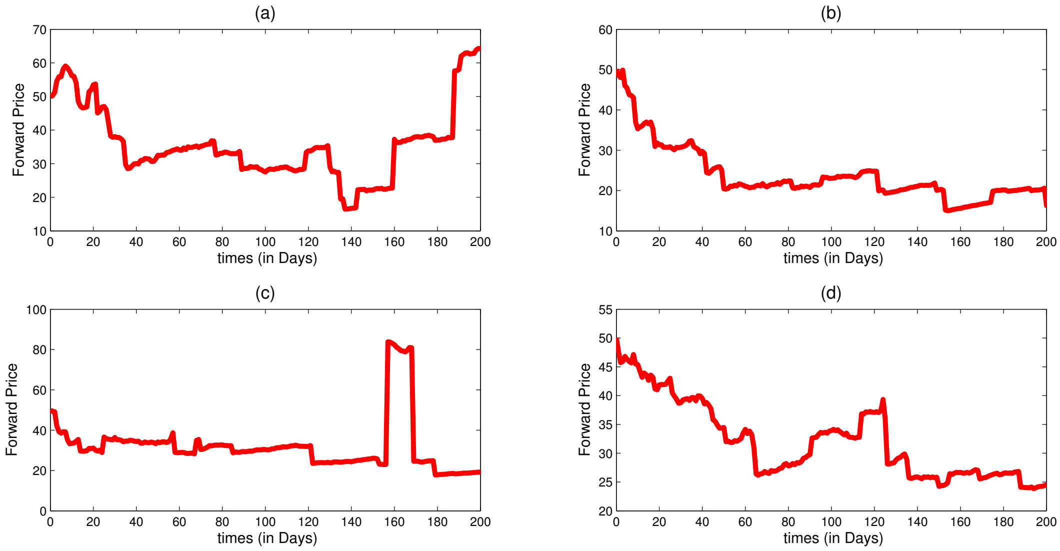

Figure 2.4: forward prices for the parameters, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\ell = 8.5$; $\sigma = \exp(-0.01t)$; $\sigma J = 0.67$; S(0)=50.

Figures 2.1 and 2.3 present four different states of the evolution of the forward price process in the absence of jumps in the spot price model. Here we observe that the forward price fluctuates around an average like the spot price. This could be justified by the fact that the forward formula obtained here is a functional of the spot price. Figures 2.2 obtained by introducing small jumps into the model show that despite the jump at the beginning, the forward price latter oscillates around an equilibrium a situation which is not observed in figure 2.4 with bigger jumps. In a nut shell, these illustrations show that our model with the mean-reversion property captures the main objective of regulation principle, which is to cap prices within a given range.

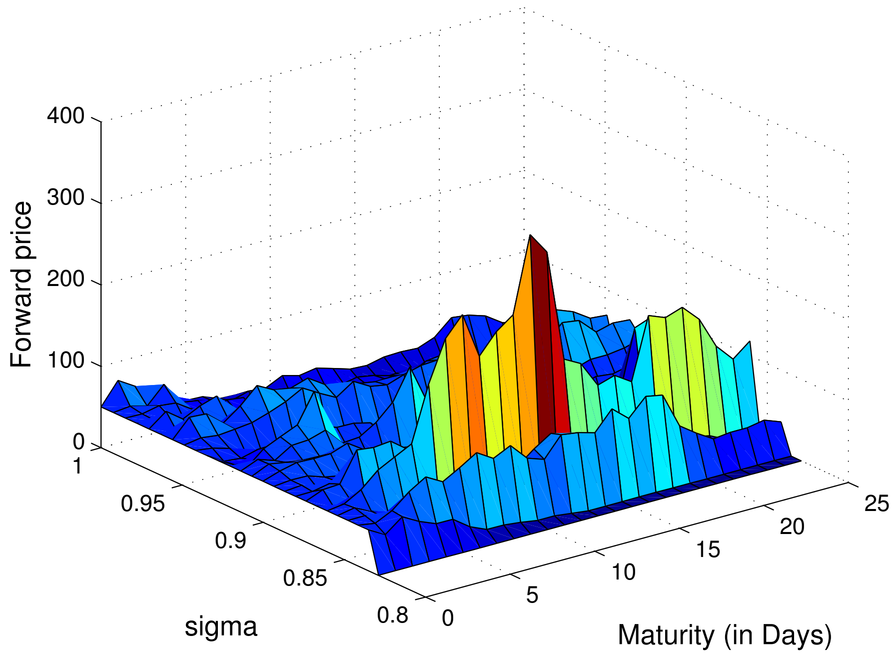

Figure 2.5: forward prices for each day for various maturity and parameter sigma, I=0.0314; G=0.01; E=0.05; H=0.001; F=0.05; $\ell = 8.5$; $\sigma = \exp(-0.01t)$; $\sigma J = 0.67$; S(0)=50.

Figures 2.5 shows that despite jumps in the prices, prices vary from a certain threshold for different maturities. We observe in figure 2.6 that when the efficiency rate factor G is more than the inflation rate factor I, forward price decrease over time. This is in accordance with the economic principle.

2.4 Option Valuation Using the Fast Fourier Transform

In this part we present an approach for determining the Fourier transform of the European call of maturity T, written on the terminal electricity spot price S_T as underlying. Let recall some definitions.

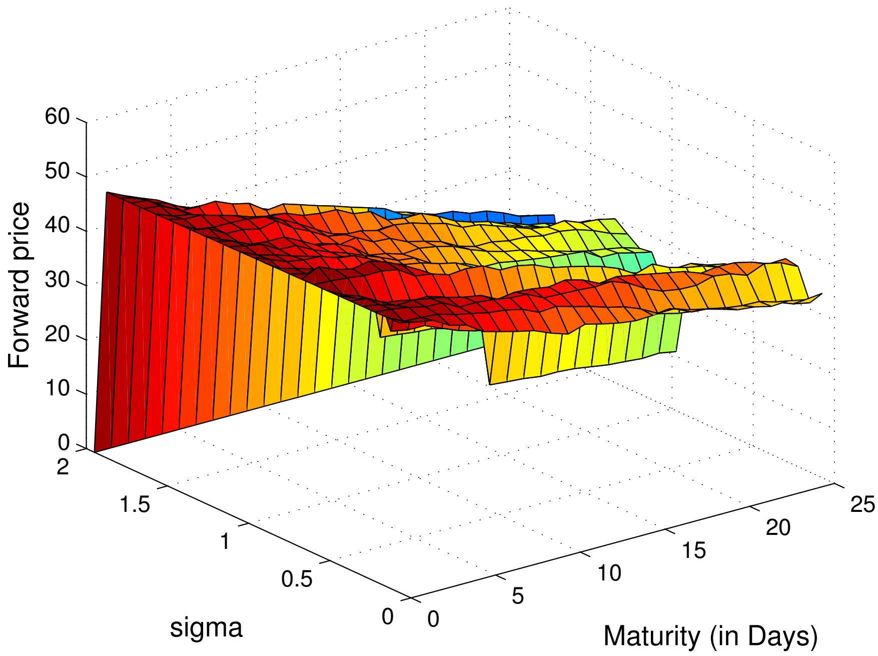

Figure 2.6: forward prices for each day for various maturity and parameter sigma I=0.0314; G=0.1; S(0)=50; $\sigma J = 0.67$; $\ell = 0.25$.

Let f be a continuous real value function that satisfies the integrability condition i.e.

$$\int_{-\infty}^{+\infty} |f(x)|dx < \infty.$$

The Fourier transform of f is defined by

$$\mathcal{F}f(u) = \int_{-\infty}^{+\infty} f(x)e^{i2\pi ux}dx, u \in \mathbb{R} \tag{2.14}$$

Given (2.14) the inverse Fourier transform allows to obtain the function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}f(u)e^{-i2\pi ux}du, x \in \mathbb{R} \tag{2.15}$$

2.4.1 Option Valuation Using Carr and Madan (1999) Approach

Under the following assumption $\int \mathbb{Q}(dx)e^{1+x} < \infty$ the European call is calculated as follows

$$C(\omega, \theta) = e^{\theta\omega}\int \mathbb{Q}(dx)\,[e^x - e^\omega]^+ \in L^1.\theta > 0 \tag{2.16}$$

The Fourier transform of (2.16) is given by

$$\mathcal{F}[C(\omega,\theta)] = -\frac{i}{2\pi}\left[\frac{1}{k-i\theta/2\pi} - \frac{1}{k-i(1+\theta)/2\pi}\right]\times \int \mathbb{Q}(dx)e^{i2\pi[k-i(1+\theta)/2\pi]x}$$

then, apply Fourier inverse transform we obtain

$$\begin{aligned}\mathcal{F}[C(\omega,\theta)] &= -\frac{i}{2\pi}\left[\frac{1}{k-i\theta/2\pi} - \frac{1}{k-i(1+\theta)/2\pi}\right] \\ &\times \Phi_Y(k-\frac{i(1+\theta)}{2\pi})\end{aligned}$$

thus, the European call option is given by

$$C(\omega) = -\frac{ie^{-\theta\omega}}{2\pi}\int dke^{-i2k\pi\omega}\Phi_Y(k-\frac{i(1+\theta)}{2\pi})\times \left[\frac{1}{k-i\theta/2\pi} - \frac{1}{k-i(1+\theta)/2\pi}\right] \tag{2.17}$$

where Φ represent the characteristic function of the process that describe the underlying asset.

2.4.2 Option Valuation Using Lewis (1999) Approach

In this part we use the approach developed in Lewis (2001) to derive the European call option. Since we have

$$e^{-\theta x}\left[e^x - e^\omega\right] \in L^1, \theta > 1 \tag{2.18}$$

Assume that there exists β such that

$$\mathbb{E}[e^{\lambda x}] < \infty, \quad \forall\lambda < \beta.$$

Then we have

$$C(\omega,\lambda) = \int [\mathbb{Q}e^{\lambda x}][e^{-\lambda x}(e^x - e^\omega)^+].$$

By owning in the same way as before the European call is derived as follow

$$C(\omega) = e^{-(\lambda-1)\omega}\int dke^{i2\pi k\omega}\Phi_Y(k-i\frac{\lambda}{2\pi})(\frac{-i}{2\pi})\times \left[\frac{1}{k+i\lambda/2\pi} - \frac{1}{k+i(\lambda-1)/2\pi}\right]. \tag{2.19}$$

the two formulas depend on the characteristic function. The process that models electricity spot prices does not have a characteristic function with an exact expression. Indeed, the process is the sum of two laws. so, to evaluate the option an approximation of the characteristic function is necessary.

2.4.3 Characteristic Function Approximation

The characteristic function is not available to give the approximation we first approximated (1.8) after set $Y = \ln S$ to simplify the later.

Let $\Delta t = t_{n+1} - t_n$, $\Delta W_n^M = W_{n+1}^M - W_n^M$, $\Delta q_n^M = q_{n+1}^M - q_n^M$, $Y_{t_n}^M = Y_n^M$ with $f(Y_t) = -\alpha(t)\left(\gamma(t)e^{-Y_t} - (1 + \lambda\frac{\sigma(t)}{\alpha(t)})\right)$ From tamed Euler scheme of (1.8) rewrote in Y is given by the following relation

$$Y_{n+1}^M = Y_n^M + \frac{\Delta t f(Y_n^M)}{1 + \Delta t|f(Y_n^M)|} + \sigma_n^M \Delta W_n^M + \ln J \Delta q_n^M. \tag{2.20}$$

Thus, the characteristic function of $Y = \ln S$ is given by

$$\mathbb{E}\left[e^{iuY_{n+1}^M}|Y_n^M\right] = e^{iu\left(Y_n^M + \frac{\Delta t f(Y_n^M)}{1+\Delta t|f(Y_n^M)|}\right)} \times$$
$$e^{-\frac{1}{2}u^2\sigma_n^M \Delta t} \times e^{l\Delta t\left(e^{(-iu-u^2)\sigma_J^2/2}-1\right)} \tag{2.21}$$

2.4.4 An Illustrative example

using the approximation of the characteristic function and an approximation by the trapezoidal method of the integrals we obtained for different approach a value of the European call option in the following table.

method	Carr Madan	Alan Lewis
strike K=30	8.1437	3.0892
strike K=50	2.6675	2.5761
strike K=70	1.5784	2.3018

Table 2.1: European call option for the parameters $r = 0.75$, $S_T = 53$, $\alpha = 0.5$, $\gamma = 1.5$, $\sigma = 0.5$, $\sigma_J = 0.1$, $l = 0.75$, $\lambda = 0.5$, $T = 1$, $N = 1000$

Table 2.1 give the call option price for different strike using both precedent approach presented in this part. The decreasing of call option when the strike increase is in accordance with reality.

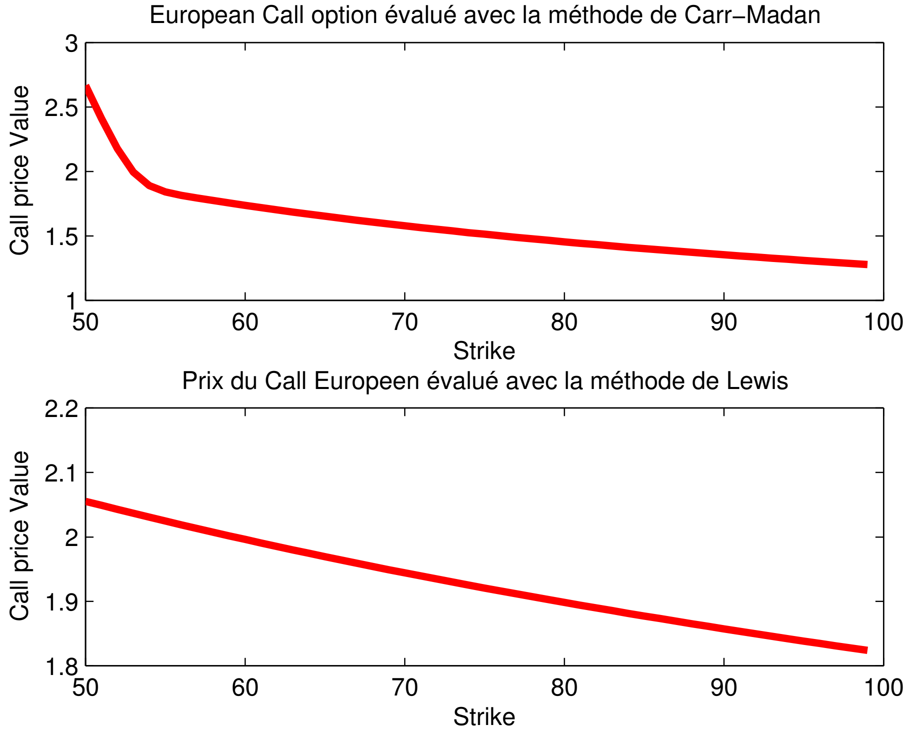

Figure 2.7: European call option in electricity market

Figure 2.7 represents the simulated call option value versus the strike in the case that the underlying is the electricity where the prices are regulated by price-cap principle and modeled by (2.1). Both curves where in the first call option formula in performed using Carr-Madan approach and where the call option formula is calculated using Lewis approach show that when the strike increase, call option value decrease which it is in accord with the practice in finance.

2.5 Partial Integro-Differential Equation for Call Option Prices

This part aims at evaluating the price of an option (Put or Call) in the regulated electricity market under risk-neutral probability, $\mathbb{Q}$, with the terminal payoff, H_T, which is given by:

$$C_t = \mathbb{E}[e^{-r(T-t)}H_T|\mathcal{F}_t], \tag{2.22}$$

where r represents a free risk discounting rate, T denotes the maturity, and K represents the strike price. Let S_T be the solution of (1.8) at T, which is the equation of the underlying. $H_T = H(S_T)$, with $H(S) = (S-K)^+$ for European Call or $H(S) = (K-S)^+$ for European Put. From the Markov property, C_t becomes

$$C(t,S) = \mathbb{E}[e^{-r(T-t)}H_T|S_t = S]. \tag{2.23}$$

Proposition 2.2. Assume that the European option C given by

$$\begin{aligned} C : (0,T)\times(0,\infty) &\to \mathbb{R} \\ (t,S) &\mapsto C(t,S) \end{aligned} \tag{2.24}$$

is $C^{1,2}$, with $\partial C/\partial S$ and $\partial^2 C/\partial S^2$ bounded, then C satisfies the partial integro-differential equation:

$$\begin{aligned} \frac{\partial C}{\partial t}(t,S) + (\alpha(t)S - \beta(t))\frac{\partial C}{\partial S}(t,S) + \frac{\sigma(t)^2S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) - rC(t,S) \\ +\ell\int_{\mathbb{R}}\nu(dx)[C(t,xS) - C(t,S)] = 0 \end{aligned} \tag{2.25}$$

on $(0,T)\times(0,\infty)$ with the terminal condition $C(T,S) = H(S),\ \ \forall S > 0$, where ℓ represents the intensity of the Poisson process under risk-neutral measure, and the measure $\nu(dx)$ is the jump size distribution.

Proof. The proof consists of applying Itô formula with jump to the martingale $\tilde{C}(t,S_t) = e^{-rt}C(t,S_t)$, then identify the drift component and set it to zero. By construction $\tilde{C}$ is a martingale. Applying the Itô formula to $\tilde{C}$ we obtain:

$$\begin{aligned} d\tilde{C}_t &= e^{-rt}\left[-rC(t,S_t) + \frac{\partial C}{\partial t}(t,S_t) + \frac{\sigma(t)^2S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_t)\right]dt \\ &\quad +e^{-rt}\frac{\partial C}{\partial S}(t,S_t)dS_t \\ &\quad +e^{-rt}\left[C(t,JS_{t^-}) - C(t,S_{t^-}) + (J-1)S_{t^-}\frac{\partial C}{\partial S}(t,S_{t^-})\right]dq_t. \end{aligned}$$

From (2.5) this equation is equivalent to

$$\begin{aligned} d\tilde{C}_t &= e^{-rt}\left[-rC(t,S_t) + \frac{\partial C}{\partial t}(t,S_t) + \frac{\sigma(t)^2S_t^2}{2}\frac{\partial^2 C}{\partial S_t^2}(t,S_t)\right]dt \\ &\quad +e^{-rt}\left[(\alpha(t)S_t - \beta(t))\frac{\partial C}{\partial S}(t,S_t)dt + \frac{\partial C}{\partial S}(t,S)S_t\sigma(t)dW_t\right] \\ &\quad +e^{-rt}\left[C(t,JS_{t^-}) - C(t,S_{t^-})\right]dq_t. \end{aligned}$$

Adding and subtracting

$$\ell\int_{\mathbb{R}}\nu(dx)(C(t,S_tx) - C(t,S_t))dt,$$

one has

$$d\tilde{C}_t = a(t)dt + dM_t,$$

where

$$\begin{aligned} a(t) &= e^{-rt}\left[\frac{\partial C}{\partial t}(t,S_t) + (\alpha(t)S_t - \beta(t))\frac{\partial C}{\partial S}(t,S_t) + \frac{\sigma(t)^2 S_t^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S_t)\right. \\ &\quad \left. -rC(t,S_t) + \ell\int_{\mathbb{R}}\nu(dx)(C(t,S_t x) - C(t,S_t))\right] \end{aligned}$$

and

$$dM_t = e^{-rt}\left[\frac{\partial C}{\partial S}(t,S)S_t\sigma(t)dW_t + (C(t,JS_{t^-}) - C(t,S_{t^-}))d\tilde{q}_t\right],$$

with $\tilde{q}_t = q_t - \ell t$. We now show that M_t is a martingale. Since the payoff function H is Lipschitz. Then, C is also Lipschitz with respect to the second variable S. Indeed:

$$\begin{aligned} |C(t,x) - C(t,y)| &= e^{-(T-t)}|\mathbb{E}[H(S_t e^{X_T - X_t} - \int_t^T \beta(s)e^{X_T - X_s}ds) \mid S_t = x] \\ &\quad -\mathbb{E}[H(S_t e^{X_T - X_t} - \int_t^T \beta(s)e^{X_T - X_s}ds) \mid S_t = y]| \\ &\leq c_1 e^{-r(T-t)}\mathbb{E}[e^{\int_t^T \alpha(s) - \frac{1}{2}\sigma(s)^2 ds + \int_t^T \sigma(s)dW_s + \int_t^T \ln J dq_s}]|x-y|, \text{ for every fixed } t. \end{aligned}$$

Since $e^{\int_t^T -\frac{1}{2}\sigma(s)^2 ds + \int_t^T \sigma(s)dW_s}$ is a martingale and we also have from assumption 1 that $\mathbb{E}[e^{\int_t^T \ln J dq_s}] = 1$, then we get

$$|C(t,x) - C(t,y)| \leq c|x-y|e^{\int_t^T \alpha(s)ds} \leq c_1|x-y|,$$

with $c_1 = ce^{\int_t^T \alpha(s)ds}$.

Therefore the predictable random function $\varphi(t,x) = C(t,xS_{t^-}) - C(t,S_{t^-})$ verifies:

$$\begin{aligned} \mathbb{E}[\int_0^T\int_{\mathbb{R}}\nu(dx)|\varphi(t,x)|^2 dt] &\leq \mathbb{E}[\int_0^T dt\int_{\mathbb{R}}\nu(dx)c_1(x^2+1)S_t^2] \\ &\leq \int_{\mathbb{R}} c_1^2(x^2+1)\nu(dx)\mathbb{E}[\int_0^T S_t^2 dt] < \infty, \end{aligned}$$

where the last inequality holds because the distribution $\nu(dx)$ of the jump sizes is assumed $\log$-normal. In effect, we have $\int_{\mathbb{R}} x^2\nu(dx) < \infty$, hence $\mathbb{E}[\int_0^T S_t^2 dt] < \infty$. Therefore, the compensated Poisson integral $\int_0^T \int_{\mathbb{R}} e^{-rt}[C(t, xS_{t-}) - C(t, S_{t-})]d\tilde{q}_t$ is a square integrable martingale. Since C is Lipschitz, $\dfrac{\partial C}{\partial S}(t,.) \in L^\infty$ and $\|\dfrac{\partial C}{\partial S}(t,.)\|_{L^\infty} \leq c_2$. Thus, $\mathbb{E}\big[\int_0^T S_t^2|\dfrac{\partial C}{\partial S}(t,S_t)|^2 dt\big] \leq c_2^2\mathbb{E}[\int_0^T S_t^2 dt] < \infty$. Furthermore, using the isometric relation and the preceding result, it follows that $\int_0^T \dfrac{\partial C}{\partial S}(t,S_t)S_t\sigma(t)dW_t$ is a square integrable martingale. Therefore, M_t is also a square integrable martingale, implying $\tilde{C}_t - M_t$ is a square integrable martingale. But $\tilde{C}_t - M_t = \int_0^t a(t)dt$ is also a continuous process with finite variation, so, from Theorem 4.13-450 in Jacod and Shiryaev (2003), one must have $a(t) = 0$ $\mathbb{Q}$-almost surely, leading to the PIDE (2.25). $\square$

Note that the smoothness (particularly the uniform boundedness of derivatives) assumption made on the European call option is not generally verified as discussed in Cont and Ekaterina (2005b). In this case, option prices should be considered as a viscosity solution of the PIDE obtained in Proposition 2.2. The following proposition gives the link between option prices and the viscosity solution of the PIDE.

Proposition 2.3. (**Option prices as viscosity solutions**)
The European option defined by (2.23) is the (unique) viscosity solution of the Cauchy problem (2.25).

Proof. Existence and uniqueness of viscosity solutions for such parabolic integro-differential equations are discussed in Alvarez and Tourin (1996) in the case (the one considered here) where ν is the finite measure.
In what follows, we propose a numerical solution to the PIDE which converges to the viscosity solution as proven in Cont and Ekaterina (2005a). $\square$

2.6 An Explicit-Implicit Difference scheme

In this section we present a numerical procedure for solving the PIDE (2.25) obtained in Proposition 2.2. Introducing the change of variable $x = \ln\dfrac{S}{S_0}$ and $\tau = T - t$ and defining: $u(\tau, x) = e^{r\tau}C(T-\tau, S_0e^x)$, we obtain:

$$\begin{aligned} u(\tau,x) &= \mathbb{E}\left[H(S_t e^{X_T - X_{T-\tau}} - \int_{T-\tau}^{T}\beta(s)e^{X_T - X_s}ds) \mid S_t = S_0e^x\right] \\ &= \mathbb{E}\left[H(Y_\tau^x)\right], \end{aligned} \tag{2.26}$$

where $Y_\tau^x = S_0 e^{x+X_T - X_{T-\tau}} - \int_{T-\tau}^{T} \beta(s) e^{X_T - X_s} ds$. We then obtain a PIDE in terms of u, given by:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \mathcal{L}u, \ on \ (0,T] \times O \\ u(0,x) = H(S_0 e^x), \quad x \in O, \quad u(\tau,x) = 0, \quad x \in O^c, \end{cases} \tag{2.27}$$

where $O \subset \mathbb{R}$ is an open interval which is not necessarily bounded,

$$\begin{aligned} \mathcal{L}u(\tau,x) &= \left(\alpha(T-\tau) - \frac{1}{2}\sigma^2(T-\tau) - \frac{\beta(T-\tau)}{S_0}\right)\frac{\partial u}{\partial x}(\tau,x) + \frac{1}{2}\sigma^2(T-\tau)\frac{\partial^2 u}{\partial x^2}(\tau,x) \\ &\quad + \ell \int_{\mathbb{R}} [u(\tau,x+y) - u(\tau,x)]\, g_{\ln J}(y) dy, \end{aligned} \tag{2.28}$$

with $g_{\ln J}$ denoting the density function of $\ln J$.

The main idea in this method is to split the operator $\mathcal{L}$ into two parts as in Cont and Ekaterina (2005b). We replace the differential part with a finite difference approximation, and the integral part with a trapezoidal quadrature approximation. We treat the integral part with an explicit time stepping in order to avoid the inversion problem of the dense matrix L_J associated to the discretization of the integral term. We then rewrite the PIDE (2.27) as follow:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = (\mathcal{L}_D + \mathcal{L}_J)u, \ on \ (0,T] \times O \\ u(0,x) = H(S_0 e^x), \quad x \in O, \quad u(\tau,x) = 0, \quad x \in O^c, \end{cases} \tag{2.29}$$

where

$$\begin{aligned} \mathcal{L}_D u(\tau,x) &= \left(\alpha(T-\tau) - \frac{1}{2}\sigma^2(T-\tau) - \frac{\beta(T-\tau)}{S_0}\right)\frac{\partial u}{\partial x}(\tau,x) \\ &\quad + \frac{1}{2}\sigma^2(T-\tau)\frac{\partial^2 u}{\partial x^2}(\tau,x), \end{aligned} \tag{2.30}$$

$$\mathcal{L}_J u(\tau,x) = \ell \int_{\mathbb{R}} [u(\tau,x+y) - u(\tau,x)]\, g_{\ln J}(y) dy. \tag{2.31}$$

In order to solve numerically the PIDE (2.29), we first localize the variables and the integral term to bounded domains..

2.6.1 Localisation to a bounded domain

To numerically solve the Cauchy problem (2.29), we first truncate the space domain to a bounded interval $(-A_l, A_r)$. Usually this leads to defining some boundary conditions at $x = -A_l$ and $x = A_r$. But here, we are in an elliptic local PIDE due to the presence of an integral term. Thus, we need to extend the function $u(\tau,.)$ to a subset $\{x+y : x \in (-A_l, A_r), y \in \operatorname{supp} g_{\ln J}\}$, where $\operatorname{supp} g_{\ln J} = \mathbb{R}_+$, is the support of $g_{\ln J}$. Let $u_A(\tau,x)$ be the solution of the following localization problem:

$$\begin{cases} \dfrac{\partial u_{l,r}}{\partial \tau} = (\mathcal{L}_D + \mathcal{L}_J)u_{l,r}, \;\; on \;\; (0,T] \times (-A_l, A_r) \\ u_{l,r}(0,x) = H(S_0 e^x), \;\; x \in (-A_l, A_r) \;\; u_{l,r}(\tau,x) = 0, \;\; x \notin (-A_l, A_r). \end{cases} \tag{2.32}$$

We will show in the following proposition that the localization error decays exponentially with the domain size A.

Proposition 2.4. Assume $\|H\|_\infty < \infty$ and $C_\tau = \mathbb{E}\left[e^{\sup\limits_{\eta\in[0,\tau]}|X_T - X_{T-\eta}|}\right] < \infty$. Let $u_{l,r}(\tau,x)$ and $u(\tau,x)$ be respectively the solutions of the Cauchy problems (2.29) and (2.32). Then

$$|u(\tau,x) - u_{l,r}(\tau,x)| \leq C_\tau \|H\|_\infty e^{-\max(A_l,A_r)+|x|}, \quad \forall x \in (-A_l, A_r) \tag{2.33}$$

where the constant C_τ does not depend on A_r and A_l.

Proof. Let $M_\tau^x = \sup\limits_{\eta\in[0,\tau]} |x + X_T - X_{T-\eta}|$. Then

$$\begin{aligned} u_{l,r}(\tau,x) &= \mathbb{E}\left[\mathbb{1}_{\{M_\tau^x < \max(A_l,A_r)\}} H(Y_\tau^x)\right] \\ \text{and} \qquad u(\tau,x) &= \mathbb{E}\left[H(Y_\tau^x)\right]. \end{aligned} \tag{2.34}$$

Hence

$$\begin{aligned} |u - u_{l,r}| &= \left|\mathbb{E}\left[H(Y_\tau^x)\mathbb{1}_{\{M_\tau^x \geq \max(A_l,A_r)\}}\right]\right| \\ &\leq \|H\|_\infty \left|\mathbb{E}\left[\mathbb{1}_{\{M_\tau^x \geq \max(A_l,A_r)\}}\right]\right| \\ &\leq \|H\|_\infty \mathbb{Q}(M_\tau^x \geqslant \max(A_l,A_r)). \end{aligned} \tag{2.35}$$

Theorem 25.18 in Sato (1999) and the fact that $\int_{\mathbb{R}} e^{|x|}\nu(dx) < \infty$ imply

$$C_\tau = \mathbb{E}\left[e^{\sup\limits_{\eta\in[0,\tau]}|X_T - X_{T-\eta}|}\right] < \infty. \tag{2.36}$$

But

$$\mathbb{Q}(M_\tau^x \geqslant \max(A_l,A_r)) \;\; = \;\; \mathbb{Q}(e^{M_\tau^x} \geqslant e^{\max(A_l,A_r)}) \tag{2.37}$$

$$\tag{2.38}$$

and, since $\sup\limits_{\eta\in[0,\tau]} |x + X_T - X_{T-\eta}| \leq \sup\limits_{\eta\in[0,\tau]} |X_T - X_{T-\eta}| + |x|$, then

$\{M_\tau^x \geqslant \max(A_l, A_r)\} \subset \left\{ \sup_{\eta\in[0,\tau]} |X_T - X_{T-\eta}| \geq \max(A_l, A_r) - |x| \right\}$. Hence

$$\mathbb{Q}\left(e^{M_\tau^x} \geqslant e^{\max(A_l,A_r)}\right) \leq \mathbb{Q}\left(e^{\sup\limits_{\eta\in[0,\tau]}|X_T - X_{T-\eta}|} \geqslant e^{\max(A_l,A_r)-|x|}\right). \tag{2.39}$$

Now, using Markov's inequality we obtain

$$\mathbb{Q}\left(e^{\sup\limits_{\eta\in[0,\tau]}|X_T - X_{T-\eta}|} \geqslant e^{\max(A_l,A_r)}\right) \leq \frac{\mathbb{E}\left[e^{\sup\limits_{\eta\in[0,\tau]}|X_T - X_{T-\eta}|}\right]}{e^{\max(A_l,A_r)-|x|}}. \tag{2.40}$$

Inserting these last inequalities in (2.35) gives the desired result. □

2.6.2 Truncation of the integral

To numerically compute the integral term of the PIDE (2.29), we need to reduce the region of integration to a bounded interval which leads to the truncation of large jumps. We then estimate the error resulting from this approximation. In effect, suppose a new process, $\tilde{S}_t$, is characterized by the fact that logarithm of the jump size, $\ln \tilde{J}$, is bounded in $[B_l, B_r]$, with the associated measure $\mathbb{1}_{\{y\in[Bl,Br]\}}\nu$, where B_l and B_r are real. We further suppose, without loss generality, that $B_l < 0$ and $B_r > 0$. In this case the corresponding solution to the associated PIDE is denoted by $\tilde{u}(\tau, x)$. We analyse, in the following proposition, the difference $|u - \tilde{u}|$.

Proposition 2.5. One has:

$$|u(\tau, x) - \tilde{u}(\tau, x)| \leq C^\tau \left(C_1 e^{-|B_l|} + C_2 e^{-B_r}\right), \tag{2.41}$$

where $C^\tau = C\left[\tau \ell S_0 + \ell \int_{T-\tau}^{T} \beta(s)(T-s)ds\right]$.

Proof. Firstly, let $\tilde{X}_t$ be a new Lévy process defined by:

$$\tilde{X}_t = \int_0^t \alpha(s) - \frac{1}{2}\sigma(s)^2 ds + \int_0^t \sigma(s)dW_s + \int_0^t \ln \tilde{J} dq_s, \tag{2.42}$$

and let:

$$\tilde{u}(\tau, x) = \mathbb{E}[H(\tilde{Y}_\tau^x)], \tag{2.43}$$

where $\tilde{Y}^x_\tau = S_0 e^{x+\tilde{X}_T-\tilde{X}_{T-\tau}} - \int_{T-\tau}^T \beta(s) e^{\tilde{X}_T-\tilde{X}_s} ds$. Setting $R_\tau = X_T - \tilde{X}_T - (X_\tau - \tilde{X}_\tau)$ and using the Lipschitz property on H, we obtain:

$$\begin{aligned}
|u(\tau,x)-\tilde{u}(\tau,x)| &= |\mathbb{E}[H(Y^x_\tau)] - \mathbb{E}[H(\tilde{Y}^x_\tau)]| \\
&\leq c_1 \mathbb{E}\Big[|S_0(e^{x+\tilde{X}_T-\tilde{X}_{T-\tau}+R_{T-\tau}} - e^{x+\tilde{X}_T-\tilde{X}_{T-\tau}}) \\
&\quad - \int_{T-\tau}^T \beta(s)(e^{(\tilde{X}_T-\tilde{X}_s+R_s} - e^{\tilde{X}_T-\tilde{X}_s} ds|\Big] \\
&\leq c_1 \Big(S_0 \mathbb{E}[e^{\tilde{X}_T-\tilde{X}_{T-\tau}} |e^{R_{T-\tau}} - 1|] \\
&\quad + \int_{T-\tau}^T \beta(s) \mathbb{E}[e^{\tilde{X}_T-\tilde{X}_s} |e^{R_s} - 1|] ds \Big). \qquad (2.44)
\end{aligned}$$

Since R_τ and $\tilde{X}_T - \tilde{X}_\tau$ are independent, we have

$$\begin{aligned}
|u(\tau,x)-\tilde{u}(\tau,x)| &\leq c_1 e^x (S_0 \mathbb{E}[e^{\tilde{X}_T-\tilde{X}_{T-\tau}}] \mathbb{E}[|e^{R_{T-\tau}} - 1|] \\
&\quad + \int_{T-\tau}^T \beta(s) \mathbb{E}[e^{\tilde{X}_T-\tilde{X}_s}] \mathbb{E}[|e^{R_s} - 1|] ds). \qquad (2.45)
\end{aligned}$$

Moreover, $\left(e^{\tilde{X}_T-\tilde{X}_{T-u}}, u \in [0,T]\right)$ being a martingale, $\mathbb{E}[e^{\tilde{X}_T-\tilde{X}_{T-\tau}}] = 1$ and $\mathbb{E}[e^{\tilde{X}_T-\tilde{X}_s}] = 1$. As a consequence,

$$|u(\tau,x)-\tilde{u}(\tau,x)| \leq c_1 e^x (S_0 \mathbb{E}[|e^{R_{T-\tau}} - 1|] + \int_{T-\tau}^T \beta(s) \mathbb{E}[|e^{R_s} - 1|] ds). \qquad (2.46)$$

Since, for every $a \in \mathbb{R}$, $|e^a - 1| = (e^a - 1) + 2(e^a - 1)^+$ and $(e^a - 1)^+ \leq |a|$, then

$$|u(\tau,x)-\tilde{u}(\tau,x)| \leq c_1 e^x (S_0 \mathbb{E}[|R_{T-\tau}|] + \int_{T-\tau}^T \beta(s) \mathbb{E}[|R_s|] ds). \qquad (2.47)$$

But:

$$\begin{aligned}
\mathbb{E}[|R_{T-\tau}|] &\leq \ell \int_{T-\tau}^T \left[-\int_{-\infty}^{Bl} y g_{\ln J}(y) dy + \int_{Br}^{+\infty} y g_{\ln J}(y) dy \right] ds \\
&\leq \tau \ell \left(-e^{-|Bl|} \int_{-\infty}^{Bl} y e^{|y|} g_{\ln J}(y) dy + e^{-|Br|} \int_{Br}^{+\infty} y e^{|y|} g_{\ln J}(y) dy \right) \\
&\leq \tau \ell S_0 \left(C_1 e^{-|Bl|} + C_2 e^{-|Br|} \right), \qquad (2.48)
\end{aligned}$$

where $C_1 = -\int_{-\infty}^{Bl} y e^{|y|} g_{\ln J}(y) dy$ and $C_2 = \int_{Br}^{+\infty} y e^{|y|} g_{\ln J}(y) dy$. Replacing (2.48) into (2.44), we get:

$$|u(\tau,x)-\tilde{u}(\tau,x)| \leq C \left[\tau \ell S_0 + \ell \int_{T-\tau}^T \beta(s)(T-s) ds \right] \left(C_1 e^{-|Bl|} + C_2 e^{-|Br|} \right), \qquad (2.49)$$

where $C = c_1 e^x$ □

From Proposition2.4 and 2.5, $\tilde{u}$ converges to u when $|B_l|$ and $|B_r|$ grow to infinity.

2.6.3 Explicit-implicit scheme

Define a uniform grid on $(0, T] \times (-A_l, A_r)$ by $\tau_n = n\Delta t$, $n = 0$,.....,M, $x_i = i\Delta x - A_l$, $i = 0, .., N$, with $\Delta t = T/M$ and $\Delta x = \dfrac{A_r + A_l}{N}$. Let (u_i^n) be the solution of the numerical scheme which we define below: Firstly, to approximate the integral terms, we use the trapezoidal quadrature rule with the same resolution Δx. Let K_l and K_r be such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$, then:

$$\int_{B_l}^{B_r} (u(\tau, x_i + y) - u(\tau, x_i)) g_{\ln J}(y) dy \simeq \sum_{j=K_l}^{K_r} \nu_j (u_{i+j} - u_i), \tag{2.50}$$

where $\nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} g_{\ln J}(y) dy$. Notice that to compute the integral term, we need to extend the solution to $[-A_l + B_l, Ar + B_r]$. Hence, we assume that this solution is zero except in $[-A_l, A_r]$. The derivatives are discretized using the finite difference method thus:

$$\begin{cases} \left(\dfrac{\partial^2 u}{\partial x^2}\right)_i \simeq \dfrac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} \\ \left(\dfrac{\partial u}{\partial x}\right)_i \simeq \begin{cases} \dfrac{u_{i+1} - u_i}{\Delta x} & \text{if } f(\tau, x) \geq 0 \\ \dfrac{u_i - u_{i-1}}{\Delta x} & \text{if } f(\tau, x) \leq 0, \end{cases} \end{cases} \tag{2.51}$$

where $f(\tau, x) = \alpha(T - \tau) - \dfrac{1}{2}\sigma^2(T - \tau) - \dfrac{\beta(T - \tau)}{S_0} e^x$.
Using (2.50) and (2.51), and supposing $f(\tau, x) < 0$, we obtain the following relation:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = (L_D u)_i^{n+1} + (L_J u)_i^n, \tag{2.52}$$

where

$$\begin{cases} (L_D u)_i^n = f(\tau_n, x_i) \dfrac{u_{i+1}^n - u_i^n}{\Delta x} + \dfrac{1}{2}\sigma^2(T - \tau_n) \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \\ (L_J u)_i^n = \sum\limits_{j=K_l}^{K_r} \nu_j (u_{i+j}^n - u_i^n). \end{cases} \tag{2.53}$$

Finally, we replace the problem (2.29) with the following time-stepping numerical scheme:

$$\begin{cases} \text{Initialisation } u_i^0 = H(S_0 e^{x_i}) & \text{if } i \in \{0, ...N - 1\} \\ \text{For n=0,...,M-1} & \\ \dfrac{u_i^{n+1} - u_i^n}{(\Delta t)} = (L_D u)_i^{n+1} + (L_J u)_i^n & \text{if } i \in \{0, .., N - 1\} \\ u_i^{n+1} = 0 & \text{if } i \notin \{0, .., N - 1\}. \end{cases} \tag{2.54}$$

After defining the numerical scheme, we study some of its important properties, particularly, consistency, monotonicity, stability and convergence.

2.6.4 Consistency

The follow proposition shows that (2.54) is consistent with (2.29).

Proposition 2.6. (Consistency)

The finite difference scheme (2.54) is locally consistent with equation (2.29): That is, $\forall\, v \in C_0^\infty([0,T]\times (A_l, A_r))$, and $\forall\, (\tau_n, x_i) \in [0,T]\times\mathbb{R}$, one has:

$$\left|\frac{v_i^{n+1}-v_i^n}{(\Delta t)} - (L_D v)_i^{n+1} - (L_J v)_i^n - \frac{\partial v}{\partial \tau}(\tau_n, x_i) - (\mathcal{L}_D + \mathcal{L}_J)v(\tau_n, x_i)\right| = r_i^n(\Delta t, \Delta x) \to 0 \tag{2.55}$$

as $(\Delta t, \Delta x) \to (0,0)$. In other words, $\exists c > 0$ such that: $|r_i^n(\Delta t, \Delta x)| \leq c(\Delta t + \Delta x)$.

Proof. Let

$$\begin{cases} a_1 = \dfrac{v_i^{n+1}-v_i^n}{\Delta t} - \dfrac{\partial v}{\partial \tau}(\tau_n, x_i) \\ a_2 = (L_D v)_i^{n+1} - \mathcal{L}_D v(\tau_n, x_i) \\ a_3 = (L_J v)_i^n - \mathcal{L}_J v(\tau_n, x_i). \end{cases} \tag{2.56}$$

Using the second order Taylor expansion with respect to τ, we obtain

$$v_i^{n+1} \approx v_i^n + \Delta t \frac{\partial v}{\partial \tau}(\tau_n, x_i) + \frac{(\Delta t)^2}{2}\frac{\partial^2 v}{\partial \tau^2}(\tau_n, x_i).$$

Replacing this relation in the first equation in (2.56) we get:

$$|a_1| = \frac{\Delta t}{2}\left|\frac{\partial^2 v}{\partial \tau^2}(\tau_n, x_i)\right| \leq \frac{\Delta t}{2}\left\|\frac{\partial^2 v}{\partial \tau^2}(\tau_n, x_i)\right\|_\infty = \frac{\Delta t}{2}M, \tag{2.57}$$

where $M = \left\|\dfrac{\partial^2 v}{\partial \tau^2}\right\|_\infty$. We now show that $|a_2|$ is bounded. In effect, using the mean-value theorem, there exists $\theta \in]\tau_n, \tau_{n+1}[$ such that:

$$\mathcal{L}_D v(\tau_{n+1}, x_i) - \mathcal{L}_D v(\tau_n, x_i) \approx \Delta t \partial_\tau \mathcal{L}_D v(\tau_n + \Delta t\theta, x_i).$$

Replacing this relation in the second equation in (2.56), we obtain:

$$|a_2| = \left|(L_D v)_i^{n+1} - \mathcal{L}_D v(\tau_{n+1}, x_i) + \Delta t \partial_\tau \mathcal{L} v(\tau_n + \Delta t\theta, x_i)\right|. \tag{2.58}$$

Next, taking Taylor expansion of v of order 4 gives:

$$\begin{aligned} v_{i+1}^{n+1} \;\approx\; & v_i^n + \Delta x \frac{\partial v}{\partial x}(\tau_{n+1}, x_i) + \frac{(\Delta x)^2}{2}\frac{\partial^2 v}{\partial x^2}(\tau_{n+1}, x_i); \\ & + \frac{(\Delta x)^3}{6}\frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, x_i) + \frac{(\Delta x)^4}{24}\frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, x_i) \end{aligned}$$

$$\begin{aligned} v_{i-1}^{n+1} &\approx v_i^{n+1} - \Delta x \frac{\partial v}{\partial x}(\tau_{n+1}, x_i) + \frac{(\Delta x)^2}{2} \frac{\partial^2 v}{\partial x^2}(\tau_{n+1}, x_i) \\ & - \frac{(\Delta x)^3}{6} \frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, x_i) + \frac{(\Delta x)^4}{24} \frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, x_i), \end{aligned}$$

hence

$$\frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{(\Delta x)^2} \approx \left((\Delta x)^2 \frac{\partial^2 v}{\partial x^2} + \frac{(\Delta x)^3}{12} \frac{\partial^4 v}{\partial x^4} \right).$$

Putting this last result in (2.58) gives:

$$\begin{aligned} |a_2| &\leq \frac{(\Delta x)^2}{6} |f(\tau_{n+1}, x_i)| \left| \frac{\partial^3 v}{\partial x^3} + \frac{(\Delta x)}{4} \frac{\partial^4 v}{\partial x^4} \right| + \frac{(\Delta x)^2}{6} \frac{\sigma^2 (T - \tau_{n+1})}{4} \left| \frac{\partial^4 v}{\partial x^4} \right| \\ & + \Delta t |\partial_\tau \mathcal{L} v(\tau_n + \Delta t \theta, x_i)|, \end{aligned} \tag{2.59}$$

since α, σ, β and f are bounded functions. Also, since the derivatives $\partial^{m+n} v \big/ \partial \tau^n \partial x^m$ are all supposed bounded, it implies:

$$|a_2| \leq (\Delta x)^2 M_1 + \Delta t M_2. \tag{2.60}$$

One also has:

$$\begin{aligned} |a_3| &= \left| \sum_{j=K_l}^{K_r} \nu_j (v_{i+j}^n - v_i^n) - \int_{B_l}^{B_r} (v(\tau, x_i + y) - v(\tau, x_i)) g_{\ln J}(y) dy \right| \\ &= \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (v_{i+j}^n - v_i^n) g_J(y) dy - \int_{B_l}^{B_r} (v(\tau, x_i + y) - v(\tau, x_i)) g_J(y) dy \right|. \end{aligned}$$

Since $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$, then we have:

$$|a_3| \leq \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (v(\tau_n, x_i + y_j) - v(\tau_n, x_i + y) g_{\ln J}(y) dy \right|,$$

and using Taylor's expansion of order one, we get:

$$|a_3| \leq \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} (y_j - y) \frac{\partial v}{\partial x}(\tau_n, x_i + \psi) g_{\ln J}(y) dy \right|, \;\; \psi \in]x_i + y, x_i + y_j[.$$

From the scheme, we have $\Delta x(j - 1/2) \leq y \leq \Delta x(j + 1/2)$, which leads to $-\dfrac{\Delta x}{2} \leq y_j - y \leq \dfrac{\Delta x}{2}$. Hence,

$$\begin{aligned} |a_3| &\leq \frac{\Delta x}{2} \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \frac{\partial v}{\partial x}(\tau_n, x_i + \psi) g_{\ln J}(y) dy \right| \\ &\leq \frac{\Delta x}{2} \left\| \frac{\partial v}{\partial x} \right\|_\infty \left| \sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} g_{\ln J}(y) dy \right| = \frac{\Delta x}{2} M_3, \end{aligned} \tag{2.61}$$

where $M_3 = \left\|\frac{\partial v}{\partial x}\right\|_\infty \left|\sum_{j=K_l}^{K_r} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} g_{\ln J}(y)dy\right|$. Finally, (2.57), (2.60) and (2.61) imply

$$|r_i^n(\Delta t, \Delta x)| \leq \Delta t\left(\frac{M}{2} + M_1\right) + \Delta x\left(\Delta x M_2 + \frac{M_3}{2}\right) \to 0$$

as $(\Delta t, \Delta x) \to (0, 0)$. $\square$

2.6.5 Stability and monotonicity

Two properties are important to show convergence to viscosity solutions: stability and monotonicity of scheme.

Definition 2.6. Stability

The scheme (2.54) is stable if, and only if, for some bounded initial conditions, its solution exists and is bounded independently of Δt and Δx, and uniformly bounded on $[0, T] \times \mathbb{R}$. That is to say,

$$\exists C > 0,\ \ \forall \Delta t > 0,\ \ \forall \Delta x > 0,\ \ i \in \mathbb{Z},\ \ n \in \{0, ..., M\},\ \ |u_i^n| \leq C.$$

We will say that a given vector v is positive if all its elements are positive. We write $u \geq v$ if $u - v \geq 0$. In this part we show the stability property of the scheme, which in turn implies the discrete comparison principle, a property which has an important interpretation in finance. This property makes possible the fact that the options values computed using our numerical scheme will check arbitrage inequalities: Inequality between payoffs leading to inequality between options values.

Proposition 2.7. (Stability and the discrete comparison principle)

If $\Delta t \leq 1 \big/ \sum_{j=K_l}^{K_r} \nu_j$, the scheme (2.54) is stable, and hence verifies the discrete comparison principle:

$$u^0 \geq v^0 \Longrightarrow \forall n \in \mathbb{N}^*,\ \ u^n \geq v^n.$$

Proof. Firstly, consider (2.54) in the form:

$$-cu_{i-1}^{n+1} + (1 + a\Delta t)u_i^{n+1} - b\Delta t u_{i+1}^{n+1} = \left(1 - \Delta t \sum_{j=K_l}^{K_r} \nu_j\right) u_i^n + \Delta t \sum_{j=K_l}^{K_r} u_{i+j}^n \nu_j, \tag{2.62}$$

where

$$\begin{cases} c &= \frac{1}{2}\frac{1}{(\Delta x)^2}\sigma^2(T - \tau_{n+1}) \geq 0 \\ a &= \frac{1}{\Delta x} f(\tau_{n+1}, x_i) + \frac{1}{(\Delta x)^2}\sigma^2(T - \tau_{n+1}) \geq 0 \\ b &= \frac{1}{\Delta x} f(\tau_{n+1}, x_i) + \frac{1}{2}\frac{1}{(\Delta x)^2}\sigma^2(T - \tau_{n+1}) \geq 0. \end{cases} \tag{2.63}$$

The positivity of a and b arises from g being positive. If g is not positive, we change the approximation of the first-order derivatives in the scheme used. In either case, one has:

$$a = b + c \quad \Rightarrow \quad a\Delta t = b\Delta t + c\Delta t \quad \Rightarrow \quad 1 + a\Delta t > (b + c)\Delta t.$$

It follows that $|1+a\Delta t| > |-c\Delta t|+|-b\Delta t|$, implying the matrix of linear system on $(u_0^{n+1}, ..., u_N^{n+1})$ has a strict dominant diagonal, hence invertible. Therefore, the solution of the linear system exists and is unique. We now show by mathematical induction that this solution is bounded. That is, if $\|H\|_\infty \leq \infty$ is the bounded initial condition, then, $\forall n \in \mathbb{N}$,

$$\|u^n\|_\infty \leq \|H\|_\infty. \tag{2.64}$$

By definition of u^0, we have $\|u^0\|_\infty \leq \|H\|_\infty$. Assume (2.64) holds for n. To show that it holds for $n+1$, we suppose on the contrary that $\|u^{n+1}\|_\infty > \|H\|_\infty$. By the definition of $\|.\|_\infty$, $\exists i_0 \in \{0, ..., n\}$ such that $|u_{i_0}^{n+1}| = \|u^{n+1}\|_\infty$, and $\forall i \in \mathbb{Z},\ |u_i^{n+1}| < |u_{i_0}^{n+1}|$.
Since $a = b + c$, we can write,

$$\|u^{n+1}\|_\infty = |u_{i_0}^{n+1}| = -c\Delta t|u_{i_0}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0}^{n+1}|. \tag{2.65}$$

Moreover, as $|u_{i_0-1}^{n+1}| < |u_{i_0}^{n+1}|$ and $|u_{i_0+1}^{n+1}| < |u_{i_0}^{n+1}|$ we have

$$\|u^{n+1}\|_\infty \leqslant -c\Delta t|u_{i_0-1}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0+1}^{n+1}|. \tag{2.66}$$

Using (2.62) and (2.66), and the fact that $\Delta t \leq 1/\sum_{j=K_l}^{K_r} \nu_j$, we obtain:

$$\begin{aligned}
\|u^{n+1}\|_\infty &\leq \left|\left(1 - \Delta t \sum_{j=K_l}^{K_r} \nu_j\right) u_{i_0}^n + \Delta t \sum_{j=K_l}^{K_r} u_{i_0+j}^n \nu_j\right| \\
&\leq \left(1 - \Delta t \sum_{j=K_l}^{K_r} \nu_j\right) |u_{i_0}^n| + \Delta t \sum_{j=K_l}^{K_r} |u_{i_0+j}^n \nu_j| \\
&\leq \left(1 - \Delta t \sum_{j=K_l}^{K_r} \nu_j\right) \|u^n\|_\infty + \Delta t \sum_{j=K_l}^{K_r} \nu_j \|u^n\|_\infty \\
&= \|u^n\|_\infty \leq \|H\|_\infty,
\end{aligned}$$

which contradicts our assumption. Hence $\|u^n\|_\infty \leq \|H\|_\infty$. $\square$

Proposition 2.8. **(Monotonicity)**
Let u^n and v^n be two solutions to (2.54) corresponding to some initial conditions f and h respectively, satisfying $f(x) \geq h(x) \ \ \forall x \in \mathbb{R}$. If $\Delta t \leq 1 / \sum_{j=K_l}^{K_r} \nu_j$, then $u^n \geq v^n$, $\forall n \in \mathbb{N}$.

Proof. Define $w^n = u^n - v^n$. We show that $w^n \geq 0 \ \ \forall n \in \mathbb{N}$. As in Proposition 2.7, we proceed by induction. By construction, we have $w_i^0 = f(x_i) - b(x_i) \geq 0$, $\forall i \in \mathbb{Z}$. Let $w^n \geq 0$, and suppose that: $\inf_{i \in \mathbb{Z}} w_i^{n+1} < 0$. Since $\forall i \in \mathbb{Z} \backslash \{0, ..., N\}$, $w_i^{n+1} = 0$, this implies that $\exists i_0 \in \{0, ..., N\}$ such that $w_{i_0}^{n+1} = \inf_{i \in \mathbb{Z}} w_i^{n+1}$. Using (2.62) and $\Delta t \leq \dfrac{1}{\sum_{j=K_l}^{K_r} \nu_j}$, we have that

$$
\begin{aligned}
\inf_{i \in \mathbb{Z}} w_i^{n+1} = w_{i_0}^{n+1} &= -c\Delta t w_{i_0}^{n+1} + (1 + a\Delta t) w_{i_0}^{n+1} - b\Delta t w_{i_0}^{n+1} \\
&\geq -c\Delta t w_{i_0-1}^{n+1} + (1 + a\Delta t) w_{i_0}^{n+1} - b\Delta t w_{i_0+1}^{n+1} \\
&= \left(1 - \Delta t \sum_{j=K_l}^{K_r} \nu_j\right) w_{i_0}^n + \Delta t \sum_{j=K_l}^{K_r} w_{i_0+j}^n \nu_j \geq 0,
\end{aligned}
$$

which is a contradiction. Therefore, $\inf_{i \in \mathbb{Z}} w_i^{n+1} \geq 0$, and hence $w^{n+1} \geq 0$. □

2.6.6 Convergence

As proved above, our scheme (2.54) is locally consistent, stable, monotone and verifies the discrete comparison principle. In the usual approach to the convergence of finite difference schemes for PDE's, consistency and stability ensure convergence under regularity assumptions on the solution. These conditions are not sufficient here because the solution may not be smooth, and higher order derivatives may not exist. This is where the notion of viscosity solutions are introduced. In the second order parabolic/elliptic PDEs Barles and Souganidis (1991) showed that for elliptic (or parabolic) PDEs, any locally consistent, stable and monotone finite difference scheme converge uniformly, on each compact subset $[0, T] \times \mathbb{R}$, to the unique continuous viscosity solution. Cont and Ekaterina (2005b) showed that the solution of a numerical scheme converges uniformly on each compact subset of $[0, T] \times \mathbb{R}$ to the unique viscosity solution even when the subsolution and the supersolution constructed using a numerical scheme may not have uniform continuity properties. The PIDE studied relies on the same assumptions as in Cont and Ekaterina (2005b), except that here, we are in the case of a finite activity measure since the sizes of the jumps is log-normal. Then we used the same technics to showed the convergence of the explicit-implicit scheme (2.54) to the viscosity solution of problem (2.29).

Proposition 2.9. **(Convergence of the explicit-implicit scheme)**
Let H be a bounded piecewise continuous initial condition, then solution $u^{(\Delta t,\Delta x)}$ of the numerical scheme converges uniformly on each compact subset of $[0,T]\times\mathbb{R}$ to the viscosity solution u of continuous problem (2.29).

Proof. Define

$$\begin{cases} \underline{u}(\tau,x) = \liminf\limits_{(\Delta t,\Delta x)\to(0,0)(t,y)\to(\tau,x)} u^{(\Delta t,\Delta x)}(t,y) \quad and \\ \overline{u}(\tau,x) = \limsup\limits_{(\Delta t,\Delta x)\to(0,0)(t,y)\to(\tau,x)} u^{(\Delta t,\Delta x)}(t,y). \end{cases} \tag{2.67}$$

The aim of this proof consists to show the following equalities $\underline{u}(\tau,x)=\overline{u}(\tau,x)=u(\tau,x)$. Before showing this equalities some preparatory results are needed.
We start by rewriting (2.54) in the following form:

$$\begin{aligned} u(\tau_n,x_i) &= F[u(\tau_n-\Delta t,.)](x_i),\, n=1,...,M,\, i\in 0,...,N, \\ u(0,x_i) &= H(S_0e^{x_i}),\, i\in 0,...,N, \\ u(\tau_n,x_i)) &= 0,\, n=0,...,M,\, \notin 0,...,N. \end{aligned} \tag{2.68}$$

One can define super and subsolution of (2.68) by the following definition

Definition 2.7. A function w is a supersolution of (2.68) if

$$\begin{aligned} w(\tau_n,x_i) &\geq F[w(\tau_n-\Delta t,.)](x_i),\, n=1,...,M,\, i\in 0,...,N, \\ w(0,x_i) &\geq H(S_0e^{x_i}),\, i\in 0,...,N, \\ w(\tau_n,x_i)) &\geq 0,\, n=0,...,M,\, \notin 0,...,N. \end{aligned} \tag{2.69}$$

A function z is a subsolution of (2.68) if

$$\begin{aligned} z(\tau_n,x_i) &\leq F[z(\tau_n-\Delta t,.)](x_i),\, n=1,...,M,\, i\in 0,...,N, \\ z(0,x_i) &\leq H(S_0e^{x_i}),\, i\in 0,...,N, \\ z(\tau_n,x_i)) &\leq 0,\, n=0,...,M,\, i\notin 0,...,N. \end{aligned} \tag{2.70}$$

To Avoid the problem of uniform continuity and smoothness which may not hold for $\underline{u}$ and $\overline{u}$ define in 2.67, it is convenient to consider smooth super and subsolutions of (2.29) and super and subsolutions of 2.68 and derived the link with $\underline{u}$ and $\overline{u}$. The following results extends the comparison principle to the super and subsolutions.

Lemma 2.2. For any supersolution w and subsolution z of (2.68) we have $z \leq u \leq w$.

Proof. For ($i \notin 0, ..., N$) or ($n = 0$ and $i \in 0, ..., N$) the above inequalities are satisfied by definition. For $n = 1, ..., M,\ \ i \in 0, ..., N$ from monotonicity of the scheme we have

$$\begin{aligned} z(\tau_n, x_i) &\leq F[z(\tau_n - \Delta t, .)](x_i) \leq F[u(\tau_n - \Delta t, .)](x_i) = u(\tau_n, x_i) \\ &= F[u(\tau_n - \Delta t, .)](x_i) \leq F[w(\tau_n - \Delta t, .)](x_i) \leq w(\tau_n, x_i). \end{aligned}$$

□

Lemma 2.3. Let w and z be a smooth supersolution and subsolution of (2.29) respectively. Then for all ϵ, there exists $\Delta > 0$ such that
$\forall \Delta t, \Delta x, \leq \Delta,\ \ \forall n \geq 0,\ \ \forall i \in \mathbb{Z},\ z(\tau_n, x_i) - \epsilon < u(\tau_n, x_i) < w(\tau_n, x_i) + \epsilon$

Proof. Choose q such that $0 < q(T + 1) < \epsilon$ and let $\tilde{w}(\tau, x) = w(\tau, x) + q(1 + \tau)$, notice that a constant function is always a solution. In fact one can see from the definition that the scheme is linear. If $i \notin 0, ..., N$, we have

$$\tilde{w}(\tau_n, x_i) = w(\tau_n, x_i) + q(\tau + 1) \geq q \geq 0. \tag{2.71}$$

If $n = 0$ and $i \in 0, ..., N$,

$$\tilde{w}(0, x_i) = w(0, x_i) + q \geq H(S_0 e^{x_i}). \tag{2.72}$$

If $n \geq 1,\ \ i \in 0, ..., N$ from the consistency of the scheme we obtain

$$\begin{aligned} \frac{\tilde{w}(\tau_n, x_i) - \tilde{w}(\tau_n - \Delta t, x_i)}{(\Delta t)} - L_D \tilde{w}(\tau_n - \Delta t, x_i) & \\ - L_J \tilde{w}(\tau_n - \Delta t, x_i) &= \frac{w(\tau_n, x_i) - w(\tau_n - \Delta t, x_i)}{(\Delta t)} \\ - L_D w(\tau_n - \Delta t, x_i) - L_J w(\tau_n - \Delta t, x_i) + q > 0 & \qquad (2.73) \\ &\longrightarrow \frac{\partial w}{\partial \tau}(\tau, x) \ - (\mathcal{L}_D + \mathcal{L}_J) w(\tau, x) + q \end{aligned}$$

as $\Delta t,\ \Delta x \longrightarrow (0, 0),\ \ (\tau_n, x_i) \longrightarrow (\tau, x)$, uniformly on $(0, T[\times O$. Therefore for any sufficiently small $\Delta > 0$, for all $\Delta t,\ \Delta x \leq \Delta$, we have

$$\tilde{w}(\tau_n, x_i) \geq F[\tilde{w}(\tau_n - \Delta t, .)](x_i),\ \ \forall n \leq 1, \forall i \in 0, ..., N). \tag{2.74}$$

Combining (2.71), (2.73) and (2.74), show that function $\tilde{w}$ is supersolution of (2.68). Indeed, Lemma 2.2 implies that

$$u(\tau_n, x_i) \leq \tilde{w}(\tau_n, x_i) + q(1+T) < w(\tau_n, x_i) + \epsilon, \ \ \forall n \geq 0, \ \forall i \in \mathbb{Z},$$

which is the desired property. The lower bound $z(\tau_n, x_i) - \epsilon$ can be proved in the same manner and then completes the proof. □

Following Lemma (2.71) and Lemma (2.73), we have the following main Lemma

Lemma 2.4. Let $\underline{u}$ and $\overline{u}$ be the function define by (2.67). For any smooth supersolution $w(\tau, x)$ and any subsolution $z(\tau, x)$ of the problem (2.29), we have for $(\tau, x) \in [0, T] \times O$,

$$z(\tau, x) \leq \underline{u}(\tau, x) \leq \overline{u}(\tau, x) \leq w(\tau, x). \tag{2.75}$$

Proof. By the definition of upper and lower limits, Lemma 2.3 implies desired property. □

After giving some properties needed we can start the proof of convergence (i.e. showed that $\underline{u} = \overline{u} = u$). If $\overline{H}$, $\underline{H}$ are smoothness functions on $\mathbb{R}$ such that $\underline{H} \leq H \leq \overline{H}$, then $w(\tau, x) = \mathbb{E}[\overline{H}(Y_\tau^x)]$ and $z(\tau, x) = \mathbb{E}[\underline{H}(Y_\tau^x)]$ are respectively a supersolution and a subsolution of the Cauchy problem (2.29). From Lemma 2.4 we obtain (2.75). Notice that If $w(\tau, x) - u(\tau, x), \ u(\tau, x) - z(\tau, x)$ could be made small this would imply that $\lim_{(\Delta t, \Delta x) \to (0,0)(\tau_n, x_i) \to (\tau, x)} u^{(\Delta t, \Delta x)}(\tau_n, x_i) = u(\tau, x)$. Indeed, it remains to construct appropriate smooth approximations $\overline{H}$ and $\underline{H}$.
Let $\zeta_1, ..., \zeta_I$ be the discontinuity points of H. We suppose that the jumps of H are bounded by δ. Given $\epsilon > 0$ and $\overline{H}$, $\underline{H}$ smooth functions that satisfied the following relations

$$\begin{aligned}
\underline{H}(x) \leq H \leq \overline{H}(x) & \qquad \forall x \in \mathbb{R}, \\
\mid \underline{H}(x) - \overline{H}(x) \mid \ \leq \ \delta & \quad \forall x \in \bigcup_{j=1}^{I} (\zeta_j - \epsilon, \zeta_j + \epsilon), \\
\mid \underline{H}(x) - \overline{H}(x) \mid \ \leq \ \epsilon & \quad \forall x \notin \bigcup_{j=1}^{I} (\zeta_j - \epsilon, \zeta_j + \epsilon).
\end{aligned}$$

We have

$$\begin{aligned} w(\tau,x)-z(\tau,x) &= \mathbb{E}[\overline{H}(Y_\tau^x)-\underline{H}(Y_\tau^x)] \\ &\leq \delta\mathbb{Q}(Y_\tau^x \in \bigcup_{j=1}^{I}(\zeta_j-\epsilon,\zeta_j+\epsilon)) + \epsilon\mathbb{Q}(Y_\tau^x \notin \bigcup_{j=1}^{I}(\zeta_j-\epsilon,\zeta_j+\epsilon)) \quad (2.76) \\ &\leq \delta\mathbb{Q}(Y_\tau^x \in \bigcup_{j=1}^{I}(\zeta_j-\epsilon,\zeta_j+\epsilon)) + \epsilon. \quad (2.77) \end{aligned}$$

Noting that $\bigcap_{\epsilon>0}\{Y_\tau^x \in \bigcup_{j=1}^{I}(\zeta_j-\epsilon,\zeta_j+\epsilon)\} = \{Y_\tau^x \in \{\zeta_1,....,\zeta_I\}\}$. Since Y_τ^x has an absolutely continuous distribution, so we have $\mathbb{Q}(\{Y_\tau^x \in \{\zeta_1,....,\zeta_I\}\}) = 0$. Consequently $\mathbb{Q}(Y_\tau^x \in \bigcup_{j=1}^{I}(\zeta_j - \epsilon,\zeta_j+\epsilon)) \longrightarrow 0$ as $\epsilon \longrightarrow 0$.
Therefore $w(\tau,x)-z(\tau,x) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ and the inequalities $z(\tau,x) \leq u(\tau,x) \leq w(\tau,x)$ together with Lemma 2.4 implies desired result which completes the proof. □

Remark 2.1. For $\tau = 0$ the scheme does not converge to the initial condition at the discontinuous points of H. This is due to the fact that $\mathbb{Q}(Y_\tau^x \in \bigcup_{j=1}^{I}(\zeta_j-\epsilon,\zeta_j+\epsilon)) \longrightarrow \mathbb{Q}(S_0e^x \in \{\zeta_1,....,\zeta_I\}) = \mathbb{1}_{\{x\in\{\ln(\frac{\zeta_1}{S_0}),...,\ln(\frac{\zeta_I}{S_0})\}\}}$. However, this has no practical interest since it is not important to compute the solution numerically at $\tau = 0$.

2.7 Numerical Result

In this section we discuss to the details of the implementation of our scheme and present numerical results and some interpretation.

Before started simulation parameters scheme are take as follow the parameters used to implement

Table 2.2: Scheme parameters

T	M	N	A_l	A_r
1	100	175	-0.096	0.079

the following results are taking independent to time as follow.

Table 2.3: Model parameters

figure	model	r	Strike	Product
2.8, 2.9, 2.10, 2.11	$\alpha = 0.015$ $\beta = 0.4$ $\sigma_J = 0.5$ $\ell = 1.5$ $\sigma = 0.5$ $S_0 = 50$	0.04	$K = 45$	Call

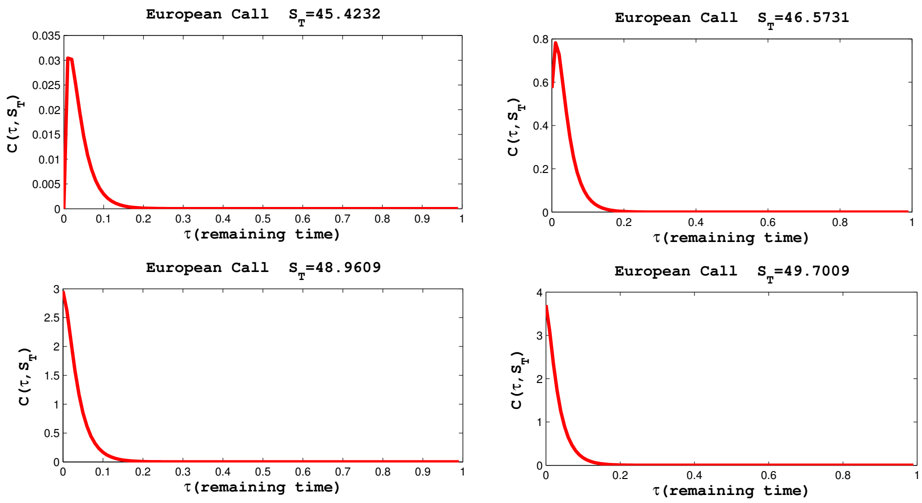

Figure 2.8: Call price for four different values of spot price at maturity versus remaining time to maturity

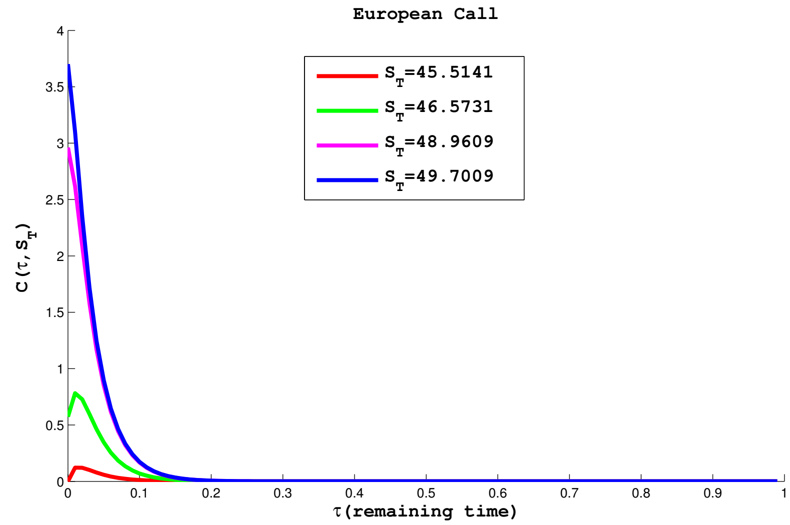

Figure 2.9: Comparison of call values for four different values of spot price at maturity versus remaining time to maturity

Figure 2.8 plots call option for different spot price of the underlying at maturity versus remaining time. Note that, as remaining time increase the mean-reverting and price-cap effect in all four case cause the call prices converge to zero under the set parameters gived above. Figure 2.9 give comparison between illustrative curve of call for four different electricity spot price at maturity. This two figures illustrate that for different initials conditions call option prices converge verse to the same value. From this we can thus, say that different properties shown theoretically are effective under the established conditions.

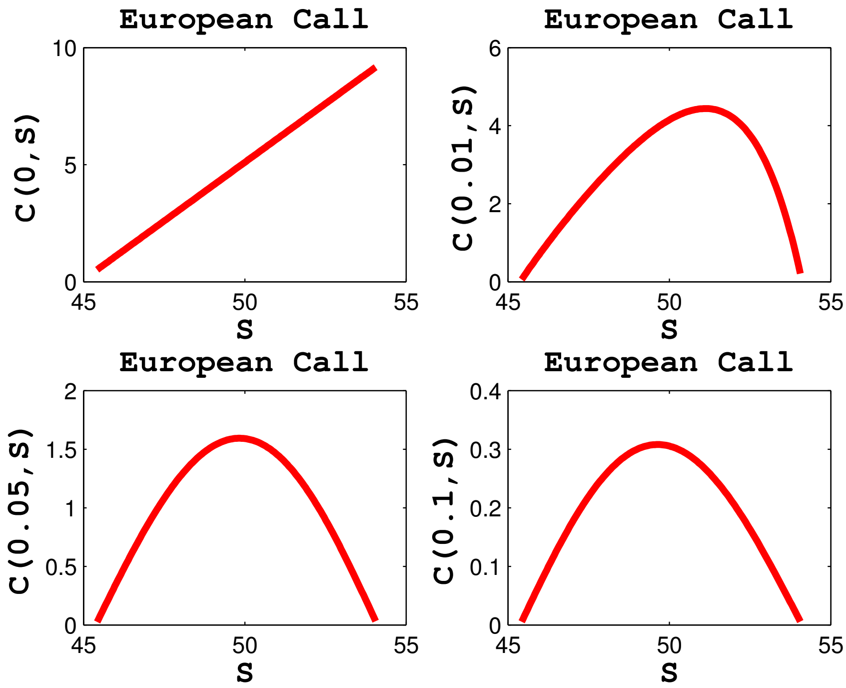

Figure 2.10: Call price for four different remaining time to maturity versus spot price

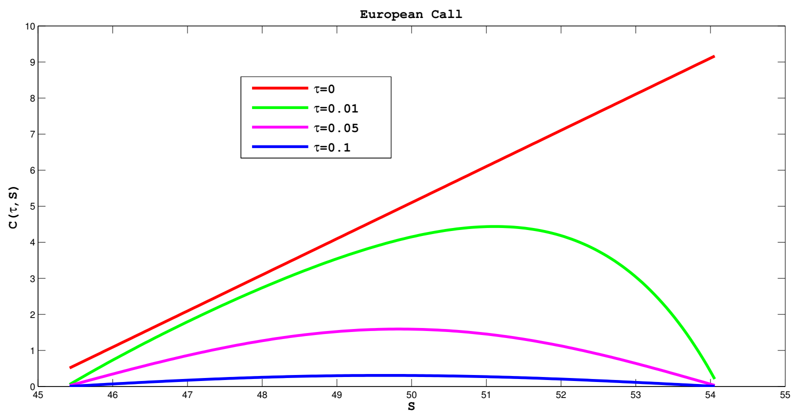

Figure 2.11: Comparison of call price for four different remaining time to maturity versus spot price

Figure 2.10 and 2.11 illustrate a reality enough close to those in the classical financial markets. In the sens that in financial market the values of call option before the maturity evolved in the form of a curve which towards to the payoff line progressively and as we approach maturity.

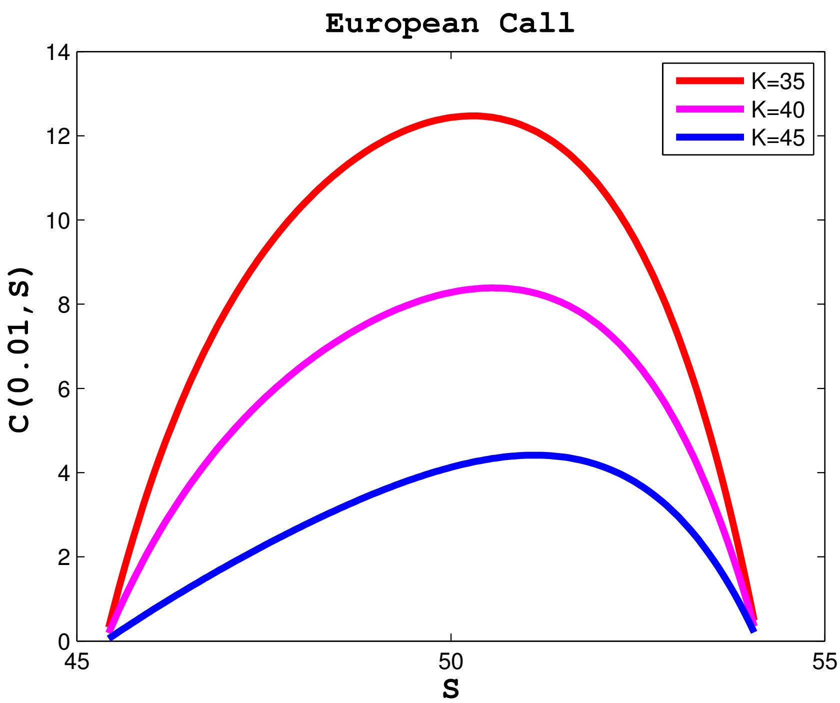

Figure 2.12: Call price for three different values of strike versus spot price S, the other parameters is unchanged as in table 2.3

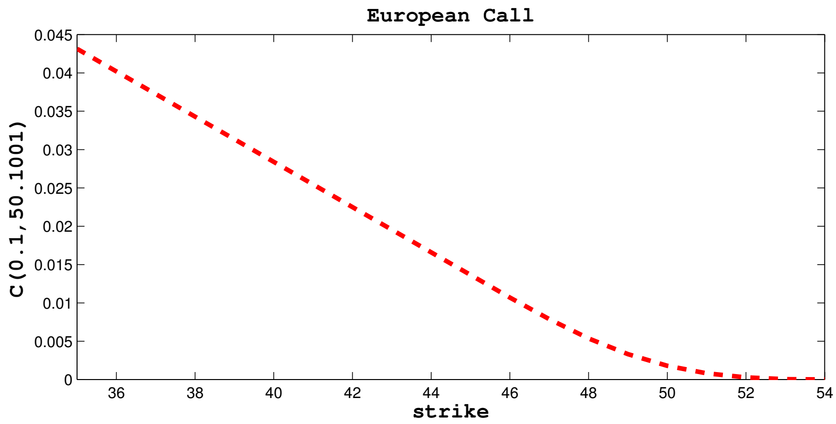

Figure 2.13: Call price versus strike price K, the other parameters is unchanged as in table 2.3

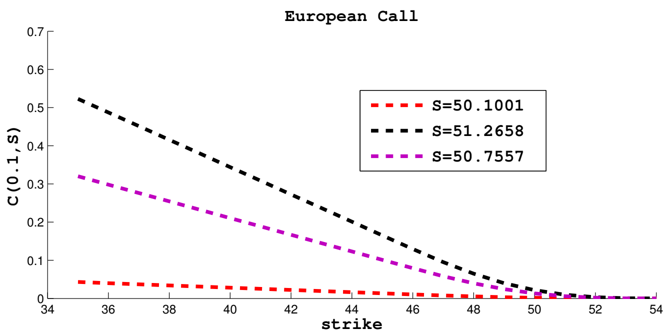

Figure 2.14: Call price for three different values of spot price S versus strike price K, the other parameters is unchanged as in table 2.3

From figures 2.12, 2.13 and 2.14 one can observed that the values of call decrease when strike price increase. This behavior of the call values is from a risk management point of view what it is wished.

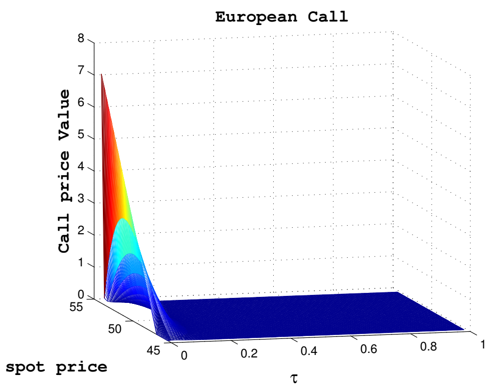

Figure 2.15: Call price versus remaining time and spot price $\sigma_J = 0.5$, $K = 45$, the other parameters is unchanged as in table 2.3

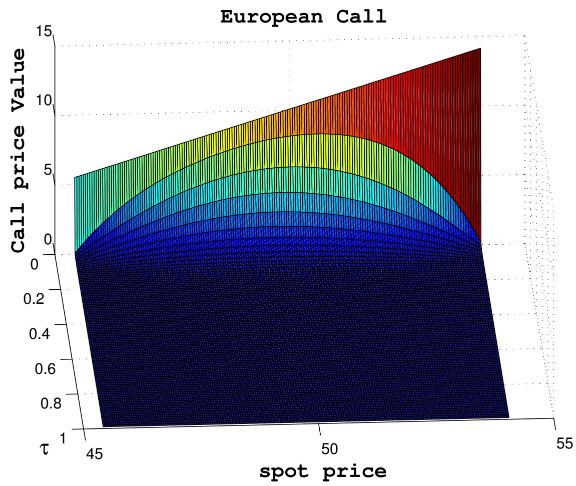

Figure 2.16: Call price versus remaining time and spot price $\sigma_J = 2.5$ and $K = 40$, the other parameters is unchanged as in table 2.3

Analysis plots of figures 2.15 and 2.16 which illustrate call option price as a function of remaining time and spot price one can said that for a large remaining time jumps effects are not perceptible and then can effect call. Whereas a small remaining time to maturity call prices increase suddenly which express the effect of jump. We must therefore say that the jump term which allows to take into account certain reality of the electricity market is not inconsiderable since that it impact on the call prices are quite noticeable.

2.8 Discussion and Conclusion

In this chapter, we first use the jump-diffusion model derived in the preceding chapter to compute a new exact formula for the forward contract price under an equivalent martingale measure, and we compare it with the one in Cartea and Figueroa (2007). Secondly, we have calculated the European call option in the regulated electricity market. Here the approaches used are the same as in Lewis (2001) and Carr and Madan (1999). We made a double approximation to evaluate the call option in the two approaches which may reduce precision. The characterization of options prices in terms of the classical solution, or, in general, in the terms of the viscosity solution of a PIDE allows the use of numerical methods to obtain efficient approximations of option values. This has been a center of research in recent times in the case of exponential Lévy models with finite arrival or infinite arrival rate of jumps. Some authors like Alvarez and Tourin (1996) use the finite difference method to approximate the PIDE solution, while others like Cont and Ekaterina (2005a) approximate viscosity solutions in the case of nonsmoothness of option prices. In both cases, success (in terms of efficient approximation) has been obtained. In this paper we used their approach to evaluate European call option when the underlying is electricity. The motivation behind our approach arose from the fact that the electricity price model presented here, by hypothesis, possesses most of the properties (as in their case) of an exponential Lévy model, and the Markov process property. We focus on the pricing of call option because put option can be deduced from the put-call using the parity formula. In a mathematical point of view, our numerical results confirm the established theoretical results. In the finance point of view, the numerical results present an interpretation which was coherent with some realities in the electricity market, when it is regulated under price cap.

CHAPTER THREE

PARAMETER ESTIMATION IN A JUMP DIFFUSION PROCESS OF ELECTRICITY PRICES REGULATED BY PRICE-CAP PRINCIPLE

Contents

In this chapter, a parameter estimation approach is proposed for the new jump diffusion process derived in chapter 1. The estimation method is based on maximum likelihood principle after approximating the transition density with the saddle point method. Standard errors of the estimated parameters are also computed.

3.1 Introduction

Parameter estimation in stochastic differential models remains a challenge in model calibration. In effect, parameters estimation in stochastic differential equations is based on the knowledge of the conditional probability density of the process. Once this conditional density is known, the discretized likelihood of the data is written as the product of individual transition densities and the parameters are obtained by maximizing this likelihood. However, the transition density does not exist in closed-form, except for a handful of cases. Hence, approximations are typically necessary. In the literature, there are four types of approximations that are generally used: Closed-form Hermite expansions of the transition density (Aït-Sahalia, 1999, 2008), Importance sampling (Pedersen, 1995; Santa, 1997; Brandt and Santa-Clara, 2002), Methods based on the exact simulation of diffusion (Beskos et al., 2006), and Approximations derived by numerically solving the Kolmogorov forward equation (Lo, 1988).

In our model, the presence of jump complicates parameters estimation. The main difficulty lies on the fact that the process is written as the sum of two different distributions for the Brownian motion and for the jump term, hence making it difficult to have an expression of the probability density. In the literature, there are two ways to approach this problem. In one, the transition function is deduced by calculating the conditional expectation of the discretized process (Jensen, 1995; Jensen and Poulsen, 2002). In the other, the one considered in this thesis, is based on the saddlepoint method introduced by Daniels (1954) and developed in Rogers and Zane (1999) and Jensen (1995).

The chapter is therefore organized as follows: We recall the model in Section 2, then some approximate transition functions in Section 3. The last section illustrates an example.

3.2 The Model

Suppose that S_t is a *càdlàg* process in a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ where $(\mathcal{F}_t)_t$ is a natural filtration of S_t. The Stochastic Differential Equation (SDE)

$$dS_t = -\alpha(t)(\gamma(t) - S_t)dt + \sigma(t)S_t dW_t + (J-1)S_t dq_t, \tag{3.1}$$

where W_t is the standard Brownian motion, and the coefficients involved are deterministic functions of time denoted as such: $\sigma(t)$ is the volatility, $\beta(t) := E(t) + H(t) - F(t)$ defines the exogenous factors, $\alpha(t) := I(t) - G(t)$ the endogenous factors;

with the following assumptions:

Assumption 1: The proportional random jump size J is log-Normally distributed, with $\mathbb{E}[J] = 1$. Hence, $\ln J \sim \mathcal{N}\left(-\frac{\sigma_J^2}{2}, \sigma_J^2\right)$.

Assumption 2: The random jump size J, $\mathrm{d}q_t$ and $\mathrm{d}W_t$ are independent.

denote the dynamic of electricity spot price in regulated context.
We will further assume that the unknown electricity parameters $\alpha(.)$, $\beta(.)$, $\sigma(.)$, ℓ and σ_J are constant.
The exact solution of (3.1) is given, using Itô formula for jump diffusion process, by

$$S_t = S_0 e^{X_t} - \int_0^t \beta(s) e^{X_t - X_s} ds, \tag{3.2}$$

where $X_t = \int_0^t \alpha(s) - \dfrac{1}{2}\sigma(s)^2 ds + \int_0^t \sigma(s) dW_s + \int_0^t \ln J dq_s$ is a Lévy process.

3.3 Parameter Estimation

This part is devoted to the estimation of electricity price parameters denoting $\Theta = (\alpha, \beta, \sigma, \sigma_J, \ell)$ using a sample discrete set of observations say $\{X_{t_1}^{obs}, X_{t_2}^{obs}, ..., X_{t_n}^{obs}\}$, of $\log$-price process. where, $X_{t_i}^{obs}$ represents the $\log$ of electricity price observe at t_i where $t_i = i\Delta$ and $\Delta > 0$ fixed. Let $p(\Delta, y \mid x; \Theta)$ the conditional density of $X_{i\Delta} = y$ given $X_{(i-1)\Delta} = x$, also called the transition function. Recalling that under Markovian hypothesis of (3.1) the $\log$-likelihood has the form

$$l_n(\Theta) = \sum_{i=1}^{n} \ln p(\Delta, y_i \mid y_{i-1}; \Theta) \tag{3.3}$$

and from this likelihood,

- Maximum Likelihood estimation (MLE) parameters $\hat{\Theta}$ are obtained by $\hat{\Theta} = arg \max l_n(\Theta)$.
- $V(\hat{\Theta})$ is obtained asymptotically using the Central Theorem limit CTL that is $\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{D} \mathcal{N}(0, I^{-1}(\Theta))$.

The variability of the estimation will be analyzed by using standard error (SE) which is it evaluated by the following formula

- The SE is then given by $\sqrt{diag(V(\hat{\Theta}))}$.

In the following we give an approximating transition density $p(\Delta, y \mid x; \Theta)$.

3.3.1 Approximating the transition density function using saddlepoint method

The objective of this part consist to approximate the transition density function $p(\Delta, y \mid x)$ of the process $\ln(S)$ using saddlepoint method with non Gaussian leading term. In what follows, we will assume that for each couple $(\Delta, x) \in \mathbb{R}_+ \times \mathbb{R}$ the probability measure $\mathbb{Q}(\Delta, ., \mid x)$ admits a probability density $p(\Delta, ., \mid x)$ with respect to Lebesgue measure and the conditional Laplace transform of the precess X is defined for $u \in [a, b] \subset \mathbb{R}$ with $a \leqslant 0$, $b \geqslant 0$. Therefore $\varphi(\Delta, u \mid x) = \mathbb{E}\left[e^{uX_{i\Delta}} \mid X_{(i-1)\Delta} = x\right]$ and the cumulant transform (cumulant generating function) of X is the function $K(\Delta, u \mid x) = \ln \varphi(\Delta, u \mid x)$.

Note that for given (Δ, x), derivatives of all order of φ and K exist. The infinitesimal generator of process (X_t) is given by:

$$L = L_D + L_J, \tag{3.4}$$

where

$$L_D f(x) = (\alpha - \frac{1}{2}\sigma^2 - \beta e^{-x})\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(x), \tag{3.5}$$

$$L_J f(x) = \ell \int_{\mathbb{R}} f(x+z) - f(x)\nu(dz) \tag{3.6}$$

To simplify the calculations we define the following functions: $\mu(x) = (\alpha - \dfrac{1}{2}\sigma^2 - \beta e^{-x})$. As we are in the presence of jumps here we will take the leading term as in Aït-Sahalia and Yu (2006) in the form: $f_0(y) = (1 - \ell\Delta)\phi(y; x + \mu(x)\Delta, \sigma^2\Delta) + \ell\Delta\nu(y - x)$ and the associated cumulant given by:

$$e^{K_0(\Delta, u|x)} = e^{ux}[(1 - \ell\Delta)e^{(\mu(x)u + \frac{1}{2}\sigma^2 u^2)\Delta} + \ell\Delta\theta(u)] \tag{3.7}$$

where $\theta(u) = \int_{\mathbb{R}} e^{uz}\nu(dz) = e^{\frac{1}{2}\sigma_J^2(u^2 - u)}$. Considering all these assumptions approximated transition density is given in the following Proposition:

Proposition 3.1. The saddlepoint approximation at the first order with a non-Gaussian base K_0 and it corresponding leanding f_0 is

$$\begin{aligned} p(\Delta, y \mid x) \;&=\; f_0(y)\left\{\Delta((1 - \Delta\ell)(\sigma^2 + \Delta g'(\hat{w})^2)e^{g(\hat{w})\Delta} + \ell\theta''(\hat{w}))((1 - \Delta\ell)e^{g(\hat{w})\Delta} + \ell\theta(\hat{w}))\right. \\ &\quad \left. - \Delta^2((1 - \Delta\ell)g'(\hat{w})^2 e^{g(\hat{w})\Delta} + \ell\theta(\hat{w}))^2\right\}^{1/2} \\ &\quad \times \frac{e^{\Delta[g(\hat{u}) + \ell(\theta(\hat{u}) - 1)] + (\hat{w} - \hat{u})(y - x)}}{\sqrt{\Delta(\sigma^2 + \ell\theta''(\hat{u}))}((1 - \Delta\ell)e^{g(\hat{w})\Delta} + \ell\Delta\theta(\hat{w}))^2}, \end{aligned} \tag{3.8}$$

where $\hat{u}$ and $\hat{w}$ are respectively solution of equation

$$y = \frac{\partial K(\Delta, \hat{u} \mid x)}{\partial u}, \tag{3.9}$$

and

$$y = \frac{\partial K_0(\Delta, \hat{w} \mid x)}{\partial w}, \tag{3.10}$$

and $g(u) = \mu(x)u + \frac{1}{2}\sigma^2 u^2$.

Proof. Since from Aït-Sahalia and Yu (2006) in continuous Markov process context as the one in this work we have

$$\begin{aligned} p(\Delta, y \mid x) &= f_0(y) e^{(K^{(1)}(\Delta, \hat{u}|x) - \hat{u}y - \{K_0(\hat{w}) - \hat{w}y\})} \\ &\times (K_0''(\hat{w}))^{1/2} \left(\frac{\partial^2 K^{(1)}(\Delta, \hat{u} \mid x)}{\partial u^2} \right)^{-1/2}. \end{aligned} \tag{3.11}$$

Deriving K_0 in first and second order we obtain

$$\frac{\partial K_0(\Delta, u \mid x)}{\partial u} = x + \frac{\left[(1 - \ell\Delta) g'(u) e^{\Delta g(u)} + \ell\theta'(u)\right] \Delta}{(1 - \ell\Delta) e^{\Delta g(u)} + \ell\Delta\theta(u)}. \tag{3.12}$$

and

$$\begin{aligned} \frac{\partial^2 K_0(\Delta, u \mid x)}{\partial u^2} &= \{\left[(1 - \ell\Delta)(\sigma^2 + g'(u)^2) e^{\Delta g(u)} + \ell\theta''(u)\right] \Delta \\ &\times \left[(1 - \ell\Delta) e^{\Delta g(u)} + \ell\Delta\theta(u)\right] - \left[(1 - \ell\Delta) g'(u) e^{\Delta g(u)} + \ell\theta'(u)\right]^2 \Delta^2\} \\ &\times \frac{1}{(1 - \ell\Delta) e^{\Delta g(u)} + \ell\Delta\theta(u)}. \end{aligned} \tag{3.13}$$

Expansion of cumulant fonction K is obtained as a Taylor series in Δ around their continuous-time limit of the expansion of Laplace transform φ. This expansion of Laplace transform in small time Δ are obtained by iterations of the infinitesimal generator of process (see Aït-Sahalia and Yu (2006)). Define as:

$$\mathbb{E}\left[e^{uX_{i\Delta}} \mid X_{(i-1)\Delta} = x\right] = \sum_{n=0}^{N} \frac{\Delta^n}{n!} L^n . e^{ux} + O(\Delta^{n+1}) \tag{3.14}$$

Applying iteration formula 3.14 the expansion of Laplace transform in Δ in order 1 is given by

$$\varphi^{(1)}(\Delta, u \mid x) = e^{ux} \left[1 + \mu(x)u + \frac{1}{2}\sigma^2 u^2 + \ell(\theta(u) - 1)\right] \Delta, \tag{3.15}$$

and associated cumulant is given by

$$K^{(1)}(\Delta, u \mid x) = ux + \left[\mu(x)u + \frac{1}{2}\sigma^2 u^2 + \ell(\theta(u) - 1)\right]\Delta. \tag{3.16}$$

Deriving K^1 in first and second order one as

$$\frac{\partial K^{(1)}(\Delta, u \mid x)}{\partial u} = x + \left[\mu(x) + \sigma^2 u + \ell\theta(u)'\right]\Delta.$$

and

$$\frac{\partial^2 K^{(1)}(\Delta, u \mid x)}{\partial u^2} = \left[\sigma^2 + \ell\theta(u)''\right]\Delta. \tag{3.17}$$

Putting together (3.7), (3.13), (3.16) and (3.17) into (3.11), leads to the result (3.8). $\square$

3.3.2 An Illustrative Example of Estimation

In this part, we estimated the vector of parameters $\hat{\Theta}$ using simulated process with the following parameters $\alpha = 0.05, \;\; \beta = 0.5, \; \sigma = 1$, we take 4 different values of σ_J which is standard deviation of jump size J suppose to be log-normal distributed in this work. This variation permit us to appreciate the effect of jump to the estimation. Different results are given in the following Tables.

Table 3.1: Estimated parameters

	σ_J	1/8		1/4		1/2		1	
	Parameter	**Est**	**SE**	**Est**	**SE**	**Est**	**SE**	**Est**	**SE**
	α	0.06	0.06	0.07	0.10	0.07	0.11	0.08	0.01
N=100	β	0.70	0.29	0.70	0.26	0.70	0.27	0.69	0.03
	σ	1.01	0.07	0.99	0.01	0.99	0.01	1.06	0.01
	α	0.05	0.29	0.05	1.56	0.05	0.95	0.07	0.02
N=250	β	0.45	0.33	0.45	0.80	0.43	0.54	0.11	0.04
	σ	1.00	0.03	1.05	1.68	0.85	1.31	1.05	0.02
	α	0.05	0.35	0.05	0.85	0.05	5.08	0.06	0.01
N=500	β	0.45	0.69	0.45	0.39	0.45	0.44	0.06	0.04
	σ	1.01	0.03	1.05	0.9	1.05	4.95	1.25	0.16

From the results of the estimated parameters in table 3.1 one observe that the saddle point method could be used to estimate the parameters α, β and σ of the equation (3.1). Overall, this approach gives better estimation of the volatility and mean reversion parameter than α which it is just a speed of the return to the mean. We observe that the precision (SE) of the estimates increase when the number of jump increase this can be justified by the fact that the presence of jumps in data makes estimation more difficult as we recently showed in Takam Soh et al. (2020). We note that the refinement of the time step improves the estimates. But the number of operations required for the estimates is quite heavy and therefore to refine further you need a more efficient machine. Summary, large jumps increase the standard deviation (SE).

3.3.3 Some Illustrative Curves

To illustrate the parameters estimation result we construct the following curves which represents one trajectory of equation (3.1) with the different parameters obtained in table 3.1 and compared them with the initials given parameters.

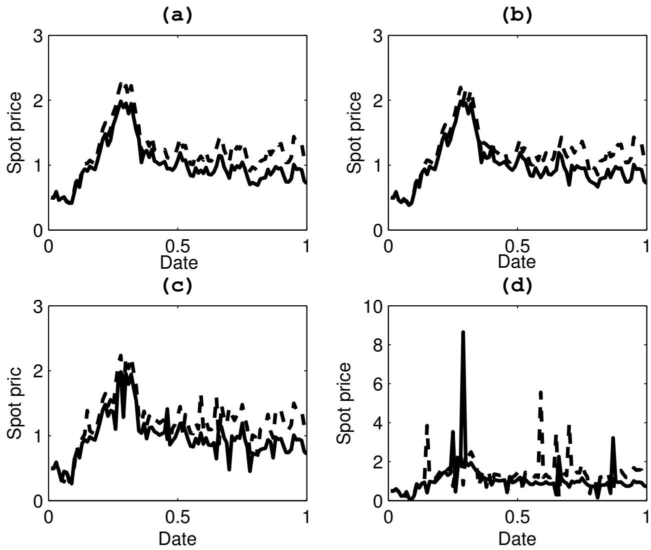

Figure 3.1: Spot price with estimated parameters (dashed line) and Spot price with reals parameters (continuous line) in case N=100

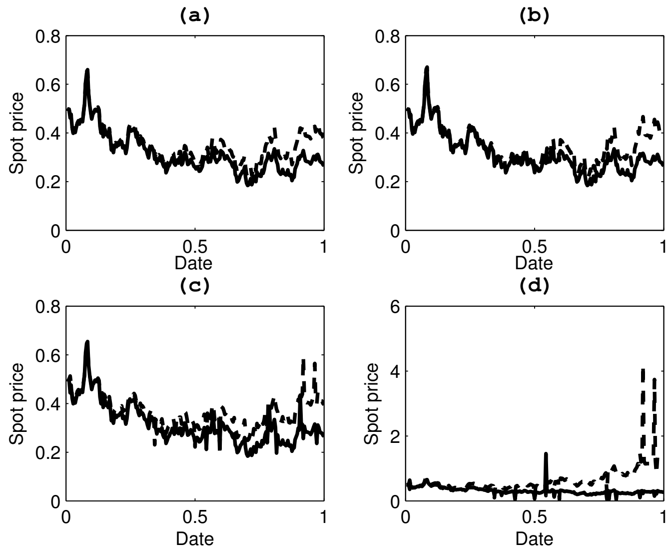

Figure 3.2: Spot price with estimated parameters (dashed line) and Spot price with reals parameters (continuous line) in case N=250

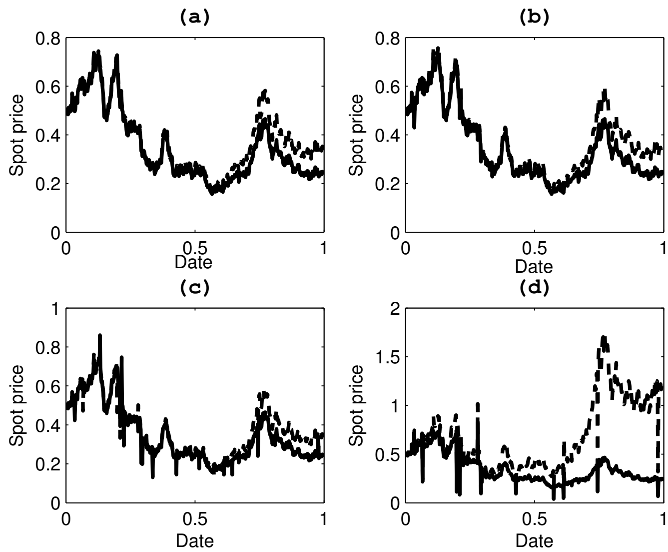

Figure 3.3: Spot price with estimated parameters (dashed line) and Spot price with reals parameters (continuous line) in case N=500

we observe that when the data increases the estimation results are more accurate when there are no jumps and less when they increase along with the jumps. This is justified by the fact that for a larger data size we have more jumps which make estimation difficult. From Figures 3.1, 3.2 and 3.3 we observe that the approximate curve is less close to the real process as the size of the jumps increases,figures 3.1c-3.1d, 3.2d and 3.3d illustrates this. The overall analysis that we can do from the curves that we obtained is that the saddle point method used to approximate the transition density of the introduce process leads to a good result. Despite the double approximation both at the process and in the transition level, one of the trajectory of the estimated parameters when we have fewer jumps in the process close to the trajectory of the process with the real parameters.

Conclusion

In this chapter, we approximated the transition density using the saddle point method. Variation in jump sizes has made it possible to highlight the influence of this method on the quality of the estimation. The method seems to be better adapted to the approximation of transition density of a process like the one we presented in 1. The approximation can be improved at the expense cost in terms of operations which could reduce the speed of convergence of the method.

Conclusion and perspectives

In this thesis we propose some solutions to the problem of pricing electricity. To achieve that we divided our work into three parts. In part one, we modeled electricity spot price regulated by the price cap principle. The second part then consists in deriving two of the most useful financial derivative which are forward contract and European option. In the last part, we calibrate the model obtained using the likelihood approach, with the transition density approximated using the saddlepoint method.

Our proposed model was constructed following Merton (1976) and is a stochastic jump diffusion equation with a new drift term which results from the fact that our model was constructed based on the price cap principle, what makes our work original. The term led us to use recent stochastic analysis tools and techniques to analyze our model and to compare it with existing models in electricity like those proposed in Cartea and Figueroa (2007). Our model, in the context of regulation by the price cap principle, takes into account some important electricity properties such as mean reversion and jumps due to sudden variations which may occur in electricity spot prices.

The calculation of forward contract options in the second part led to a closed form formula which can be handled using analysis tools like measure theory, and the Markov property in conditional probability. On the contrary, calculation of the European option led us to a partial integro-differential equation (PIDE). By drawing inspiration from the works of Cont and Ekaterina (2005b) and Alvarez and Tourin (1996), we showed that the PIDE obtained admits a viscosity solution which, in this case, is our call option. The PIDE obtained was solved using a non classical explicit-implicit numerical scheme. The first step was to perform a numerical analysis (consistency, stability and convergence) of the scheme. A mathematical difficulty encountered in the analysis is the fact that we obtained rather a Lévy functional, contrary to Cont and Ekaterina (2005a).

The calibration of our constructed model was performed using the saddlepoint method rather than a classical estimation method. This choice resulted from the presence of jumps which induce a difficulty in determining the transition density in the model. The scarcity of real data led us to test the proposed method on simulated data. The results showed that our model succeeded in capturing the price cap.

At the end of our work on the price-cap pricing modelling, we observe that the jumps of prices disrupt the mean reverting trend in the dynamic of the prices, which, in general, is of interest to consumers in the case of low value of this mean. It is clear that this reduces the action of the regulator (government) who aims at keeping prices relatively low. It should be remembered that taking into account jumps in price evolution makes it possible to take into account sudden price changes due to failures in the production system. For example, a failure could be a breakdown or a rise in engine speed caused by the drop in flow at the hydroelectric dam. As a recommendation, governments that have adopted this pricing principle should anticipate failures in the electricity generation system. For example, through the construction of water retention dams like in Cameroon.

A limitation in our model is in its non handling of seasonality that may occur in electricity prices, like when regional markets are disappearing in favor of inter-regional markets. In our future work, we intend to use time-series tools to account for seasonality that may occur in the given data. Also, in electricity market restructuring, electricity derivatives play an important role in establishing price signals, providing price discovery, facilitating effective risk management, and inducing capacity investments in generation and transmission. Thus, many exotic forms of electricity options can meet specific needs for hedging and speculation. Future research should also focus on other options like the Asian option in regulated electricity market. We will also, in future research, explore the path of calibrating our model using data obtained from the observation of option values through the regularization method. Adapting our model to the revenu cap principle is also envisaged in the future.

Appendix

Let USC (respectively, LSC) denote the class of upper semicontinuous (respectively, lower semicontinuous) function on $[0, T] \times \mathbb{R}$. and by $C_p^+([0, T] \times \mathbb{R})$ a set of measurable function on $[0, T] \times \mathbb{R}$ with polynomial growth of degree p at plus infinity and bounded on $[0, T] \times \mathbb{R}^-$):

$$g \in C_p^+([0, T] \times \mathbb{R}) \Longleftrightarrow \exists C > 0, |g(t, x)| \leq C(1 + |x|^p 1_{x>0}). \tag{3.18}$$

Definition 3.1. (Viscosity solution)

A function $v \in USC$ is a viscosity subsolution of (2.29) if for any (test function) $\phi \in C^2([0, T] \times \mathbb{R}) \cap C_p^+([0, T] \times \mathbb{R})$ and any global maximum point $(\tau, x) \in [0, T] \times \mathbb{R}$ of $v - \phi$, the following properties are verified:

$$\text{if } (\tau, x) \in [0, T] \times O, \ \frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x) \leq 0, \tag{3.19}$$

$$\text{if } \tau = 0, x \in O^c, \min\{\frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x), v(\tau, x) - H(S_0 e^x)\} \leq 0,$$

$$\text{if } \tau \in (0, T], x \in \partial O, \min\{\frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x), v(\tau, x)\} \leq 0,$$

$$\text{if } x \notin O^c, \ v(\tau, x) \leq 0. \tag{3.20}$$

A function $v \in LSC$ is a viscosity supersolution of (2.29) if for any (test function) $\phi \in C^2([0, T] \times \mathbb{R}) \cap C_p^+([0, T] \times \mathbb{R})$ and any global minimum point $(\tau, x) \in [0, T] \times \mathbb{R}$ of $v - \phi$ the following properties are verified:

$$\text{if } (\tau, x) \in [0, T] \times O, \ \frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x) \geq 0, \tag{3.21}$$

$$\text{if } \tau = 0, x \in O^c, \min\{\frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x), v(\tau, x) - H(S_0 e^x)\} \geq 0,$$

$$\text{if } \tau \in (0, T], x \in \partial O, \min\{\frac{\partial \phi}{\partial \tau}(\tau, x) - (\mathcal{L}_D + \mathcal{L}_J)\phi(\tau, x), v(\tau, x)\} \geq 0,$$

$$\text{if } x \notin O^c, \ v(\tau, x) \geq 0. \tag{3.22}$$

A function v is called a viscosity solution of (2.29) if it is both a subsolution and a supersolution. In this case, v is continuous on $(0, T] \times \mathbb{R}$

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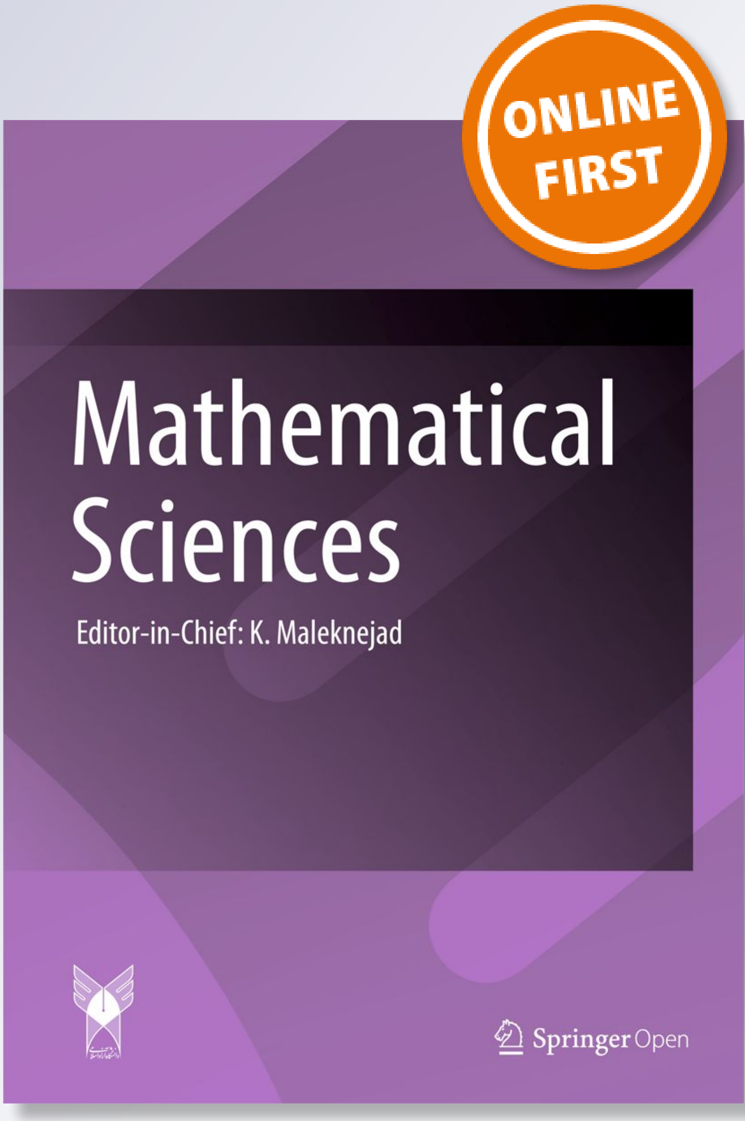

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ORIGINAL RESEARCH

A jump-diffusion model for pricing electricity under price-cap regulation

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Abstract
In this paper, we derive a new jump-diffusion model for electricity spot price from the "Price-Cap" principle. Next, we show that the model has a non-classical mean-reverting linear drift. Moreover, using this model, we compute a new exact formula for the price of forward contract under an equivalent martingale measure and we compare it to Cartea et al. (Appl Math Finance 12(4):313–335, 2005) formula.

Keywords Model · Electricity market · Price-cap regulation · Spot price · Forward price

Introduction

According to [26], no economical development is possible without the availability of energy. Accepting this reality, several governments consider electricity as one of their main priorities. For example in France, electricity is recognized by the law as a basic necessity. Its price is determined by regulated tariffs made by the government [4]. Several changes have been operated in electricity sector. Regulation and deregulation are the main mechanisms which caused these changes observed in electricity sector. The aims were to create a competitive economical environment in which the producers, the investors and the large part of consumers would get their satisfaction.

The introduction of deregulation induced many consequences. One main consequence was the high variation in price which encouraged the development of a new breed of financial products in electricity markets. These new products may help cover both physical and financial risks on the new market. Therefore, there has been an important research effort devoted to electricity price modeling for derivative pricing. Due to the non-storable nature of electricity, the challenge of the researchers was the development of a completely satisfying methodology that would help to obtain realistic and robust models. Two standard approaches have often been used to handle this problem in the literature. The first consists in modeling directly the forward curve dynamics and deduces the spot price [2, 5]. The second approach starts from a spot price model to derive future prices as the expectation of the spot price under a risk-neutral probability. Relevant contributions have been made by [8, 20] in pricing energy derivatives and electricity. They took into account seasonality and mean reversion. However, their model did not take into account the huge and non-negligible observed spikes in the market. Further [3, 6] were among the first to consider price spikes using jump-diffusion models. Similar works were done in [10, 14, 15, 24, 25, 30]. Regular increase in electricity prices and crises observed in the unregulated market are a point of focus in the media and raised the question of regulation. Moreover, direct link between the price of electricity and the national strategy of poverty reduction motivated governments to limit electricity prices, so that it can be more accessible. This leads to the reintroduction of price regulation by most governments. For instance, [19] was the first to propose the price-cap regulation to British government. Several works were done to study effect and impact of regulation in electricity network and the wholesale electricity market [12, 16].

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The main contribution of this paper is, using the rate of increase given by the price-cap, to construct a spot price model in regulated electricity market. In addition to well-known specific features of electricity such as mean-reverting and spikes, our proposed model captures important characteristics of price-cap regulation: inflation rate and efficiency rate. Furthermore, we compute explicitly the forward price in this electricity model.

The rest of this paper is structured as follows: In section two, we present a review of some recent electricity price models. Section three deals with the formulation of our model. Moreover, in section three, we discuss the mean-reversion feature of our model and further compare it with [3] model. In section five, we present some numerical simulations of our model to illustrate our theoretical results and support our discussion.

Review of some recent electricity models

Several models on electricity price dynamics have been proposed in the literature, among which the jump-diffusion model. Merton [23] was among the first to work on this class of models. His first model was developed to describe the dynamic return on equity. This model was progressively extended by [3, 8, 20, 24].

Schwartz [8] considered spot prices as a stochastic process with two components represented by

$$P_t = f(t) + X_t;\ t \in [0, \infty); \tag{1}$$

where f is a deterministic differentiable function and X_t is the stochastic component satisfying

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}W_t, \tag{2}$$

with $\alpha > 0$ representing the speed of mean reversion, $X(0) = x_0$ being the initial condition and W being a standard Brownian motion. Applying Itô's formula on (2), they obtained the following dynamic for spot price

$$\mathrm{d}P_t = \alpha(a(t) - P_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{3}$$

where

$$a(t) = \frac{1}{\alpha} f'(t) + f(t).$$

In the same paper, they also considered log of spot prices, i.e., $\ln P_t = f(t) + Y_t$ where Y_t follows process (2). In this case, they obtained

$$\mathrm{d}P_t = \alpha(b(t) - \ln P_t)P_t \mathrm{d}t + \sigma \mathrm{d}W_t, \tag{4}$$

where

$$b(t) = \frac{1}{\alpha}\left(\frac{\sigma^2}{2} + f'(t)\right) + f(t).$$

This model captures mean-reverting feature which is one of the main characteristics of electricity, but the model has failed to take into account the spikes which can occur in electricity markets. Cartea and Figueroa [3] in the same context of deregulation markets extended (4) by adding a jump term and obtained a mean-reverting and jump-diffusion model. They supposed that the spot price process is in the form $\ln S_t = g(t) + Y_t$, where g is a seasonal deterministic function and that Y_t follows a stochastic process given by

$$\mathrm{d}Y_t = -\alpha Y_t \mathrm{d}t + \sigma \mathrm{d}W_t + \ln J \mathrm{d}q_t. \tag{5}$$

Using Itô's formula and equation (5), [3] obtained the following model:

$$\mathrm{d}S_t = \alpha(\rho(t) - \ln S_t)S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}W_t + S_t(J - 1)\mathrm{d}q_t, \tag{6}$$

where

$$\rho(t) = \frac{1}{\alpha}\left(g'(t) + \frac{1}{2}\sigma^2(t)\right) + g(t),$$

J is the proportional size of jump and q_t is the Poisson process. Hence, on contrary to [3, 8] in their model considered the non-constant volatility, jump and deterministic part of spot price as a seasonal function of time.

Model derivation and the main result

Our model is partly inspired from the electricity price-cap regulation proposed by Littlechild [19] that we recall as follow.

Price-cap market regulation

The price-cap regulation is an economical principle which aims to establish an incentive scheme for the regulated market. A key objective is to enable companies to maximize the well-being while seeking to maximize their own interests, see [1]. Its principle is to cap the market price. The main components of the price cap include the efficiency factor (G), for transferring the gains to consumers through the reduction of costs; the inflation rate (I), which drives the price changes; the exogenous factors such as customer portion of earnings' sharing (E), service quality penalties (H) and flow-through and uncontrollable costs, if any (F). ENMAX [11] proposed price-cap formula:

$$\frac{P_i - P_{i-1}}{P_{i-1}} = I_i - G_i + \left(\frac{-E_i - HS_i + F_i}{P_{i-1}}\right), \tag{7}$$

where P_i represents the current year's price and P_{i-1} preceding year price. Later, we would be inspired by the economic formula (7) to model the drift of the model.

Spot price modeling procedure

The daily (resp. weekly and monthly) change in price is the difference between today's price (resp. this week's price and this month's price) and yesterday's price (resp. last week's price and last month's price). In general, one denotes a change over a given time period $\mathrm{d}t$ by $\mathrm{d}S_t$. For a daily change, we therefore have $\mathrm{d}t = \frac{1}{365}$, $\mathrm{d}t = \frac{1}{52}$ for weekly change and $\mathrm{d}t = \frac{1}{12}$ for a monthly change. The change in price $\mathrm{d}S_t$ over a given time period $\mathrm{d}t$ is the sum of two components: the "drift" term and the stochastic (or "random") term, that is,

$\mathrm{d}S =$ drift term + stochastic term.

The drift term represents the portion of the movement in the spot price S, which we expect to see with certainty. This term is proportional to the time period over which the change in the price is measured. That is,

drift $\propto \mathrm{d}t$.

The stochastic term represents the portion of the change that is random and cannot be predicted. This term is proportional to the increment $\mathrm{d}W_t$ of standard Brownian motion, which is normally distributed with mean zero and variance $\mathrm{d}t$ (see [28, pp. 377–380]. That is,

stochastic term $\propto \mathrm{d}W_t, \quad \mathrm{d}W_t \sim \mathcal{N}(0, \mathrm{d}t)$.

The main result

Before stating the following theorem, let us recall that a *càdlàg* stochastic process is the right continuous with left-limit stochastic process.

Theorem 1 *Suppose that the spot price S_t is a càdlàg process in a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}\right)$ where $(\mathcal{F}_t)_t$ is a natural filtration of S_t. Assume the following conditions:*

(i) *for a small time interval Δt, the change in the electricity price is proportional to Δt,*
(ii) *the inflation rate I always differs from the efficiency factor G,*
(iii) *the stock prices jumps from the previous value S_{t^-} to a next value JS_{t^-} where J is the proportional size of the random jump assumed log-normally distributed, i.e., $\ln J \sim \mathcal{N}(m_J, \sigma_J^2)$ with $\mathbb{E}[J] = 1$,*
(iv) *the change before and after the jumps is driven by increments $\mathrm{d}q_t$ of a Poisson process q_t defined by*

$$\mathrm{d}q_t = \begin{cases} 1, & \text{with probability } \ell \mathrm{d}t \\ 0, & \text{with probability } 1 - \ell \mathrm{d}t, \end{cases}$$

where ℓ is the intensity or frequency of the process. Then, the price-cap principle (7) yields the stochastic differential equation (SDE) below

$$\mathrm{d}S_t = -\alpha(t)(\gamma(t) - S_t)\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t + (J-1)S_t\mathrm{d}q_t, \tag{8}$$

where W_t is the standard Brownian motion and the coefficients involved are deterministic functions of time denoted as such: $\sigma(t)$ is the volatility, $\beta(t) := E(t) + H(t) - F(t)$ defines the exogenous factors, $\alpha(t) := I(t) - G(t)$, and $\gamma(t) := \beta(t)/\alpha(t)$.

Proof For a small time interval Δt, the change in the electricity price is proportional to Δt and the expected change is by (7) therefore we have

$$\Delta S = \left[S_t(I(t) - G(t)) - E(t) - H(t) + F(t)\right]\Delta t, \tag{9}$$

where $\Delta S_t = S_{t+\Delta t} - S_t$. For $\Delta t \to 0$, we obtain

$$\mathrm{d}S_t = \left[S_t(I(t) - G(t)) - E(t) - H(t) + F(t)\right]\mathrm{d}t. \tag{10}$$

To take into account market volatility in the model, the stochastic, or random, contribution to the change in the spot price is represented by $\sigma(t)S_t\mathrm{d}W_t$ (see [28, pp. 103–104]). Hence, we obtain the following SDE

$$\mathrm{d}S_t = \left[S_t(I(t) - G(t)) - \beta(t)\right]\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t. \tag{11}$$

Next, to capture the market shocks we add the jump term in (11) using [3] idea as follows. We suppose that the stock prices jump from the previous value S_{t^-} to a next value JS_{t^-} where J is the proportional size of the random jump assumed log-normally distributed such that $\mathbb{E}(J) = 1$ this assumption is motivated by the fact that under regulation we want that the risk of the market shocks fluctuates around unit. Next, we assume that the term $(J-1)S_{t^-}$, which give the change before and after the jumps, is driven by increments $\mathrm{d}q_t$ of a Poisson process. Hence, from equation (11), setting $\alpha(t) := I(t) - G(t)$, $\gamma(t) := \beta(t)/\alpha(t)$, we finally obtain the SDE (8). □

Mean-reversion condition

A mean-reverting process has a drift term that brings the variable being pulled back to some equilibrium. This feature is captured by one stochastic differential equation if the following definition is verified.

Definition 1 (*Condition* (A_3) *of* [22]) Consider a jump-diffusion process Y_t with a differentiable drift function $\mu(.)$.

If

$$\limsup_{|Y_t|\to\infty} \frac{| Y_t + \mu(Y_t) |}{| Y_t |} < 1,$$

then Y_t is mean-reverting.

From this definition, we have the following proposition

Proposition 1 *The jump-diffusion model (8) is mean-reverting.*

Proof It is straightforward and is based on the fact that from an economic point of view, $\beta(t)$ is bounded on $[0, T]$ and we have $| 1 + \alpha(t) | < 1$ for all $t \in [0, T]$. □

Regulated electricity forward price

Computation of regulated electricity forward price

The price at time t of the forward expiring at time T (i.e., $\mathbf{F}(t, T)$) is obtained as the expected value of the spot price under an equivalent $\mathbb{Q}$-martingale measure, conditional on the information set available up to time t, precisely

$$\mathbf{F}(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right].$$

where $\mathbb{E}_t^{\mathbb{Q}}$ represents the conditional expectation knowing a natural filtration of S_t under the risk-neutral probability Q. To incorporate the non-opportunity of arbitrage in the model, we use the same approach as in [20] and [3], which consists of incorporating a market price of risk in the drift, to obtain

$$\hat{\gamma}(t) = \gamma(t) - \lambda\frac{\sigma(t)}{\alpha(t)}S_t, \tag{12}$$

where λ denotes the market price of risk per unit risk linked to the state variable S_t. This market price of risk to be calibrated from market information pins down the choice of one particular martingale measure. Recall that when a market subjected to that measure, the opportunity of arbitrage is theoretically excluded in this market. Hence, under this equivalent martingale measure SDE (8) becomes

$$\mathrm{d}S_t = -\alpha(t)\big(\hat{\gamma}(t) - S_t\big)\mathrm{d}t + \sigma(t)S_t\mathrm{d}\widehat{W}_t + (J-1)S_t\mathrm{d}q_t; \tag{13}$$

substituting (12) in (13), we obtain

$$\begin{aligned}\mathrm{d}S_t = -\alpha(t)\left(\gamma(t) - \left(1+\lambda\frac{\sigma(t)}{\alpha(t)}\right)S_t\right)\mathrm{d}t + \sigma(t)S_t\mathrm{d}\widehat{W}_t \\ + (J-1)S_t\mathrm{d}q_t,\end{aligned} \tag{14}$$

where $\mathrm{d}\widehat{W}$ is the increment of a Brownian motion in the $\mathbb{Q}$-martingale measure specified by the choice of λ.

The next addresses the forward price computations.

Proposition 2 *Assume that J, the increments of q_t and W_t, are independent. Under the risk-neutral or martingale measure $\mathbb{Q}$ and Novikov hypothesis, i.e., $\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\int_0^t\sigma(s)^2\mathrm{d}s}\right] < \infty$, electricity forward price under regulated market is given by*

$$\mathbf{F}(t, T) = S_t\mathrm{e}^{\int_t^T(\alpha(s)+\lambda\sigma(s))\mathrm{d}s} - \int_t^T \beta(s)\mathrm{e}^{\int_s^T(\alpha(u)+\lambda\sigma(u))\mathrm{d}u}\mathrm{d}s. \tag{15}$$

Before proving Proposition 2, let us first prove the following lemmas.

Lemma 1 *The solution of equation (14) is the process $(S_t, 0 \le t \le T)$ defined by*

$$S_t = Z_t\left(S_0 - \int_0^t \beta(s)Z_s^{-1}\mathrm{d}s\right),$$

where $Z_t = \mathrm{e}^{\left(\int_0^t(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s+\int_0^t\sigma(s)\mathrm{d}\widehat{W}_s+\int_0^t\ln J\mathrm{d}q_s\right)}$.

Proof To solve equation (14), we consider a process Z, solution of the following equation

$$\mathrm{d}Z_t = Z_t\left(\alpha(t)\left(1+\lambda\frac{\sigma(t)}{\alpha(t)}\right)\mathrm{d}t + \sigma(t)\mathrm{d}\widehat{W}_t + (J-1)\mathrm{d}q_t\right)$$
$$\text{and } Z_0 = 1.$$

Applying Itô formula with jumps stated in 7.10 [9], we obtain
$Z_t = Z_0\mathrm{e}^{\left(\int_0^t(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s+\int_0^t\sigma(s)\mathrm{d}\widehat{W}_s+\int_0^t\ln J\mathrm{d}q_s\right)}$.

Now, let us set $f(S_t, Z_t) = \frac{S_t}{Z_t}$. By applying Itô formula with jumps one more, we obtain

$$\begin{aligned}\frac{S_t}{Z_t} &= \frac{S_0}{Z_0} + \int_0^t \frac{1}{Z_s}\Big[((\lambda(s)\sigma(s)+\alpha(s))\mathrm{d}s + \sigma(s)\mathrm{d}\widehat{W}_s \\ &\quad +(J-1)\mathrm{d}q_s)S_s - \beta(s)\mathrm{d}s\Big] \\ &\quad - \int_0^t \frac{S_s}{(Z_s)^2}Z_s((\lambda(s)\sigma(s)+\alpha(s))\mathrm{d}s \\ &\quad + \sigma(s)\mathrm{d}\widehat{W}_s + (J-1)\mathrm{d}q_s) \\ &\quad + \frac{1}{2}\left(\int_0^t 2\frac{S_s}{(Z_s)^3}(\sigma(s)Z_s)^2\mathrm{d}s - \frac{2}{(Z_s)^2}\sigma(s)^2S_sZ_s\mathrm{d}s\right).\end{aligned} \tag{16}$$

The development of (16) leads to

$$\frac{S_t}{Z_t} = \frac{S_0}{Z_0} + \int_0^t \frac{S_s}{Z_s}((\lambda(s)\sigma(s) + \alpha(s))\mathrm{d}s + \sigma(s)\mathrm{d}\widehat{W}_s + (J-1)\mathrm{d}q_s) - \int_0^t \frac{1}{Z_s}\beta(s)\mathrm{d}s - \int_0^t \frac{S_s}{Z_s}((\lambda(s)\sigma(s) + \alpha(s))\mathrm{d}s + \sigma(s)\mathrm{d}\widehat{W}_s + (J-1)\mathrm{d}q_s) + \int_0^t \frac{S_s}{Z_s}\sigma(s)^2\mathrm{d}s - \frac{S_s}{Z_s}\sigma(s)^2\mathrm{d}s. \tag{17}$$

By observing that the sum of the first and fourth term of equation (17) is equal to zero and since $Z_0 = 1$ we obtain

$$\frac{S_t}{Z_t} = S_0 - \int_0^t Z_s^{-1}\beta(s)\mathrm{d}s.$$

Finally, we obtain

$$S_t = Z_t S_0 - Z_t \int_0^t Z_s^{-1}\beta(s)\mathrm{d}s.$$

This ends the proof. □

Furthermore, the solution of (14) at T starting at t is given by

$$S_T = \frac{Z_T}{Z_t}S_t - Z_T\int_t^T \alpha(s)\mu(s)Z_s^{-1}\mathrm{d}s, \tag{18}$$

where $Z_T = Z_t \mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_t^T \sigma(s)\mathrm{d}\hat{W}_s + \int_t^T \ln J \mathrm{d}q_s\right)}$.

Lemma 2 *If J is a log-normal distributed process with $\mathbb{E}[J] = 1$ and q a Poisson process, then* $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\int_t^T \ln J_s \mathrm{d}q_s}] = 1$.

Proof Firstly, we use differentiation method to compute $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\int_0^t \ln J_s \mathrm{d}q_s}]$.

Let us define L_t such that

$$L_t \equiv \mathrm{e}^{\int_0^t \ln J_s \mathrm{d}q_s}, \equiv \mathrm{e}^{m_t} \tag{19}$$

where m_t is

$$m_t = \int_0^t \ln J_s \mathrm{d}q_s,$$

or equivalently

$$\mathrm{d}m_t = \ln J_t \mathrm{d}q_t. \tag{20}$$

In order to write the dynamic followed by L_t for process define in (20) we use the generalization form of Itô's lemma [9]. The SDE verified by L_t is

$$L_t = L_0 + \int_0^t L_s \ln J_s \mathrm{d}q_s - \int_0^t L_s \ln J_s \mathrm{d}q_s + \int_0^t L_s(\mathrm{e}^{\ln J_s} - 1)\mathrm{d}q_s = 1 + \int_0^t L_s(\mathrm{e}^{\ln J_s} - 1)\mathrm{d}q_s. \tag{21}$$

Then, from (21) we obtain

$$\mathbb{E}[L_t] = 1 + \int_0^t \mathbb{E}[L_s](\mathbb{E}[\mathrm{e}^{\ln J_s}] - 1)\ell\mathrm{d}s = 1.$$

Alternatively, we can remark that $\int_0^t \ln J_s \mathrm{d}q_s = \sum_{i=0}^{q_t} \ln J_s$ which is the particular case of Lévy process with the moment-generating function. Using Lévy–Khintchine representation, we have

$$\mathbb{E}\left[\mathrm{e}^{iu\int_0^t \ln J_s \mathrm{d}q_s}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{iu\int_0^t \ln J \mathrm{d}q_s}|q_t\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\varphi(u)^{q_t}\right]\right] = \mathrm{e}^{t\ell(\varphi(u)-1)}. \tag{22}$$

where φ is the moment-generating function of the jump $\ln J$. Evaluating (22) at $u = -i$ leads to desired result. □

Proof of Proposition 2 Before stating let us recall that forward price formula is given by $\mathbf{F}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right]$. By substituting S_T with (18), we obtain

$$\mathbf{F}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[S_T\right] = S_t\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_t^T \sigma(s)\mathrm{d}\hat{W}_s + \int_t^T \ln J_s \mathrm{d}q_s\right)}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[Z_t\mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_t^T \sigma(s)\mathrm{d}\hat{W}_s + \int_t^T \ln J_s \mathrm{d}q_s\right)} \times \int_t^T \alpha(s)\mu(s)Z_s^{-1}\mathrm{d}s\right]. \tag{23}$$

We first compute $\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s + \int_t^T \sigma(s)\mathrm{d}\hat{W}_s + \int_t^T \ln J_s \mathrm{d}q_s\right)}\right] \equiv A.$

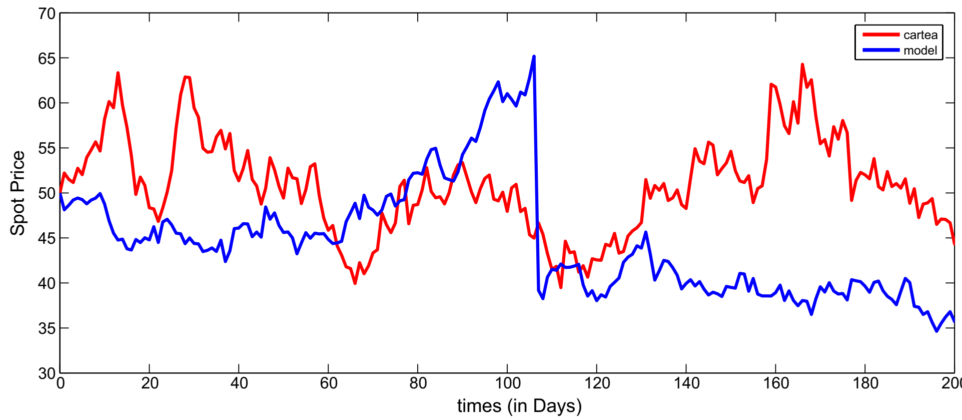

Fig. 1 Spot prices for the parameters, $I = 0.0314$; $G = 0.01$; $E = 0.05$; $H = 0.001$; $F = 0.05$; $\ell = 2.85$; $\sigma = 0.75$; $\sigma J = 0.67$; $S(0) = 50$

From independence between J, $\mathrm{d}q_t$ and $\mathrm{d}W_t$, we obtain

$$
\begin{aligned}
A &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s+\int_t^T\sigma(s)\mathrm{d}\hat{W}_s\right)}\right]\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\int_t^T\ln J\mathrm{d}q_s}]\\
&= \mathrm{e}^{\int_t^T(\alpha(s)+\lambda\sigma(s))\mathrm{d}s}\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\int_t^T\ln J\mathrm{d}q_s}]
\end{aligned}
\tag{24}
$$

We now compute $\mathbb{E}_t^{\mathbb{Q}}\left[Z_t\mathrm{e}^{\left(\int_t^T(\lambda\sigma(s)+\alpha(s)-\frac{1}{2}\sigma(s)^2)\mathrm{d}s+\int_t^T\sigma(s)\mathrm{d}\hat{W}_s+\int_t^T\ln J\mathrm{d}q_s\right)}\int_t^T\alpha(s)\gamma(s)Z_s^{-1}\mathrm{d}s\right] \equiv A_1$.

Replacing Z_s^{-1} by its expression, using independence between J, $\mathrm{d}q_t$ and $\mathrm{d}W_t$ and Fubini theorem [29], we obtain

$$
\begin{aligned}
A_1 &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T\alpha(s)\gamma(s)\mathrm{e}^{\left(\int_s^T(\lambda\sigma(u)+\alpha(u)-\frac{1}{2}\sigma(u)^2)\mathrm{d}u+\int_s^T\ln J\mathrm{d}q_u+\int_s^T\sigma(u)\mathrm{d}\hat{W}_u\right)}\mathrm{d}s\right]\\
&= \int_t^T\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\int_s^T\ln J\mathrm{d}q_u}]\alpha(s)\gamma(s)\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{\left(\int_s^T(\lambda\sigma(u)+\alpha(u)-\frac{1}{2}\sigma(u)^2)\mathrm{d}u+\int_s^T\sigma(u)\mathrm{d}\hat{W}_u\right)}\right]\mathrm{d}s\\
&= \int_t^T\alpha(s)\gamma(s)\mathrm{e}^{\int_s^T(\alpha(u)+\lambda\sigma(u))\mathrm{d}u}\mathrm{d}s.
\end{aligned}
\tag{25}
$$

By replacing finally (24) and (25) in (23), we obtain the forward price

$$
\mathbf{F}(t,T) = S_t\mathrm{e}^{\int_t^T(\alpha(s)+\lambda\sigma(s))\mathrm{d}s} - \int_t^T\beta(s)\mathrm{e}^{\int_s^T(\alpha(u)+\lambda\sigma(u))\mathrm{d}u}\mathrm{d}s.
$$

This completes the proof. □

Analytical comparison with forward price in Cartea et al. (2005)

Recall that forward price obtained in [3] is given by

$$
F(t,T) = G(T)\left(\frac{S(t)}{G(t)}\right)^{\mathrm{e}^{-\alpha(T-t)}}
\mathrm{e}^{\int_t^T\frac{1}{2}\sigma^2(s)\mathrm{e}^{-2\alpha(T-s)}-\lambda\sigma(s)\mathrm{e}^{-\alpha(T-s)}\mathrm{d}s+\int_t^T\xi(\sigma_J,\alpha,T,s)\ell\mathrm{d}s-\ell(T-t)}
$$

The forward price formula (15) derived in this work is an affine function of the spot price S_t, Unlike in the works of [3], where they have obtained a power function of the spot price. This is what justifies the presence of fewer jumps in the forward prices. This is in line with the fact that we are in a regulating context where prices are likely to undergo less variations.

Some illustrative curves of spot and forward price in regulated electricity market

This section deals with the numerical simulations of the forward price in order to illustrate some meaningful behaviors of the model and in comparison with the model developed in [3]. The proposed simulations also aim at highlighting the fundamental role of some particular parameters in the outcomes of the prices. For the numerical computation of spot and forward prices, we approximated the integrals using the trapezium and the Stratonovich integration methods. The parameters used in the simulations are plausible relative to those used in the literature.

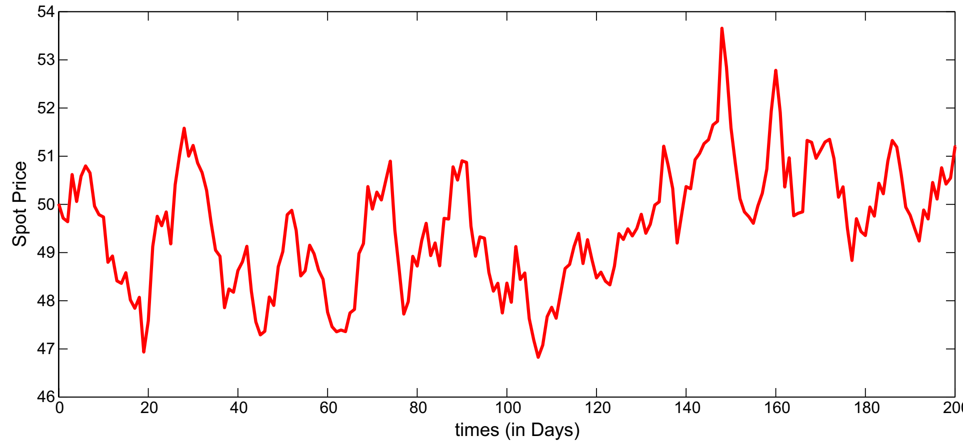

Fig. 2 Spot prices for the parameters, $I = 0.0314$; $G = 0.01$; $E = 0.05$; $H = 0.001$; $F = 0.05$; $\ell = 8.85$; $\sigma = 0.75$; $\sigma J = 0.67$; $S(0) = 50$

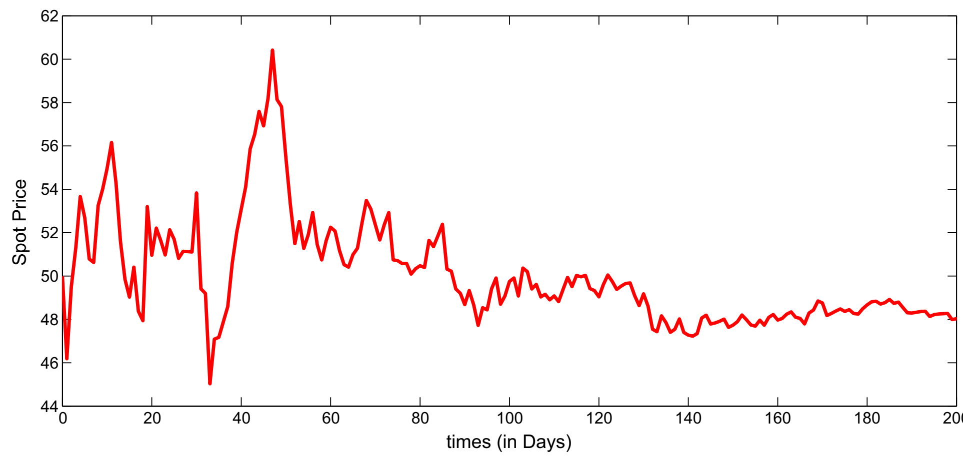

Fig. 3 Spot prices for the parameters, $I = 0.0314;G = 0.01$; $E = 0.05;H = 0.001;F = 0.05$; $\ell = 1.5$; $\sigma = \exp(-0.015t)$; $\sigma J = 0.67;S(0) = 50$

Figure 1 shows a simulated spot price compared to the spot price in [3] without the seasonal part. One can observe that the proposed model captures some characteristics discussed in the regulated market such as mean reversion, a property also observed in Figs. 2 and 3 confirming the theoretical results. It is further relevant to discuss that in our model, the frequency of jumps is less than in the deregulated market. Figures 4 and 5 present four different states of the evolution of the forward price process in the absence of jumps in the spot price model. Here, we observe that the forward price fluctuates around an average like the spot price. This could be justified by the fact that the forward formula obtained here is a functional of the spot price. Figure 6 obtained by introducing small jumps into the model shows that despite the jump at the beginning, the forward price latter oscillates around an equilibrium a situation which is not observed in Fig. 7 with bigger jumps. In a nut shell, these illustrations show that our model with the mean-reversion property captures the main objective of regulation principle, which is to cap prices within a given range.

Figure 8 shows that despite jumps in the prices, prices vary from a certain threshold for different maturities. We observe in Fig. 9 that when the efficiency rate factor G is more than the inflation rate factor I, forward price decreases over time. This is in accordance with the economic principle.

Conclusion

In this paper, we have proposed a new model of spot price in regulated electricity market. The principal aim was to propose the forward price in regulated electricity market using economic principle price cap. It is also motivated by the fact that incentives regulation in public utilities, especially in electricity field, becomes more prevalent. The proposed model leads to non-classical Ornstein–Uhlenbeck process due to non-constant speed of the mean reversion. The determination of the exact solution permits us to derive the explicit expression of the forward price. An important topic for further research is to use historical data to calibrate the introduced model.

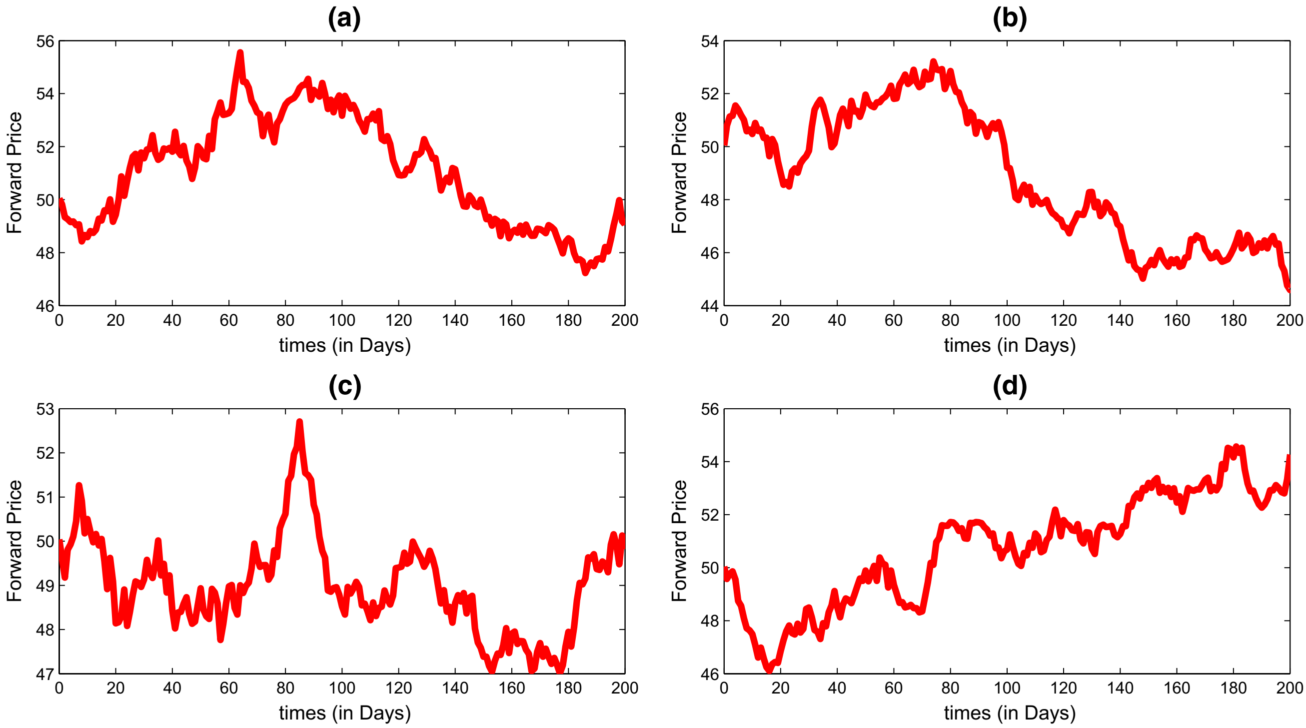

Fig. 4 Forward price for the parameters, $I = 0.0314; G = 0.01; E = 0.05; H = 0.001; F = 0.05; \sigma = 0.75; \sigma J = 0.67; S(0) = 50$

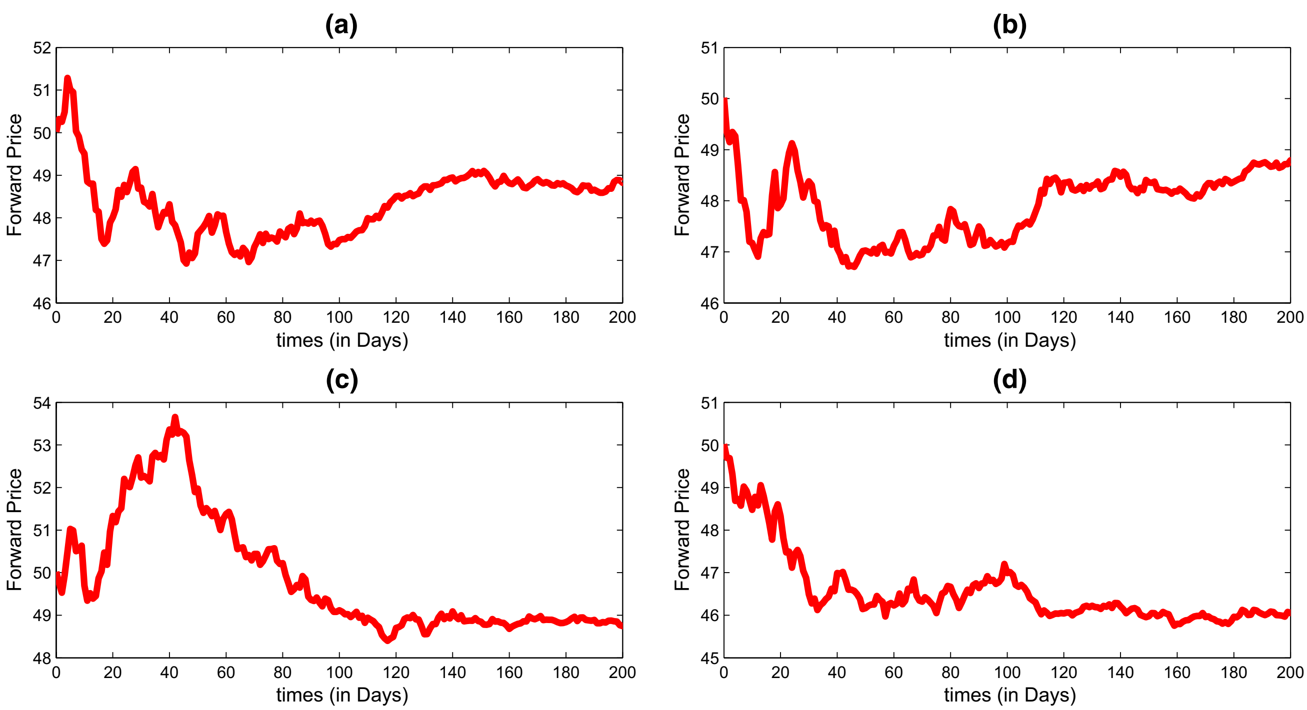

Fig. 5 Forward prices for the parameters, $I = 0.0314; G = 0.01; E = 0.05; H = 0.001; F = 0.05; \sigma = \exp(-0.01t); \sigma J = 0.67; S(0) = 50$

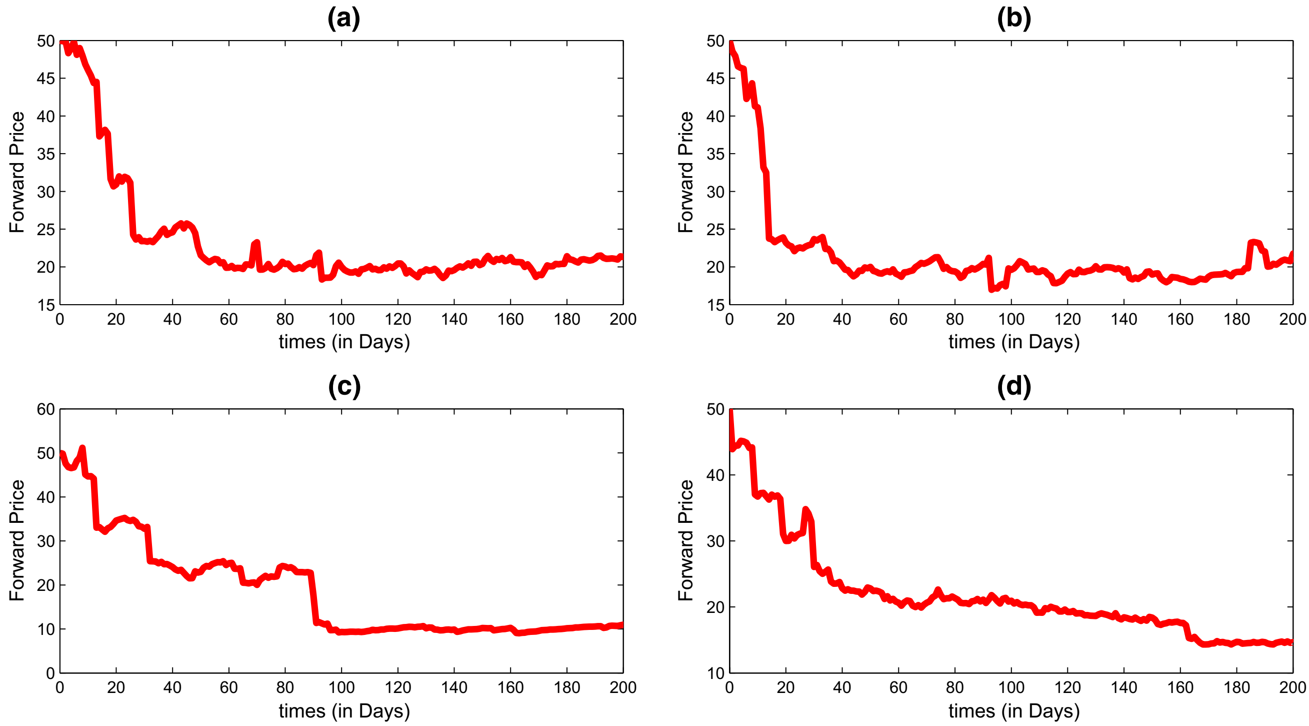

Fig. 6 Forward prices for the parameters, $I = 0.0314; G = 0.01;\ E = 0.05; H = 0.001;\ F = 0.05; \ell = 8.5;\ \sigma = 0.75;\ \sigma J = 0.67; S(0) = 50$

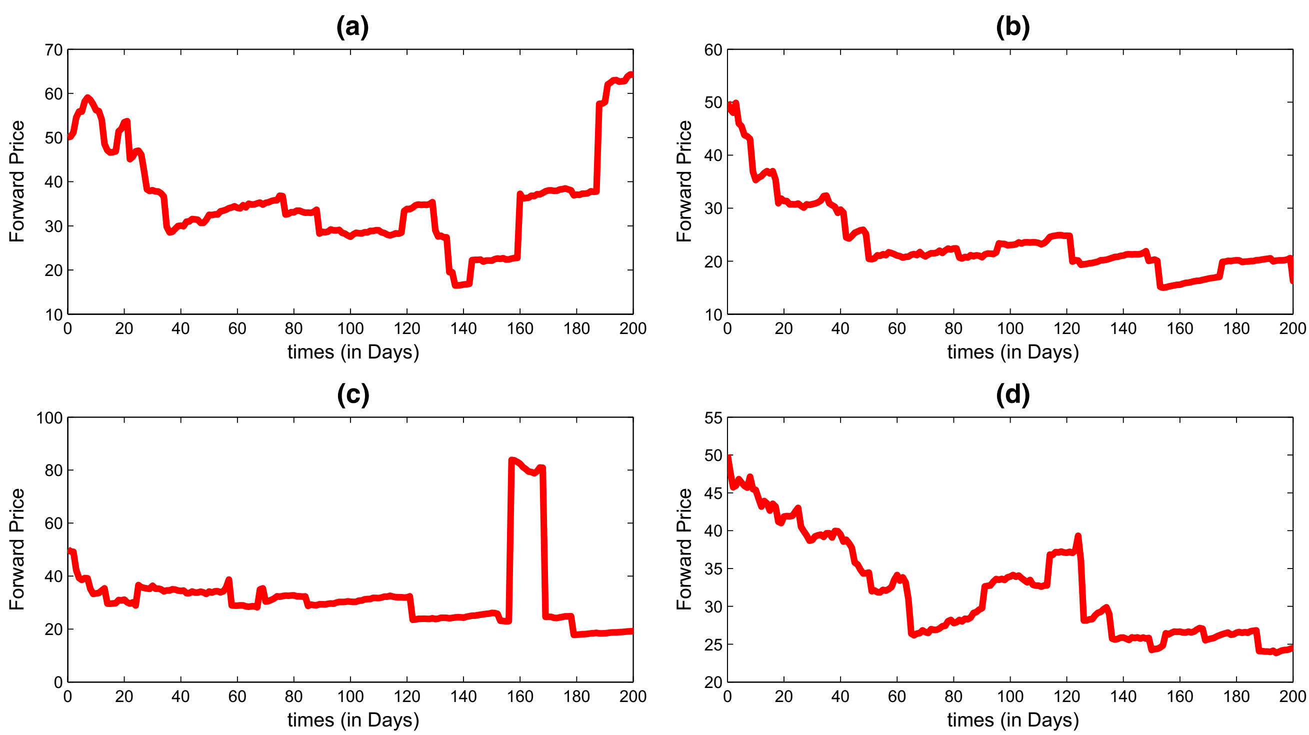

Fig. 7 Forward prices for the parameters, $I = 0.0314; G = 0.01;\ E = 0.05; H = 0.001;\ F = 0.05; \ell = 8.5;\ \sigma = \exp(-0.01t);\ \sigma J = 0.67; S(0) = 50$

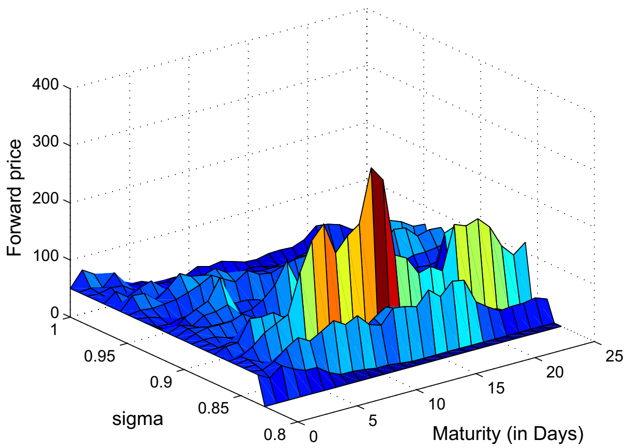

Fig. 8 Forward prices for each day for various maturities and parameter sigma, $I = 0.0314; G = 0.01;$ $E = 0.05; H = 0.001;$ $F = 0.05; \ell = 8.5; \sigma = \exp(-0.01t); \sigma J = 0.67; S(0) = 50$

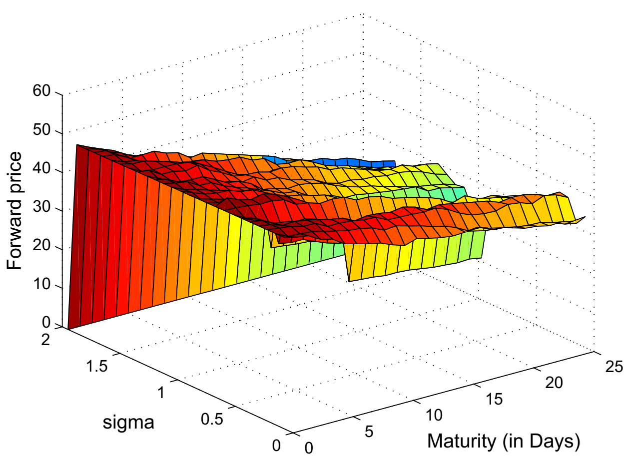

Fig. 9 Forward prices for each day for various maturities and parameter sigma $I = 0.0314; G = 0.1; S(0) = 50; \sigma J = 0.67; \ell = 0.25$

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