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ON ANSCOMBE'S PARADOX AND RELATED TOPICS

THÈSE DE DOCTORAT/PhD

Par : **OUAMBO KAMDEM Monge Kleber**

Sous la direction de
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ATTESTATION DE CORRECTION

Nous soussignés, membres du jury lors de la soutenance de thèse de Doctorat Ph.D de Monsieur **OUAMBO KAMDEM Monge Kleber**, étudiant à l'Université de Yaoundé I sous le matricule **04U190,** attestons que la thèse intitulée **« On Anscombe's paradox and related topics »**, présentée en soutenance publique le vendredi 17 Juillet 2020 à 14 heures dans la salle R110 du bloc pédagogique par le candidat, a été corrigée conformément à nos recommandations.

En foi de quoi, la présente attestation lui est établie et délivrée pour servir et valoir ce que de droit.

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THÈSE DE DOCTORAT/PhD

Intitulée

ON ANSCOMBE'S PARADOX AND RELATED TOPICS

Présentée et soutenue à l'université de Yaoundé I

le 17 juillet 2020

Par

OUAMBO KAMDEM Monge Kleber

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DECLARATION

I hereby declare that this submission is my own work and to the best of my knowledge, it contains no materials previously published or written by another person. No material which to a substantial extent has been accepted for the award of any other degree or diploma at The University of Yaoundé I or any other educational institution except where due acknowledgement is made in this thesis. Any contribution made to the research by others, with whom i have worked at The University of Yaoubndé I or elsewhere, is explicitly cited in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in style, presentation and linguistic expression is acknowledge.

OUAMBO KAMDEM Monge Kleber

DÉDICACE

Je dédie ce mémoire à mon feu papa M. KAMDEM Jean et ma maman chérie Mme KAMDEM Bernadette..

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ABSTRACT

The present work is a contribution to the study of the well-known Anscombe's paradox first noted in Social Choice Theory by Anscombe (1976). Roughly speaking, this surprising voting event refers to the possibility that, at the end of a series of votes on a given finite set of proposals, more than the half of the voters are each frustrated on more than the half of the proposals although each decision on each proposal is supported by more than the half of voters. Since this paradox can be encountered in real voting situations, especially in parliaments, congresses or senates, it deserves a detailed study for a better understanding of the circumstances of its appearance or to identify some conditions to avoid it. This is mainly the object of our thesis. Indeed, several questions about the Anscombe's paradox remain open. For instance, when an absolute majority of favorable votes is required for the adoption of each proposal, it is not clear whether or not there are a number of voters and a number of proposals that preclude all occurencies of the current paradox. Moreover, when the decision rule is a qualified majority or any other binary decision rule distinct from the majority rule, no study to the best of our knowledge has yet been made to check whether it is still possible for more than the half of the voters to be frustrated each on more than the half of the proposals. In the first part of our investigations, we provide answers to these two concerns. Known domains of individual votes, such as single-switch domains from Laffond and Lainé (2006), that discard the possibility of observing the Anscombe's paradox are polynomial in cardinality. To the question of whether there exist exponential improvements of such domains, we provide a positive answer in the second part of our study. Finally, to check how real life practices impact on the occurencies of the Anscombe's paradox, we mimic a parliamentary organization with a leading party, an opposition party and possible freethinkers; and provide necessary and sufficient conditions to avoid the paradox in such contexts.

RÉSUMÉ

Cette thèse est une contribution à l'étude du paradoxe d'Anscombe abordé en théorie du choix social pour la première fois par Anscombe (1976). Ce paradoxe renvoie à toute situation de vote où, de façon surprenante, il est possible qu'au terme d'une succession de votes sur un ensemble fini et fixé de propositions (lois, amendements, ...), plus de la moitié des votants soient frustrés chacun sur plus de la moitié des propositions alors que chaque décision sur chaque proposition est l'opinion d'une majorité de votants. Ce paradoxe qu'on peut observer en situation réel de vote notamment dans les parlements, les congrès ou les sénats, mérite une étude approfondie pour une meilleure compréhension des circonstances de son apparition ou pour identifier des conditions permettant de l'éviter. C'est effectivement l'object de notre thèse. En effet, plusieurs questions sur le paradoxe d'Anscombe sont restées pendantes à nos jours. D'une part, lorsque la règle de vote est une majorité qualifiée, aucune étude à notre connaissance n'a encore été faite pour vérifier l'apparition ou non de ce paradoxe. C'est aussi le cas pour toute autre règle de décision binaire distincte de toute règle majoritaire. Dans un premier temps, nous généralisons l'étude du paradoxe d'Anscombe à une règle binaire quelconque. D'autre part, lorsque les votes individuels sont issus de certains domaines tels que les domaines "single-switch" de Laffond and Lainé (2006), il n'est plus possible d'observer le paradoxe d'Anscombe. Seulement les seuls domaines ayant cette propriété dans la littérature sont de cardinalité polynomiale. Nous améliorons ce résultat, dans la deuxième partie de notre travail, en montrant qu'il existe de tels domaines de cardinalité exponentielle. Enfin, pour vérifier comment les pratiques courantes de vote impactent sur la possibilité d'observer le paradoxe d'Anscombe, nous modélisons une organisation parlementaire avec un parti majoritaire, un parti d'opposition et d'éventuellement libres-penseurs. Des conditions nécessaires et suffisantes sont alors fournies pour éviter le paradoxe d'Anscombe en supposant que la logique des partis est respectée.

INTRODUCTION

For a decision over two alternatives or any "Yes-No" voting, as it is the case in referenda or amendment processes within legislatures of democratic nations, the majority rule is a frontrunner among most often used voting rules. It simply states that between two competing options, the option that benefits from the support of more than half of the voters is the winning outcome. Although for two alternatives, May (1952) characterizes the majority rule by means of very appealing axioms, some surprising scenarios arise under majority rule theatres. One of those unexpected voting scenarios we are interested in - first noted by Anscombe (1976) and later illustrated by Gorman (1978) - is quoted in the literature as the Anscombe's paradox: given a session of several proposals, applying the majority rule on each proposal may result in outcomes with which a majority of voters are frustrated on a majority of proposals (they disagree with the majority decision); for some extensions, see Wagner (1984) ; Saari (2001) and Nurmi (1999) for a discussion and several related issues. For similar paradoxes, interested readers are referred to Ostrogorski (1902), Laffond and Lainé (2012), or Nurmi and Meskanen (2000) for some alternative views and specific analysis on majority voting paradoxes; or Gehrlein and Lepelley (2010b) for a probabilistic analysis of a panel of paradoxical voting outcomes with more than two alternatives.

Should one cares about the Anscombe's paradox? There are known surprising and undesired aspects of Anscombe's paradox that may be encountered within voting bodies where the majority rule is used for ratification of amendments or for adoption of policies. Among such aspects is the phenomenon of logrolling. It consists for voters in exchanging their votes in order to secure appropriate decisions on some proposals; see Tullock (1959) for an analysis of the majority voting as a method of allocating ressources; Tullock (1961) and Downs (1961) for a discussion; Kau and Rubin (1979) for a real-life context study; or Nurmi (2015) for recent illustrations. Another comment on possible consequences of Anscombe's paradox is due to Lagerspetz (1996a) who notes what may look like a mistake by pollsters when from polls, "a majority of the citizens agrees with every decision, and simultaneously, a majority of the citizens complains that most decisions are made against its will". In a conceptual viewpoint, Saari (2001) has shown the link between Anscombe's paradox and the Condorcet triplet. Significant frequencies of observing the Anscombe's paradox have being reported by Mbih and Valeu (2016), showing that this paradox is not an infrequent event. However this paradox can be mainly viewed as a social resentment over a series of majority decisions (a social planner concern) as compared to a social outcome inconsistency (a normative concern) such as the intransitivity of the majority rule; see Condorcet (1785) or Gehrlein and Fishburn (1976).

Many contributions have been undertaken mainly to describe some Anscombe's paradox free domains. There are two main approaches on this line of inquiry which we refer to as intra-profile-conditions and domain-conditions respectively:

Firstly, a vote profile is any collection of individual opinions that specifies the choice (Yes or No) of each voter on each proposal. The intra-profile-conditions approach consists in giving necessary or sufficient conditions on profiles at which Anscombe's paradox never occurs. For example, Wagner (1983) shows that when only vote profiles at which the majority decision is on average the opinion of at least threefourths of the voters are considered, occurrences of Anscombe's paradox are precluded; other intra-profile-conditions are given by Laffond and Lainé (2013) where the authors, using a measure of the level of unanimity among voters' opinions, provide sufficient conditions for avoiding a generalized Anscombe's paradox.

Secondly, the opinion of a voter is a list that states the choice of that voter on each possible proposal while a preference domain is a nonempty subset of individual opinions that describes all possible observable opinions. The domain-conditions approach consists in looking for preference domains that, independently of the total number of voters, do not yield any profile at which Anscombe's paradox occurs. The notion of single-switch preferences from Laffond and Lainé (2006) is certainly the most attractive and simple illustration of Anscombe's paradox free domains. The domain-conditions approach is also known in Social Choice Theory as domain restriction framework; see for example Kalai and Muller (1977) or Chatterji et al. (2013) for preference domains that overcome Arrow's impossibility result on social welfare functions (Arrow (1951); Andjiga et al. (2011); and Mossel and Tamuz (2012)).

The two research boulevards above overshadow a third one. Indeed, the Anscombe's paradox is a two-fold event which include a decision rule and a domain of individual opinions. Our first concern relays on how the decision rule impacts the possibility of observing the Anscombe's paradox. On this issue, we redefine the Anscombe's paradox following two key points: the rule and the majority. On the one hand, the Anscombe's paradox for a given binary voting rule occurs when each member of a coalition of more than the half of the voters are each frustrated on more than the half of proposals. In other hand, coalition of more than the half of the voters may not be decisive, as it is the case with simple games; see Taylor and Zwicker (1999) or Moulen and Diffo (2001b). In such contexts, a qualified Anscombe's paradox appears when the members of a winning coalition are each frustrated on more than the half of the proposals. Considering these two generalizations, we prove that exhibiting the Anscombe's paradox is not a proper downside of the majority rule, but a common flaw of almost all binary voting rules.

Now, the domain condition is simply a restriction of individual preferences. As an illustration, single-switchness refers to the existence of an ordering of proposals along which the votes of each individual may be listed from the first ranked proposal to the list-ended proposal while switching from Yes to No (or conversely) at most once. Laffond and Lainé (2006) who introduce single-switch domains observe that these domains which are polynomial in cardinality are rather drastic restriction of individual preferences. To address this issue of constructing non polynomial domains that preclude the possibility of observing the Anscombe's paradox, we adopt here a similar but different way of generating individual opinions over the set of proposals by providing a tight sufficient condition that rules out Anscombe's paradox. More precisely, we assume that: (i) for each proposal, an *arbitrary standard* - adoption ($+1$) or rejection (-1) - exists; and (ii) there exists a subset of proposals, called *unifying proposals*, such that each voter deviates from the issue-specific standards over these unifying proposals only on a limited number of issues. Our settings thus include three major parameters namely the set of all proposals, the subset of all unifying proposals and the maximum number k (called *barometer of consensus*) of unifying proposals on which a voter may deviate from common standards. The corresponding *unifying preference domain* is the set of all vote vectors that differ from the vector of common standards on at most k unifying proposals. Clearly, the existence of unifying proposals does not guarantee that no majority of voters will ever be frustrated on a majority of proposals. Our main result provides necessary and sufficient conditions on a unifying preference domain to rule out the possibility of observing Anscombe's paradox.

Finally, the intra-profile condition is simply a description of all observable vote profiles which may not be derived from a cartesian product of a given domain of individual opinions. Such an intra-profile condition is obviously implies by the organization of voters in many voting bodies such as parliaments where there are several political tendencies and each voter votes according to the ideology of the party he/she belongs to. To address this issue of measuring how real life practices impact on the occurrences of Anscombe's paradox, we mimic a parliamentary organization with a leading party, an opposition party and possible freethinkers. Moreover, we assume that only parties introduce proposals to vote on; and that the members of a given party all vote for the adoption of the proposals initiated by that party, but may have distinct opinions on other proposals. Meanwhile, independent voters are freethinkers on all proposals. We then obtain consensual voting environments; and provide necessary and sufficient conditions to avoid all occurrences of Anscombe's paradox.

A detailed presentation of what we just present above includes four chapters as follows. In Chapter 1, we introduce some basic notions about Anscombe's paradox and identify all combinations of the total number of voters and the total number of proposals for which the majority rule does not exhibit Anscombe's paradox. In Chapter 2, we explore the existence of Anscombe's paradox when the decision rule is a qualified majority or any other binary decision rule distinct from the majority rule. In Chapter 3, we construct unifying preference domains with exponential cardinality and prove a domain restriction result by providing necessary and sufficient conditions for which unifying preference domains do not exhibit Anscombe's paradox. And in Chapter 4, we construct consensual voting environments and prove an intra-profile condition result by providing necessary and sufficient conditions for which consensual voting environments do not exhibit Anscombe's paradox.

CHAPTER 1

ANSCOMBE'S PARADOX : PRELIMINARIES AND CONCERNS

In this Chapter, basic notions related to Anscombe's paradox are presented in Section 1.1, some other related paradoxes in Section 1.2 and some major contributions in analyzing this paradox in Section 1.3. Our concerns in the next Chapters are outlined in Section 1.4

1.1 Preliminaries

1.1.1 The context

Consider a set $N = \{1, 2, \dots, n\}$ of n individuals (voters, congressmen, representatives, ...) involved in a process that consists in several Yes-No votes over a set $\mathcal{M} = \{a^1, a^2, \dots, a^m\}$ of m mutually independent proposals (laws, projects, referenda, ...). Any non empty subset of voters is called a coalition. Given a proposal a^j, each voter may either approve a^j (he/she votes for the adoption of a^j); or disapprove a^j (he/she votes for the rejection of a^j). A rule is designed to derive from each possible profile of individual opinions on a^j a social decision that consists in adopting or rejecting a^j.

Definition 1.1.1.

1. *The opinion of voter i is an m-tuple $X_i = (X_i^a)_{a=a^1,a^2,\dots,a^m}$ such that $X_i^a = +1$ if i approves a or $X_i^a = -1$ if i disapproves a.*

   The vector X_i will also be called i's vector of votes.

2. Given a proposal a, the vector of votes on a is an n-tuple $X^a = (X_i^a)_{i=1,\ldots,n}$ such that $X_i^a = +1$ if individual i approves a and $X_i^a = -1$ if individual i disapproves a.

Remark 1.1.1.

- An **individual vector of votes** can be seen as a mapping that associates each proposal with -1 or $+1$. Since proposals are initially labeled, the set of all individual vectors of votes with m proposals will be identified with $\{-1;+1\}^m$, the m^{th} cartesian power of $\{-1;+1\}$.

- Similarly, when we collect all individual votes on a proposal, say a, we obtain a **vector of votes on proposal** a which is an n-tuple $X^a = (X_i^a)_{i=1,2,\ldots,n}$ with all entries in $\{-1;+1\}$ such that $X_i^a = +1$ if voter i approves proposal a while $X_i^a = -1$ if voter i disapproves proposal a.

As above, the set of all vectors of votes on a proposal with n voters will be identified with $\{-1;+1\}^n$. When each voter in N is endowed with an individual vector of votes over proposals in $\mathcal{M}$, one obtains a matrix $X = (X_i^a)_{i\in N, a\in\mathcal{M}}$ such that column i describes individual i's vector of votes (that is X_i) while row a describes the vector of votes on proposal a (that is X^a or simply X^j when $a = a^j$ for an existing labelling).

Definition 1.1.2. *A vote profile (or profile for short) is any collection $X = (X_i^a)_{i\in N, a\in\mathcal{M}}$ of n individual vectors of votes, one for each voter.*

Profiles will sometimes be represented using a table as in Example 1.1.1 below.

Example 1.1.1. *Let $\mathcal{M} = \{a^1, a^2, a^3\}$ and $N = \{1,2,3,4,5,6\}$. Below is the representation of a profile by a table:*

	X_1	X_2	X_3	X_4	X_5	X_6
a^1	-1	$+1$	$+1$	-1	-1	-1
a^2	$+1$	$+1$	$+1$	$+1$	-1	$+1$
a^3	-1	$+1$	$+1$	-1	$+1$	-1

With respect to this profile, it appears for example that

- *Voter $1's$ vector of votes is $X_1 = (-1,+1,-1)$;*

- The vector of votes on proposal a^3 is $X^{a^3} = (-1, +1, +1, -1, +1, -1)$.

For each possible profile on m proposals, a social outcome is derived provided that a binary voting rule is chosen to describe how a social decision on each proposal is obtained. We assume that a predetermined rule is applied to each proposal and is then sufficient to obtain the social outcome at all possible profiles; in this case, the social outcome at a profile consists in the collection of all social decisions associated with each of the m vectors of votes on proposals extracted from that profile.

Definition 1.1.3. *A binary voting rule is a mapping* $R : \{-1, +1\}^n \to \{-1, +1\}$ *that associates each possible vector of votes on a proposal with a social decision such that given a proposal* a *and a vector of votes* X^a *on* a,

- $R(X^a) = +1$ *means that proposal* a *is adopted (or equivalently, the social decision on* a *is adoption);*
- $R(X^a) = -1$ *means that proposal* a *is rejected (or equivalently, the social decision on* a *is rejection).*

Note that binary voting rules are designed for decisions on proposals. Hereafter, $R[X]$ stands for the collection $R[X] = (R(X^a))_{a\in\mathcal{M}}$ of social decisions given a profile X and a voting rule R.

Definition 1.1.4. *Given a binary voting rule* R*, the vector of decision for a profile* $X = (X_i^a)_{i\in N, a\in\mathcal{M}}$ *is the* m*-tuple* $R[X] = \left(R\left(X^{a^1}\right), R\left(X^{a^2}\right), \ldots, R\left(X^{a^m}\right)\right)$ *of all social decisions on the* m *proposals.*

Note that $R[X]$ is a notation since an argument of a binary voting rule is a vector of votes on a proposal; but not a profile. Before we continue, here is an example of a binary voting rule.

Definition 1.1.5. *Given a coalition* S*, the* S*-oligarchy rule is the binary voting rule denoted by* O_S *and defined by:*

$$\forall x \in \{-1, 1\}^n, \; O_S(x) = \begin{cases} +1 & \text{if } S \subset \{i \in N \;:\; x_i = +1\} \\ -1 & \text{otherwise} \end{cases}$$

According to the S-oligarchy rule, a proposal is adopted if and only if all members of S, the oligarchy, vote for its adoption. For example when $S = \{1, 5\}$; the vector of decision associated with the profile in Example 1.1.1 is $O_S(X) = (-1, -1, -1)$.

Definition 1.1.6. *A binary voting rule R is an oligarchy if $R = O_S$ for some coalition S.*

Among usual binary voting rules, some has gained a lot of attention. That is the case for the majority which is undoubtedly one of the most often used voting rules especially for referenda or for amendment processes within parliaments or other voting bodies of democratic nations. The next section is devoted to the presentation of this particular binary voting rule.

1.1.2 The majority rule

According to the majority rule, a proposal is adopted if more than half of voters vote for its adoption; otherwise it is rejected. More formally,

Definition 1.1.7. *The majority rule is the voting rule MR defined by*

$$\forall v \in \{-1, +1\}^n, \quad MR(v) = \begin{cases} +1 & \text{if } |\{i \in N : v_i = +1\}| > \frac{n}{2} \\ -1 & \text{otherwise} \end{cases}$$

Remark 1.1.2. *From the above definition,*

$$MR(v) = +1 \iff \sum_{i \in N} v_i > 0 \text{ and } MR(v) = -1 \iff \sum_{i \in N} v_i \leq 0.$$

Moreover, given a profile $X = (X_i^a)_{i \in N, a \in \mathcal{M}}$, the majority decision on X is the m-tuple $MR[X] = \left(MR\left(X^{a^1}\right), MR\left(X^{a^2}\right), \dots, MR\left(X^{a^m}\right)\right)$ of all majority decisions on the m proposals.

Hereafter, in the table of a vote profile, the last column is added to represent the social decision on each proposal.

Example 1.1.2. *Let $\mathcal{M} = \{a^1, a^2, a^3\}$ and $N = \{1, 2, 3, 4, 5, 6\}$. Consider the vote profile in Example 1.1.1.*

	X_1	X_2	X_3	X_4	X_5	X_6	$MR[X]$
a^1	-1	$+1$	$+1$	-1	-1	-1	-1
a^2	$+1$	$+1$	$+1$	$+1$	-1	$+1$	$+1$
a^3	-1	$+1$	$+1$	-1	$+1$	-1	-1

When the voting rule is the majority rule, this table shows that proposal a^2 is collectively adopted while proposals a^1 and a^3 are collectively rejected.

1.1.3 The Anscombe's paradox

Anscombe (1976) notes that, while using the majority rule, there exist some vote profiles such that there are more voters who disagree with the majority decision on a majority of proposals than voters who agree with the majority decision on a majority of proposals. Such a voting event is referred to as the Anscombe's paradox. Below is a formal definition of this paradox.

Definition 1.1.8. *Consider a vote profile X, a voter i and a proposal a.*

1. *Voter i is* **frustrated** *on proposal a if he or she disagrees with the majority decision on that proposal, that is,*

$$X_i^a \neq MR(X^a).$$

2. *Voter i is* **frustrated** *at X if the vote of i differs from the majority decision on more than one half of the proposals, that is,*

$$|\{a \in \mathcal{M} \ : \ i \text{ frustrated on a}\}| > \frac{m}{2}.$$

With seven proposals for example, a voter is frustrated if his or her vote differs from the majority decision on at least four proposals; equivalently his or her vote coincides with the majority decision on at most three proposals.

Definition 1.1.9. *Under the majority rule, any coalition S of more than one half of voters is called a majority, that is,*

$$|S| > \frac{n}{2}.$$

Remark 1.1.3. *Note that when members of a majority all share an opinion on a given proposal, this opinion is the majority decision on that proposal. Therefore all members of a majority cannot be frustrated on the same proposal.*

Definition 1.1.10. *Given a vote profile X, the* **Anscombe's paradox** *holds at X if the members of a majority are frustrated on a majority of proposals, that is,*

$$|\{i \in N \ : \ i \text{ frustrated}\}| > \frac{n}{2}.$$

We also say that X exhibits Anscombe's paradox.

In the table of a vote profile we sometime add a new row. In this row, see example 1.1.3 below, the entry is "yes" if the corresponding voter is frustrated and "no" otherwise. The proportion of frustrated voters appears in the last cell.

Example 1.1.3. *The following vote profile exhibits the Anscombe's paradox.*

	X_1	X_2	X_3	X_4	X_5	X_6	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1	-1
a^2	$+1$	$+1$	-1	$+1$	-1	$+1$	$+1$
a^3	-1	$+1$	-1	$+1$	$+1$	-1	-1
	no	*yes*	*yes*	*yes*	*yes*	*no*	$4/6$

In fact, four voters (2, 3, 4 and 5) out of six are each frustrated on two proposals out of tree. This shows that a majority of voters are frustrated on a majority of proposals. At this vote profile, Anscombe's paradox occurs.

Definition 1.1.11. *Given a vote profile X, the support of X is the set denoted by $Supp(X)$ and made up of all vectors of votes that appear in X, that is,*

$$Supp(X) = \{X_i \ : \ i \in N\}.$$

Given $v \in Supp(X)$, denote by N_v the set of all voters who report v in X, that is,

$$N_v = \{i \in N \ : \ X_i = v\}.$$

Remark 1.1.4. *Given $v \in Supp(X)$, if a voter in N_v is frustrated on a proposal, so are all voters in N_v.*

Below is provided a threshold on the cardinality of the support of a vote profile up to which the Anscombe's paradox does not appear. The following lemma will be useful.

Lemma 1.1.1. *Given a vote profile X, $x, y \in Supp(X)$, $i_1 \in N_x$ and $i_2 \in N_y$.*

If i_1 and i_2 are both frustrated on a majority of proposals, then all voters in $N_x \cup N_y$ are simultaneously frustrated on some proposals.

Proof.

Consider a vote profile X, $x, y \in Supp(X)$, $i_1 \in N_x$ and $i_2 \in N_y$. Suppose that i_1 and i_2 are both frustrated on a majority of proposals. Then the set of proposals at which i_1 and i_2 are frustrated overlap. Let a be such a proposal. Then i_1 and i_2 are both frustrated on a. By Remark 1.1.4, all voters in $N_x \cup N_y$ are simultaneously frustrated on proposal a. ■

Theorem 1.1.

Let X be a vote profile.

If $|Supp(X)| \leq 3$ then X does not exhibit the Anscombe's paradox.

Proof.

Let X be a vote profile such that $|Supp(X)| \leq 3$.

When $|Supp(X)| = 1$, all voters unanimously agree on each proposal and then also agree with the majority decision on each proposal. Obviously, X does not exhibit the Anscombe's paradox whenever $|Supp(X)| = 1$.

Now suppose that $|Supp(X)| \in \{2, 3\}$. To show that X does not exhibit the Anscombe's paradox, suppose on the contrary that the Anscombe's paradox holds at X. By definition, there exists a majority S of voters who disagree with the majority decision on more than $\frac{m}{2}$ proposals.

First assume that $|Supp(X)| = 2$ and pose $Supp(X) = \{x, y\}$ with $x, y \in \{-1, +1\}^m$. Note that $N = N_x \cup N_y$. If $S \cap N_x \neq \emptyset$ and $S \cap N_y \neq \emptyset$ then by Lemma 1.1.1, all voters in $N = N_x \cup N_y$ are frustrated on some proposals; which is impossible. Therefore $S \cap N_x = \emptyset$ or $S \cap N_y = \emptyset$. But if $S \cap N_x = \emptyset$ then $S \subset N_y$ and therefore all voters in S are frustrated on the same proposals; which is impossible since S is a majority. In both cases, an impossibility occurs. Then X does not exhibit the Anscombe's paradox for $|Supp(X)| = 2$.

Now assume that $|Supp(X)| = 3$ and pose $Supp(X) = \{x, y, z\}$ with $x, y, z \in \{-1, +1\}^m$. Note that

$$N = N_x \cup N_y \cup N_z.$$

Since S is a majority, $S \backslash N_v \neq \emptyset$ for each $v \in Supp(X)$; otherwise all voters in S would be frustrated on the same proposals as mentioned in Remark 1.1.4. Moreover suppose that $S \subset N_v \cup N_{v'}$ for some $v, v' \in Supp(X)$. Then $S \cap N_v \neq \emptyset$ and $S \cap N_{v'} \neq \emptyset$. By Lemma 1.1.1 all voters in $N_v \cap N_{v'}$ are simultaneously frustrated on some proposals; so are voters in S. This is impossible by Remark 1.1.4. Therefore $S \backslash (N_v \cap N_{v'}) \neq \emptyset$

for all $v, v' \in Supp(X)$. Equivalently, $S \cap N_v \neq \emptyset$ for all $v \in \{x, y, z\}$. Without loss of generality assume that $|N_x| \geq |N_y| \geq |N_z|$. Then $|N_x| + |N_y| > \frac{n}{2}$. Since $S \backslash N_x \neq \emptyset$ and $S \backslash N_y \neq \emptyset$, all voters in $N_x \cup N_y$ are simultaneously frustrated on some proposals. This is impossible by Remark 1.1.4. Each possible cases concurs to an impossibility. Thus X does not exhibit the Anscombe's paradox for $|Supp(X)| = 3$. ■

The condition $|Supp(X)| \leq 3$ is sufficient to guarantee that X does not exhibit the Anscombe's paradox. But when $|Supp(X)| \geq 4$, things are somewhat scarttered.

Proposition 1.1.1. *Assume that $m = 4$ and consider a vote profile X. Then X does not exhibit the Anscombe's paradox for each of the following conditions:*

a) $|Supp(X)| = 4$;

b) $n \leq 5$.

Proof.
Suppose that $m = 4$ and let X be a vote profile that meets condition a) or condition b). To prove that X does not exhibit the Anscombe's paradox suppose the contrary. Then there exists a majority coalition S such that each member of S is frustrated on at least three proposals. Since $m = 4$, the set of proposals is $\mathcal{M} = \{a^1, a^2, a^3, a^4\}$. By Remark 1.1.3, all voters in S are not frustrated on a^1. Thus there exists a voter $i_1 \in S$ who is not frustrated on a^1. Voter $i_1 \in S$ is frustrated on at least three proposals. Therefore i_1 is frustrated on a^2, a^3 and a^4. By Remark 1.1.4, all voter in $N_{X_{i_1}}$ are also frustrated on a^2, a^3 and a^4; but not on a^1. Note that this shows that voters in S who are not frustrated on a^1 report the same vector of votes X_{i_1}. Similarly for $a^j \in \{a^2, a^3, a^4\}$ there exists some voter i_j such that i_j is frustrated on all proposals except a^j. So are all voter in $N_{X_{i_j}}$. It follows that for $j = 1, 2, 3, 4$, only voters in $N_{X_{i_j}}$ among voters in S agree with the majority decision on a^j. Moreover,

$$S = N_{X_{i_1}} \cup N_{X_{i_2}} \cup N_{X_{i_3}} \cup N_{X_{i_4}} \tag{1.1}$$

i) Suppose that $|Supp(X)| = 4$. Then $Supp(X) = \{X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}\}$. Therefore $S = N$ by (1.1). This implies that for $j = 1, 2, 3, 4$, $|N_{X_{i_j}}| \geq \frac{n}{2}$ since only voters in $N_{X_{i_j}}$ agree with the majority decision on a^j. Recalling that $N_{X_{i_j}}$ for $j = 1, 2, 3, 4$ are disjoint subsets, it follows that $N_{X_{i_1}} \cup N_{X_{i_2}} = N$ and thus $N_{X_{i_3}} \cup N_{X_{i_4}} = \emptyset$. This is a contradiction since $N_{X_{i_3}} \neq \emptyset$.

ii) Suppose that $n \leq 5$.

Note that for $n \leq 5$, i_2, i_3 and i_4 constitute a majority coalition and each disagrees with the majority decision on a^1. This is a contradiction by Remark 1.1.3. ■

Proposition 1.1.2. *Given m the total number of proposals and n the total number of voters,*

a) If $n \geq 4$ and ($m = 3$ or $m \geq 5$), then there exists a vote profile X such that $|Supp(X)| = 4$ and X exhibits the Anscombe's paradox.

b) If $n \geq 6$ and $m = 4$, then there exists a vote profile X such that $|Supp(X)| = 5$ and X exhibits the Anscombe's paradox.

Proof.

a) Suppose that $n \geq 4$ and ($m = 3$ or $m \geq 5$). Then there exist two integers p and t such that

$$m = 3p + t$$

with

$$p \geq 1,\ t \in \{0, 1, 2\} \text{ and } (p \neq 1 \text{ or } t \neq 1).$$

Pose $\lambda = \left\lfloor \frac{n}{2} \right\rfloor$ and $\delta = \left\lfloor \frac{n-1}{2} \right\rfloor$. Note that $n = \lambda + \delta + 1$. Let x, y, z and u be four vectors of votes defined as follows: for each proposal a^j with $j = 3k + s$, $s \in \{0, 1, 2, 3\}$:

	x_{3k+s}	y_{3k+s}	z_{3k+s}	u_{3k+s}
$s = 1$	-1	$+1$	$+1$	-1
$s = 2$	$+1$	-1	$+1$	-1
$s = 3$	$+1$	$+1$	-1	-1

Consider a profile X obtained from a partition of N into four subsets N_x, N_y, N_z and N_u such that

$$Supp(X) = \{x, y, z, u\},\ |N_x| = \lambda - 1,\ |N_y| = 1,\ |N_z| = 1 \text{ and } |N_u| = \delta.$$

Profile X is as follows:

	X_{N_x}	X_{N_y}	X_{N_z}	X_{N_u}	$MR[X]$
a^1	-1	$+1$	$+1$	-1	-1
a^2	$+1$	-1	$+1$	-1	-1
a^3	$+1$	$+1$	-1	-1	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{3k-2}	-1	$+1$	$+1$	-1	-1
a^{3k-1}	$+1$	-1	$+1$	-1	-1
a^{3k}	$+1$	$+1$	-1	-1	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Pose $S = N_x \cup N_y \cup N_z$. Note that $|S| = \lambda + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Thus S is a majority coalition. Also note that each voter in S is frustrated on exactly two proposals given three consecutive proposals a^{3k+1}, a^{3k+2} and a^{3k+3}. Therefore for $t = 0$ each voter in S is frustrated on

$$2p = \frac{2}{3}m > \frac{m}{2}$$

proposals. Therefore X exhibits the Anscombe's paradox. Moreover $t = 1$ or $t = 2$, each voter in S is frustrated on

$$2p + t - 1 = \frac{2}{3}m + \frac{t}{3} - 1$$

proposals. Since $t \geq 1$ and $m \neq 4$ then $m \geq 5$ and

$$\frac{2}{3}m + \frac{t}{3} - 1 \geq \frac{2}{3}m - \frac{2}{3} = \frac{m}{2} + \frac{m-4}{6} > \frac{m}{2}.$$

This proves that each member of S is frustrated on more than $\frac{m}{2}$ proposals. Hence X exhibits the Anscombe's paradox.

b) Suppose that $m = 4$ and $n \geq 6$. Pose $\lambda = \left\lfloor \frac{n}{2} \right\rfloor$ and $\delta = \left\lfloor \frac{n-1}{2} \right\rfloor$. Note that $n = \lambda + \delta + 1$. Moreover $\lambda \geq 3$ since $n \geq 6$. Consider the five vectors of votes x, y, z, u and v that appear in the table below and consider a partition of N with respect to the set of voters reporting each of these vectors of votes in such a way that

$$|N_x| = \lambda - 2,\ |N_y| = 1,\ |N_z| = 1,\ |N_u| = 1,\ |N_v| = \delta$$

The corresponding vote profile X satisfies $Supp(X) = 5$ and is as follows:

	X_{N_x}	X_{N_y}	X_{N_z}	X_{N_u}	X_{N_v}	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1
a^2	$+1$	-1	$+1$	$+1$	-1	-1
a^3	$+1$	$+1$	-1	$+1$	-1	-1
a^4	$+1$	$+1$	$+1$	-1	-1	-1

Let $S = N_x \cup N_y \cup N_z \cup N_u$. Clearly, $|S| = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and each voter in S is frustrated on 3 proposals. Therefore X exhibits the Anscombe's paradox. ■

Theorem 1.2.

Given $n \geq 2$ and $m \geq 3$, the majority rule does not exhibit the Anscombe's paradox if and only if

$$(n \leq 3) \text{ or } (m = 4 \text{ and } n \leq 5).$$

The following table summarizes the conditions of Theorem 1.2 by identifying all combinations of n and m for which the majority rule does not exhibit the Anscombe's paradox.

m \ n	2	3	4	5	$n > 5$
3	No	No	Yes	Yes	Yes
4	No	No	No	No	Yes
$m \geq 5$	No	No	Yes	Yes	Yes

No : no Anscombe's paradox

Yes : Anscombe's paradox appears

Proof.

Given $n \geq 2$ and $m \geq 3$, suppose that the majority rule does not exhibit the Anscombe's paradox. To prove that $(n \leq 3)$ or $(m = 4$ and $n \leq 5)$, assume the contrary. Then $(n \geq 4)$ and $(m \neq 4$ or $n \geq 6)$. There are two possible cases. Either $(n \geq 4$ and $m \neq 4)$ or $n \geq 6$. In both cases, a contradiction holds from Proposition 1.1.2.

Conversely suppose that $(n \leq 3)$ or $(m = 4$ and $n \leq 5)$. First suppose that $n \leq 3$. Then for all possible vote profile X, $|Supp(X)| \leq 3$. By Theorem 1.1 the majority rule does not exhibit the Anscombe's paradox. Now suppose that ($m = 4$ and $n \leq 5$). By Proposition 1.1.1 the majority rule does not exhibit the Anscombe's paradox. ■

1.2 Related voting paradoxes

There is an abundant literature on voting paradoxes. A nice round-up with intuitive comments is provided by Nurmi (1999); or Chatterji et al. (2013) for detailed analysis of their respective levels of quantitative significance in terms of propagability measurement of their occurrences; see also Gehrlein and Lepelley (2010a) for other technical investigation. In this section, we only focus on voting paradoxes that deals with voting on several independent Yes-No proposals as in the case of Anscombe's paradox.

1.2.1 Ostrogorski paradox

Assume that each proposal is an issue (educational topics, health care program, security management, ...) in such a way that the set of all proposals can be seen as a (political) platform. Also assume that proposals are independent issues in such that each voter only cares about the total number of issues he or she accepts on a platform.

Definition 1.2.1. *Consider a platform $\mathcal{M}$, a vote profile X and a voter i.*

i) Voter i (globally) approves of the platform if the number of issues he or she accepts is greater than the number of issues he or she rejects; that is

$$\sum_{a\in\mathcal{M}} X_i^a > 0.$$

Voter i (globally) disapproves of the platform otherwise.

ii) To adopt the platform, there are two possible approaches:

a) **A single vote straightforward (SVS) on the platform at whole**:

The platform is collectively adopted if a majority of voters adopt it; that is

$$\left|\left\{i \in N \ : \ \sum_{a\in\mathcal{M}} X_i^a > 0\right\}\right| > \frac{n}{2}.$$

The platform is rejected otherwise.

b) **A series of independent votes (SIV), one proposal at a time**:

The platform is collectively adopted if a majority of proposals on the platform are adopted; that is

$$\sum_{a\in\mathcal{M}} MR_a(X) > 0.$$

The platform is rejected otherwise.

While voting on a platform, the two processes (SVS) and (SIV) described above may yield different outcome under two distincts scenarii referred to as the Ostrogorski (1902) below.

Definition 1.2.2. *The Ostrogorski paradox occurs at some profile of individual vectors of votes if:*

i) a majority of voters approve of the platform under (SVS) while a majority of issues are rejected by a majority of voters under (SIV); that is

$$\left|\left\{i \in N \ : \ \sum_{a\in\mathcal{M}} X_i^a > 0\right\}\right| > \frac{n}{2} \text{ and } \sum_{a\in\mathcal{M}} MR_a(X) < 0.$$

or

ii) a majority of voters disapprove of the platform under (SVS) while a majority of issues are adopted by a majority of voters under (SIV); that is

$$\left|\left\{i \in N \ : \ \sum_{a\in\mathcal{M}} X_i^a < 0\right\}\right| > \frac{n}{2} \text{ and } \sum_{a\in\mathcal{M}} MR_a(X) > 0.$$

The Ostrogorski's paradox then refers to voting situations for which the two approaches (SVS) and (SIV) while voting on a platform with several proposals diverge, the outcome of each process being supported by a majority of voters.

Example 1.2.1. *Let $N = \{1,2,3,4,5\}$, $\mathcal{M} = \{a^1, a^2, a^3\}$ and consider the vote profile that follows:*

	X_1	X_2	X_3	X_4	X_5	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	$+1$
a^2	$+1$	-1	$+1$	-1	-1	-1
a^3	$+1$	$+1$	-1	-1	-1	-1
	$+1$	$+1$	$+1$	-1	-1	

Note that, the last line in this table indicates whether a voter approves of the platform or not. More precisely, $(+1)$ in the last line of a column means that the corresponding voter adopts the platform while (-1) stands in case of a disapprobation. For the current vote profile,

- in the (SVS) approach, voters 1, 2 and 3 vote for the adoption of the platform while voters 4 and 5 vote for its rejection. Thus the platform is adopted in this approach with a majority of three voters out of five.
- in the (SIV) approach, proposal a^1 is adopted while proposal a^2 and a^3 are both rejected. Thus a majority of two proposals out of three are rejected on the platform; meaning that the platform is now rejected.

Clearly, the Ostrogorski's paradox occur on this vote profile.

Remark 1.2.1. *Although Ostrogorski's paradox and Anscombe's paradox are both unpleasant voting circumstances, they are rather distinct voting realities under the majority rule for votes over platforms. Indeed :*

- *in Example 1.1.3, the Anscombe's paradox holds; but the vote profile does not exhibit Ostrogorski's paradox. In fact,*
  - *under the (SVS) approach, voters 1, 3, 5 and 6 disapprove of the platform which is then rejected;*
  - *under the (SIV) approach, the platform is also rejected since proposals a^1 and a^3 are both rejected.*

  Therefore the two processes concur to the same social outcome: rejecting the platform. Clearly this is not an instance of Ostrogorski's paradox.
- *in Example 1.2.1, the vote profile X exhibits the Ostrogorski's paradox but not Anscombe's paradox; since only voter 1 is frustrated on a majority of proposals.*
- *Nevertheless, there exist some vote profiles on which both Anscombe's paradox*

and Ostrogorski's paradox occur; that the case of the following vote profile:

	X_1	X_2	X_3	X_4	X_5	$MR[X]$
a^1	-1	-1	$+1$	$+1$	$+1$	$+1$
a^2	-1	$+1$	-1	$+1$	$+1$	$+1$
a^3	$+1$	-1	-1	$+1$	$+1$	$+1$
Accep. or no	-1	-1	-1	$+1$	$+1$	
Frust. or no	*yes*	*yes*	*yes*	*no*	*no*	$3/5$

In the (SVS) approach, voters 1, 2 and 3 vote for the rejection of the platform while voters 4 and 5 vote for its adoption. Thus the platform is rejected in this approach with a majority of three voters out of five. In the (SIV) approach, all the three proposals a^1 a^2 and a^3 are adopted. Thus the platform is adopted. Hence the Ostrogorski's paradox occurs.

Moreover, for this vote profile, a majority of voters (say 1, 2 and 3) are frustrated; thus the Anscombe's paradox holds.

1.2.2 Cyclical majority decisions

An alternative way for a voting committee to decide on a platform is as follows:

- A default decision or a statu quo, say $w = (w_1, w_2, \dots, w_m)$, is considered and provides a decision w_j on each proposal a^j;
- if no amendment of w is proposed by a majority of voters, then w is the final decision on the platform. In case an amendment w' of w is proposed by a majority of voters, then w' becomes the new statu quo and the process continues.

In this process, each voter i is endowed with a preference relation $\succ_i$ over possible collective decisions defined as follows:

Definition 1.2.3. *Let X be a vote profile and $x, y \in \{-1, +1\}^m$ be two vectors of votes. An individual i prefers x to y and we write $x \succ_i y$ if the total number of proposals on which i is in agreement with x is greater than the total number of proposals on which i is in agreement with y; more formally,*

$$x \succ_i y \text{ if } |\{a \in \mathcal{M},\ X_i^a = x^a\}| > |\{a \in \mathcal{M},\ X_i^a = y^a\}|.$$

An amendment w' of a statu quo w by a majority S of voters occurs when each voter from S prefers w' to w. This consideration introduces a majority relation $\succ_{Maj}$ over possible vectors of decisions on the platform as follows:

Definition 1.2.4. *Let X be a vote profile and $x, y \in \{-1, +1\}^m$ be two vectors of votes. Vector x defeats y if the total number of individuals who prefer x to y is greater than the total number of voters who prefer y to x; that is*

$$x \succ_{Maj} y \iff |\{i \in N,\ x \succ_i y\}| > |\{i \in N,\ y \succ_i x\}|.$$

Remark 1.2.2. *Note that individual preference relations over all possible vectors of decisions on a platform are acyclic and transitive binary relations. Nevertheless, as shown in the next example, the majority relation obtained by aggregating these acyclic and transitive binary relations may result in a cyclical and intransitive binary relation.*

Example 1.2.2. *Consider the following vote profile*

	X_1	X_2	X_3	X_4	X_5
a^1	$+1$	-1	-1	$+1$	$+1$
a^2	-1	$+1$	-1	$+1$	$+1$
a^3	-1	-1	$+1$	$+1$	$+1$

Possible agreements of individual opinions with the eight possible vector of decision are presented below with the following disposals:

- *Following Lagerspetz (1996b), new lines are inserted, each associated with a vector of decisions;*
- *Given a voter i and a vector of decision $d = (d^1, d^2, d^3)$, the cell corresponding to line d and column X_i gives to total number of agreements between X_i and d; that is:*

$$|\{a^j \in \mathcal{M},\ X_i^j = d^j\}|.$$

Vector d of decisions	X_1	X_2	X_3	X_4	X_5	$MR[X]$
	$+1$	-1	-1	$+1$	$+1$	$+1$
	-1	$+1$	-1	$+1$	$+1$	$+1$
	-1	-1	$+1$	$+1$	$+1$	$+1$
	Agreement of X_i with d					
$(+1,+1,+1)$	1	1	1	3	3	
$(+1,+1,-1)$	2	2	0	2	2	
$(+1,-1,+1)$	2	0	2	2	2	
$(-1,+1,+1)$	0	2	2	2	2	
$(+1,-1,-1)$	3	1	1	1	1	
$(-1,+1,-1)$	1	3	1	1	1	
$(-1,-1,+1)$	1	1	3	1	1	
$(-1,-1,-1)$	2	2	2	0	0	

It follows from this table that $(+1,+1,+1) \succ_{Maj} (-1,-1,+1) \succ_{Maj} (-1,-1,-1) \succ_{Maj} (+1,+1,+1)$. Then the majority relation $\succ_{Maj}$ is acyclic and not transitive.

Definition 1.2.5. Let $x \in \{-1;+1\}^m$ be a vector of votes, the opposite of x is a vector of vote denote by $-x$ such that :

$$\forall a \in \mathcal{M},\ x^a = +1 \iff -x^a = -1.$$

Laffond and Lainé (2013) provide a bridge between Anscombe's paradox and the majority relation.

Proposition 1.2.1 (Laffond and Lainé (2013)).

The Anscombe's paradox holds at a vote profile X if and only if $-MR[X] \succ_{Maj} MR[X]$.

Proposition 1.2.2. Assume that each majority decision at a vote profile X is supported by a majority coalition.

If X exhibits Anscombe's paradox then the majority relation is cyclical at X. The converse is false.

Proof.

Consider a vote profile X. Assume that each majority decision at X is supported by a majority coalition; that is for each $a^j \in \mathcal{M}$, the coalition

$$S_j = \{i \in N \ : \ X_i^j = MR(X^j)\}$$

contains more than the half of voters. Suppose that X exhibits Anscombe's paradox. We prove that the majority relation at X is necessary cyclical. To see this, note that since Anscombe's paradox occurs at X, there exists a majority coalition S_0 such that each member of S_0 prefers $-MR[X]$ to $MR[X]$; that is $-MR[X] \succ_{Maj(X)} MR[X]$. Now, let ${}^0Y = -X$ and for $1 \leq k \leq m$, denote by ${}^kY$ the vote profile at which each voter votes as in X on each proposal a^j with $1 \leq j \leq k$ and votes as in $-X$ on each proposal a^j with $k < j \leq m$; that is, for all $i \in N$,

$${}^k\mathrm{Y}_i^j = \begin{cases} X_i^j & \text{if } 1 \leq j \leq k, \\ -X_i^j & \text{otherwise.} \end{cases}$$

Note that ${}^{k+1}Y$ and ${}^kY$ differ only on proposal a^k. Since S_{k+1} is a majority coalition, $MR({}^kY^{k+1}) = -MR(X^{k+1})$ and $MR({}^{k+1}Y^{k+1}) = MR(X^{k+1})$. Therefore voters in S_{k+1} prefer $MR[{}^{k+1}Y]$ to $MR[{}^kY]$; that is $MR[{}^{k+1}Y] \succ_{Maj(X)} MR[{}^kY]$. It then follows that

$$MR[{}^mY] \succ_{Maj(X)} MR[{}^{m-1}Y] \succ_{Maj(X)} \cdots \succ_{Maj(X)} MR[{}^1Y] \succ_{Maj(X)} MR[-X] \succ_{Maj(X)} MR[X].$$

Since ${}^mY = X$, it follows that the majority relation at X is cyclical.

To see that the converse is false, let $\mathcal{M} = \{a^1, a^2, \ldots, a^7\}$, $N = \{1, 2, 3, 4, 5\}$ and X the vote profile define as

	X_1	X_2	X_3	X_4	X_5	$MR[X]$
a^1	$+1$	-1	-1	$+1$	$+1$	$+1$
a^2	-1	$+1$	-1	$+1$	$+1$	$+1$
a^3	-1	-1	$+1$	$+1$	$+1$	$+1$
a^4	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^5	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^6	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^7	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
	no	*no*	*no*	*no*	*no*	0/5

It follows from this table that Anscombe's paradox does not hold at this profile. However it can be easily checked that $(+1, +1, +1, +1, +1, +1, +1) \succ_{Maj(X)}$

$(-1,-1,+1,+1,+1,+1,+1) \succ_{Maj(X)} (-1,-1,-1,+1+1,+1,+1) \succ_{Maj(X)}$ $(+1,+1,+1,+1,+1,+1,+1)$. Therefore the majority relation $\succ_{Maj}$ is cyclical. ■

1.2.3 Log-rolling

A log-rolling practice occurs when frustrated members in a coalition of voters strategically exchange their votes to support each other while voting on proposals in such a way that each member of the coalition is no more frustrated.

To formalize this notion and given a vote profile X, X_S is the collection of vectors of votes for all members of a coalition S while X_{-S} is the collection of vectors of votes for all members of $N \backslash S$; that is

$$X_S = (X_i)_{i \in S} \text{ and } X_{-S} = (X_i)_{i \in N \backslash S}.$$

Moreover, $f_i(X, Y_S)$ denotes the total number of proposals on which a voter i is frustrated when at X, all members of S vote according to Y_S while all members of $N \backslash S$ vote according to X_{-S}.

Definition 1.2.6. *Let X be a vote profile. The log-rolling takes place at X if for some coalition $S \in 2^N$ and for some vote profile Y_S for voters in S, the following holds*

$$f_i(X, Y_S) < \frac{m}{2} < f_i(X, X_S) \quad \forall i \in S.$$

Example 1.2.3. *Consider the following vote profile X on which voters in coalition $S = \{2,3,4,5\}$ are frustrated.*

	X_1	X_2	X_3	X_4	X_5	X_6	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1	-1
a^2	$+1$	$+1$	-1	$+1$	-1	$+1$	$+1$
a^3	-1	$+1$	-1	$+1$	$+1$	-1	-1
	no	*yes*	*yes*	*yes*	*yes*	*no*	4/6

Now, suppose that coordination and binding agreements are feasible in such a way that voter 5 shifts her vote on proposal a^1 (from rejection to adoption), voters 2 and 4 turn their votes on proposal a^2 to rejection while voter 3 accepts to vote for the adoption of proposal a^3. Then the new profile is as follows:

	X_1	Y_2	Y_3	Y_4	Y_5	X_6	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^2	$+1$	-1	-1	-1	-1	$+1$	-1
a^3	-1	$+1$	$+1$	$+1$	$+1$	-1	$+1$
	yes	*no*	*no*	*no*	*no*	*yes*	

By so doing, members of the coalition S come out better of since none of them is no more frustrated with respect to the new vector of decisions. Clearly, this is an occurrence of log-rolling. Also note that the initial vote profile exhibits Anscombe's paradox while after log-rolling no majority of voters disagree with the majority decision on a majority of proposals.

Proposition 1.2.3. *Anscombe's paradox implies log-rolling; but the converse is false.*

Proof.

Let X be a vote profile on which the Anscombe's paradox holds. We prove that log-rolling also occurs at X. In fact, since Anscombe's paradox holds at X, there exists a coalition S of more than half of the voters each frustrated on X. Let Y_S be a vote profile such that for all $i \in S$, $Y_i = -MR[X]$. Since S is a majority, then $MR[(X_{-S}, Y_S)] = -MR[X]$. By assumption, the Ancombe's paradox holds at X. Thus by proposition 1.2.1, we have $\forall i \in S$, $-MR[X] \succ_i MR[X]$. Therefore X is vulnerable to log-rolling.

Conversely, there exists some vote profile at which log-rolling may occur without Anscombe's paradox. To see this, consider the following vote profile:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1	$+1$	-1	$+1$	$+1$
a^2	-1	-1	$+1$	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$
a^3	$+1$	$+1$	-1	$+1$	$+1$	-1	-1	$+1$	-1	$+1$
a^4	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	-1	-1	$+1$
a^5	$+1$	$+1$	-1	$+1$	-1	$+1$	$+1$	-1	-1	$+1$
	no	*no*	*no*	*no*	*no*	*yes*	*yes*	*yes*	*yes*	$4/9$

Clearly, this vote profile does not exhibit Anscombe's paradox. However, voters in $S = \{6, 7, 8, 9\}$ now coordinate their votes to vote each against all proposals. The new vote profile - say Y - is as follows:

	X_1	X_2	X_3	X_4	X_5	Y_6	Y_7	Y_8	Y_9	$MR[(X_{-S}, Y)]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1	-1
a^2	-1	-1	$+1$	$+1$	$+1$	-1	-1	-1	-1	-1
a^3	$+1$	$+1$	-1	$+1$	$+1$	-1	-1	-1	-1	-1
a^4	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1	-1	-1
a^5	$+1$	$+1$	-1	$+1$	-1	-1	-1	-1	-1	-1
	yes	*yes*	*yes*	*yes*	*yes*	*no*	*no*	*no*	*no*	$5/9$

Note that each voter $i \in S$ prefers the new outcome $MR[Y]$ with three agreements with X_i to the initial outcome $MR[X]$ with only two agreements with X_i. Thus the vote profile X exhibits log-rolling but not Anscombe's paradox. ■

1.3 Anscombe's paradox in the literature

Many investigations have been undertaken to explain how to avoid Anscombe's paradox. The two main approaches that emerge include the intra-profile-conditions and the domain-conditions. On the one hand, the domain-conditions advocated by Laffond and Lainé (2006) consists in finding a subset D of the set $\{-1, +1\}^m$ of all possible vectors of votes such that the Anscombe's paradox never holds whenever each voter picks up his/her vector of votes from D. This is in fact a wellknown concern in Social Choice theory; see also Dietrich and List (2010). On the other hand, the intra-profil-conditions consists in giving necessary or sufficient conditions to identify vote profiles at which the Anscombe's paradox never holds; see Wagner (1983) and Wagner (1984). We present in the next sections some contributions to these two lines of inquiry.

1.3.1 Single switch preference

The notion of single switch preferences was introduced by Laffond and Lainé (2006) and appears to be the most famous illustration of domain-conditions on Anscombe's paradox.

Definition 1.3.1. *A preference domain with m proposals is a nonempty subset D of $\{-1; +1\}^m$ that consists of all observation vectors of votes.*

The notion of preference domain captures the possibility for voters to share some commun values that have the effect to rule out some vectors of votes that are not compatible with those values.

Definition 1.3.2. *Given a preference domain D, a vector of votes v is admissible if $v \in D$; and a vote profile X is admissible if each individual vector of votes X_i is admissible; that is*

$$\forall i \in N, \quad X_i \in D.$$

To avoid confusion (if any), we also said that the vote profile X is D-admissible.

A preference domain D then has the effect of restricting the set of vote profiles from $(\{-1;+1\}^m)^n$ to D^n and $Supp(X) \subseteq D$.

Definition 1.3.3.

1. *A vector of votes $x = (x^a)_{1 \leq a \leq m} \in \{-1;+1\}^m$ is single-switch if there exists $a^0 \in \{1, 2, \ldots, m\}$ such that*

$$(1 \leq a \leq a^0 \iff x^a = -1) \text{ or } (1 \leq a \leq a^0 \iff x^a = 1).$$

2. *A preference domain D is single-switch if each of its vectors of votes is single-switch.*

Example 1.3.1. *The following vote profile is single-switch.*

	X_1	X_2	X_3	X_4	X_5
a^1	$+1$	$+1$	-1	$+1$	-1
a^2	$+1$	$+1$	-1	-1	$+1$
a^3	$+1$	-1	$+1$	-1	$+1$
a^4	$+1$	-1	$+1$	-1	$+1$

The switch is a^1 for X_1; a^2 for both X_2 and X_3; a^1 for both X_4 and X^5.

An interesting issue consists in checking whether a given preference domain discards all possibilities of observing the Anscombe's paradox. More formally,

Definition 1.3.4. *A preference domain D is Anscombe's paradox free if for all $n \geq 3$, there is no admissible vote profile $X \in D^n$ such that a majority of voters disagree with the social decision on a majority of proposals.*

The following result holds.

Theorem 3.3 (Laffond and Lainé (2006)).

All single-switch domains are Anscombe's paradox free. ♣

Definition 1.3.5. *A preference domain D is symmetric if for all vector of votes x in D, its opposite $-x$ is in D.*

Proposition 1.3.1. *The domain of all single-switch vectors of votes with m proposals is symmetric and its cardinality is $2m$.*

Remark 1.3.1. *With m issues, we have 2^m possible vectors of votes. However the maximal cardinality of a single-switch domain is $2m$. A single-switch domain then represents a severe restriction of individual preferences. This is the major critic on single-switch domains.*

1.3.2 Prevailing majorities and the Anscombe's (α, β, γ)-paradox

A simple way of extending the majority principle amounts to require a proportion of more than α favorable vote for a proposal to be adopted; one then obtains the α-majority rule defined below.

Definition 1.3.6. Let $\alpha \in \left]\frac{1}{2}; 1\right[$.

The α-majority rule is the mapping MR^α from $\{-1,1\}^n$ to $\{-1,1\}$ defined by:

$$\forall x \in \{-1,1\}^n,\ MR^\alpha(x) = \begin{cases} +1 & \text{if } |\{i \ : \ x_i = +1\}| > \alpha n, \\ -1 & \text{otherwise.} \end{cases}$$

The α-majority decision over a vote profile X is the collection denoted by $MR^\alpha[X] = (MR^\alpha(X^a))_{a\in\mathcal{M}}$ that assigns to each proposal a its majority decision derived from the collection X^a of individual votes over a.

Intuitively, according to the α-majority rule, a proposal is adopted if the total number of individuals voting for its adoption is greater than α of the voters.

Definition 1.3.7. Let $X \in D^n$ and $\alpha, \beta, \gamma \in \left]\frac{1}{2}; 1\right[$.

1. A voter i is (α, β)**-frustrated** on X if the vectors of votes of i differ from the α-majority decision at X on more than a proportion β of the issues. i.e.,

$$|\{a \in \mathcal{M} \ : \ X_i^a \neq MR^\alpha(X^a)\}| > \beta m.$$

2. The **Anscombe's** (α, β, γ)**-paradox** holds at X if more than a proportion γ of the voters are (α, β)-frustrated. i.e.

$$|\{i \in N \ : \ i \ \alpha, \beta\text{-frustrated}\}| > \gamma n.$$

Example 1.3.2. Let $\alpha = \frac{2}{3}$, $\beta = \frac{1}{2}$ and $\gamma = \frac{3}{4}$. Consider the following vote profile.

	X_1	X_2	X_3	X_4	X_5	$MR^\alpha[X]$
x^1	$+1$	$+1$	-1	$+1$	$+1$	$+1$
x^2	-1	-1	-1	$+1$	$+1$	-1
x^3	-1	$+1$	$+1$	$+1$	-1	-1
x^4	-1	$+1$	$+1$	-1	$+1$	-1
x^5	-1	$+1$	-1	$+1$	$+1$	-1
	no	*yes*	*yes*	*yes*	*yes*	4/5

Voters 2, 3, 4 and 5 constitute a proposition of $0.8 > \frac{3}{4}$ of the voters and are each $\left(\frac{2}{3}, \frac{1}{2}\right)$-frustrated on three proposals. Therefore the (α, β, γ)-paradox occurs.

Definition 1.3.8. Given a vote profile X and a proposal a, the α**-prevailing coalition** is the set of all voters who are in agreement with the α-majority decision $MR^\alpha[X]$ on a; that is

$$\{i \in N \ : \ X_i^a = MR^\alpha(X^a)\}$$

We denote by

$$\eta_a(X, \alpha) = |\, \{i \in N \ : \ X_i^a = MR^\alpha(X^a)\} \,|$$

the total number of voters in the α-prevailing coalition on proposal a at X and by

$$\eta(X, \alpha) = \frac{1}{m} \sum_{a \in \mathcal{M}} \eta_a(X, \alpha)$$

the average fraction of voters in α-prevailing coalitions across all proposals.

The following theorem give us a sufficient condition to avoid the (α, β, γ)-paradox. This intra-profile-conditions type is called the rule of $(1 - \beta\gamma)$.

Theorem 3.4 (Wagner (1984)).

If X is a vote profile such that $\eta(X,\alpha) \geq (1-\beta\gamma)n$, then the Anscombe's (α,β,γ)-paradox does not hold at X. ♣

This result states that when the prevailing coalition, across all proposals, contains on average $(1-\beta\gamma)n$ voters, the set of voters who are frustrated under the α-majority rule on more than βm proposals cannot exceed γn. This result is the generalization of the rule of Three-Fourth (Wagner (1983) for $\beta = \gamma = \frac{1}{2}$: the Anscombe's paradox never holds at a vote profile whenever the prevailing coalition, across all proposals, contains on average 75% of voters.

1.3.3 Hamming distance and the Anscombe's γ-paradox

Laffond and Lainé (2013) consider the simple majority rule ($\alpha = \frac{1}{2}$) and the Anscombe's γ-paradox which occurs when the total number of frustrated individuals is greater than γn. The innovation here comes from a new tool used by the authors: the Hamming distance between two vectors of votes. The result provided is an intra-profile condition based on the maximal Hamming distance r between two vectors of votes that precludes the Anscombe's γ-paradox: this paradox never holds at a vote profile whenever the Hamming distance between any couple of vectors of votes is less than r.

Definition 1.3.9. *Given $\gamma \in \left[\frac{1}{2};1\right[$ and a vote profile X, the Anscombe's γ-paradox occurs at X if the $\left(\frac{1}{2},\frac{1}{2},\gamma\right)$-paradox occurs at X.*

Definition 1.3.10. *Let $x, y \in \{-1;+1\}^m$ be two vectors of votes. The* **Hamming distance** *$d(x,y)$ between x and y is defined by :*

$$d(x,y) = \frac{1}{2}\sum_{a\in\mathcal{M}} |x^a - y^a|.$$

It is wellknown that the Hamming distance is a distance on $\{-1;+1\}^m$. Intuitively, the Hamming distance between two vectors of votes x and y is the total number of proposals on which x and y differ. Therefore, it permits a straightforward reformulation of the Anscombe's paradox as stated below:

PROPOSITION 1.3.2. *The Anscombe's paradox holds at a vote profile X if and only if*

$$\left|\left\{i \ : \ d(X_i, MR[X]) > \frac{m}{2}\right\}\right| > \frac{n}{2}.$$

Hereafter and given a vote profile X, let

$$r_X = \max\left\{\frac{d(X_i, X_j)}{m}, \ i, j = 1, \ldots, n\right\}$$

be the relative maximal Hamming distance between two vectors of votes in X. Laffond and Lainé (2013) show that the values of r_X for which the vote profile X does not exhibit the Anscombe's paradox depend on γ and is denoted by $r_X(\gamma)$.

Theorem 3.5 (Laffond and Lainé (2013)).

Let $\gamma \in \left[\frac{1}{2}; 1\right[$ and X a vote profile such that

$$r_X(\gamma) \leq \frac{\sqrt{\gamma} - \gamma}{1 - \gamma}.$$

then the Anscombe's γ-paradox never holds at X.

For $\gamma = \frac{1}{2}$ the Anscombe's paradox never holds at a vote profile when the maximal relative Hamming distance between any two vectors of votes is least than 41.4% of the proposals.

1.4 Our concerns

As presented above, several authors have paid a lot of attention on the study of the Anscombe's paradox. Before we continue, it is good to clearly state what are our concerns; or in other words, what are the questions that guide our contribution in the next chapters.

1.4.1 Are there binary voting rules that preclude Anscombe's paradox?

Almost all papers on Anscombe's paradox focus on the use of majority rules. It is worth nothing that this paradox relies on two key parameters: the decision rule and the set of majorities (coalitions endowed with the power to secure the adoption of any proposal unanimously supported by all of its members). Our first question is whether using alternative binary voting rules (other than the majority rule) would result in avoiding all occurrences of the Anscombe's paradox in the sense that for all possible vote profiles, no majority coalition exists such that each member disagrees with the collective decision on more than the half of proposals. Of course, such a framework is a generalization of the Anscombe's paradox from majority rules to any other binary voting rule on proposals. Furthermore, we initiate a second generalization by considering voting contexts in which the decision rule as well as the notion of majority coalition capture distinct features than those of the majority rule. This is the case with simple games which generalizes the majority rule as well as the notion of the majority coalitions.

1.4.2 Is there any non polynomial domain restriction that is Anscombe's paradox free?

Any vote profile with single-switch vectors of votes does not exhibit the Anscombe's paradox. With m proposals, there are 2^m possible vectors of votes; but only $2m$ such vectors of votes are single-switch. Single-switchness is then a polynomial (in size) domain restriction that is free of the Anscombe's paradox. This is also the case with the domain restriction from Laffond and Lainé (2006) based on a threshold of Hamming distance between two vectors of votes. To the best of our knowledge these are the two most known prominent domains that preclude all occurrences of Anscombe's paradox. We aim at providing domain restrictions that are non polynomial and still immune from exhibiting the Anscombe's paradox.

1.4.3 Does the organization of voters in parties impact the possibility of observing the Anscombe's paradox?

Conditions by Wagner (1983) or by Laffond and Lainé (2006) identify some sets of vote profiles that do not exhibit Anscombe's paradox. These intra-profile conditions describe some relationship between vectors of votes in the same profile with no information on how such relationship may arise. We assume that voters are members of political parties and thus share some common values within parties. In this configuration we aim at providing the necessary and sufficient condition to avoid the Anscome's paradox by following the logical of the parties.

CHAPTER 2

ANSCOMBE'S PARADOX : GENERALIZATIONS TO BINARY VOTING RULES

Depending on the nature of the proposals, there are several alternatives to the majority rule. For example, constitutional issues generally require a qualified majority. Moreover, individual votes may be weighted from some exogenous considerations such as the voting right shares of a representative, the total numbers of shares of a shareholder, . . . In this chapter, the Anscombe's paradox is generalized in two different ways: (i) the rule changes but a majority is any coalition with more than the half of the voters (see section 2.1) ; and (ii) the rule changes and a majority refers to a coalition endowed with a power of imposing the adoption of any proposal unanimously supported by all of its members (see section 2.2).

2.1 Anscombe's paradox and binary voting rules

Under the majority rule, the Anscombe's paradox holds when more than the half of the voters are each frustrated on more than the half of the proposals. We now consider a generic binary voting rule and check whether it is still possible to find vote profiles at which more than the half of the voters are each frustrated on more than the half of the proposals.

2.1.1 Anscombe's paradox with standard majorities

We recall that a binary voting rule is a mapping $R : \{-1;+1\}^n \to \{-1;+1\}$ that associates each possible vector of votes on a proposal with a social decision such that given a proposal a and a vector of votes X^a on a, $R(X^a) = +1$ if proposal a is collectively adopted and $R(X^a) = -1$ otherwise.

Definition 2.1.1. *Given a binary voting rule, a standard majority is any coalition of more than the half of the voters.*

Frustration for a voter under a binary voting rule also refers to the possibility for that voter to disagree with the collective decision. More formally,

Definition 2.1.2. *Consider a binary voting rule R, a vote profile X, a voter i and a proposal a.*

1. *Voter i is R-frustrated on proposal a if he/she disagrees with the collective decision on that proposal; that is*

$$X_i^a \neq R(X^a).$$

2. *Voter i is R-frustrated at X if he/she disagrees with the collective decision on more than the half of the proposals; that is*

$$|\{a \in \mathcal{M} \ \ : \ \ i \ R\text{-frustrated on a}\}| > \frac{m}{2}.$$

Definition 2.1.3. *Given a number n of voters and a vector of votes $v \in \{-1,+1\}^n$. When n is even, a* **tie** *occurs at v if half of the voters vote for the adoption while the other half vote against. In this case, we say that v is* polarized.

Now, we can state our first generalization of the Anscombe's paradox to any binary voting rule.

Definition 2.1.4. *Giving a binary voting rule R, a vote profile X exhibits the R-Anscombe's paradox if all members of a standard majority are each R-frustrated on more than the half of the proposals; that is*

$$|\{i \in N \ \ : \ \ i \ R\text{-frustrated}\}| > \frac{n}{2}.$$

The binary voting rule R exhibits the Anscombe's paradox if some vote profile exhibits the R-Anscombe's paradox.

Example 2.1.1. *Define the binary voting rule R by requiring favorable votes of both voter 1 and 2 to be necessary and sufficient for the adoption of a proposal; that is*

$$\forall a \in \mathcal{M}, \quad R(X^a) = \begin{cases} +1 & \text{if } X_1^a = X_2^a = +1 \\ -1 & \text{otherwise} \end{cases}$$

Let X be the following vote profile:

	X_1	X_2	X_3	X_4	X_5	$R[X]$
a^1	-1	$+1$	$+1$	$+1$	-1	-1
a^2	$+1$	$+1$	-1	-1	-1	$+1$
a^3	-1	-1	$+1$	-1	$+1$	-1

It appears that voters 3, 4 and 5 are R-frustrated at X. Therefore X exhibits the R-Anscombe's paradox.

In the precedent example, it is not surprising that voters 3, 4 and 5 are R-frustrated at the vote profile X provided. Clearly, these three voters are null voters (their opinions never count for a decision) under the binary voting rule R; that is their opinions never count.

2.1.2 Vulnerability to the standard Anscombe's paradox

In Chapter 1, it has been shown that for some combinations of n and m, the majority rule is immune to Anscombe's paradox. That is the case when there are at most three voters; or there are exactly four proposals and less than six voters. We are now interested whether it is possible to construct alternative binary voting rules that are not vulnerable to the Anscombe's paradox. To do this, we classify binary voting rules into two classes. In the first class, we consider all binary voting rules that are minority sensitive in the sense that, at some vectors of votes, a standard majority disagrees with the collective decision; such a decision is then the opinion of a standard minority coalition (a coalition of less than the half of the voters). In the second class, we consider all binary voting rules that never support the opinion of a standard minority coalition.

Definition 2.1.5. *A binary voting rule R is minority sensitive if for some vectors of votes $x \in \{-1;+1\}^m$, the collective decision $R(x)$ is supported only by a minority coalition; that is*

$$|\{i \in N, \quad x_i = R(x)\}| < \frac{n}{2}.$$

Clearly the majority rule is not minority sensitive.

Remark 2.1.1. *By definition, a binary voting rule R is minority sensitive if for some vectors of votes $x \in \{-1;+1\}^m$, all members of a standard majority disagree with the collective decision; that is*

$$|\{i \in N, \quad x_i \neq R(x)\}| > \frac{n}{2}.$$

Proposition 2.1.1. *Any binary voting rule that is minority sensitive is vulnerable to the Anscombe's paradox.*

Proof.
Let R be a binary voting rule that is minority sensitive. Then there exists a vector of votes x such that all members of a standard majority coalition S disagree with the collective decision $R(x)$. Consider a vote profile X such that for each proposal a, $X^a = x$. Clearly all voters in S are each frustrated on all proposals. Therefore X exhibits the R-Anscombe's paradox. ■

Remark 2.1.2. *Let R be a binary voting rule that is not minority sensitive. Note that for all vector of votes $x \in \{-1;+1\}^m$ and for all $a \in \{-1;+1\}$,*

$$|\{i \in N, \ x_i = a\}| > \frac{n}{2} \Rightarrow R(x) = MR(x) = a. \tag{2.1}$$

It follows that $R = MR$ when n is odd.

Proposition 2.1.2. *Let R be a binary voting rule that is not minority sensitive. If n is odd then $R = MR$ and R exhibits the Anscombe's paradox if and only if $n \geq 7$ or ($m \neq 4$ and $n \geq 5$).*

Proof.
See Remark 2.1.2 and Theorem 1.2 of Chapter 1. ■

As seen above, a binary voting rule that is not minority sensitive coincides with the majority rule when the total number of voters n is odd. But for even value of n, R may be differ from the majority rule on some vectors of votes where a tie holds between -1 and $+1$.

PROPOSITION 2.1.3. *Let R be a binary voting rule that is not minority sensitive and n is even. If $n \geq 10$ and ($m = 3$ or $m \geq 5$) or $n \geq 12$ and $m = 4$ then R exhibits the Anscombe's paradox.*

Proof.

Let $R : \{-1; +1\}^n \to \{-1; +1\}$ be binary voting rule that is not minority sensitive. Suppose that n is even and consider the following cases:

Case 1: $n \geq 12$ and $m = 4$. Pose $n = 2k$ and consider the vector of votes $x \in \{-1, +1\}^n$ such that a tie occur at x. Without loss of generality, suppose that $x_1 = x_2 = \cdots = x_k \neq R(x) = \delta \in \{-1, +1\}$. Consider the following vote profile where the first $k + 1$ voters are each in disagreement with $R(X)$ on at least three proposals.

		X_1	X_2	X_3	X_4	X_5	X_6	X_7	$\cdots$	X_k		$\cdots$	$\cdots$	X_n	$R[X]$
x	a^1	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$\cdots$	$-\delta$	δ	δ	$\cdots$	δ	δ
	a^2	δ	δ	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$\cdots$	$-\delta$	$-\delta$	δ	$\cdots$	δ	δ
	a^3	$-\delta$	$-\delta$	δ	δ	$-\delta$	$-\delta$	$-\delta$	$\cdots$	$-\delta$	$-\delta$	δ	$\cdots$	δ	δ
	a^4	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	$-\delta$	$\cdots$	$-\delta$	$-\delta$	δ	$\cdots$	δ	δ

Case 2: $n \geq 10$. Then there exist two integers p and r such that $n = 4p + r$ with $r = 0, 2$ and $p \geq 2$ (with $r = 2$ if $p = 2$). Consider a vote profile X obtained from a partition of N into four subsets N_1, N_2, N_3 and N_4 such that $|N_1| = p$, $|N_2| = p - \delta$, $|N_3| = p - \delta$ and $|N_4| = p + 2$ with $\delta = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r = 2. \end{cases}$

Since $m = 3$ or $m \geq 5$, there exist two integers q and t such that $m = 3q + t$ with $q \geq 1$, $t \in \{0, 1, 2\}$ and ($q \neq 1$ or $t \neq 1$). We define the vote profile X as another one of the proof of the Proposition 1.1.2.

	X_{N_1}	X_{N_2}	X_{N_3}	X_{N_4}	$R[X]$
a^1	-1	$+1$	$+1$	-1	-1
a^2	$+1$	-1	$+1$	-1	-1
a^3	$+1$	$+1$	-1	-1	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{3k-2}	-1	$+1$	$+1$	-1	-1
a^{3k-1}	$+1$	-1	$+1$	-1	-1
a^{3k}	$+1$	$+1$	-1	-1	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Pose $S = N_1 \cup N_2 \cup N_3$. We have $|S| = 3p - 2\delta > \frac{n}{2}$. Hence S is a majority coalition and we show as proof of the Proposition 1.1.2 that each voter of S disagrees with $R[X]$ on a majority of proposals. Therefore R is weakly vulnerable to Anscombe's paradox in X. ■

For all $x \in \{-1;+1\}^n$, we denote by $F(R,x) = \{i \in N,\ x_i \neq R(x)\}$ the set of voters who disagree with the outcome $R(x)$.

Proposition 2.1.4. *Let R be a binary voting rule that is not minority sensitive. Assume that there are six or eight voters and $m = 3$ or $m \geq 5$. Then R exhibits the Anscombe's paradox.*

Proof. Consider the vector of votes $x \in \{-1;+1\}^n$ such that a tie occur at x. That is $|F(R,x)| = \frac{n}{2}$.

Pose $E(x) = \left\{y \in \{-1;+1\}^n\ :\ |F(R,y)| = \frac{n}{2} \text{ and } |F(R,x) \cap F(R,y)| = \frac{n}{2} - 1\right\}$. Without loss of generality, suppose that $F(R,x) = \{1,2,3\}$ if $n = 6$ or $F(R,x) = \{1,2,3,4\}$ if $n = 8$.

Pose $R(x) = \delta \in \{-1;+1\}$. There are two possibles cases:

Cases 1: $E(x) \neq \emptyset$. Let $y \in E(x)$ and pose $R(y) = \gamma \in \{-1;+1\}$. Without loss of generality, suppose that $F(R,x) \cap F(R,y) = \{1,2\}$ if $n = 6$ or $F(R,x) \cap F(R,y) = \{1,2,3\}$ if $n = 8$. Since $m \geq 3$, there exist two integers p and t such that $m = 3p + t$ with $p \geq 1,\ t \in \{0,1,2\}$ and ($p \neq 1$ or $t \neq 1$). Consider the following vote profile X

		X_1	X_2	X_3	X_4	X_5	X_6	$R[X]$
x	a^1	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ
y	a^2	$-\gamma$	$-\gamma$	γ	$-\gamma$	γ	γ	γ
z	a^3	δ	δ	$-\delta$	$-\delta$	δ	δ	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	a^{3k-2}	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ
y	a^{3k-1}	$-\gamma$	$-\gamma$	γ	$-\gamma$	γ	γ	γ
z	a^{3k}	δ	δ	$-\delta$	$-\delta$	δ	δ	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

		X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	$R[X]$
x	a^1	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
y	a^2	$-\gamma$	$-\gamma$	$-\gamma$	γ	$-\gamma$	γ	γ	γ	γ
z	a^3	δ	δ	δ	$-\delta$	$-\delta$	δ	δ	δ	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	a^{3k-2}	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
y	a^{3k-1}	$-\gamma$	$-\gamma$	$-\gamma$	γ	$-\gamma$	γ	γ	γ	γ
z	a^{3k}	δ	δ	δ	$-\delta$	$-\delta$	δ	δ	δ	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Pose $S = \{1, 2, 3, 4\}$ if $n = 6$ or $S = \{1, 2, 3, 4, 5\}$ if $n = 8$ and remark that each voter in S disagrees with $R[X]$ on exactly two proposals given the three consecutive vectors x, y and z. By following the same reasoning of the Claim of Proposition 1.1.2, we show that R weakly exhibits the Anscombe's paradox in X.

Cases 2: $E(x) = \emptyset$. Consider the vectors $u = (-\delta, -\delta, \delta, -\delta, \delta, \delta)$, $v = (\delta, -\delta, \delta, \delta, -\delta, -\delta)$, $w = (-\delta, -\delta, -\delta, \delta, -\delta, \delta, \delta, \delta)$ and $h = (\delta, \delta, -\delta, \delta, \delta, -\delta, -\delta, -\delta)$ note that if $R(u) = \delta$, then $F(R, x) \cap F(R, u) = \{1, 2\}$ hence $u \in E(x)$, which is impossible then $R(u) = -\delta$. At same $R(v) = \delta$, $R(w) = -\delta$ and $R(h) = \delta$. We have the following vote profile X

		X_1	X_2	X_3	X_4	X_5	X_6	$R[X]$
x	a^1	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ
u	a^2	$-\delta$	$-\delta$	δ	$-\delta$	δ	δ	$-\delta$
v	a^3	δ	$-\delta$	δ	δ	$-\delta$	$-\delta$	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	a^{3k-2}	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ
u	a^{3k-1}	$-\delta$	$-\delta$	δ	$-\delta$	δ	δ	$-\delta$
v	a^k	δ	$-\delta$	δ	δ	$-\delta$	$-\delta$	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

		X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	$R[X]$
x	a^1	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
w	a^2	$-\delta$	$-\delta$	$-\delta$	δ	$-\delta$	δ	δ	δ	$-\delta$
h	a^3	δ	δ	$-\delta$	δ	δ	$-\delta$	$-\delta$	$-\delta$	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	a^{3k-2}	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
w	a^{3k-1}	$-\delta$	$-\delta$	$-\delta$	δ	$-\delta$	δ	δ	δ	$-\delta$
h	a^{3k}	δ	δ	$-\delta$	δ	δ	$-\delta$	$-\delta$	$-\delta$	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Each voter of $S = \{2, 3, 5, 6\}$ if $n = 6$ or $S = \{3, 4, 6, 7, 8\}$ if $n = 8$ disagrees with $R[X]$ on more than $\frac{m}{2}$ proposals. Then R weakly exhibits the Anscombe's paradox in X. ∎

Proposition 2.1.5. *Let R be a binary voting rule that is not minority sensitive. Assume that there are exactly four voters and $m = 3$ or $m \geq 5$. The following conditions are equivalent*

A_1) R exhibits the Anscombe's paradox.

A_2) There exists a voter $i \in N$, such that for all pairs of voters $\{j; k\} \subset N \backslash \{i\}$ there exists a vector $x \in \{-1; +1\}^4$ such that $F(R, x) = \{j; k\}$.

Proof. Suppose that $N = \{1, 2, 3, 4\}$ and there exist two integers p and t such that

$$m = 3p + t$$

with

$$p \geq 1,\ t \in \{0, 1, 2\} \text{ and } (p \neq 1 \text{ or } t \neq 1).$$

$\Leftarrow$) Suppose that there exists a voter - say 1 - such that for all $\{i; j\} \subset N \backslash \{1\}$, $F(R, x) = \{i; j\}$ for some $x \in \{-1; +1\}^4$.

Hence there exist three vectors x, y and z in $\{-1; +1\}^4$ such that

$$x_2 = x_3 \neq R(x) = \delta;\ y_2 = y_4 \neq R(y) = \gamma;\ z_3 = z_4 \neq R(z) = \varepsilon.$$

Consider a vote profile X such that given three consecutive proposals a^{3k+1}, a^{3k+2} and a^{3k+3} we have $X^{a^{3k+1}} = x$, $X^{a^{3k+2}} = y$ and $X^{a^{3k+3}} = y$. Then voter 2 is frustrated on

a^{3k+1} and a^{3k+2}; voter 3 is frustrated on a^{3k+1} and a^{3k+3}; voter 4 is frustrated on a^{3k+2} and a^{3k+3}. Then the Anscombe's paradox holds.

Then the vote profile X is defined as follows.

	X_{i_1}	X_{i_2}	X_{i_3}	X_{i_4}	$R[X]$
a^1	δ	$-\delta$	$-\delta$	δ	δ
a^2	γ	$-\gamma$	γ	$-\gamma$	γ
a^3	ε	ε	$-\varepsilon$	$-\varepsilon$	ε
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{3k+1}	δ	$-\delta$	$-\delta$	δ	δ
a^{3k+2}	γ	$-\gamma$	γ	$-\gamma$	γ
a^{3k+3}	ε	ε	$-\varepsilon$	$-\varepsilon$	ε
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\Rightarrow$) Conversely, suppose that for all $i \in N$, there exists a pair $\{j;k\} \subset N\backslash\{i\}$ such that $F(R,x) \neq \{j;k\}$ for all $x \in \{-1;+1\}^4$ i.e. $x_j \neq x_k$ or $R(x) = x_j$ for all $x \in \{-1;+1\}^4$. Suppose that R exhibits the Anscombe's paradox. Then there exist a vote profile X and a coalition S such that: $|S| = 3$ and each voter in S is frustrated on a majority of proposals. For $i \notin S$, there exist two voters - say j and k - in $N\backslash\{i\}$ such that

$$\forall x \in \{-1;+1\}^4,\; x_j \neq x_k \text{ or } R(x) = x_j.$$

Hence for all $a \in \mathcal{M}$, $X_j^a = X_k^a = R(X^a)$ or $X_j^a \neq X_k^a$. Let F_j respectively F_k be the set of all proposals on which voter j respectively k is frustrated. It follows that $F_j \cap F_k = \emptyset$ and since $j,k \in S$,

$$|F_j| + |F_k| > \frac{m}{2} + \frac{m}{2} = m$$

contradiction. Therefore R does not exhibit the Anscombe's paradox. ■

Proposition 2.1.5 characterizes all binary voting rules that exhibit the Anscombe's paradox with four voters and m proposals with $m = 3$ or $m \geq 5$. The next proposition provides a relative simple condition that identifies all binary voting rules free of this paradox under the same settings on the number of voters and the number of proposals.

Proposition 2.1.6. *Let R be a binary voting rule that is not minority sensitive.*

Assume that there are exactly four voters and $m = 3$ or $m \geq 5$. The following conditions are equivalent

1. *For all voter $i \in N$, there exists a pair of voters $\{j, k\} \subset N\backslash\{i\}$, such that for all $x \in \{-1; +1\}^4$*
$$x_j = x_k \Rightarrow R(x) = x_j.$$
2. *There exists a coalition S such that $|S| = 3$ and for all $x \in \{-1; +1\}^4$,*
$$R(x) = \delta \iff |\{i \in S,\ x_i = \delta\}| \geq 2.$$

Proof.

$\Leftarrow$) Suppose that there exists a coalition S such that $|S| = 3$ and for all $x \in \{-1; +1\}^4$, $R(x) = \delta \iff |\{i \in S,\ x_i = \delta\}| \geq 2$. Pose $N\backslash S = \{l\}$.

Let $i \in N$ and pose $\{j, k\} = N\backslash\{i, l\}$. We have $j, k \in S$. By definition of R, $\forall x \in \{-1; +1\}^4$, if $x_j = x_k$ then $R(x) = x_j$.

$\Rightarrow$) Suppose that for all voter $i \in N$, there exists a pair of voters $\{j, k\} \subset N\backslash\{i\}$, such that for all $x \in \{-1; +1\}^4$, $x_j = x_k \Rightarrow R(x) = x_j$.

Pose $N = \{i_1, i_2, i_3, i_4\}$.

For all $i_t \in N$, $t = 1, 2, 3, 4$ we can denote by S_t the set of two voters which satisfies the hypothesis. It suffice to prove that there exists a player $i_0 \in N$ such that for all $t = 1, 2, 3, 4$, $i_0 \notin S_t$. Note that
$$\forall t, t' \in \{1, 2, 3, 4\},\ S_t \cap S_{t'} \neq \emptyset; \tag{2.2}$$
otherwise $S_t \cap S_{t'} = \emptyset$ and there exists a vector $x \in \{-1, +1\}^4$ such that $R(x) = -1$ and $R(x) = +1$.

Now, $S_1 \subset \{i_2, i_3, i_4\}$. Then $S_1 \in \{\{i_2, i_3\}; \{i_2, i_4\}; \{i_3, i_4\}\}$. Without loss of the generality suppose that $S_1 = \{i_2, i_3\}$ hence $N\backslash S_1 = \{i_1, i_4\} \neq S_t$ for $t \neq 1$ by Equation (2.2). Hence $S_2 \in \{\{i_1, i_3\}; \{i_3, i_4\}\}$.

- If $S_2 = \{i_1, i_3\}$ then $S_3 \neq N\backslash S_2 = \{i_2, i_4\}$ and $S_3 \neq \{i_1, i_4\}$. Hence $S_3 = \{i_1, i_2\}$ and $S_4 \in \{S_1, S_2, S_3\}$. Therefore $i_0 = i_4$ and $S = \{i_1, i_2, i_3\}$.
- If $S_2 = \{i_3, i_4\}$ then $S_3 \neq N\backslash S_2 = \{i_1, i_2\}$ and $S_3 \neq \{i_1, i_4\}$. Hence $S_3 = \{i_2, i_4\}$ and $S_4 \in \{S_1, S_2, S_3\}$. Therefore $i_0 = i_1$ and $S = \{i_2, i_3, i_4\}$.

We can conclude that $\forall x \in \{-1; +1\}^4$, $x_{i_0} \neq R(x)$ and $S = N\backslash\{i_0\}$. ■

Proposition 2.1.6 tells us that avoiding the Anscombe's paradox with four voters and m proposals with $m = 3$ or $m \geq 5$ makes one of the four voters a null voter.

Proposition 2.1.7. *Let R be a binary voting rule that is not minority sensitive. Assume that there are exactly six voters and four proposals. The following conditions are equivalent:*

A_1) R does not exhibit the Anscombe's paradox.

A_2) For all pairs $\{i;j\} \subset N$, there exists $\{k,l,t\} \subset N\backslash\{i,j\}$ such that for all $x \in \{-1;+1\}^6$ and for all $\delta \in \{-1,+1\}$, $R(x) = \delta$ whenever $x_k = x_l = x_t = \delta$.

Proof.

Let $N = \{1,2,3,4,5,6\}$ and $\mathcal{M} = \{a^1, a^2, a^3, a^4\}$

$\Rightarrow$) Suppose that (A_2) is false. Then there exists a pair of voters $\{i,j\} \subset N$, such that for all $\{k,l,t\} \subset N\backslash\{i,j\}$, there exists a vector $x \in \{-1;+1\}^6$, such that $F(R,x) = \{k,l,t\}$.

Pose $i = i_1$, $j = i_2$ and $N\backslash\{i,j\} = \{i_3, i_4, i_5, i_6\}$. Then

$$\{k,l,t\} \subset \{i_3,i_4,i_5,i_6\} = N\backslash\{i_1,i_2\},$$

there exists a vector $x \in \{-1;+1\}^6$, such that $F(R,x) = \{k,l,t\}$. By applying this when $\{k,l,t\}$ is $\{i_3,i_4,i_5\}$, $\{i_3,i_4,i_6\}$, $\{i_3,i_5,i_6\}$ or $\{i_4,i_5,i_6\}$ respectively, one can define a vote profile X as follows:

	X_{i_1}	X_{i_2}	X_{i_3}	X_{i_4}	X_{i_5}	X_{i_6}	$R[X]$
a^1	δ	δ	$-\delta$	$-\delta$	$-\delta$	δ	δ
a^2	γ	γ	$-\gamma$	$-\gamma$	γ	$-\gamma$	γ
a^3	ε	ε	$-\varepsilon$	ε	$-\varepsilon$	$-\varepsilon$	ε
a^4	α	α	α	$-\alpha$	$-\alpha$	$-\alpha$	α

where

$$\delta, \alpha, \gamma, \varepsilon \in \{-1,+1\}.$$

It appears that such a vote profile X exhibits the Anscombe's paradox since voters i_3, i_4, i_5 and i_6 are frustrated on a majority proposals. Therefore (A_1) does not hold; meaning that (A_1) implies (A_2).

$\Leftarrow$) Suppose that A_2 holds. We prove that R does not exhibits the Anscombe's paradox. On the contrary, suppose that R exhibits the Anscombe's paradox. Then there exist a vote profile X and a coalition S such that $|S| \geq 4$ with each voter in S being frustrated on a majority of proposals; this implies that each member of S is frustrated on at least three proposals. Since R is not minority sensitive, then for each of the four proposals, there exists a unique voter in S who is not frustrated on that proposal (since each voter in S disagrees with the collective decision on at least three proposals).

Denote by i_t the voter in S who is not frustrated on proposal a^t, $t = 1, 2, 3, 4$ and pose $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$. By construction, note that i_1 is necessary frustrated on proposals a^2, a^3 and a^4; i_2 is frustrated on proposals a^1, a^3 and a^4; i_3 is frustrated on proposals a^1, a^2 and a^4; and i_4 is frustrated on proposals a^1, a^2 and a^3. Since i_2, i_3 and i_4 are all frustrated on proposal a^1, then i_5 and i_6 are not frustrated on a^1. Similarly, i_5 and i_6 are not frustrated on the proposals a^2, a^3 and a^4. Then the vote profile X is as follows:

	X_{i_1}	X_{i_2}	X_{i_3}	X_{i_4}	X_{i_5}	X_{i_6}	$R[X]$
a^1	δ	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ
a^2	$-\gamma$	γ	$-\gamma$	$-\gamma$	γ	γ	γ
a^3	$-\varepsilon$	$-\varepsilon$	ε	$-\varepsilon$	ε	ε	ε
a^4	$-\alpha$	$-\alpha$	$-\alpha$	α	α	α	α

It appears that i_5 and i_6 are such that for all $\{k, l, t\} \subset N \backslash \{i_5, i_6\}$ there exists a vector $x \in \{-1; +1\}^6$, such that $x_k = x_l = x_t = w$ and $R(x) \neq w$ for some $w \in \{-1, +1\}$. A contradiction arises. Therefore R does not exhibit the Anscombe's paradox. ∎

Proposition 2.1.7 is somewhat technical as is was the case with Proposition 2.1.5. As in Proposition 2.1.6, we deduce some simple binary voting rules by nullifying the opinion of a voter as in the following corollary.

Corollary 2.1.1. *Consider a binary voting rule R with exactly six voters and four proposals that is not minority sensitive. If there exists a coalition S such that $|S| = 5$ and for all $x \in \{-1; +1\}^6$ and for all $\delta \in \{-1, +1\}$,*

$$R(x) = \delta \iff |\{i \in S,\ x_i = \delta\}| \geq 3.$$

then R does not exhibits the Anscombe's paradox.

Proof.

Suppose that there exists a coalition S such that $|S| = 5$ and for all $x \in \{-1;+1\}^6$ and for all $\delta \in \{-1,+1\}$, $R(x) = \delta \iff |\{i \in S,\ x_i = \delta\}| \geq 3$.

We have, for all $\{i,j\} \subset N$, there exists $\{k,l,t\} \subset S \cap (N \setminus \{i,j\})$ such that for all $x \in \{-1;+1\}^6$ and for all $\delta \in \{-1,+1\}$, $x_k = x_l = x_t = \delta$ imply that $R(x) = \delta$ by definition of R. Therefore R does not exhibits the Anscombe's paradox by proposition 2.1.7. ■

The precedent corollary just provides a class of binary voting rules that are free of the Anscombe's paradox with exactly six voters and four proposals. But this class is not exhaustive since Proposition 2.1.7 covers more voting rules. To see this, consider the following example:

Example 2.1.2. *To construct a binary voting rule that is free of Anscombe's paradox, we follow Proposition 2.1.7. We first associate each pair $\{i,j\} \subset N$ with a triplet*
$\{k,l,t\} \subset N \setminus \{i,j\}$ as shown in the Table 2.1 below.

Let

$$P = \{\{4,5,6\},\{1,3,5\},\{2,3,6\},\{1,2,4\},\{3,4,5\},\{1,3,4\},\{1,5,6\}\}$$

and define the binary voting rule R by:

$$R(x) = \begin{cases} \mathrm{MR}(x) & \text{if } |\{t \in N \ : \ x_t = -1\}| \neq |\{t \in N \ : \ x_t = +1\}| \\ \delta & \text{if } |\{t \in N \ : \ x_t = -1\}| = 3 \text{ and } \{t \in N \ : \ x_t = \delta\} \in P \\ +1 & \text{otherwise.} \end{cases}$$

It follows from Table 2.1 that R satisfies condition (A_2) of Proposition 2.1.7. Therefore R does not exhibit the Anscombe's paradox.

To prove that the binary voting rule R we just construct is out of the scope of Corollary 2.1.1, we identify for all coalition S such that $|S| = 5$ and for some $\delta \in \{-1,+1\}$, a vector of votes $x \in \{-1,+1\}^6$ such that $R(x) = \delta$ and $|\{t \in S \ : \ x_t = \delta\}| < 3$; such a vector is provided in Table 2.2 below.

Clearly, R does not exhibit the Anscombe's paradox and does not satisfy the condition of Corollary 2.1.1 which is then sufficient but not necessary.

One can easily check that there is no null voter under the binary voting rule provided in Example 2.1.2 with six voters and four proposals. This contrasts with the case of four

Table 2.1:

$\{i,j\}$	$\{k,l,t\}$
$\{1,2\}$	$\{4,5,6\}$
$\{1,3\}$	$\{4,5,6\}$
$\{2,3\}$	$\{4,5,6\}$
$\{4,5\}$	$\{2,3,6\}$
$\{1,4\}$	$\{2,3,6\}$
$\{1,5\}$	$\{2,3,6\}$
$\{4,6\}$	$\{1,3,5\}$
$\{2,4\}$	$\{1,3,5\}$
$\{2,6\}$	$\{1,3,5\}$
$\{5,6\}$	$\{1,2,4\}$
$\{3,5\}$	$\{1,2,4\}$
$\{3,6\}$	$\{1,2,4\}$
$\{1,6\}$	$\{3,4,5\}$
$\{2,5\}$	$\{1,3,4\}$
$\{3,4\}$	$\{1,5,6\}$

voters and $m = 3$ or $m \geq 5$ where avoiding Anscombe's paradox requires the nullification of the opinion of a voter. Moreover, Proposition 2.1.7 also tells us that it is still possible to avoid Anscombe's paradox by an appropriate tie-breaking rule; this is a positive result as compared to Theorem 1.2 in Chapter 1 where the majority rule exhibits the Anscombe's paradox with six voters and four proposals (due to a specified tie-breaking rule).

Proposition 2.1.8. *Let R be a binary voting rule that is not minority sensitive. Assume that there are exactly four voters and four proposals. Then R does not exhibit the Anscombe's paradox.*

Proof.

Suppose that $m = n = 4$ and that R exhibits the Anscombe's paradox. Then there exist a vote profile X and a coalition S such that $|S| \geq 3$ and each voter in S is frustrated on a majority of proposals. Since R is not minority sensitive and S is a majority coalition, it follows that for each proposal, there exists a voter in S who

Table 2.2:

S	Vector $x = (x_1, x_2, x_3, x_4, x_5, x_6)$	$R(x)$	$\lvert\{t \in S\ :\ x_t = R(x)\}\rvert$
$\{1,2,3,4,5\}$	$(+1,-1,-1,+1,+1,-1)$	-1	2
$\{1,2,3,4,6\}$	$(-1,-1,+1,+1,+1,-1)$	$+1$	2
$\{1,2,3,5,6\}$	$(-1,-1,+1,+1,+1,-1)$	$+1$	2
$\{1,2,4,5,6\}$	$(-1,-1,+1,+1,+1,-1)$	$+1$	2
$\{1,3,4,5,6\}$	$(+1,-1,-1,+1,+1,-1)$	-1	2
$\{2,3,4,5,6\}$	$(-1,+1,-1,+1,-1,+1)$	-1	2

is not frustrated on that proposal. Denote by i_t the voter in S who is not frustrated on proposal a^t, $t = 1, 2, 3$. Moreover each voter in S is frustrated on at least three proposals out of four. Therefore i_1, i_2 and i_3 are distinct voters. Thus $S \subseteq \{i_1, i_2, i_3\}$.

For the fourth proposal a^4, there exists a voter $i_s \in S$, $1 \leq s \leq 3$ who is not frustrated on a^4 otherwise the collective decision on a^4 would be the opinion of a minority. Therefore voter i_s is not frustrated on a^s; nor on a^4. This is a contradiction since by assumption, i_s is frustrated on at least three proposals out of four.

This proves that R does not exhibit the Anscombe's paradox. ■

Denote by $D(R)$ the set of all coalitions S such that $S = F(R, v)$ for some polarized vector of votes v; that is $D(R) = \{S : S = F(R, v)$ for some polarized vector of votes $v \in \{-1, +1\}^n\}$.

Remark 2.1.3. *Given a binary voting rule R with n voters. It follows from the definition of $D(R)$ that $D(R) = \emptyset$ when n is odd. Furthermore, when n is even, it holds that for all coalitions S of cardinality $\frac{n}{2}$, $S \in D(R)$ or $N \backslash S \in D(R)$.*

Proposition 2.1.9. *Let R be a binary voting rule that is not minority sensitive. Assume that there are exactly eight voters and four proposals. Then R is weakly vulnerable to Anscombe's paradox.*

Proof. Consider a binary voting rule R that is not minority sensitive. Assume that there are exactly eight voters and four proposals. The proof consists of two steps.

Step 1: Suppose that there exists some $\{i_1, i_2, i_3, i_4, i_5\} \subset N$ such that

$$\text{For some } \{i_1, i_2, i_3, i_4, i_5\} \subset N,\ \{i_1, i_2, i_3, i_4\}, \{i_1, i_2, i_3, i_5\}, \{i_1, i_2, i_4, i_5\} \in D(R). \quad (2.3)$$

Then there exists three polarized vectors of votes $v, v', v'' \in \{-1,+1\}^8$ such that $F(R,v) = \{i_1,i_2,i_3,i_4\}$, $F(R,v') = \{i_1,i_2,i_3,i_5\}$ and $F(R,v'') = \{i_1,i_2,i_4,i_5\}$. Pose $R(v) = \delta$, $R(v) = \delta'$ and $R(v) = \delta''$. Consider the following profile:

	X_{i_1}	X_{i_2}	X_{i_3}	X_{i_4}	X_{i_5}	X_{i_6}	X_{i_7}	X_{i_8}	$R[X]$
a^1	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
a^2	$-\delta'$	$-\delta'$	$-\delta'$	δ'	$-\delta'$	δ'	δ'	δ'	δ'
a^3	$-\delta''$	$-\delta''$	δ''	$-\delta''$	$-\delta''$	δ''	δ''	δ''	δ''
a^4	δ	δ	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ

Note that $R\left(X^{a^4}\right) = \delta$ since R is not minority sensitive. In profile X, each member of $S = \{i_1,i_2,i_3,i_4,i_5\}$ disagrees with $R[X]$ on three proposals out of four. Therefore R is weakly vulnerable to Anscombe's paradox.

Step 2: We prove that equation (2.3) necessary holds for some $\{i_1,i_2,i_3,i_4,i_5\} \subset N$. To see this, suppose on the contrary that equation (2.3) holds for no $\{i_1,i_2,i_3,i_4,i_5\} \subset N$; that is

$$\text{For all } \{i_1,i_2,i_3,i_4,i_5\} \subset N,\ \{i_1,i_2,i_3,i_4\},\{i_1,i_2,i_3,i_5\} \in D(R) \Longrightarrow \{i_1,i_2,i_4,i_5\} \notin D(R). \tag{2.4}$$

We start by proving that equation (2.4) necessarily implies the following:

$$\text{For all } \{i_1,i_2,i_3,i_4,i_5\} \subset N,\ \{i_1,i_2,i_3,i_4\} \in D(R) \Longrightarrow \{i_1,i_2,i_3,i_5\} \notin D(R). \tag{2.5}$$

Indeed, suppose on the contrary that there exist five voters $i_1,i_2,i_3,i_4,i_5 \in N$ such that $S_1 = \{i_1,i_2,i_3,i_4\}$ and $S_2 = \{i_1,i_2,i_3,i_5\}$. Pose $N\backslash(S_1 \cup S_2) = \{i_6,i_7,i_8\}$. Since $\{i_1,i_2,i_3,i_4\} \in D(R)$ and $\{i_1,i_2,i_3,i_5\} \in D(R)$, it follows by equation (2.4) that $\{i_2,i_3,i_4,i_5\} \notin D(R)$, $\{i_1,i_3,i_4,i_5\} \notin D(R)$, $\{i_1,i_2,i_4,i_5\} \notin D(R)$. Therefore, it follows by Remark 2.1.3 that $S_3 = \{i_1,i_6,i_7,i_8\} \in D(R)$, $S_4 = \{i_2,i_6,i_7,i_8\} \in D(R)$ and $S_5 = \{i_3,i_6,i_7,i_8\} \in D(R)$. By iterating the use of both equation (2.4) and Remark 2.1.3, we prove that this necessarily leads to a contradiction in five stages as follows:

s1) $S_3, S_4 \in D(R) \Rightarrow T_1 = \{i_1,i_2,i_7,i_8\}, T_2 = \{i_1,i_2,i_6,i_8\}, T_3 = \{i_1,i_2,i_6,i_7\} \notin D(R) \Rightarrow S_6 = \{i_3,i_4,i_5,i_6\}, S_7 = \{i_3,i_4,i_5,i_7\}, S_8 = \{i_3,i_4,i_5,i_8\} \in D(R)$.

s2) $S_3, S_5 \in D(R) \Rightarrow T_4 = \{i_1, i_3, i_7, i_8\}, T_5 = \{i_1, i_3, i_6, i_8\}, T_6 = \{i_1, i_3, i_6, i_7\} \notin D(R) \Rightarrow S_9 = \{i_2, i_4, i_5, i_6\}, S_{10} = \{i_2, i_4, i_5, i_7\}, S_{11} = \{i_2, i_4, i_5, i_8\} \in D(R)$.

s3) $S_4, S_5 \in D(R) \Rightarrow T_7 = \{i_2, i_3, i_7, i_8\}, T_8 = \{i_2, i_3, i_6, i_8\}, T_9 = \{i_2, i_3, i_6, i_7\} \notin D(R) \Rightarrow S_{12} = \{i_1, i_4, i_5, i_6\}, S_{13} = \{i_1, i_4, i_5, i_7\}, S_{14} = \{i_1, i_4, i_5, i_8\} \in D(R)$.

s4) $S_8, S_{14} \in D(R) \Rightarrow T_{10} = \{i_1, i_3, i_5, i_8\}, T_{11} = \{i_1, i_3, i_4, i_8\}, T_{12} = \{i_1, i_3, i_4, i_5\} \notin D(R) \Rightarrow S_{15} = \{i_2, i_4, i_6, i_7\}, S_{16} = \{i_2, i_5, i_6, i_7\}, S_{17} = \{i_2, i_6, i_7, i_8\} \in D(R)$.

s5) $S_9, S_{10} \in D(R) \Rightarrow T_{13} = \{i_2, i_5, i_6, i_7\} \notin D(R)$.

Note that $T_{13} = S_{16}$, $T_{13} \notin D(R)$ at Stage $(s4)$ and $S_{16} \in D(R)$ at Stage $(s5)$. A contradiction holds. Therefore, equation (2.5) holds for all $\{i_1, i_2, i_3, i_4, i_5\} \subset N$.

To conclude, note that by Remark 2.1.3, $\{1,2,3,4\} \in D(R)$ or $\{5,6,7,8\} \in D(R)$. Without lost of generality, suppose that $\{1,2,3,4\} \in D(R)$. By equation (2.5), $\{1,2,3,5\} \notin D(R)$ and $\{1,2,3,8\} \notin D(R)$. Therefore by Remark 2.1.3, it necessarily holds that $\{4,6,7,8\} \in D(R)$ and $\{4,5,6,7\} \in D(R)$. This stands in contradiction with (2.5).

We just prove that equation (2.4) is false; that is, equation (2.3) necessarily holds. ■

Proposition 2.1.10. *Let R be a binary voting rule that is not minority sensitive. Assume that there are exactly ten voters and four proposals. Then R is weakly vulnerable to Anscombe's paradox.*

Proof. Consider a binary voting rule R that is not minority sensitive. Assume that there are exactly ten voters and four proposals. The proof consists of two steps.

Step 1: Suppose that there exists some $\{i_1, i_2, i_3, i_4, i_5, i_6\} \subset N$ such that

$$\{i_1, i_2, i_3, i_4, i_5\}, \{i_1, i_2, i_3, i_4, i_6\} \in D(R). \tag{2.6}$$

Then there exists two polarized vectors of votes $v, v' \in \{-1,+1\}^{10}$ such that $F(R,v) = \{i_1, i_2, i_3, i_4, i_5\}$ and $F(R,v') = \{i_1, i_2, i_3, i_4, i_6\}$. Pose $R(v) = \delta$ and $R(v) = \delta'$. Consider the following profile:

	X_{i_1}	X_{i_2}	X_{i_3}	X_{i_4}	X_{i_5}	X_{i_6}	X_{i_7}	X_{i_8}	X_{i_9}	$X_{i_{10}}$	$R[X]$
a^1	$-\delta$	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ	δ
a^2	$-\delta'$	$-\delta'$	$-\delta'$	$-\delta'$	δ'	$-\delta'$	δ'	δ'	δ'	δ'	δ'
a^3	$-\delta$	$-\delta$	δ	δ	$-\delta$	$-\delta$	δ	δ	δ	δ	δ
a^4	δ	δ	$-\delta$	$-\delta$	$-\delta$	$-\delta$	δ	δ	δ	δ	δ

Note that $R\left(X^{a^3}\right) = R\left(X^{a^4}\right) = \delta$ since R is not minority sensitive. In profile X, each member of $S = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ disagrees with $R[X]$ on three proposals out of four. Therefore R is weakly vulnerable to Anscombe's paradox.

Step 2: We prove that equation (2.6) necessary holds for some $\{i_1, i_2, i_3, i_4, i_5, i_6\} \subset N$. To see this, suppose on the contrary that equation (2.6) holds for no $\{i_1, i_2, i_3, i_4, i_5, i_6\} \subset N$; that is for all $\{i_1, i_2, i_3, i_4, i_5, i_6\} \subset N$,

$$\{i_1, i_2, i_3, i_4, i_5\} \in D(R) \Longrightarrow \{i_1, i_2, i_4, i_4, i_6\} \notin D(R). \tag{2.7}$$

Recall that by Remark 2.1.3, $\{1, 2, 3, 4, 5\} \in D(R)$ or $\{6, 7, 8, 9, 10\} \in D(R)$. Without lost of generality, suppose that $\{1, 2, 3, 4, 5\} \in D(R)$. It follows by equation (2.7) that $\{1, 2, 3, 4, 6\} \notin D(R)$ and $\{1, 2, 3, 4, 7\} \notin D(R)$. Therefore, Remark 2.1.3 implies that $\{5, 7, 8, 9, 10\} \in D(R)$ and $\{5, 6, 8, 9, 10\} \in D(R)$. This holds in contradiction to equation (2.7). ■

2.1.3 Avoiding the standard Anscombe's paradox: A summary

In the previous section, we answer the question: given n voters and m proposals, is it possible to construct binary voting rules that are free of Anscombe's paradox? Binary voting rules that are minority sensitive have been discarded; see Proposition 2.1.1. For binary voting rules that are not minority sensitive, several cases have been examined and the following table 2.3 summarizes what is the answer for each possible combination of n and m with the following legend:

No: all binary voting rules that are not minority sensitive do not exhibit Anscombe's paradox;

Neutral: some binary voting rule that is not minority sensitive exhibits Anscombe's paradox;

Yes: all binary voting rules that are not minority sensitive exhibit Anscombe's paradox.

Table 2.3: Avoiding the weak Anscombe's paradox: a summary

m \ n	2	3	4	5	6	$n \geq 7$
3	No	No	Neutral	Yes	Yes	Yes
4	No	No	No	No	Neutral	Yes
$m \geq 5$	No	No	Neutral	Yes	Yes	Yes

2.2 Anscombe's paradox and simple games

Under the majority rule, each majority coalition is endowed with the power to impose an opinion which is unanimously shared by its members. But this is not the case for some other binary voting rules. For example, consider a dictatorial voting rule at which a voter, say i, dictates his/her opinion on all proposals. Then such a rule is minority sensitive and then exhibits the standard Anscombe's paradox. But it is obvious that voter i who holds the entire power of decision is never frustrated; only powerless voters can be frustrated: it is not surprising that all members of a powerless standard majority coalition are frustrated on more than the half of the proposals.

In this section, we consider binary voting rules for which some coalitions S enjoy the power of decision in the sense that a proposal is adopted whenever it is supported by all members of S. Simple voting games are common tools usually used in such a context; (see Shapley 1962 or Moulen and Diffo 2001a). Now, an occurrence of the Anscombe's paradox refers to a situation where all members of a winning coalition (a coalition endowed with the power of decision) are frustrated on more than the half of the proposals.

2.2.1 Simple voting games

Hereafter, we denote by 2^N the set of all coalitions when the set of voters is N.

Definition 2.2.1. Let $W \subset 2^N$. The couple $G = (N, W)$ is a simple voting game if

i) $N \in W$;

ii) $\forall S, T \in 2^N, (S \subset T \text{ and } S \in W) \Rightarrow T \in W$;

iii) $\forall T \in 2^N,\ T \in W \Rightarrow N\backslash T \notin W$.

In this case, a coalition $S \in W$ is said to be a winning coalition.

Simple voting games are used for many decision making contexts. To do this, a binary voting rule is associated with each simple voting game. For example, a proposal is adopted when it is supported by a winning coalition; and is rejected otherwise.

Definition 2.2.2. *Given a simple voting game $G = (N, W)$, the binary voting rule associated with G is denoted by R_G and is defined for all vectors of votes x by*

$$R_G(x) = \begin{cases} +1 & \text{if } \{i \in N \mid x_i = +1\} \in W \\ -1 & \text{otherwise} \end{cases}$$

Given a vote profile X, the vector $G[X] = (R_G(X^a))_{a\in\mathcal{M}}$ denotes the vector of decision under the binary voting rule R_G; we also say that $G[X]$ is the vector of decision under the simple voting game G. Hereafter we identify a simple voting game to its binary voting rule defined above.

Definition 2.2.3. *Let $G = (N, W)$ be a simple voting game.*

A winning coalition $S \in W$ is minimal if

$$\forall i \in S, \quad S\backslash\{i\} \notin W.$$

The set of all minimal winning coalitions will be denoted by $\overline{W}$.

Example 2.2.1. *Let $N = \{1, 2, 3, 4, 5, 6\}$, $\overline{W} = \big\{\{1, 2\}, \{1, 4\}, \{1, 6\}\big\}$ and $\mathcal{M} = \{a^1, a^2, a^3\}$. Here follows a vote profile and the corresponding vector of decision under the simple voting game G.*

	X_1	X_2	X_3	X_4	X_5	X_6	$G[X]$
a^1	-1	-1	$+1$	$+1$	$+1$	$+1$	-1
a^2	-1	$+1$	$+1$	$+1$	$+1$	$+1$	-1
a^3	$+1$	-1	-1	-1	-1	-1	-1

Clearly,

- *voter 1 is a G-veto player;*
- *voters 3 and 5 are all dummy;*

- $G[X] = (-1, -1, -1)$ *is distinct from the majority decision vector* $MR[X] = (+1, +1, -1)$. *Although five voters out of six have voted for the adoption of* a^2, *this proposal is rejected since* $\{2, 3, 4, 5, 6\}$ *is not a winning coalition.*

As with binary voting rules, a simple game may present a particular structure of winning coalitions.

Definition 2.2.4. *A simple game* $G = (N, W)$ *is oligarchic if there exists a coalition* O *such that*

$$\forall S \in 2^N,\ S \in W \iff O \subset S.$$

Definition 2.2.5. *A simple game* $G = (N, W)$ *is dictatorial if there exists a player* i_0 *such that*

$$\forall S \in 2^N,\ S \in W \iff i_0 \in S.$$

In the next section, we study the possibility for the members of a winning coalition to be all frustrated on a majority of proposals.

2.2.2 Anscombe's paradox for simple games

Under a simple voting game, the notion of majority coalition now refers to the notion of winning coalition which may contains less than the half of the voters; but is endowed with the power to impose all decisions unanimously shared by all of its members.

Definition 2.2.6. *Let* X *be a vote profile.*

1. *A voter* i *is* G**-frustrated** *on a proposal* a *if he/she disagrees with the decision on that proposal; that is*

$$X_i^a \neq R_G(X^a).$$

2. *A voter* i *is* G**-frustrated** *on a vote profile* X *if* i*'s vector of votes differs from the vector of collective decisions on more than the half of the proposals; that is*

$$|\{a \in \mathcal{M} \ :\ \ i \text{ G-frustrated on } a\}| > \frac{m}{2}.$$

Note that being G-frustrated given a simple voting game G is equivalent to be R_G frustrated where R_G is the binary voting rule associated to G.

DEFINITION 2.2.7. *Let $G = (N, W)$ be a simple voting game and $\mathcal{M}$ a set of proposals.*

1. *The game G exhibits the qualified Anscombe's paradox (Q-Anscombe's paradox) on $\mathcal{M}$ if there exists a vote profile X such that the set of G-frustrated voters is a winning coalition; that is*

$$\{i \in N \;:\; i\ G\text{-frustrated}\} \in W.$$

2. *The simple game $G = (N, W)$ is Q-Anscombe's paradox free if G does not exhibit the Q-Anscombe's paradox on all possible set $\mathcal{M}$ of proposals.*

Note that a simple voting game may exhibit the Anscombe's paradox but not the qualified Anscombe's paradox. This is obviously the case with a dictatorial simple voting game which is minority sensitive.

The next example illustrates an occurrence of the qualified Anscombe's paradox.

EXAMPLE 2.2.2. *Let $G = (N, W)$ with $N = \{1, 2, 3, 4, 5\}$ and $\overline{W} = \big\{\{1, 2, 3\}, \{2, 4\}\big\}$. The simple voting game G exhibits the Q-Anscombe's paradox on $\mathcal{M} = \{a^1, a^2, a^3\}$ as shown in the vote profile below.*

	X_1	X_2	X_3	X_4	X_5	$G[X]$
a^1	$+1$	$+1$	-1	-1	$+1$	-1
a^2	$+1$	-1	$+1$	$+1$	$+1$	-1
a^3	-1	$+1$	$+1$	-1	-1	-1
	yes	*yes*	*yes*	*no*	*yes*	

In fact, individuals 1, 2, 3 and 5 are G-frustrated at X. Since $\{1, 2, 3\} \in \overline{W}$, then $\{1, 2, 3, 5\} \in W$. Hence G exhibits the Q-Anscombe's paradox on $\mathcal{M}$.

PROPOSITION 2.2.1. *A dictatorial simple game $G = (N, W)$ is Q-Anscombe's paradox free.*

Proof.
Suppose that $G = (N, W)$ is dictatorial. Then there exists a player i_0 such that $\overline{W} = \{\{i_0\}\}$. It is clear that for all sets $\mathcal{M}$ of proposals and for all profiles of vote X on $\mathcal{M}$, we have $\forall a \in \mathcal{M},\ X^a_{i_0} = R_G(X^a)$. Therefore player i_0 is never G-frustrated on a proposal under the simple voting game G. Thus the set of G-frustrated voters is never a winning coalition. The Q-Anscombe's paradox never occurs with respect to G. ∎

Theorem 2.6.

Let $G = (N, W)$ be a simple voting game and $\mathcal{M}$ be a set of proposals. The following assertions are equivalent

(A_1) G does not exhibit the Q-Anscombe's paradox on $\mathcal{M}$.

(A_2) $(\forall S \in \overline{W},\ |S| \leq 2)$ or $(m = 4$ and $\forall S \in \overline{W},\ |S| \leq 3)$.

Proof.

$\Leftarrow$) Suppose that a simple voting game $G = (N, W)$ meets (A_2). For each player $i \in N$, let F_i be the set of all proposals on which player i is G-frustrated.

Now, suppose that G exhibits the Q-Anscombe's paradox. Then there exist a vote profile X and a winning coalition $T \in W$ such that all players in T are G-frustrated on a majority of proposals. Since $T \in W$, there exists a coalition $S \in \overline{W}$ such that $S \subseteq T$. All voters in S are G-frustrated at X.

There are three possible cases:

- **Case 1:** $|S| = 1$. Then G is dictatorial and does not exhibit the Q-Anscombe's paradox as shown in Proposition 2.2.1. A contradiction holds.

- **Case 2:** $|S| = 2$. Then there exist two players i, j such that $S = \{i, j\}$. Voters i and j are then frustrated on a majority of proposals; that is

$$|F_i| > \frac{m}{2} \text{ and } |F_j| > \frac{m}{2}.$$

  It follows that $F_i \cap F_j \neq \emptyset$. Let $a \in F_i \cap F_j$. Therefore i and j are G-frustrated on a. A contradiction occurs since $\{i, j\} \in W$.

- **Case 3:** $|S| \geq 3$. By assumption, $|S| = 3$ and $m = 4$. Each voter in S is frustrated on at least three proposals. Since S is a winning coalition, then for each proposal, there exists a voter in S who is not frustrated on that proposal. Denote by i_t the voter in S who is not frustrated on proposal a^t, $t = 1, 2, 3$. Note that i_1, i_2 and i_3 are distinct voters otherwise a voter in S would be frustrated on at most two proposals. Thus $S = \{i_1, i_2, i_3\}$. For the fourth proposal a^4, there exists a voter $i_s \in S$, $1 \leq s \leq 3$ who is not frustrated on a^4 otherwise a winning coalition would be frustrated on a proposal. Therefore voter i_s is not frustrated on a^s; nor on a^4. This is a contradiction since by assumption, i_s is frustrated on at least three proposals out of four.

This proves that R does not exhibit the Q-Anscombe's paradox. Hence (A_1) holds.
$\Rightarrow$) Suppose that (A_2) is false. Then there exists a coalition $S \in \overline{W}$ such that

$$|S| = s, \ (s > 3) \text{ or } (s = 3 \text{ and } m \neq 4).$$

Let $m = ks + r$ with $r \in \{0, 1 \ldots, s-1\}$ and

$$\mathcal{M} = \bigcup_{t=1}^{k+1} \mathcal{M}_t$$

where $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{k+1}$ are disjoint subsets of proposals such that

$$|\mathcal{M}_{k+1}| = r \text{ and } |\mathcal{M}_t| = s, \ t = 1, 2, \ldots, k.$$

Consider the vote profile X defined progressively as follows: all voters in N vote for the rejection of $a^{(t-1)s+l} \in \mathcal{M}_t$, $1 \leq t \leq s$, except voters in $S\backslash\{l\}$. This is summarized in the following tables:

- **Over proposals in $\mathcal{M}_1$ assuming $k \geq 1$:**

	X_1	$\cdots$	X_l	$\cdots$	X_s	X_{s+1}	$\cdots$	X_n	$G[X]$
a^1	-1	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^l	$+1$	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^s	$+1$	$\cdots$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	-1

- **Over proposals in $\mathcal{M}_t$ assuming $k \geq t$:** votes on $\mathcal{M}_t$ mimic votes on $\mathcal{M}_1$.

	X_1	$\cdots$	X_l	$\cdots$	X_s	X_{s+1}	$\cdots$	X_n	$G[X]$
$a^{(t-1)s+1}$	-1	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{(t-1)s+l}$	$+1$	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{(t-1)s+s}$	$+1$	$\cdots$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	-1

- **Over proposals in $\mathcal{M}_{k+1}$:**

	X_{i_1}	$\cdots$	X_{i_l}	$\cdots$	X_{i_r}	$\cdots$	X_{i_s}	$X_{i_{s+1}}$	$\cdots$	X_{i_n}	$G[X]$
a^{sk+1}	-1	$\cdots$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{ks+l}	$+1$	$\cdots$	-1	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	-1

First observe that only a proper subset of S vote for the adoption of each proposals. Since S is a minimal winning coalition, such a proper subset of S is losing. Therefore each proposal is rejected.

Now, also note that each voter $l \in S$ votes for the adoption of all proposals except for proposals $a^{(t-1)s+l} \in \mathcal{M}$, $1 \leq t \leq k+1$. There are two possible cases:

- Assume that $r = 0$. Then each member of S is in agreement with $G[X]$ on at most k proposals. Moreover, $k \geq 1$ since $m = ks$. Since $s \geq 3$, it follows that

$$\frac{m}{2} - k = \frac{ks}{2} - k = \frac{k(s-2)}{2} > 0$$

  Therefore each member of S agrees with $G[X]$ on at most $k < \frac{m}{2}$ proposals. Hence each member of S disagrees with $G[X]$ on at least $m - k > \frac{m}{2}$ proposals. Recalling that S is a winning coalition, we conclude that G exhibits the qualified Anscombe's paradox.

- Assume that $r \neq 0$. Then each member of S is in agreement with $G[X]$ on at most $k+1$ proposals. To see that $k+1 < \frac{m}{2}$, suppose on the contrary that $k+1 \geq \frac{m}{2} = \frac{ks+r}{2}$. Then we have $k(2-s) \geq r-2$. Since $s > 2$, it follows that $k(2-s) \leq 0$. Therefore $r \in \{1, 2\}$.

  - **Case 1:** Suppose that $r = 2$. Then $k(2-s) \geq 0$. Since $2 - s < 0$, this implies that $k = 0$ and $m = ks + r = 2$. A contradiction holds since $m \geq 3$.

  - **Case 2:** Suppose that $r = 1$. Then $k > 0$ otherwise $m = 1$ which is contradictory. Therefore $-1 \leq k(2-s) < 0$. It follows that $k(2-s) = -1$. Hence $s = 3$ and $k = 1$. Therefore $s = 3$ and $m = ks + r = 4$ which leads to a contradiction to our assumption.

In both cases for $r \neq 0$, it appears that $k+1 < \frac{m}{2}$. Therefore each member of S agrees with $G[X]$ on at most $k+1 < \frac{m}{2}$ proposals. Hence each member of S disagrees with $G[X]$ on at least $m-k-1 > \frac{m}{2}$ proposals. Therefore G exhibits the qualified Anscombe's paradox.

In both possible cases, G exhibits the qualified Anscombe's paradox. ■

Avoiding the Q-Anscombe's paradox given a simple voting game requires a specific structure on winning coalitions. We now explore some consequences that result from such structures.

Remark 2.2.1. *Let $G = (N, W)$ be a simple voting game.*

A voter $i \in N$ is a null voter if no minimal winning coalition contains voter i. That is

$$\forall S \in \overline{W}, \quad i \notin S.$$

Definition 2.2.8. *Let $G = (N, W)$ be a simple voting game.*

A voter i is locally decisive if there exists a minimal winning coalition in G which contains voter i. That is

$$\exists S \in \overline{W}, \quad i \in S.$$

Note that a locally decisive voter is simply a voter who is not a null voter in the game.

Definition 2.2.9. *Let $G = (N, W)$ be a simple voting game.*

A voter i is a vetoer if all minimal winning coalition contain voter i. That is

$$\forall S \in \overline{W}, \quad i \in S.$$

Given a vote profil X, and a vetoer i. If i vote for the rejection of a proposal $a \in \mathcal{M}$ then the proposal a is collectively rejected. That is

$$X_i^a = -1 \Rightarrow R_G(X^a) = -1.$$

Corollary 2.2.1. *Let $G = (N, W)$ be a simple voting game. The following assertions are equivalent*

(A_1) *For all possible set $\mathcal{M}$ of proposals, G does not exhibit the Q-Anscombe's paradox on $\mathcal{M}$.*

(A_2) $\forall S \in \overline{W},\ |S| \leq 2.$

Proof.
Straightforward from Theorem 2.6. ∎

Here below, we classify simple voting games from Corollary 2.2.1.

Proposition 2.2.2. *Let $G = (N, W)$ be a simple voting game such that for all $S \in \overline{W}$, $|S| = 2$. The following assertions are equivalent*

1. *G admits no vetoer.*
2. *There exist three voters $i, j, k \in N$ such that $\overline{W} = \{\{i,j\},\{i,k\},\{j,k\}\}$.*

Proof.

$\Rightarrow$) Suppose that G admits no vetoer. We prove that $\overline{W} = \{\{i,j\},\{i,k\},\{j,k\}\}$ for some $i, j, k \in N$. Note that $|\overline{W}| \notin \{1,2\}$ otherwise G would admit a vetoer since two winning coalitions always overlap. Therefore $\overline{W} \geq 3$. Since S_1, S_2 and S_3, are three distincts winning coalitions of cardinality two then $S_1 \cap S_2 = \{i_1\}$, $S_1 = \{i_1, i_2\}$ and $S_2 = \{i_1, i_3\}$. Moreover S_1 and S_3 overlap.

Suppose that $S_1 \cap S_3 = \{i_1\}$. Recalling that S_1, S_2 and S_3 are distinct minimal winning coalitions, it follows that $S_3 = \{i_1, i_4\}$ where i_1, i_2, i_3 and i_4 are distinct voters. We prove that this is not possible by showing that i_1 is necessary a vetoer. To see this, consider a minimal winning coalition $S \in \overline{W} \backslash \{S_1, S_2, S_3\}$ and suppose that $i_1 \notin S$. Therefore $S \cap S_1 = \{i_2\}$, $S \cap S_2 = \{i_3\}$ and $S \cap S_3 = \{i_4\}$. Since i_2, i_3 and i_4 are distinct voters and $|S| = 2$, a contradiction arrives.

We just show in the precedent paragraph that $S_1 \cap S_3 \neq \{i_1\}$. Therefore $S_1 \cap S_3 = \{i_2\}$. Since $S_2 \cap S_3 \neq \emptyset$ and $S_2 \neq S_3$, then $S_3 = \{i_2, i_3\}$. This prove that $\{i_1, i_2\}, \{i_1, i_3\}, \{i_2, i_3\} \in \overline{W}$.

Conversely, consider $S \in \overline{W}$ and suppose that $S \notin \{S_1, S_2\}$. By assumption S_1 and S are two distinct minimal winning coalitions. Therefore $S_1 \cap S$ is a singleton. First suppose that $S_1 \cap S = \{i_1\}$. Thus $S \cap S_2 = \{i_1\}$ and $i_2, i_3 \notin S$. Therefore $S \cap S_3 = \emptyset$ and a contradiction arises. Therefore $S \cap S_1 = \{i_2\}$. This implies that $S \cap S_2 = \{i_3\}$. Therefore $S = \{i_2, i_3\} = S_3$. This prove that $S \in \{S_1, S_2\}$ or $S = S_3$. It then follows $\overline{W} \subset \{S_1, S_2, S_3\}$.

$\Leftarrow$) Suppose that there exist three voters $i, j, k \in N$ such that $\overline{W} = \{\{i,j\},\{i,k\},\{j,k\}\}$. It is clear that G admits no vetoer. ■

Remark 2.2.2 (Classification of all games of Corollary 2.2.1).

According to Proposition 2.2.2, the set $\overline{W}$ of minimal winning coalitions of a simple voting games from Corollary 2.2.1 is as follows.

- *If $|\overline{W}| = 1$, then there exist two voters i, j such that, $\overline{W} = \{\{i,j\}\}$. In this case, G is oligarchic.*
- *If $|\overline{W}| = 2$, then there exist three voters i, j and k such that $\overline{W} = \{\{i,j\},\{i,k\}\}$. In this case, i is the unique vetoer; j and k are locally decisive; and all remaining voters are null voters.*
- *If $|\overline{W}| = 3$ then $\overline{W}$ is defined as follow*
  - *There exist three voters i, j and k such that, $\overline{W} = \{\{i,j\},\{i,k\},\{j,k\}\}$. In this case G admits no vetoer; i, j and k are locally decisive; and remaining voters are null voters.*
  - *There exist four voters i, j, k and l such, $\overline{W} = \{\{i,j\},\{i,k\},\{i,l\}\}$. In this case, i is the unique vetoer; j, k and l are locally decisive; and all remaining voters are null voters.*
- *If $|\overline{W}| \geq 4$, then G admits a unique vetoer; and all other voters except null voters are locally decisive.*

As shown above, simple voting games which are, independently of the number of proposals, immune to Q-Anscombe's paradox are singular ones: the size of a minimal winning coalition for such games is at most 2; and there is necessary a unique vetoer as soon as there are at least four minimal winning coalitions (no matter the size of the set N of all voters).

According to Theorem 2.6 and given a simple voting game, the following table summarizes what is the answer for each possible combination of n and m with the following legend:

No: all simple voting games do not exhibit Anscombe's paradox;

Neutral: some simple voting games exhibits Anscombe's paradox;

m \ n	2	3	$n \geq 4$
3	No	Neutral	Neutral
4	No	No	Neutral
$m \geq 5$	No	Neutral	Neutral

Combinations of n and m for which some binary voting rules exhibit Anscombe's paradox while other do not are such that

$$(n = 3 \text{ and } m \neq 4) \text{ or } n \geq 4.$$

For such couples (n, m), Remark 2.2.2 classify simple voting game that do not exhibit the Anscombe's paradox.

CHAPTER 3

ANSCOMBE'S PARADOX FREE UNIFYING PREFERENCE DOMAINS

In this chapter, we identify some Anscombe's paradox free domains of individual preferences, those are preference domains that yield only profiles at which the Anscombe's paradox never occurs. To achieve this, we assume that among proposals some may be special as they deal with crucial issues such as sovereignty, war against terrorism, constitutional amendments, electoral dispositions, ... By patriotism or common-sense, voters can unite across the prevailing political divide to deal with such proposals in such a way that each voter deviates from the issue-specific standards (default opinions for voters) over some unifying proposals only on a limited number of issues. Section 3.1 presents some generalities and preliminary results on the newly introduced notion of unifying preference domain. Section 3.2 provides necessary and sufficient conditions under which a unifying preference domain is free of Anscombe's paradox. In section 3.3, we prove that our notion of stable unifying preference domain combined with the notion of single-switchness of Laffond and Lainé (2006) completely characterizes all Anscombe's paradox free domains with exactly three proposals.

3.1 The unifying voting environment and the Anscombe's paradox

Recall that a preference domain D is Anscombe's paradox free if for all $n \geq 3$, there is no admissible vote profile $X \in D^n$ such that a majority of voters disagree with the social decision on a majority of proposals. Hereafter, we propose the construction of a family of such domains.

3.1.1 Unifying voting context

To present the intuition we develop in this chapter, suppose that voters are in a community where:

(i) For each proposal, an arbitrary standard ($+1$ or -1) exists. By collecting all standards, one obtains a vector $x^* \in \{-1, +1\}^m$ of common standards;

(ii) there exist a subset $\mathcal{U}$ of $u \geq 0$ proposals and an integer $k \geq 0$ such that each voter deviates from the issue-specific standards x^* over these u proposals on at most $k \geq 0$ issues.

We check whether or not the majority rule still exhibits the Anscombe's paradox under such voting environments. We now provide below formal definitions of the parameters in the description above.

Without lost of generality, we assume that

- $\mathcal{U} = \{a_1, a_2, \ldots, a_u\}$;
- v_j is the vote on proposal a_j with respect to a given vector of votes $v \in \{-1, +1\}^m$.

Definition 3.1.1. *A vector of common standards is any vector* $x^* \in \{-1, +1\}^m$.

Given a vector of common standards x^* and a proposal a_j, ${x^*}_j$ is the default vote on proposal a_j. Note that a vector of common standards does not impose any restriction on individual opinions. We assume that there is a chosen vector x^* of common standards.

Most importantly, each individual opinion is set to $+1$ if it agrees with the corresponding common standard; and to -1 otherwise. The same convention is observed for the majority decision on a proposal; that is, the majority decision on a proposal is set to

+1 if a majority of voters agrees with the corresponding common standard; and to -1 otherwise.

Definition 3.1.2. *A unifying voting context is a triplet $(\mathcal{M}, \mathcal{U}, k)$ where*

- *$\mathcal{M}$ is the set of all proposals;*
- *$\mathcal{U}$ is a subset of $\mathcal{M}$;*
- *k is a non negative integer.*

In this case, $\mathcal{U}$ will be called the set of unifying proposals; k the barometer of consensus; and the quadruplet $(N, \mathcal{M}, \mathcal{U}, k)$, a unifying voting environment.

The set $\mathcal{U}$ of unifying proposals can be viewed as a subset of proposals of some critical importance; and the barometer k is an a priori maximal number of unifying proposals on which a voter may deviate. This is clearly a restriction of individual opinions as soon as k is greater than or equal to 1.

Definition 3.1.3. *A vector of votes v is* **admissible** *given $(\mathcal{M}, \mathcal{U}, k)$ if v and the vector of common standards x^* differ over the u unifying proposals of $\mathcal{U}$ on at most k proposals; that is*

$$|\{j \leq u : v_j \neq x_j^*\}| \leq k.$$

The set of all such admissible vectors of votes is called the unifying preference domain associated with $(\mathcal{M}, \mathcal{U}, k)$.

Example 3.1.1. *Consider the unifying voting context $(\mathcal{M}, \mathcal{U}, k)$ where $\mathcal{M} = \{a_1, a_2, a_3\}$, $\mathcal{U} = \{a_1, a_2\}$ and $k = 1$. There are exactly six admissible vectors of votes listed below*

$$(+1,+1,+1), (+1,+1,-1), (-1,+1,+1), (-1,+1,-1), (+1,-1,+1) \text{ and } (+1,-1,-1).$$

To see this, note that under the unifying voting context $(\mathcal{M}, \mathcal{U}, k)$, each voter can deviate from the common standards on $\mathcal{U} = \{a_1, a_2\}$ only on at most one proposal. Thus there are three possibilities left on vote combinations on $\mathcal{U}$. Considering the two other possibilities for votes on a_3, one obtain exactly six ways a voter may follow.

Obviously the vector of votes $(-1,-1,+1)$ is not admissible since the votes proposed deviate from common standards on two unifying proposals.

Remark 3.1.1. *Note that all vectors of votes are admissible when the set on unifying proposals $\mathcal{U} = \emptyset$ or the barometer $k = |\mathcal{U}|$.*

Similarly, the notion of admissible vote profile is defined as follows.

Definition 3.1.4. *A vote profile X is admissible given $(N, \mathcal{M}, \mathcal{U}, k)$ if for each voter i, X_i is an admissible vector of votes.*

We denote by $\mathcal{D}(m, u, k)$ the set of all admissible vectors of votes and by $\mathcal{D}^n(m, u, k)$ the corresponding unifying preference domain with the convention that, there are m proposals $a_1, a_2, \ldots, a_m$ and u unifying proposals $a_1, a_2, \ldots, a_u$ while the barometer of consensus is k.

3.1.2 The Anscombe's paradox and unifying voting environments

Note that unifying voting contexts, and thus unifying preference domains, do not include the set of voters. We are interested in unifying preference domains that are free of Anscombe's paradox no matter the set of voters.

Before we continue, let us point that a unifying preference domain $\mathcal{D}(m, u, k)$ is Anscombe's paradox free if for all $n \geq 3$, there is no vote profile X such that each voter deviates from the vector of common standards over the u unifying proposals of $\mathcal{U}$ on at most k proposals while a majority of voters disagree with the social decision on a majority of proposals.

Definition 3.1.5. *The unifying voting context $(\mathcal{M}, \mathcal{U}, k)$ is stable if the corresponding unifying domain $\mathcal{D}(m, u, k)$ is Anscombe's paradox free.*

Hereafter in the table of a profil, the unifying proposals is listed in the upper part of the table and separated from non unifying proposals by a double-line as in Example 3.1.2 below.

Example 3.1.2. *Consider the unifying preference domain $\mathcal{D}(5, 1, 0)$ with five proposals, one unifying proposal and the barometer $k = 0$. Then the vote profile X below* with seven voters is admissible.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$MR[X]$
a^1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^2	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	$+1$
a^3	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	$+1$
a^4	$+1$	$+1$	$+1$	-1	-1	$+1$	-1	$+1$
a^5	-1	-1	-1	$+1$	$+1$	$+1$	-1	-1
	no	*no*	*no*	*yes*	*yes*	*yes*	*yes*	$4/7$

With the convention just stated and independently of the vector of common standards, $X_1^1 = +1$ means that voter 1 follows the common standard on proposal a^1 while $X_1^5 = -1$ means that voter 1 deviates from the common standard on proposal a^5. Four voters (4, 5, 6 and 7) out of seven are each frustrated on three proposals out of five. This shows that a majority of voters are frustrated on a majority of proposals. At this admissible vote profile, Anscombe's paradox occurs. Thus the unifying preference domain $\mathcal{D}(5, 1, 0)$ is not Anscombe's paradox free.

3.1.3 Majority rule and unifying preference domains

Note that the set of proposals on which a voter i is frustrated and its cardinality are respectively:

$$\{a^j : MR(X^j) \neq X_i^j\} \text{ and } |\{a^j : MR(X^j) \neq X_i^j\}| = \frac{1}{2}\sum_{j=1}^{m} |MR(X^j) - X_i^j|.$$

Now we introduce further notations to explore the set of proposals on which a voter is frustrated and which are useful in the sequel. Given a vote profile $X \in \mathcal{D}^n(m, u, k)$, we denote by

$$\mathcal{O}(X) = \{a^j : j \leq u \text{ and } MR(X^j) = -x_j^*\}$$

the set of all unifying proposals at which each majority decision differs from the corresponding common standard and by

$$\mathcal{O}_i(X) = \{a^j : j \leq u \text{ and } X_i^j = -x_j^*\}$$

the set of all unifying proposals at which individual i deviates from common standards. For each subset I of the set $\{a^1, a^2, ..., a^u\}$ of unifying proposals, X_I is the set of all voters who deviate from common standards over unifying proposals only for proposals in I; that is

$$X_I = \{i \in N : \mathcal{O}_i(X) = I\}.$$

Note that $(X_I)_{I \subseteq \{a^1,a^2,...,a^u\}}$ is a collection of disjoint subsets of N the union of which is the set $\{1,2,...,n\}$ of all voters. Similarly,

$$\mathcal{O}'(X) = \{a^j : j > u \text{ and } MR(X^j) = -x^*_j\}$$

refers to the set of all non unifying proposals for each of which the majority decision differs from the corresponding common standard and

$$\mathcal{O}'_i(X) = \{a^j : j > u \text{ and } X^j_i = -x^*_j\}$$

refers to the set of all non unifying proposals at which voter i deviates from each common standard. When there is no ambiguity, we simply write $\mathcal{O}$ instead of $\mathcal{O}(X)$; the same observation is valid for $\mathcal{O}_i(X)$, $\mathcal{O}'(X)$ and $\mathcal{O}'_i(X)$.

Example 3.1.3. *Consider the unifying preference domain $\mathcal{D}(5,3,2)$ and the following admissible vote profile with seven voters.*

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$MR[X]$
a^1	-1	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
a^2	$+1$	-1	$+1$	$+1$	-1	-1	-1	-1
a^3	-1	-1	-1	$+1$	-1	$+1$	$+1$	-1
a^4	$+1$	$+1$	$+1$	-1	-1	$+1$	-1	$+1$
a^5	-1	$+1$	-1	$+1$	-1	$+1$	-1	-1
	no	*no*	*no*	*yes*	*no*	*no*	*no*	1/7

Here $\mathcal{O} = \{a^2,a^3\}$ and $\mathcal{O}' = \{a^5\}$, $\mathcal{O}_1 = \{a^1,a^3\}$ and $\mathcal{O}'_1 = \{a^5\}$, $\mathcal{O}_2 = \{a^2,a^3\}$ and $\mathcal{O}'_2 = \emptyset$, ... Non empty sets from the collection $(X_I)_{I \subseteq \{a^1,a^2,a^3\}}$ are $X_{\{a^1,a^3\}} = \{1\}$, $X_{\{a^2,a^3\}} = \{2,5\}$, $X_{\{a^3\}} = \{3\}$, $X_\emptyset = \{4\}$ and $X_{\{a^2\}} = \{6,7\}$.

Note that a voter, say i, can be frustrated given a proposal, say a^j, both when the majority decision is x^*_j and he/she has voted $-x^*_j$ (equivalently $a^j \in \mathcal{O}_i \backslash \mathcal{O}$), as well as when the majority decision is $-x^*_j$ and he/she has voted x^*_j (equivalently $a^j \in \mathcal{O} \backslash \mathcal{O}_i$). Therefore the set of unifying proposals on which voter i is frustrated is given by :

$$\mathcal{F}_i = (\mathcal{O}_i \backslash \mathcal{O}) \cup (\mathcal{O} \backslash \mathcal{O}_i).$$

Denote by f_i the cardinality of $\mathcal{F}_i$. Note that $\mathcal{O}_i \backslash \mathcal{O}$ and $\mathcal{O} \backslash \mathcal{O}_i$ are disjoint subsets of unifying proposals and that $|\mathcal{O}_i \backslash \mathcal{O}| = |\mathcal{O}_i| - |\mathcal{O}_i \cap \mathcal{O}|$. Thus

$$f_i = |\mathcal{O}_i \backslash \mathcal{O}| + |\mathcal{O} \backslash \mathcal{O}_i| = |\mathcal{O}_i| + |\mathcal{O}| - 2|\mathcal{O}_i \cap \mathcal{O}|. \tag{3.1}$$

Similarly, the set of non unifying proposals on which voter i is frustrated is :

$$\mathcal{F}'_i = (\mathcal{O}'_i \backslash \mathcal{O}') \cup (\mathcal{O}' \backslash \mathcal{O}'_i).$$

We also denote by f'_i the cardinality of $\mathcal{F}'_i$. As above, we have

$$f'_i = |\mathcal{O}'_i \backslash \mathcal{O}'| + |\mathcal{O}' \backslash \mathcal{O}'_i| = |\mathcal{O}'_i| + |\mathcal{O}'| - 2|\mathcal{O}'_i \cap \mathcal{O}'|. \tag{3.2}$$

Then voter i is frustrated on exactly

$$\frac{1}{2} \sum_{j=1}^{m} |MR(X^j) - X_j| = f_i + f'_i$$

proposals. Furthermore, voter i is frustrated on a majority of proposals when

$$f_i + f'_i > \frac{1}{2} m. \tag{3.3}$$

Remark 3.1.2. *It is worth mentioning that when two voters are frustrated on the same proposals, those voters have the same opinion on that proposal. Therefore, a majority of voters can not be frustrated on the same proposal.*

3.2 Stability of unifying voting environments

When we consider a unifying preference domain $\mathcal{D}(m, u, k)$ with $k = u$, all vectors of votes are admissible. It is well known that under the majority rule with at least three proposals, there exist some vote profiles at which a majority of voters is frustrated on a majority of proposals. Rephrasing using the current framework: under the majority rule, there is no unifying preference domain of the form $\mathcal{D}(m, u, u)$ with $m \geq 3$ which is Anscombe's paradox free.

3.2.1 Necessary and sufficient stability conditions for $k = 0$

The next result characterizes unifying preference domains $\mathcal{D}(m, u, 0)$ that does not face Anscombe's paradox; that is the extreme case when each individual vote on each unifying proposal coincides with the common standard.

Proposition 3.2.1. *A unifying preference domain $\mathcal{D}(m,u,0)$ is Anscombe's paradox free if and only if $u \geq \frac{1}{2}m - 1$.*

Proof.

$\Rightarrow$) Suppose that $u < \frac{1}{2}m - 1$. Then $m - u = 1 + p$ with $p > \frac{1}{2}m$. Let $n = 2p+1$, $S = \{1, 2, \ldots, p+1\}$ and $T = \{p+2, p+3, \ldots, 2p+1\}$. Consider the vote profile $X \in \mathcal{D}^n(m,u,0)$ defined as follow:

(i) each voter follows common standards on all unifying proposals, that is

$$X_i^j = x_j^*,\ \forall j \in \{1, 2, \ldots, u\},\ \forall i \in \{1, 2, \ldots, 2p+1\};$$

(ii) each voter in T follows common standards on all non unifying proposals, that is

$$X_i^j = x_j^*,\ \forall j \in \{u+1, u+2, \ldots, m\},\ \forall i \in T;$$

and

(iii) each voter $i \in S$ deviates from common standards on all non unifying proposals except proposal a^{u+i}, that is

$$(X_i^j = -x_j^* \text{ if } j \neq u{+}i \text{ and } X_{i,u+i} = x_j^* \text{ if } j = u{+}i),\ \forall i \in S,\ \forall j \in \{u{+}1, u{+}2, \ldots, m\}.$$

	X_1	X_2	X_3	$\cdots$	X_p	X_{p+1}	X_{p+2}	$\cdots$	X_{2p+1}	$MR[X]$
a^1	$+1$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
a^u	$+1$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	$\cdots$	$+1$	$+1$
a^{u+1}	$+1$	-1	-1	$\cdots$	-1	-1	$+1$	$\cdots$	$+1$	$+1$
a^{u+2}	-1	$+1$	-1	$\cdots$	-1	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
a^{u+p+1}	-1	-1	-1	$\cdots$	-1	$+1$	$+1$	$\cdots$	$+1$	$+1$
	yes	*yes*	*yes*	$\cdots$	*yes*	*yes*	*no*	$\cdots$	*no*	$\frac{p+1}{2p+1}$

Clearly, $X \in \mathcal{D}^n(m,u,0)$ and each voter in S is frustrated on p proposals. Since $|S| > \frac{n}{2}$ and $p > \frac{1}{2}m$, the unifying preference domain $\mathcal{D}(m,u,0)$ is not Anscombe's paradox free.

$\Leftarrow$) Suppose that $u \geq \frac{1}{2}m - 1$ and assume that the unifying preference domain $\mathcal{D}(m, u, 0)$ is not Anscombe's paradox free. Then there exists some total number n of voters and a vote profile $X \in \mathcal{D}^n(m, u, 0)$ such that a majority S ($|S| > \frac{n}{2}$) of voters are frustrated on a majority of proposals. By assumption, voters are unanimous on each unifying proposal. Thus each voter agrees with the majority decision on the u unifying proposals. Thus $u < m$; otherwise no voter is frustrated on any proposal. Consider a non unifying proposal a^j. Since S contains more than the half of voters, then by Remark 3.1.2, at least one voter in S, say i_0, agrees with the majority decision on a^0. Voter i_0 then agrees with the majority decision on at least $u + 1$ proposals. Since i_0 is frustrated on a majority of proposals, then $u + 1 < \frac{1}{2}m$. A contradiction as $u \geq \frac{1}{2}m - 1$. ■

When each voter follows on each unifying proposal the corresponding common standard, Proposition 3.2.1 informally states that, the corresponding domain is Anscombe's paradox free provided that the total number of unifying proposals is greater than or equal to the half of the proposals, minus one proposal. As shown in the next example, the condition $|u| \geq \frac{1}{2}m - 1$ from Proposition 3.2.1 is no more sufficient to guarantee the stability of unifying preference domains with a positive barometer of consensus $k \geq 1$.

Example 3.2.1. *Consider the unifying preference domain $\mathcal{D}(5, 2, 1)$ and the following 5-voter vote profile*

	X_1	X_2	X_3	X_4	X_5	$MR[X]$
a^1	$+1$	-1	-1	$+1$	$+1$	$+1$
a^2	-1	$+1$	$+1$	$+1$	$+1$	$+1$
a^3	$+1$	-1	-1	$+1$	$+1$	$+1$
a^4	-1	$+1$	-1	$+1$	$+1$	$+1$
a^5	-1	-1	$+1$	$+1$	$+1$	$+1$
	yes	*yes*	*yes*	*no*	*no*	$3/5$

Since Anscombe's paradox occurs at this admissible vote profile, the unifying preference domain $\mathcal{D}(5, 2, 1)$ is not Anscombe's paradox free although the condition $|u| \geq \frac{1}{2}m - 1$ from Proposition 3.2.1 is satisfied.

To deal with the general case, we need some preliminary results.

3.2.2 Analyzing disagreements on unifying proposals

Given a unifying preference domain and an admissible vote profile, the next result provides the maximum number of unifying proposals on each of which the majority decision differs from the corresponding common standard.

Proposition 3.2.2. *Given a unifying preference domain $\mathcal{D}(m,u,k)$ and an admissible vote profile, there are at most $2k$ unifying proposals on which each majority decision differs from the corresponding common standard. That is*

$$|\mathcal{O}| \in \{0,1,2,\ldots,2k\}.$$

Proof.

Consider $X \in \mathcal{D}^n(m,u,k)$. Recall that for each subset I of U and for each $a^j \in I \cap \mathcal{O}$, the vote of each voter in X_I on a^j and the majority decision on a^j are both equal to $-x_j^*$. Thus, for each $j \in \{1,2,\ldots,u\}$, we have $\sum\limits_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I}} |X_I| \geq \frac{n}{2}$. Considering all possible $a^j \in \mathcal{O}$ implies,

$$\sum_{a^j\in\mathcal{O}} \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I}} |X_I| \geq |\mathcal{O}|\frac{n}{2} \tag{3.4}$$

But

$$\sum_{a^j\in\mathcal{O}} \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I}} |X_I| = \sum_{I\subseteq\mathcal{O}} \sum_{a^j\in I\cap\mathcal{O}} |X_I| = \sum_{I\subseteq\mathcal{O}} |I\cap\mathcal{O}||X_I| \tag{3.5}$$

Since X is admissible, $|X_I| = 0$ for all $I \subseteq \mathcal{O}$ such that $|I| > k$. Therefore,

$$\sum_{\substack{I\subseteq\mathcal{O}\\ |I|\leq k}} |X_I| = n \text{ and } \sum_{I\subseteq\mathcal{O}} |I\cap\mathcal{O}||X_I| = \sum_{\substack{I\subseteq\mathcal{O}\\ |I|\leq k}} |I\cap\mathcal{O}||X_I| \leq k \sum_{\substack{I\subseteq\mathcal{O}\\ |I|\leq k}} |X_I| \tag{3.6}$$

By equation (3.4), (3.5) and (3.6), we deduce that

$$k \sum_{\substack{I\subseteq\mathcal{O}\\ |I|\leq k}} |X_I| = nk \geq |\mathcal{O}|\frac{n}{2} \tag{3.7}$$

Hence $|\mathcal{O}| \leq 2k$. ∎

Note that although each individual deviates from common standards on at most k unifying proposals, Proposition 3.2.2 states that the collection of majority decisions may differ from common standards on up to $2k$ unifying proposals. The following proposition shows that by increasing only the total number of unifying proposals from an Anscombe's paradox free unifying preference domain, we obtain an Anscombe's paradox free unifying preference domain. Intuitively, increasing the set of unifying proposals and maintaining the barometer of consensus results in increasing the total number of unifying proposals on which each voter follows common standards.

Proposition 3.2.3. *Let $\mathcal{D}(m,u,k)$ and $\mathcal{D}(m,u',k)$ be two unifying preference domains such that $u \leq u'$. If $\mathcal{D}(m,u,k)$ is Anscombe's paradox free, then so is $\mathcal{D}(m,u',k)$.*

Proof.

Assume that $u \leq u'$ and suppose that $\mathcal{D}(m,u,k)$ is Anscombe's paradox free while $\mathcal{D}(m,u',k)$ is not. Then for some total number n of voters, there exists some admissible vote profile $X \in \mathcal{D}^n(m,u',k)$ such that a majority of voters are frustrated on a majority of proposals. Recall that unifying proposals at $\mathcal{D}(m,u',k)$ are the u' first proposals. As $u \leq u'$, all unifying proposals at $\mathcal{D}(m,u,k)$ are also unifying proposals at $\mathcal{D}(m,u',k)$. Since X is admissible on $\mathcal{D}(m,u',k)$, X is also admissible on $\mathcal{D}(m,u,k)$. Therefore $X \in \mathcal{D}^n(m,u,k)$ and a majority of voters are frustrated on a majority of proposals at X. Thus $\mathcal{D}(m,u,k)$ is not Anscombe's paradox free. A contradiction. ■

3.2.3 Necessary and sufficient stability conditions for $k \neq 0$

Given the total number m of proposals and a barometer of consensus k, we now prove that the set of unifying proposals should be large enough to guarantee that $\mathcal{D}(m,u,k)$ is Anscombe's paradox free. We do this by providing an upper bound of u up to which $\mathcal{D}(m,u,k)$ is not an Anscombe's paradox free domain.

Proposition 3.2.4. *Given an integer $k \geq 1$, a unifying preference domain $\mathcal{D}(m,u,k)$ is not Anscombe's paradox free whenever $u < \frac{1}{2}m + 2k - 2$.*

Proof.

Let $\mathcal{D}(m,u,k)$ be a unifying preference domain and u^* be the smallest integer greater than or equal to $\frac{1}{2}m + 2k - 2$. Suppose that $k \geq 1$ and $u < u^*$. Note that $u < u^*$ is equivalent to $u < \frac{1}{2}m + 2k - 2$.

To prove that $\mathcal{D}(m,u,k)$ is not Anscombe's paradox free, it is sufficient, by applying Proposition 3.2.3, to prove that $\mathcal{D}(m,u,k)$ is not Anscombe's paradox free for $u = u^* - 1$.

Now, assume that $u = u^* - 1$. We set $m = 2m' + r$ with $r \in \{0, 1\}$; $p = m' + 1$; $n = 2p + 1$; $q = p - 2$ if $r = 0$ and $q = p - 1$ if $r = 1$. Note that $u^* = q + 2k - 1$ and $u = q + 2k - 2$. It is easy to check that $m - u = p - 2k + 2$. Since $u \leq m$, we have $m - u \geq 0$ and thus $p \geq 2k - 2$. This implies $p + k \leq p + 2k - 1 \leq 2p + 1 = n$. Consider the vote profile X defined as follows:

- for all $i = 1, \ldots, p$,

$$X_i^j = \begin{cases} x_j^* & \text{if } j \in \{1, \ldots, q\} \cup \{q+1, \ldots, q+k-1\} \cup \{q+2k, \ldots, m\} \\ -x_j^* & \text{if } j \in \{q+k+1, \ldots, q+2k-1\} \end{cases}$$

- for all $i_l = p + l$, $l = 1, \ldots, k-1$

$$X_{i_l,j} = \begin{cases} x_j^* & \text{if } j \in \{1, \ldots, q\} \cup \{q+l\} \cup \{q+k, \ldots, q+2k-1\} \\ -x_j^* & \text{if } j \in \{q+1, \ldots, q+l-1, q+l+1, \ldots, q+k-1\} \cup \{q+2k, \ldots, m\} \end{cases}$$

- for all $i_e = p + k + e$, $e = 1, \ldots, k$

$$X_{i_e,j} = \begin{cases} x_j^* & \text{if } j \in \{1 \ldots, q\} \cup \{q+k, \ldots, q+k+e-1, q+k+e+1, \ldots, q+2k-1\} \\ -x_j^* & \text{if } j \in \{q+1, \ldots, q+k-1\} \cup \{q+k+e\} \cup \{q+2k, \ldots, m\} \end{cases}$$

- for all $i_t = p + k + t$, $t = 1, \ldots, p - 2k + 2$

$$X_{i_t,j} = \begin{cases} x_j^* & \text{if } j \in \{1 \ldots, q\} \cup \{q+k, \ldots, q+2k-1\} \cup \{q+2k-1+t\} \\ -x_j^* & \text{if } j \in \{q+1, \ldots, q+k-1\} \cup \{q+2k, \ldots, q+2k+t-1, q+2k+t+1, \ldots, m\} \end{cases}$$

Here follows the matrix form of X with $l = 1, \ldots, k-1$; $e = 1, \ldots, k$ and $t = 1, \ldots, p - 2k + 2$.

	X_1	$\cdots$	X_p		$\cdots$	X_{p+l}	$\cdots$	X_{p+k-1}		$\cdots$	X_{p+k+e}	$\cdots$	X_{p+2k-1}		$\cdots$	$X_{p+2k-1+t}$	$\cdots$	X_{2p+1}	$MR[X]$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^q	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$
a^{q+1}	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^{q+l}	$+1$	$\cdots$	$+1$	-1	$\cdots$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^{q+k-1}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	$+1$
a^{q+k}	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^{q+k+e}	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	-1	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^{q+2k-1}	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	-1	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1
a^{q+2k}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	$+1$	$\cdots$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$a^{q+2k-1+t}$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	$+1$	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$\cdots$	$+1$	$+1$
	No			Yes					Yes					Yes					$\frac{p+1}{2p+1}$

There is no difficulty to check that:

(i) the votes of each voter differ from common standard on at most k unifying proposals; and that

(ii) each voter $i \in \{p+1, \ldots, 2p+1\}$ is frustrated on exactly $m-q-1$ proposals.

Since $m-q-1 = \frac{1}{2}m+1$ for $r=0$ and $m-q-1 = \frac{1}{2}m+\frac{3}{2}$ for $r=1$, it follows that X is an admissible vote profile at which a majority of voters are frustrated on a majority of proposals. Hence $\mathcal{D}(m,u,k)$ is not an Anscombe's paradox free domain. ■

As shown in the following theorem, the bound provided in Proposition 3.2.4 identifies the minimum number of unifying proposals that characterizes unifying preference domains that are Anscombe's paradox free.

Theorem 2.7 (Andjiga et al. (2017)).

A unifying preference domain $\mathcal{D}(m,u,k)$ is Anscombe's paradox free if and only if $u \geq \frac{1}{2}m+2k-2$. ♣

As in the proof of Proposition 3.2.4, let u^* be the smallest integer greater than or equal to $\frac{1}{2}m + 2k - 2$. Note that $u \geq \frac{1}{2}m + 2k - 2$ is equivalent to $u \geq u^*$ meanwhile $u < \frac{1}{2}m + 2k - 2$ is equivalent to $u < u^*$. Since $\frac{1}{2}m + 2k - 2$ is not an integer for odd m, u^* will be used in the sequel. To ease the proof of Theorem 2.7, we introduce some useful and instructive lemmas. Given a unifying preference domain with u^* unifying proposals, the first lemma states that for all admissible vote profiles, Anscombe's paradox does not occur whenever the total number of unifying proposals at which the majority decision differs from common standards is less than the barometer of consensus.

Lemma 3.2.1. *Let $\mathcal{D}(m, u, k)$ be a unifying preference domain such that $u = u^*$ and let X be an admissible vote profile. If $0 \leq |\mathcal{O}| \leq k - 1$, then there is no majority of voters who are frustrated on a majority of proposals.*

Proof.

Consider a unifying preference domain $\mathcal{D}(m, u, k)$ such that $u = u^*$ and an admissible vote profile X such that $0 \leq |\mathcal{O}| \leq k - 1$. Assume that a majority of voters - say S - are frustrated on a majority of proposals. Indeed, we prove that this necessarily leads to a contradiction.

Consider a voter $i \in S$. Recall that by definition, $u^* \geq \frac{1}{2}m + 2k - 2$. Therefore the total number of non unifying proposals is such that

$$m - u^* \leq \frac{1}{2}m - 2k + 2 \tag{3.8}$$

Since X is an admissible vote profile, we have $|\mathcal{O}_i| \leq k$. By assumption $|\mathcal{O}| \leq k-1$. Therefore (3.1) implies $f_i \leq 2k-1$. Moreover i is frustrated on a majority of proposals. Then by (3.3), we deduce that the total number of non unifying proposals on which voter i is frustrated is

$$f'_i > \frac{1}{2}m - f_i \geq \frac{1}{2}m - 2k + 1 \tag{3.9}$$

By (3.8), it follows that $f'_i > m - u^* - 1$. That is $f'_i \geq m - u^*$. Hence $f'_i = m - u^*$. Thus, voter i is frustrated on all non unifying proposals. Since S contains a majority of voters, the set of non unifying proposals is empty by Remark 3.1.2 (otherwise a majority of voters would be frustrated on the same proposal). Clearly, $m - u^* = f'_i = 0$ and (3.8) becomes $\frac{1}{2}m - 2k + 1 \geq -1$ while (3.9) induces

$$0 > \frac{1}{2}m - f_i \geq \frac{1}{2}m - 2k + 1 \geq -1.$$

Therefore

$$f_i > \frac{1}{2}m \geq 2k - 2. \tag{3.10}$$

This proves that $f_i \geq 2k-1$ and thus $f_i = 2k-1$. By (3.1), $|\mathcal{O}_i| \geq f_i - |\mathcal{O}|$ and since by assumption $|\mathcal{O}| \leq k-1$, we deduce that $|\mathcal{O}_i| \geq k$. Therefore $|\mathcal{O}_i| = k$ because X is an admissible vote profile. Taking this into account in (3.1) yields

$$2|\mathcal{O}_i \cap \mathcal{O}| \leq k + |\mathcal{O}| - f_i \leq 0$$

as $|\mathcal{O}| \leq k-1$ by assumption. This implies $|\mathcal{O}_i \cap \mathcal{O}| = 0$ and $|\mathcal{O}| = f_i - k = k-1$. We conclude that $\mathcal{O}_i \cap \mathcal{O} = \emptyset$ for each voter in S. Therefore each voter in S disagrees with the majority decision on all proposals in $\mathcal{O}$. Note that by (3.10), $2k > \frac{1}{2}m + 1$. This proves that $k > 1$ and that $\mathcal{O}$ is not empty. By Remark 3.1.2, a contradiction holds. ■

Given a unifying preference domain with u^* unifying proposals, the next lemma says that for all admissible vote profiles, any voter who is frustrated on a majority of proposals is necessarily frustrated on at least $2k-1$ proposals where k is the barometer of consensus.

Lemma 3.2.2. *Let $\mathcal{D}(m, u, k)$ be a unifying preference domain such that $u = u^*$. If a voter is frustrated on a majority of proposals given an admissible vote profile X, then he/she is frustrated on at least $2k-1$ unifying proposals.*

Proof. Consider a unifying preference domain $\mathcal{D}(m, u, k)$ such that $u = u^*$ and an admissible vote profile X. Assume that a voter i is frustrated on a majority of proposals. As shown in (3.8),

$$m - u^* \leq \frac{1}{2}m - 2k + 2.$$

Thus f_i' is such that $f_i' \leq \frac{1}{2}m - 2k + 2$. Since i is frustrated on a majority of proposals, it follows from (3.3) that $f_i' > \frac{1}{2}m - f_i$. Hence

$$\frac{1}{2}m - 2k + 2 > \frac{1}{2}m - f_i.$$

Therefore $f_i \geq 2k-1$. ■

Given a unifying preference domain with u^* unifying proposals, the lemma that follows provides a majoration of the total number of unifying proposals on which a voter deviates from common standards but agrees with the majority decision provided that this voter is frustrated on a majority of proposals and that the majority decision differs from common standards on at least k unifying proposals; k is the barometer of consensus.

Lemma 3.2.3. *Let $\mathcal{D}(m,u,k)$ be a unifying preference domain such that $u = u^*$. If a voter i is frustrated on a majority of proposals at an admissible vote profile X such that $|\mathcal{O}| \geq k$, then among unifying proposals in $\mathcal{O}$, voter i agrees with the majority decision on at most $\frac{|\mathcal{O}|-k+1}{2}$ proposals. That is $|\mathcal{O} \cap \mathcal{O}_i| \leq \frac{|\mathcal{O}|-k+1}{2}$.*

Proof.

Consider a unifying preference domain $\mathcal{D}(m,u,k)$ such that $u = u^*$ and an admissible vote profile X such that $|\mathcal{O}| \geq k$. Assume that voter i is frustrated on a majority of proposals. By Lemma 3.2.2, voter i is frustrated on at least $2k-1$ unifying proposals. Thus (3.1) implies that $2k-1 \leq |\mathcal{O}| + |\mathcal{O}_i| - 2|\mathcal{O} \cap \mathcal{O}_i|$. Since X is an admissible vote profile, $|\mathcal{O}_i| \leq k$. Therefore $2k-1 \leq |\mathcal{O}| + k - 2|\mathcal{O} \cap \mathcal{O}_i|$. Hence $|\mathcal{O} \cap \mathcal{O}_i| \leq \frac{|\mathcal{O}|-k+1}{2}$. ∎

Given a unifying preference domain with u^* unifying proposals, we now use Lemma 3.2.2 and Lemma 3.2.3 to prove in the following lemma that for all admissible vote profiles, Anscombe's paradox does not occur whenever the total number of proposals on which the majority decision differs from common standards is greater than or equal to the barometer of consensus.

Lemma 3.2.4. *Let $\mathcal{D}(m,u,k)$ be a unifying preference domain such that $u = u^*$. If X is an admissible vote profile such that $|\mathcal{O}| \geq k$, there is no majority of voters who are frustrated on a majority of proposals.*

Proof.

Consider a unifying preference domain $\mathcal{D}(m,u,k)$ such that $u = u^*$ and an admissible vote profile X such that $|\mathcal{O}| \geq k$. Assume that a majority of voters - say S - are frustrated on a majority of proposals. We prove that this necessarily leads to a contradiction.

First, assume that $\mathcal{O}_i \cap \mathcal{O} = \emptyset$ for all $i \in S$. That is $X_i^j = x_j^*$ for all $a^j \in \mathcal{O}$ and for all $i \in S$. Since $|\mathcal{O}| \geq k$ and $MR(X^j) = -x_j^*$ for all $a^j \in \mathcal{O}$, a contradiction arises by

Remark 3.1.2. Hereafter $\mathcal{O}_i \cap \mathcal{O} \neq \emptyset$ for some $i \in S$.

Secondly, assume that $|\mathcal{O}| = k$. Since each voter in S is frustrated on a majority of proposals, then by Lemma 3.2.3, for all $i \in S$, $|\mathcal{O}_i \cap \mathcal{O}| \leq \frac{1}{2}$ and thus $\mathcal{O}_i \cap \mathcal{O} = \emptyset$. A contradiction holds as we just prove.

Finally, assume that $|\mathcal{O}| \in \{k+1, \ldots, 2k\}$. Let $|\mathcal{O}| = d$. For each subset I of $\mathcal{O}$, we denote by y_I the total number of voters $i \in S$ such that $\mathcal{O}_i \cap \mathcal{O} = I$ and by z_I the total number of voters $i \notin S$ such that $\mathcal{O}_i \cap \mathcal{O} = I$. Moreover, we denote by s^t the sum of all y_I such that $|I| = t$ and by n^t the sum of all z_I such that $|I| = t$. Note that s^t is the total number of voters in S who deviate from common standards on exactly t proposals in $\mathcal{O}$ while n^t is the total number of voters out of S who deviate from common standards on exactly t proposals in $\mathcal{O}$.

We pose $p = \max\limits_{i \in S} |\mathcal{O}_i \cap \mathcal{O}|$ and for each $a^j \in \mathcal{O}$, we denote by Δ_j the difference between the total number of voters who agree with the common standard on a^j and the total number of voters who disagree with common standard on a^j. Note that by assumption, $p \geq 1$. By definition of $\mathcal{O}$, the majority decision on each proposal in $\mathcal{O}$ differs from the corresponding common standard. This implies $\Delta_j \leq 0$ for all $a^j \in \mathcal{O}$. Therefore

$$\sum_{a^j \in \mathcal{O}} \Delta_j \leq 0. \tag{3.11}$$

To complete the proof, we show that (3.11) leads to a contradiction by reorganizing $\sum\limits_{a^j \in \mathcal{O}} \Delta_j$ in an appropriate way to highlight the role of parameters s^t and n^t we introduce above. For this purpose, consider $a^j \in \mathcal{O}$:

$$\begin{aligned} \Delta_j &= \sum_{I \subseteq \mathcal{O}: a^j \notin I} y_I + \sum_{I \subseteq \mathcal{O}: a^j \notin I} z_I - \sum_{I \subseteq \mathcal{O}: a^j \in I} y_I - \sum_{I \subseteq \mathcal{O}: a^j \in I} z_I \\ &= \sum_{t=0}^{p} \left(\sum_{\substack{I \subseteq \mathcal{O}: \\ a^j \notin I, |I| = t}} y_I - \sum_{\substack{I \subseteq \mathcal{O}: \\ a^j \in I, |I| = t}} y_I \right) + \sum_{t=0}^{k} \left(\sum_{\substack{I \subseteq \mathcal{O}: \\ a^j \notin I, |I| = t}} z_I - \sum_{\substack{I \subseteq \mathcal{O}: \\ a^j \in I, |I| = t}} z_I \right) \end{aligned}$$

By considering all proposals a^j from $\mathcal{O}$, we have

$$\begin{aligned}
\sum_{a^j\in\mathcal{O}}\Delta_j &= \sum_{a\in\mathcal{O}}\sum_{t=0}^{p}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\notin I,|I|=t}} y_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I,|I|=t}} y_I\right) + \sum_{a^j\in\mathcal{O}}\sum_{t=0}^{k}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\notin I,|I|=t}} z_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I,|I|=t}} z_I\right)\\
&= \sum_{t=0}^{p}\sum_{a^j\in\mathcal{O}}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\notin I,|I|=t}} y_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I,|I|=t}} y_I\right) + \sum_{t=0}^{k}\sum_{a^j\in\mathcal{O}}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\notin I,|I|=t}} z_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ a^j\in I,|I|=t}} z_I\right)\\
&= \sum_{t=0}^{p}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}}(|\mathcal{O}|-t)y_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}} ty_I\right) + \sum_{t=0}^{k}\left(\sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}}(|\mathcal{O}|-t)z_I - \sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}} tz_I\right)\\
&= \sum_{t=0}^{p}(|\mathcal{O}|-2t)\sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}} y_I + \sum_{t=0}^{k}(|\mathcal{O}|-2t)\sum_{\substack{I\subseteq\mathcal{O}:\\ |I|=t}} z_I
\end{aligned}$$

Using $d = |\mathcal{O}|$, we reorganize the previous expression of $\sum_{a^j\in\mathcal{O}}\Delta_j$ as follows:

$$\begin{aligned}
\sum_{a^j\in\mathcal{O}}\Delta_j &= \sum_{t=0}^{p}(d-2t)s^t + \sum_{t=0}^{k}(d-2t)n^t\\
&= \sum_{t=0}^{p-1}(2p-2t)s^t + \sum_{t=0}^{p-1}(d-2p)s^t + (d-2p)s^p + \sum_{t=0}^{k}(d-2t)n^t\\
&= \sum_{t=0}^{p-1}(2p-2t)s^t + \sum_{t=0}^{p}(d-2p)s^t + \sum_{t=0}^{k}(d-2t)n^t\\
&= \sum_{t=0}^{p-1}(2p-2t)s^t + (d-2p)|S| + \sum_{t=0}^{k}(d-2t)n^t
\end{aligned}$$

Recall that $\sum_{t=0}^{k} n_t = |N\backslash S|$ and that S contains more than the half of the voters. Thus we can rewrite $|S| = |N\backslash S| + \rho = \rho + \sum_{t=0}^{k} n^t$ with $\rho = 2|S| - n > 0$ to obtain the following:

$$\sum_{a^j\in\mathcal{O}}\Delta_j = 2ps^0 + \sum_{t=1}^{p-1}(2p-2t)s^t + (d-2p)\rho + 2\sum_{t=0}^{k}(d-t-p)n^t \tag{3.12}$$

By Lemma 3.2.3,

$$d-p \geq d - \frac{d-k+1}{2} = \frac{d+k-1}{2} \text{ and } d-2p \geq k-1 \geq 0.$$

Since $d \geq k+1$ by assumption, then it follows from $d-p \geq \frac{d+k-1}{2}$ that $d-p \geq k$. Therefore $d-p-t \geq 0$ for all $t \in \{0,1,\ldots,k\}$. Taking this into account in (3.12), we conclude that $\sum_{a^j \in \mathcal{O}} \Delta_j \geq 0$. Hence by (3.11), we have $\sum_{a^j \in \mathcal{O}} \Delta_j = 0$. Since $p \geq 1$ by assumption and $\rho > 0$, then we deduce that $s^0 = 0$ and $d-2p = 0$. By definition, $s^0 = 0$ means that for all $i \in S$, $|\mathcal{O}_i \cap \mathcal{O}| \neq 0$. Now recall that $d-2p \geq k-1 \geq 0$. We deduce that $d-2p = k-1 = 0$ and that $k=1$. Since k is the barometer of consensus, it follows that $p=1$, $d=2$ and $|\mathcal{O}_i \cap \mathcal{O}| = 1$ for all $i \in S$. Since $d = |\mathcal{O}| = 2$ and $|\mathcal{O}_i| = |\mathcal{O}_i \cap \mathcal{O}| = 1$ for all $i \in S$, it follows from (3.1) that for all $i \in S$, $f_i = 1$ and by (3.3), $f_i' > \frac{1}{2}m - 1$. Since $k=1$, then u^* is by definition such that

$$u^* \geq \frac{1}{2}m - 2k - 2 = \frac{1}{2}m.$$

Thus

$$\frac{1}{2}m - 1 < f_i' \leq m - u^* \leq \frac{1}{2}m.$$

Hence for all $i \in S$, $f_i' = m - u^*$ is the greatest integer less than or equal to $\frac{1}{2}m$. Thus all voters in S are frustrated on all non unifying proposals. By remak 3.1.2, a contradiction holds. ∎

Proof.

[*Proof of Theorem* 2.7.]

$\Rightarrow$) See Proposition 3.2.4.

$\Leftarrow$) Suppose that $u \geq u^*$. By Lemma 3.2.1 and Lemma 3.2.4, $\mathcal{D}(m, u^*, k)$ is Anscombe's paradox free. By Proposition 3.2.3, $\mathcal{D}(m, u, k)$ is also Anscombe's paradox free. ∎

When all proposals are unifying proposals, Theorem 2.7 becomes:

Corollary 3.2.1. *A unifying preference domain $\mathcal{D}(m, m, k)$ is Anscombe's paradox free if and only if $k \leq \frac{1}{4}m + 1$.*

Proof.

Straightforward from Theorem 2.7 by observing that $m \geq u^*$ is equivalent to $m \geq \frac{1}{2}m + 2k - 2$. ∎

Here follow three final remarks on the cardinality of unifying preference domains, the relationship with single-switch domains and the rule of three-fourths respectively.

Remark 3.2.1. Given two unifying preference domains $\mathcal{D}(m,u,k)$ and $\mathcal{D}(m,u',k')$, it can be easily checked that:

(i) if $u \leq u'$ and $k = k'$, then $|\mathcal{D}(m,u,k)| \geq |\mathcal{D}(m,u',k')|$; and

(ii) if $u = u'$ and $k \leq k'$, then $|\mathcal{D}(m,u,k)| \leq |\mathcal{D}(m,u',k')|$.

Furthermore the cardinality of $\mathcal{D}(m,u,k)$ is given by

$$|\mathcal{D}(m,u,k)| = 2^{m-u} \sum_{l=0}^{k} \binom{l}{u}.$$

Although $|\mathcal{D}(m,u,k)|$ is exponential, we have checked for small values of m that the ratio $\lambda(m)/2^m$ tend to 0, where $\lambda(m)$ is the maximum cardinality of a unifying preference domain that is Anscombe's paradox free; see Table 3.1 below where u^{**} and k^{**} are such that $\lambda(m) = |\mathcal{D}(m,u^{**},k^{**})|$ and $\mathcal{D}(m,u^{**},k^{**})$ is Anscombe paradox free with maximum cardinality. As in the case of single-switch domains, unifying preference domains that are Anscombe's paradox free are also strong restrictions on individual opinions.

Remark 3.2.2. It is obvious that unifying preference domains overlap with single-switch domains. To see this, (i) rearrange the components of the vector x^* of common standards such that one switches from $+1$ to -1 at most once over unifying proposals; (ii) consider the vector of votes ω that consists simultaneously in following all common standards on unifying proposals and rejecting all non unifying proposals. While each single-switch domain contains both ω and $-\omega$, a unifying preference domain that is Anscombe's free is only guaranteed to contain ω; but may be of exponential cardinality. Therefore single-switchness and unifying preference domains are rather distinct approaches that describe possible Anscombe's paradox free domains (none being an extension of the other).

Remark 3.2.3. When we assume that all proposals are unifying, it appears that the barometer of consensus for each unifying preference domain that is Anscombre's paradox free is not greater than one fourth of the proposals by more than one unit; meaning that the total number of proposals at which each voter follows common standards is not less than the three-fourths of the proposals by more than one unit. However, the average fraction of voters, across all proposals, may still be less than

Table 3.1: *Maximum cardinality of a unifying preference domain that is Anscombe's paradox free (%)*

m	u^{**}	k^{**}	$\lambda(m)/2^m$
3	2	1	75.00%
4	2	1	75.00%
5	5	2	50.00%
6	5	2	50.00%
7	6	2	34.38%
8	8	3	36.33%
9	9	3	25.39%
10	9	3	25.39%
20	20	6	5.76%
30	29	8	1.21%
40	40	11	0.32%
50	49	13	0.07%

three-fourths. In these cases, the corresponding profiles are now out of the scope of the rule of three-fourths while the conditions provided here are sufficient (of course not necessary) to rule out all occurrences of Anscombe's paradox. To see this, consider for example the vote profile in the table below where we have inserted an extra column to the vote profile in order to give the fraction of the voters who agree with the majority decision on each motion.

	X_1	X_2	X_3	X_4	X_5	$MR[X]$	
a^1	$+1$	$+1$	$+1$	-1	-1	$+1$	3/5
a^2	$+1$	$+1$	-1	$+1$	-1	$+1$	3/5
a^3	$+1$	-1	-1	$+1$	$+1$	$+1$	3/5
a^4	-1	-1	$+1$	$+1$	$+1$	$+1$	3/5
a^5	-1	$+1$	$+1$	-1	$+1$	$+1$	3/5
	no	*no*	*no*	*no*	*no*	0/5	

Each voter deviates from common standards on exactly two proposals out of five. Thus this vote profile is admissible given that all proposals are unifying and the barometer of consensus is $k = 2$. We can check that the condition from Corollary 3.2.1 is satisfied and that the majority rule does not exhibit Anscombe's paradox. But on average across all the five proposals, sixty percent of the voters agree with the majority decision.

This is of course in accordance with Wagner who notes (see Wagner, 1983, p. 307) that not requiring on average the assent of at least three-fourths of the voters leaves place to the possibility of observing Anscombe's paradox. We simply argue that the conditions provided in Theorem 2.7 (or even in Corollary 3.2.1) identifies some of these remaining vote profiles not affected by the paradox.

3.3 Identifying Anscombe's paradox free domains for $m = 3$

It is worth noticing that a domain that is Anscombe's paradox free needs not be a unifying preference domain. This is the case with single-switch domains. Nevertheless, we prove here that with exactly three proposals, a preference domain is Anscombe's paradox free if and only if it is a subset of a stable unifying preference domain or a subset of a single-switch domain.

3.3.1 Domain representation and equivalent domains

We recall that a preference domain is a nonempty subset of $\{-1; +1\}^m$ from which each voter picks up his/her vector of votes. To ease our analysis, we need some general notations. Mainly, we assume that vectors of votes in a preference domain D is numbered in such a way that $D = \{x_1, x_2, \ldots, x_d\}$ where $d = |D|$. This allows us to give D a matrix representation as $D = (x_j^a)_{\substack{a=1,\ldots,m \\ j=1,\ldots,d}}$ where x_i^a is the opinion on proposal a according to the vector of vote x_j.

As in Laffond and Lainé (2006), the notion of equivalent domains provided in the next definition will be useful in our exploration.

Definition 3.3.1. *Let $\mathcal{M}'$ be a subset of $\mathcal{M}$, σ a permutation of $\mathcal{M}$ and $D = (x_j^a)_{\substack{a=1,\ldots,m \\ j=1,\ldots,d}}$ a preference domain.*

i) The $\mathcal{M}'$-relabeling of D is the preference domain $D^{\mathcal{M}'} = (y_j^a)_{\substack{a=1,\dots,m\\ j=1,\dots,d}}$ obtained by reversing approvals and disapprovals regarding proposals in $\mathcal{M}'$ in each element of D; that is

$$x_i^a = 1 \iff y_i^a = -1,\ \forall a \in \mathcal{M}' \text{ and } x_i^a = y_i^a,\ \forall a \notin \mathcal{M}'.$$

ii) The σ-permutation of D is the preference domain $D^{\sigma} = (y_j^a)_{\substack{a=1,\dots,m\\ j=1,\dots,d}}$ obtained by permuting the positions of the proposals according to σ; that is

$$y^a = x^{\sigma(a)}, \quad a \in \mathcal{M}.$$

iii) Two preference domains D and D' are equivalent if there exist a subset $\mathcal{M}'$ of $\mathcal{M}$ and a permutation σ of $\mathcal{M}$ such that D' is obtained from D by the σ-permutation of the $\mathcal{M}'$-relabeling of D; that is

$$D' = \left(D^{\mathcal{M}'}\right)^{\sigma}.$$

In this case, D' is simply denoted by $D' = D^{\mathcal{M}',\sigma}$.

Similarly, a vector of votes $Y \in D^{\mathcal{M}',\sigma}$ will be denoted by $Y = X^{\mathcal{M}',\sigma}$ if $X \in D$ is such that for each voter i, $\{Y_i\} = \{X_i\}^{\mathcal{M}',\sigma}$.

Example 3.3.1. Let $\mathcal{M} = \{a_1, a_2, a_3, a_4\}$ and consider the following preference domains

$$D = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x^1 & -1 & +1 & -1 & +1 & -1 \\ x^2 & +1 & -1 & +1 & -1 & +1 \\ x^3 & +1 & +1 & -1 & +1 & -1 \\ x^4 & -1 & -1 & +1 & +1 & -1 \end{array} \quad \text{and} \quad D' = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x^1 & +1 & +1 & -1 & +1 & -1 \\ x^2 & +1 & +1 & -1 & -1 & +1 \\ x^3 & +1 & -1 & +1 & -1 & +1 \\ x^4 & +1 & -1 & +1 & -1 & +1 \end{array}$$

Here x^j refers to the vector of votes on proposal a^j while x_k is the k^{th} vector of votes in the domain D.

Let $\mathcal{M}' = \{a_1, a_4\}$ and σ be the permutation of $\mathcal{M}$ defined by $\sigma(1) = 3$, $\sigma(2) = 4$,

$\sigma(3) = 1$ and $\sigma(4) = 2$. Then the $\mathcal{M}'$-relabeling of D is given by:

$$D^{\mathcal{M}'} = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x^1 & +1 & -1 & +1 & -1 & +1 \\ x^2 & +1 & -1 & +1 & -1 & +1 \\ x^3 & +1 & +1 & -1 & +1 & -1 \\ x^4 & +1 & +1 & -1 & -1 & +1 \end{array}$$

Now the σ-permutation of $D^{\mathcal{M}'}$ is

$$\left(D^{\mathcal{M}'}\right)^{\sigma} = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x^3 & +1 & +1 & -1 & +1 & -1 \\ x^4 & +1 & +1 & -1 & -1 & +1 \\ x^1 & +1 & -1 & +1 & -1 & +1 \\ x^2 & +1 & -1 & +1 & -1 & +1 \end{array}$$

Clearly, $D^{\mathcal{M}',\sigma} = D'$. It then appears that D and D' are equivalent preference domains.

Proposition 3.3.1. *Assume that the total number of voters is odd and suppose that there exists a D-admissible vote profile at which the majority rule exhibits the Anscombe's paradox.*

If a preference domain D' is equivalent to D, then there also exists a D'-admissible vote profile at which the majority rule exhibits the Anscombe's paradox.

Proof.

Suppose that $D' = D^{\mathcal{M}',\sigma}$ and that there exists a vote profile X with an odd number of voters such that $Supp(X) \subseteq D$ and that the majority rule exhibits the Anscombe's paradox at X. Denote by S the set of all voters who are frustrated on a majority of proposals and by $Y = X^{\mathcal{M}'}$ the vote profile obtained from X after the $\mathcal{M}'$-relabeling of proposals.

By the definition of the $\mathcal{M}'$-relabeling of proposals, $MR(Y^a) = MR(X^a)$ since each majority decision with an odd number of voters is supported by more than the half of the voters. Therefore, each voter is frustrated on the same set of proposals in X as in Y. Therefore voters in S are each frustrated on a majority of proposals in Y.

Let $Z = X^{\mathcal{M}',\sigma}$. By definition, $Z \in D' = D^{\mathcal{M}',\sigma}$ and $Z = Y^{\sigma}$ is the vote profile obtained from Y by the σ-permutation of proposals. Moreover, for each proposal a, the vector of individual votes Z^a on a in Z is identical to the vector of individual votes $Y^{\sigma(a)}$ on $\sigma(a)$ in Y. Therefore, each voter is frustrated on a proposal a in Z if and only if he/she is frustrated on $\sigma(a)$ in Y. Thus each voter is frustrated on the same number of proposals in Y as in Z. By assumption, voters in S are frustrated on a majority of proposals in Z. Since Z is D'-admissible, Z is a D'-admissible vote profile at which the majority rule exhibits the Anscombe's paradox. ■

To continue, we further describe each row in a preference domain of a given cardinality in terms of the total number of +1-entries it contains.

Definition 3.3.2. *Given a preference domain D of cardinality d, the type of a row x of D containing exactly k entries equal to $+1$ is the integer $[x] = \min(k, d-k)$.*

Example 3.3.2. *In a preference domain of cardinality 7, the vector*

$x = (+1, +1, -1, +1, -1, -1, +1)$ is a row of type $[x] = 3$.

Note that x and $-x$ have the same type.

Definition 3.3.3. *Let D be a preference domain of cardinality d. The type of D is the collection $[D] = \left(t_1^{k_1}, t_2^{k_2}, \ldots, t_s^{k_s}\right)$ of all weighted types of rows in D such that exactly k_l rows in D are of type t_l with $t_1 \leq t_2 \leq \cdots \leq t_s$.*

The class of all preference domains of cardinality d having the same type with D is the set of all preference domains having type $[D]$ and containing each exactly d vectors; it will be denoted by $C_{d,t_1^{k_1},t_2^{k_2},\ldots,t_s^{k_s}}$.

Example 3.3.3. *Let D be the preference domain defined as follows:*

	x_1	x_2	x_3	x_4
x^1	$+1$	-1	$+1$	-1
x^2	-1	$+1$	$+1$	$+1$
x^3	-1	-1	-1	$+1$

The vectors x^1, x^2 and x^3 are respectively of types 2, 1 and 1. Thus D is of type $[D] = (1^2, 3^1)$ and belongs to the class $C_{4,1^2,3^1}$.

Proposition 3.3.2. *If two domains are equivalent then they have the same type.*

Proof.

Straightforward from definition. ■

The following proposition presents all possible classes given a preference domain with 3 proposals of cardinality at least 3.

Proposition 3.3.3. Let $D \subseteq \{-1,1\}^3$.

i) if $|D| = 8$ then $D \in C_{8,4^3}$.

ii) If $|D| = 7$ then $D \in C_{7,3^3}$.

iii) If $|D| = 6$ then $D \in C_{6,2^2,3^1} \cup C_{6,2^1,3^2} \cup C_{6,3^3}$.

iv) If $|D| = 5$ then $D \in C_{5,1^1,2^2} \cup C_{5,2^3}$.

v) If $|D| = 4$ then $D \in C_{4,0^1,2^2} \cup C_{4,1^3} \cup C_{4,1^2,2^1} \cup C_{4,1^1,2^2} \cup C_{4,2^3}$.

vi) If $|D| = 3$ then $D \in C_{3,0^1,1^2} \cup C_{3,1^3}$.

Proof.

Let D be a preference domain with 3 proposals a_1, a_2 and a_3 and $x^j \in D$ for some $j \in \{1,2,3\}$. The vector x^j contains k $+1$-entries and l -1-entries such that $k + l = |D|$ and $k \in \{0,1,2,3,4\}$.

1. For $|D| = 8$, $k = l = 4$ and $[x^j] = \min(4,4) = 4$. Thus $D \in C_{8,4^3}$.

2. For $|D| = 7$, $(k,l) \in \{(4,3),(3,4)\}$ and $[x^j] = 3$. Thus $D \in C_{7,3^3}$.

3. For $|D| = 6$, $(k,l) \in \{(4,2),(3,3),(2,4)\}$ and $[x^j] \in \{2,3\}$. Moreover, three rows can not be of type 2 at the same time, otherwise two columns of the domain would coincide. Thus $D \in C_{6,2^2,3^1} \cup C_{6,2^1,3^2} \cup C_{6,3^3}$.

4. For $|D| = 5$, $(k,l) \in \{(4,1),(3,2),(2,3),(1,4)\}$ and $[x^j] \in \{1,2\}$. Moreover, two rows can not be of type 1 at the same time, otherwise two columns of the domain would coincide. Thus $D \in C_{5,1^1,2^2} \cup C_{5,2^3}$.

5. For $|D| = 4$, $(k,l) \in \{(4,0),(3,1),(2,2),(1,3),(0,4)\}$ and $[x^j] \in \{0,1,2\}$. Moreover, two rows can not be of type 0 at the same time, otherwise two columns of the domain would coincide. Similarly all the the three types can not simultaneously occur. Thus $D \in C_{4,0^1,2^2} \cup C_{4,1^3} \cup C_{4,1^2,2^1} \cup C_{4,1^1,2^2} \cup C_{4,2^3}$.

6. For $|D| = 3$, $(k, l) \in \{(3, 0), (2, 1), (1, 2), (0, 3)\}$ and $[x^j] \in \{0, 1\}$. As above, two rows can not be of type 0 at the same time. Thus $D \in C_{3,0^1,1^2} \cup C_{3,1^3}$. ∎

3.3.2 Anscombe's paradox free domains with three proposals

We are now ready to state and prove our characterization result for three proposals.

Theorem 3.8.

Given a preference domain $D \subset \{-1, 1\}^3$, if D is Anscombe's paradox free then D is unifying preference domain or D is equivalent to a single-switch preference domain. ♣

Proof. Pose $\{-1, +1\}^3 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

Table 3.2:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x^1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1
x^2	$+1$	$+1$	-1	-1	$+1$	$+1$	-1	-1
x^3	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1

Given $k_1, k_2, \ldots, k_l \in \{1, \ldots, 8\}$, let $D^{k_1 k_2 \ldots k_l} = \{-1; 1\}^3 \backslash \{x_{k_1}, x_{k_2}, \ldots, x_{k_l}\}$. Suppose that $D \subseteq \{-1, +1\}^3$ is Anscombe's paradox free and that $|D| \geq 3$. We show that D is a single-switch preference domain or D is a unifying preference domain.

1. Suppose that $|D| = 7$. By Proposition 3.3.3, $D \in C_{7,3^3}$. Moreover, $C_{7,3^3} = \{D^1, D^2, D^3, D^4, D^5, D^6, D^7, D^8\}$.

   We first prove that all domains in $C_{7,3^3}$ are equivalent. To see this, pose $D^1 = T$. Then using the relabeling operation, we have $D^2 = T^{\{a^3\}}$, $D^3 = T^{\{a^2\}}$, $D^4 = T^{\{a^2,a^3\}}$, $D^5 = T^{\{a^1\}}$, $D^6 = T^{\{a^1,a^3\}}$, $D^7 = T^{\{a^1,a^2\}}$, $D^8 = T^{\{a^1,a^2,a^3\}}$. Hence all domains of $C_{7,3^3}$ are equivalent.

We now prove that D^7 is not Anscombe's paradox free.

$$D^7 = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_8 \\ \hline +1 & +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & -1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 & +1 & -1 & -1 \end{array}$$

Let X be a vote profile with eleven voters and three proposals defined as follow $X_1 = x_1$, $X_2 = x_2$, $X_3 = X_4 = X_5 = x_3$, $X_6 = x_4$, $X_7 = X_8 = x_5$, $X_9 = x_6$, $X_{10} = X_{11} = x_8$. It holds that $MR[X] = (+1, -1, +1)$. Each voter of the majority coalition $\{2, 7, 8, 9, 10, 11\}$ is frustrated on a majority of proposals. Thus the Anscoombe's paradox holds at X. Since $n = 11$ is odd and by Proposition 3.3.1, D^7 is not Anscombe's paradox free as well as the seven other domains from $C_{7,3^3}$.

2. Suppose that $|D| = 6$ then $D \in C_{6,2^2,3^1} \cup C_{6,2^1,3^2} \cup C_{6,3^3}$. Note that D is obtained by deleting two vectors from $\{-1; 1\}^3$.

   (a) Suppose that $D \in C_{6,2^1,3^2}$. D is obtained by deleting in $\{-1, +1\}^3$ a pair of vectors that form a domain of type $(0^1, 1^2)$. Thus by Table 3.3, we have $C_{6,2^1,3^2} = \{D^{14}, D^{16}, D^{17}, D^{23}, D^{25}, D^{28}, D^{35}, D^{38}, D^{46}, D^{47}, D^{58}, D^{67}\}$.

Table 3.3: All pairs of vectors that form a domain of type $(0^1, 1^2)$

x_1	x_4	x_1	x_6	x_1	x_7	x_2	x_3	x_2	x_5	x_2	x_8	x_3	x_5	x_3	x_8	x_4	x_6	x_4	x_7	x_5	x_8	x_6	x_7
+1	+1	+1	−1	+1	−1	+1	+1	+1	−1	+1	−1	+1	−1	+1	−1	+1	−1	+1	−1	−1	−1	−1	−1
+1	−1	+1	+1	+1	−1	+1	−1	+1	+1	+1	−1	−1	+1	−1	−1	−1	+1	−1	−1	+1	−1	+1	−1
+1	−1	+1	−1	+1	+1	−1	+1	−1	+1	−1	−1	+1	+1	+1	−1	−1	−1	−1	+1	+1	−1	−1	+1

We first prove that all domains in $C_{6,2^1,3^2}$ are equivalent. To see this, pose $D^{14} = T$. Then using the relabeling operation and the permutation, we have

$D^{16} = T^{\sigma}$ with $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$.

$D^{17} = T^{\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{23} = T^{\{a_3\}}$.

$D^{25} = T^{\{a_3\},\sigma}$ with $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$.

$D^{28} = T^{\{a_1\},\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{35} = T^{\{a_2\},\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{38} = T^{\{a_1\},\sigma}$ with $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$.

$D^{46} = T^{\{a_1,a_2\},\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{47} = T^{\{a_1,a_3\},\sigma}$ with $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$.

$D^{58} = T^{\{a_1\}}$. $D^{28} = T^{\{a_1,a_3\}}$. Hence all domains of $C_{6,2^1,3^2}$ are equivalent.

We now prove that D^{14} exhibits the Anscombe's paradox.

$$\text{Let } D^{14} = D_{(24),(33),(33)} = \begin{array}{cccccc} x_2 & x_3 & x_5 & x_6 & x_7 & x_8 \\ \hline +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & +1 & +1 & -1 & -1 \\ -1 & +1 & +1 & -1 & +1 & -1 \end{array}$$

Let X be a vote profile with nine voters and three proposals defined as follow $X_1 = x_2$, $X_2 = x_2$, $X_3 = x_3$, $X_4 = x_3$, $X_5 = x_3$, $X_6 = x_5$, $X_7 = x_6$, $X_8 = x_7$, $X_9 = x_8$. It holds that $MR[X] = (+1, -1, +1)$. Each voter of the majority coalition $\{1, 2, 6, 7, 8\}$ is frustrated on a majority of proposals. Thus the Anscoombe's paradox holds at X. Since $n = 9$ is odd and by proposition 3.3.1 D^{14} is not Anscombe's paradox free as well as the other domain from $C_{6,2^1,3^2}$.

(b) Suppose that $D \in C_{6,3^3}$. D is obtained by deleting in $\{-1, +1\}^3$ a pair of vectors that form a domain of type (1^3). Thus by Table 3.4, we have $C_{6,3^3} = \{D^{18}, D^{27}, D^{36}, D^{45}\}$

Table 3.4: All pairs of vectors that form a domain of type (1^3)

x_1	x_8	x_2	x_7	x_3	x_6	x_4	x_5
$+1$	-1	$+1$	-1	$+1$	-1	$+1$	-1
$+1$	-1	$+1$	-1	-1	$+1$	-1	$+1$
$+1$	-1	-1	$+1$	$+1$	-1	-1	$+1$

We first prove that all domains in $C_{6,3^3}$ are equivalent. To see this, pose $D^{18} = T$. Then using the relabeling operation, we have $D^{27} = T^{\{a^3\}}$, $D^{36} = T^{\{a^2\}}$, $D^{45} = T^{\{a^2,a^3\}}$. Hence all domains of $C_{6,3^3}$ are equivalent.

$$\text{Since } D^{36} = \begin{array}{cccccc} x_1 & x_2 & x_4 & x_5 & x_7 & x_8 \\ \hline +1 & +1 & +1 & -1 & -1 & -1 \\ +1 & +1 & -1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 \end{array} \text{ is single-switch, all domain of } C_{6,3^3}$$

are Anscombe's paradox free as stated in Laffond and Lainé (2006).

(c) Suppose that $D \in C_{6,2^2,3^1}$. D is obtained by deleting in $\{-1,+1\}^3$ a pair of vectors that form a domain of type $(0^2, 1^1)$. Thus by Table 3.5,

$$C_{6,2^2,3^1} = \{D^{12}, D^{13}, D^{15}, D^{24}, D^{26}, D^{34}, D^{37}, D^{48}, D^{56}, D^{57}, D^{68}, D^{78}\}$$

Table 3.5: All pairs of vectors that form a domain of type $(0^2, 1^1)$

x_1	x_2	x_1	x_3	x_1	x_5	x_2	x_4	x_2	x_6	x_3	x_4	x_3	x_7	x_4	x_8	x_5	x_6	x_5	x_7	x_6	x_8	x_7	x_8
$+1$	$+1$	$+1$	$+1$	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1	-1	-1	-1	-1	-1	-1
$+1$	$+1$	$+1$	-1	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1	-1	-1	-1	-1	-1	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1
$+1$	-1	$+1$	$+1$	$+1$	$+1$	-1	-1	-1	-1	$+1$	-1	$+1$	$+1$	-1	-1	$+1$	-1	$+1$	$+1$	-1	-1	$+1$	-1

We first prove that all domains in $C_{6,2^2,3^1}$ are equivalent. To see this, pose $D^{12} = T$. Then using the relabeling operation and the permutation, we have

$D^{13} = T^{\sigma}$ with $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.

$D^{15} = T^{\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{24} = T^{\{a_2\},\sigma}$ with $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.

$D^{26} = T^{\{a_1\},\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{34} = T^{\{a_2\}}$.

$D^{37} = T^{\{a_2\},\sigma}$ with $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$.

$D^{48} = T^{\{a_1,a_2\},\sigma}$ with $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$.

$D^{56} = T^{\{a_1\}}$.

$D^{57} = T^{\{a_1\},\sigma}$ with $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.

$D^{68} = T^{\{a_1,a_2\},\sigma}$ with $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.

$D^{78} = T^{\{a_1,a_2\}}$. Hence all domains of $C_{6,2^2,3^1}$ are equivalent.

Since $D^{78} = \begin{array}{c|cccccc} & x_1 & x_2 & x_5 & x_6 & x_3 & x_4 \\ \hline a^1 & +1 & +1 & -1 & -1 & +1 & +1 \\ a^2 & +1 & +1 & +1 & +1 & -1 & -1 \\ a^3 & +1 & -1 & +1 & -1 & +1 & -1 \end{array}$ is a unifying preference domain with $\mathcal{U} = \{a^1, a^2\}$ and $k = 1$ then by Theorem 2.7, D^{78} does not exhibit the Anscombe's paradox. According to the definition of unifying preference domains, by a relabeling of domain D^{78}, we obtain another unifying preference domain with $u = 2$ and $k = 1$. Thus each preference domain of $C_{6,2^2,3^1}$ is a unifying preference domain and does not exhibit the Anscombe's paradox.

Till now, we have shown that if $|D| \geq 6$ then D is Anscombe's paradox free if and only if $D \in C_{6,2^2,3^1} \cap C_{6,3^3}$.

3. Suppose that $|D| = 5$ then $D \in C_{5,1^1,2^2} \cup C_{5,2^3}$

   (a) Assume that $D \in C_{5,1^1,2^2}$.

   We show that if $D \in C_{5,1^1,2^2}$ then there exists a preference domain D' in $C_{6,2^2,3^1}$ such $D \subset D'$.

   D is obtained by deleting in Table 3.2 a triplet of type $(0^1, 1^2)$. Thus,

   $C_{5,1^1,2^2} = \{D^{123}, D^{124}, D^{125}, D^{126}, D^{134}, D^{135}, D^{137}, D^{156}, D^{234}, D^{246}, D^{248}, D^{256}, D^{157}, D^{268}, D^{347}, D^{348}, D^{357}, D^{378}, D^{468}, D^{478}, D^{567}, D^{568}, D^{578}, D^{678}\}$

   - Since $\{1, 2\}$ is the subset of $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}$, then $D^{123}, D^{124}, D^{125}, D^{126}$ are sub-domains of D^{12}.

   - Since $\{1, 3\}$ is the subset of $\{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 7\}$, then $D^{134}, D^{135}, D^{137}$ are sub-domains of D^{13}.

   - Since $\{1, 5\}$ the subset of $\{1, 5, 6\}, \{1, 5, 7\}$, then D^{156}, D^{157} are sub-domains of D^{15}.

   - In the same way $D^{234} \subset D^{34}, D^{246} \subset D^{24}, D^{248} \subset D^{24}, D^{256} \subset D^{56}, D^{268} \subset D^{68}$, $D^{347} \subset D^{34}, D^{348} \subset D^{34}, D^{357} \subset D^{57}, D^{378} \subset D^{78}, D^{468} \subset D^{68}, D^{478} \subset D^{48}, D^{567} \subset D^{57}, D^{568} \subset D^{68}, D^{578} \subset D^{57}, D^{678} \subset D^{68}$.

   Therefore all domains of $C_{5,1^1,2^2}$ are unifying preference domains.

   (b) Suppose that $D \in C_{5,2^3}$.

   Since vectors in each domain from $C_{5,2^3}$ have the same type, there are some domains of $C_{5,2^3}$ which contain a pair of opposite vectors. We can pose $C_{5,2^3} = C^1_{5,2^3} \cup C^2_{5,2^3}$ with $C^1_{5,2^3}$ and $C^2_{5,2^3}$ defined as follow:

   - D belongs to $C^1_{5,2^3}$ if there exists $x \in D$ such that $\forall y \in D \backslash \{x\}, \ -y \notin D \backslash \{x\}$; meaning that $D \backslash \{x\}$ is not symmetric.

   - $D \in C^2_{5,2^3}$ if there exists $x \in D$ such that $\forall y \in D \backslash \{x\}, \ -y \in D \backslash \{x\}$; meaning that $D \backslash \{x\}$ is symmetric.

   • We show that all domains from $C^1_{5,2^3}$ exhibits the Anscombe's paradox.

A domain of $C^1_{5,2^3}$ is obtained by deleting in Table 3.2 a triplet $\{x, y, z\}$ of type (1^3) such that $\forall t \in \{x, y, z\}, -t \notin \{x, y, z\}$.

Thus: $C^1_{5,2^3} = \{D^{146}, D^{147}, D^{167}, D^{235}, D^{238}, D^{258}, D^{358}, D^{467}\}$. Note that each domain of $C^1_{5,2^3}$ is a sub-domain of a domain of $C_{6,2^1,3^2}$. Indeed $D^{146} \subset D^{14}, D^{147} \subset D^{14}, D^{167} \subset D^{16}, D^{235} \subset D^{23}, D^{238} \subset D^{23}, D^{258} \subset D^{58}, D^{358} \subset D^{58}, D^{467} \subset D^{46}$. Since all domains of $C_{6,2^1,3^2}$ are equivalent, then all domains of $C^1_{5,2^3}$ are equivalent. We now prove that D^{146} exhibits the Anscombe's paradox.

$$D^{146} = \begin{array}{ccccc} x_2 & x_3 & x_5 & x_7 & x_8 \\ \hline +1 & +1 & -1 & -1 & -1 \\ +1 & -1 & +1 & -1 & -1 \\ -1 & +1 & +1 & +1 & -1 \end{array}$$

Let X be a vote profile with seven voters and three proposals defined as follow $X_1 = X_2 = x_2$, $X_3 = X_4 = x_3$, $X_5 = x_5$, $X_6 = x_7$, $X_7 = x_8$. It holds that $MR[X] = (+1, -1, +1)$. Each voter of the majority coalition $\{1, 2, 5, 8\}$ is frustrated on a majority of proposals. Thus the Anscombe's paradox holds at X. Since $n = 7$ is odd and by proposition 3.3.1, D^{146} is not Anscombe's paradox free as well as the other domains from $C^1_{5,2^3}$.

- We show that all domains of $C^2_{5,2^3}$ are single-switch.

A domain of $C^2_{5,2^3}$ it is obtained by deleting in Table 3.2 a triplet that contains a pair of opposite vectors.

Thus

$C^2_{5,2^3} = \{D^{127}, D^{128}, D^{136}, D^{138}, D^{145}, D^{148}, D^{158}, D^{168}, D^{178}, D^{236}, D^{237}, D^{245}, D^{247}, D^{257}, D^{267}, D^{278}, D^{345}, D^{346}, D^{356}, D^{367}, D^{368}, D^{456}, D^{457}, D^{458}\}$

Note that by deleting a triplet that contains a pair of opposite vectors, we obtain a sub-domain of a domain in $C_{6,3^3}$. Therefore each domain of $C^2_{5,2^3}$ does not exhibit the Anscombe's paradox and is equivalent to a single switch domain.

4. Suppose that $|D| = 4$, then $D \in C_{4,0^1,2^2} \cup C_{4,1^3} \cup C_{4,1^2,2^1} \cup C_{4,1^1,2^2} \cup C_{4,2^3}$.

For some integer $k_1, k_2, \ldots, k_l$ of $\{1, 2, \ldots, 8\}$, denote by ${}^{k_1 k_2 \cdots k_l}D$ the sub-domain

${}^{k_1 k_2 \cdots k_l}D = \{x_{k_1}, x_{k_2}, \ldots, x_{k_l}\}$ of the full domain in Table 3.2.

(a) We prove that all domains of $C_{4,0^1,2^2}$ are unifying preference domain.

According to Table 3.2, we have $C_{4,0^1,2^2} = \{{}^{1234}D, {}^{1256}D, {}^{1357}D, {}^{2468}D, {}^{3478}D, {}^{5678}D\}$. Hence,

- Since $\{7, 8\}$ is not contained in $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}$, then ${}^{1234}D$ and ${}^{1256}D$ are sub-domains of D^{78}.

- Since $\{2, 4\}$ is not contained in $\{1, 3, 5, 7\}, \{5, 6, 7, 8\}$, then ${}^{1357}D$ and ${}^{5678}D$ are sub-domains of D^{24}.

In the same way ${}^{2468}D \subset D^{13}$ and ${}^{3478}D \subset D^{15}$.

(b) We show that all domains from $C_{4,1^3}$ are unifying preference domains.

We have $C_{4,1^3} = \{{}^{1235}D, {}^{1246}D, {}^{1347}D, {}^{1567}D, {}^{2348}D, {}^{2568}D, {}^{3578}D, {}^{4678}D\}$ and ${}^{1235}D \subset D^{48}$, ${}^{1246}D \subset D^{57}$, ${}^{1347}D \subset D^{68}$, ${}^{1567}D \subset D^{48}$, ${}^{2348}D \subset D^{57}$, ${}^{2568}D \subset D^{13}$, ${}^{3578}D \subset D^{12}$, ${}^{4678}D \subset D^{13}$. Thus each preference domain of $C_{4,1^3}$ is contained in a unifying preference domain of $C_{6,2^2,3^1}$.

(c) We show that all domain of $C_{4,1^2,2^1}$ are unifying preference domain.

As above, $C_{4,1^2,2^1} = \{{}^{1245}D, {}^{1237}D, {}^{1236}D, {}^{1248}D, {}^{1257}D, {}^{1268}D,$ ${}^{1345}D, {}^{1348}D, {}^{1356}D, {}^{1378}D, {}^{1568}D, {}^{1578}D, {}^{2346}D, {}^{2347}D, {}^{2456}D, {}^{2478}D, {}^{2567}D, {}^{2678}D, {}^{3457}D,$ ${}^{3468}D, {}^{3567}D, {}^{3678}D, {}^{4568}D, , {}^{4578}D\}$

Thus,

${}^{1245}D \subset D^{37}, {}^{1237}D \subset D^{68}, {}^{1236}D \subset D^{57}, {}^{1248}D \subset D^{57}, {}^{1257}D \subset D^{34}, {}^{1268}D \subset D^{34}, {}^{1345}D \subset D^{26}, {}^{1348}D \subset D^{26}, {}^{1356}D \subset D^{24}, {}^{1378}D \subset D^{24}, {}^{1568}D \subset D^{24}, {}^{1578}D \subset D^{24}, {}^{2346}D \subset D^{57}, {}^{2347}D \subset D^{15}, {}^{2456}D \subset D^{13}, {}^{2478}D \subset D^{13}, {}^{2567}D \subset D^{13}, {}^{2678}D \subset D^{13}, {}^{3457}D \subset D^{26}, {}^{3468}D \subset D^{12}, {}^{3567}D \subset D^{12}, {}^{3678}D \subset D^{12}, {}^{4568}D \subset D^{12}, {}^{4578}D \subset D^{12}$. Thus each preference domain of $C_{4,1^2,2^1}$ is contained in a unifying preference domain of $C_{6,2^2,3^1}$.

(d) We show that all domains of $C_{4,1^1,2^2}$ are unifying preference domains.

We have :

$C_{4,1^1,2^2} = \{{}^{1238}D, {}^{1247}D, {}^{1258}D, {}^{1267}D, {}^{1346}D, {}^{1358}D, {}^{1367}D, {}^{1456}D, {}^{1457}D, {}^{1468}D, {}^{1478}D,$ ${}^{1678}D, {}^{2345}D, {}^{2356}D, {}^{2357}D, {}^{2368}D, {}^{2378}D, {}^{2458}D, {}^{2467}D, {}^{2578}D, {}^{3458}D, {}^{3467}D, {}^{3568}D, {}^{4567}D\}$

and

${}^{1238}D \subset D^{57}, {}^{1247}D \subset D^{56}, {}^{1258}D \subset D^{37}, {}^{1267}D \subset D^{34}, {}^{1346}D \subset D^{57}, {}^{1358}D \subset D^{26}, {}^{1367}D \subset D^{24}, {}^{1456}D \subset D^{78}, {}^{1457}D \subset D^{26}, {}^{1468}D \subset D^{57}, {}^{1478}D \subset D^{26}, {}^{1678}D \subset D^{24}, {}^{2345}D \subset D^{68}, {}^{2356}D \subset D^{78}, {}^{2357}D \subset D^{68}, {}^{2368}D \subset D^{57}, {}^{2378}D \subset D^{56}, {}^{2458}D \subset D^{13}, {}^{2467}D \subset D^{13}, {}^{2578}D \subset D^{13}, {}^{3458}D \subset D^{12}, {}^{3467}D \subset D^{12}, {}^{3568}D \subset D^{12}, {}^{4567}D \subset D^{12}$. Thus each preference domain of $C_{4,1^1,2^2}$ is contained in a unifying preference domain of $C_{6,2^2,3^1}$.

(e) We show that all preference domains of $C_{4,2^3}$ which do not exhibit the Anscombe's paradox are single-switch.

Since vectors in each domain of $C_{4,2^3}$ have the same type, we partition $C_{4,2^3}$ in two subsets $C^1_{4,2^3}$ and $C^2_{4,2^3}$ of $C_{4,2^3}$ such that :

- $D \in C^1_{4,2^3}$ if for all $x \in D$, $-x \notin D$.

- $D \in C^2_{4,2^3}$ if for all $x \in D$, $-x \in D$.

• $C^1_{4,2^3} = \{{}^{2358}D, {}^{1467}D\}$ and it is clear that these two domains are equivalent since each vector of ${}^{2358}D$ is the opposite of a vector of ${}^{1467}D$.

$$
{}^{2358}D = \begin{array}{cccc} x_2 & x_3 & x_5 & x_8 \\ \hline +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & +1 & -1 \end{array}
$$

Let X be a vote profile defined as follow $X_1 = x_2$, $X_2 = X_3 = x_3$, $X_4 = x_5$, $X_5 = x_8$. It holds that $MR[X] = (+1, -1, +1)$ and each voters of $\{1, 4, 5\}$ which is a majority is frustrated on a majority of proposals. Thus the Anscombe's paradox holds on ${}^{2358}D$.

• $C^2_{4,2^3} = \{{}^{1278}D, {}^{1368}D, {}^{1458}D, {}^{2367}D, {}^{2457}D, {}^{3456}D\}$.

It holds that ${}^{1278}D \subset D^{45}, {}^{1368}D \subset D^{45}, {}^{1458}D \subset D^{36}, {}^{2367}D \subset D^{45}, {}^{2457}D \subset D^{36}, {}^{3456}D \subset D^{18}$. Hence each domain of $C^2_{4,2^3}$ is a subset of a domain of single-switch domain of $C_{6,3^3}$. Therefore all those domains are single-switch.

5. Suppose that $|D| = 3$, then $D \in C_{3,0^1,1^2} \cup C_{3,1^3}$. We show that D is a subset of a domain which is a unifying preference domain or is single-switch; and then does not exhibit the Anscombe's paradox by Theorem 1.1 of Chapter 1.

• $C_{3,0^1,1^2} = \{{}^{123}D, {}^{124}D, {}^{125}D, {}^{126}D, {}^{134}D, {}^{135}D, {}^{137}D, {}^{156}D, {}^{157}D, {}^{234}D, {}^{246}D, {}^{248}D, {}^{256}D, {}^{268}D, {}^{347}D, {}^{348}D, {}^{357}D, {}^{378}D, {}^{468}D, {}^{478}D, {}^{567}D, {}^{568}D, {}^{578}D, {}^{678}D\}$ Note that

- ${}^{123}D, {}^{124}D, {}^{125}D, {}^{126}D, {}^{134}D, {}^{135}D, {}^{156}D, {}^{234}D, {}^{246}D, {}^{256}D$ are sub-domains of D^{78}.

- ${}^{137}D, {}^{248}D, {}^{347}D, {}^{348}D, {}^{378}D, {}^{478}D$ are sub-domains of D^{56}

- ${}^{157}D, {}^{268}D, {}^{567}D, {}^{568}D, {}^{578}D, {}^{678}D$ are sub-domains of D^{34}

- ${}^{357}D, {}^{468}D \subset D^{12}$.

Therefore, all domains of $C_{3,0^1,1^2}$ are unifying preference domains since each of them is contained in a unifying preference domain from $C_{6,2^2,3^1}$.

• As above, $C_{3,1^3} = C^1_{3,1^3} \cup C^2_{3,1^3}$;

$C^1_{3,1^3} = \{{}^{146}D, {}^{147}D, {}^{167}D, {}^{235}D, {}^{238}D, {}^{258}D, {}^{358}D, {}^{467}D\}$;

${}^{146}D \subset D^{57}$, ${}^{147}D \subset D^{68}$, ${}^{167}D \subset D^{34}$, ${}^{235}D \subset D^{34}$, ${}^{238}D \subset D^{57}$, ${}^{258}D \subset D^{34}$, ${}^{358}D \subset D^{12}$ and ${}^{467}D \subset D^{12}$. Therefore, all domains of $C^1_{3,1^3}$ are unifying preference domains since each of them is contained in a unifying preference domain from $C_{6,2^2,3^1}$.

• Similarly, $C^2_{3,1^3} = \{{}^{127}D, {}^{128}D, {}^{136}D, {}^{138}D, {}^{145}D, {}^{148}D, {}^{158}D, {}^{168}D, {}^{178}D, {}^{236}D, {}^{237}D, {}^{245}D, {}^{247}D, {}^{257}D, {}^{267}D, {}^{278}D, {}^{345}D, {}^{346}D, {}^{356}D, {}^{367}D, {}^{368}D, {}^{456}D, {}^{457}D, {}^{458}D\}$.

Note that:

- ${}^{127}D, {}^{128}D, {}^{136}D, {}^{138}D, {}^{168}D, {}^{178}D, {}^{278}D, {}^{368}D$ are sub-domains of D^{45}.

- ${}^{145}D, {}^{148}D, {}^{158}D, {}^{458}D$ are sub-domains of D^{27}

- ${}^{236}D, {}^{245}D, {}^{237}D, {}^{247}D, {}^{257}D, {}^{267}D^{345}D, {}^{346}D, {}^{356}D, {}^{367}D, {}^{456}D, {}^{457}D$ are sub-domains of D^{18}.

Therefore, all domains of $C^2_{3,1^3}$ are single-switch domains since each of them is contained in a single switch domain from $C_{6,3^3}$.

We conclude that for all preference domains D in $\{-1,1\}^3$ which does not exhibit the Anscombe's paradox, there exists a preference domain $D' \in C_{6,2^2,3^1} \cup C_{6,3^3}$ such that $D \subseteq D'$. ■

All possible preference domains with three proposals have been scrutinized in the proof of Theorem 3.8. The following corollary is a summary of the conclusion that hold in each possible cases.

Corollary 3.3.1. *Let $D \subset \{-1,1\}^3$ be a domain of cardinal d.*

i) If $d = 3$, then D is Anscombe's paradox free.

ii) If $d = 4$ then D is Anscombe's paradox free if and only if

$D \in C_{4,0^1,2^2} \cup C_{4,1^3} \cup C_{4,1^2,2^1} \cup C_{4,1^1,2^2}$ or $D \in C_{4,2^3}$ and $(\forall x \in D,\ -x \notin D)$.

iii) If $d = 5$ then D is Anscombe's paradox free if and only if $D \in C_{5,1^1,2^2}$ or $D \in C_{5,2^3}$ and $\exists x \in D$ such that $\forall y \in D \backslash \{x\},\ -y \in D \backslash \{x\}$.

iv) If $d = 6$ then D is Anscombe's paradox free if and only if $D \in C_{6,2^2,3^1} \cup C_{6,3^3}$.

v) $D \in \{-1, 1\}^3$ is Anscombe's paradox free if and only if there exists a domain $D' \in C_{6,2^2,3^1} \cup C_{6,3^3}$ such that $D \subseteq D'$.

CHAPTER 4

CONSENSUAL VOTING ENVIRONMENTS AND THE ANSCOMBE'S PARADOX

In this chapter, we mimic the functioning of some real voting bodies such as a parliament, a congress, a board of directors, ... In such voting environments, the simplest political divide includes a leading party, an opposition party (or a coordination) and possibly some freethinkers. This implies an intra-profile dependence among voters in the leading party as well as among voters in the opposition coordination. The main question we address here is whether the Anscombe's paradox still occurs in such a real life world. Basic definitions around consensual voting environments are provided in Section 4.1. Stable consensual voting environments on which the majority rule never exhibits the Anscombe's paradox over the set of all admissible vote profiles are characterized in Section 4.2 when the leading party submits almost a majority of proposals; and Section 4.3 is devoted to stable voting environments of other types.

4.1 Consensual voting environments

4.1.1 Majority, opposition and admissible vote profiles

We suppose that the set N of voters is partitioned in three subsets N_1, N_2 and N_3. For example, N_1 may represent the set of voters of the ruling party (a party with more than the half of representatives), N_2 the set of voters in the opposition party (the challenging party) and N_3 the set of independent voters (voters who are involved in no party). We also suppose that only parties N_1 and N_2 may submit proposals. We denote by M^j the set of all proposals initiated by N_j. Furthermore, we assume that voters in the same party all support all proposals from their party; but may have distinct opinions on other proposals. Meanwhile, independent voters are freethinkers on all proposals.

Definition 4.1.1. *A consensual voting environment is a quintet* $(N_1, N_2, N_3, M^1, M^2)$ *where*

- (M^1, M^2) *is called the agenda;*
- N_1, N_2 *and* N_3 *are respectively the leading party, the opposition party and the set of independent voters in such a way that* N_1, N_2 *and* N_3 *are disjoint subsets of* N *with* $N = N_1 \cup N_2 \cup N_3$ *and* $|N_1| > |N_2| + |N_3|$;
- M^1 *and* M^2 *are disjoint subsets of proposals such that* $\mathcal{M} = M^1 \cup M^2$.

As mentioned above, there are some political values that voters in the same party share. This implies the existence of a discipline party for N_1 and N_2 respectively; in such a way that voters in N_j all vote for the adoption of all proposals in M^j for $j \in \{1,2\}$. A vector of votes of a voter in N_j, given $j = 1, 2$, is admissible when this party discipline is observed.

Definition 4.1.2. *A profile of votes* X *is admissible given a consensual voting environment* $(N_1, N_2, N_3, M^1, M^2)$ *if voters in* N_j *all vote for the adoption of all proposals in* M^j, $j = 1, 2$; *that is*

$$\forall j \in \{1,2\},\ \forall a \in M^j,\ \forall i \in N_j, X_i^a = +1.$$

Clearly, the vector of votes of an independent voter $i \in N_3$ suffers no restriction and may takes any vector of votes from $\{-1, +1\}^m$.

Hereafter, we pose $|M^j| = m_j$ and $|N_j| = n_j$;

Example 4.1.1. *Consider* $\mathcal{M} = \{a^1, a^2, a^3, a^4, a^5, a^6, a^7\}$, $M^1 = \{a^1, a^2, a^3, a^4\}$, $M^2 = \{a^5, a^6, a^7\}$, $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5\}$ *and* $N_3 = \{6, 7\}$. *Let* X *be the vote profile presented in the table below:*

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
a^1	$+1$	$+1$	$+1$	-1	$+1$	-1	-1
a^2	$+1$	$+1$	$+1$	-1	-1	$+1$	-1
a^3	$+1$	$+1$	$+1$	-1	$+1$	-1	-1
a^4	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1
a^5	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$	-1
a^6	-1	$+1$	-1	$+1$	$+1$	-1	-1
a^7	$+1$	-1	-1	$+1$	$+1$	-1	-1

All voters in N_1 vote for the adoption of all proposals in $M^1 = \{a^1, a^2, a^3, a^4\}$ while all voters in N_2 vote for the adoption of all proposals in $M^2 = \{a^5, a^6, a^7\}$. Therefore X is an admissible vote profile under voting environment $(N_1, N_2, , N_3, M^1, M^2)$.

Definition 4.1.3. A consensual voting environment $(N_1, N_2, N_3, M^1, M^2)$ is stable if the majority rule never exhibits the Anscombe's paradox over the set of all admissible vote profiles; that is, given that X an admissible vote profile, MR does not exhibit the Anscombe's paradox at X.

Example 4.1.2. Consider $\mathcal{M} = \{a^1, a^2, a^3, a^4, a^5\}$, $M^1 = \{a^1, a^2\}$, $M^2 = \{a^3, a^4, a^5\}$, $N_1 = \{1, 2, 3, 4\}$, $N_2 = \{5\}$ and $N_3 = \{6, 7\}$. Then the following vote profile X is admissible and MR exhibits the Anscombe's paradox at X.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	$MR(X)$
a^1	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
a^2	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
a^3	$+1$	$+1$	-1	-1	$+1$	$+1$	-1	$+1$
a^4	$+1$	$+1$	-1	-1	$+1$	-1	$+1$	$+1$
a^5	-1	$+1$	-1	-1	$+1$	$+1$	$+1$	$+1$
	no	*no*	*yes*	*yes*	*no*	*yes*	*yes*	$4/7$

Indeed, each voters of $\{3, 4, 6, 7\}$ which is a majority is frustrated on a majority of proposals. The consensual voting environment $(N_1, N_2, N_3, M^1, M^2)$ is not stable.

Remark 4.1.1. Since $|N_1| > \frac{n}{2}$ and $\forall a \in M^1$, $X_i^a = +1$, for any admissible vote profile X, each voter i in N_1 is not frustrated on all proposals of M^1.

Being admissible under a consensual voting environment does not immune a vote profile to allow the possibility of observing the Anscombe's paradox. In what follow, we aim at identifying consensual voting environments that are stable.

4.1.2 Majority and agenda configurations

As we assume that the leading party N_1 contains more than the half of the voters, then $|N_1| = \lfloor \frac{n}{2} \rfloor + \delta$ for some non negative integer δ. We then distinguish:

- exact majorities when $\delta = 1$; that is when N_1 losses the majority as soon as a voter in N_1 gives up;
- extra-unit majorities when $\delta = 2$; for N_1 to loss the majority, at least two of its members must give up;
- multiple-unit majorities when N_1 is a superset of a two-unit majority.

More formally,

Definition 4.1.4. *A consensual voting environment $(N_1, N_2, N_3, M^2, M^2)$ has:*

- *an exact majority if $|N_1| = \lfloor \frac{n}{2} \rfloor + 1$;*
- *an extra-unit majority if $|N_1| = \lfloor \frac{n}{2} \rfloor + 2$;*
- *a multiple-unit majority if $|N_1| \geq \lfloor \frac{n}{2} \rfloor + 3$.*

Agenda in consensual voting environment may be of four types depending on whether the leading party submits more than the half of proposals or not.

Definition 4.1.5. *A consensual voting environment $(N_1, N_2, N_3, M^1, M^2)$ has:*

- *an agenda of type I if $|M^1| \geq \lfloor \frac{m-1}{2} \rfloor + 1$;*
- *an agenda of type II if $|M^1| = \lfloor \frac{m-1}{2} \rfloor$;*
- *an agenda of type III if $|M^1| = \lfloor \frac{m-1}{2} \rfloor - 1$;*
- *an agenda of type IV if $|M^1| \leq \lfloor \frac{m-1}{2} \rfloor - 2$.*

Remark that when the consensual voting environment $(N_1, N_2, N_3, M^2, M^2)$ has an agenda of type II, III or IV, the set of opposition party is nonempty since $|M^2| \neq \emptyset$. In the next sections, we study how the type of the majority and the type of the agenda in a voting environment influence its stability. To achieve this, we need further notation as follows:

$|N_1| = \lfloor \frac{n}{2} \rfloor + 1 + d$ and since $\mathcal{M} = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m-1}{2} \rfloor + 1$, $|M^2| = \lfloor \frac{m}{2} \rfloor + 1 + r$

where d and r are integers such that $d \geq 0$ and $-\lfloor \frac{m_2}{2} \rfloor - 1 \leq r \leq \lfloor \frac{m_2-1}{2} \rfloor$.

4.2 Consensual voting environments with an agenda of type I or II

In this section we consider the voting environment with an agenda of type I or II that means that the number of proposals submitted by the opposition is lower to the one submitted by the leading party or the opposition submitted an exact majority of proposals. Under these conditions we regard the stability in two cases when the set of independent voters is empty and nonempty.

4.2.1 Agenda of type I or II with no independent voter

When the set of independent voters is empty, the consensual voting environment with an agenda of type I or II is stable as shown in the following result.

Proposition 4.2.1. *A consensual voting environment (N_1, N_2, M^1, M^2) with an agenda of type I or II but no independent voter is stable.*

Proof.

Consider an admissible vote profile X in a consensual voting environment (N_1, N_2, M^1, M^2) with an agenda of type I or II but no independent voter. We have $|M^1| \geq \lfloor \frac{m-1}{2} \rfloor$.

- Suppose that $|M^1| < \lfloor \frac{m-1}{2} \rfloor$. Since each voter in N_1 is frustrated on at most $m_2 = m - m_1 < \lfloor \frac{m}{2} \rfloor + 1$, then no voter in N_1 is frustrated on a majority of proposals. Moreover N_2 is not a majority. Therefore no majority of voters may be frustrated on a majority of proposals. The Anscombe's paradox does not occur at X.

• Suppose that $|M^1| = \left\lfloor \frac{m-1}{2} \right\rfloor$ then $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 1$. To prove that the Anscombe's paradox does not occurs at X, suppose the contrary holds. Then there exists a majority S of voters who are each frustrated on a majority of proposals. Since $n_2 < \frac{n}{2}$, it follows that S and N_1 overlap. Moreover, each voter in $S \cap N_1$ is frustrated on all proposals of M^2. Now, $m_1 < \frac{m}{2}$. Thus voters in $S \cap N_2$ are all frustrated on at least one proposal in M^2 since all voters in N_2 vote all for the adoption of each proposal in M^2. Let a be such a proposal. Recalling that $S = (S \cap N_1) \cup (S \cap N_2)$, it follows that all voter in S are each frustrated on a. Thus a contradiction holds since S is a majority. ■

4.2.2 Agenda of type I or II with a nonempty set of independent voters

Given a consensual voting environment, we now prove that when the set of independent voters is nonempty and the opposition submits less proposals than the leading party, the environment is stable.

Proposition 4.2.2. *A voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type I is stable.*

Proof.

In a consensual voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type I with a nonempty set of independent voters, consider an admissible vote profile X. Then $|M^1| \geq \left\lfloor \frac{m-1}{2} \right\rfloor + 1$ and $|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor$. Since each voter in N_1 is frustrated on at most $m_2 = m - m_1 < \frac{m}{2}$ proposals, then no voter in N_1 is frustrated on a majority of proposals. Moreover $N_2 \cup N_3$ is not a majority. Therefore no majority of voters may be frustrated on a majority of proposals. The Anscombe's paradox does not occur at X. ■

Proposition 4.2.3. *A voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type II and a nonempty set of independent voters is stable if and only if*

$$(|N_2| = 1,\ |N_3| = 2,\ |M^2| = 3,\ m \text{ and } n \text{ are even}) \text{ or } (|N_3| = 1).$$

Proof.

Necessity: Consider a stable voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type II and a nonempty set of independent voters. We have $m_1 = \lfloor \frac{m-1}{2} \rfloor$ and $n_3 \geq 1$. Then $m_2 = \lfloor \frac{m}{2} \rfloor + 1$. Note that $n_2 + n_3 < n_1$. Thus there exist n_1' and n_1'' such that $n_1 = n_1' + n_1''$ and $n_2 + n_1' = \lfloor \frac{n}{2} \rfloor - 1$. Pose $N_1 = \{1_1, \ldots, 1_{n_1'}\} \cup \{2_1, \ldots, 2_{n_1''}\}$, $N_2 = \{3_1, \ldots, 3_{n_2}\}$, $N_3 = \{4_1, \ldots, 4_{n_3}\}$. To prove that ($|N_2| = 1$ and $|N_3| = 2$ and $|M^2| = 3$, m and n are even) or ($|N_3| = 1$), suppose the contrary and distinguish two cases.

Case 1: m is even.

Then ($|N_3| \geq 2$ and $|N_2| \geq 2$) or ($|N_3| \geq 3$) or ($|M^2| \geq 4$ and $|N_3| \geq 2$) or ($|N_3| \geq 3$ and n odd). We have the following subcases:

Subcase 1.1: ($|N_3| \geq 3$, $|N_2| \geq 2$) or ($|N_3| \geq 2$ and n odd). Pose $T = \{4_1, 4_2\}$ and note that $n_1'' + n_3 = \lfloor \frac{n-1}{2} \rfloor + 2$. Consider the following vote profile

	X_{1_1}	$\cdots$	$X_{1_{n_1'}}$	X_{2_1}	$\cdots$	$X_{2_{n_1''}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	X_{4_1}	X_{4_2}	X_{4_3}	$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	-1	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1}	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	-1	-1	$\cdots$	-1	$+1$
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	-1	$\cdots$	-1	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	-1	$\cdots$	-1	-1
a^{m_1+3}	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	-1	$+1$	$\cdots$	$+1$	$+1$
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	-1	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	-1	-1	$\cdots$	-1	-1

Subcase 1.2: $|N_3| \geq 2$ and $|M^2| \geq 4$. In this case, pose $T = \{4_1, 4_2\}$ and use the following matrix in the vote profile above.

	X_{1_1}	$\cdots$	$X_{1_{n_1'}}$	X_{2_1}	$\cdots$	$X_{2_{n_1''}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	X_{4_1}	X_{4_2}	X_{4_3}	$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^{m_1+1}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	-1	$\cdots$	-1	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	-1	$\cdots$	-1	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	-1	$\cdots$	-1	-1
a^{m_1+4}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	-1	$\cdots$	-1	-1

Subcase 1.3: $|N_3| \geq 3$. In this case, pose $T = \{4_1, 4_2, 4_3\}$, reset n_1' such that $n_2 + n_1' = \lfloor \frac{n}{2} \rfloor - 2$. Then $n_1'' + n_3 = \lfloor \frac{n-1}{2} \rfloor + 3$. Use the following matrix in the vote profile above in **Subcase 1.1**.

	X_{1_1}	$\cdots$	$X_{1_{n_1'}}$	X_{2_1}	$\cdots$	$X_{2_{n_1''}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	X_{4_1}	X_{4_2}	X_{4_3}		$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^{m_1+1}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	$+1$	-1	$\cdots$	-1	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	$+1$	-1	$\cdots$	-1	-1
a^{m_1+4}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	-1	$\cdots$	-1	-1

Let $S = \{1_1, \ldots, 1_{n'_1}\} \cup \{3_1, \ldots, 3_{n_2}\} \cup T$, $|S| = n'_1 + n_2 + |T| = \lfloor \frac{n}{2} \rfloor + 1$. Each voter in $\{1_1, \ldots, 1_{n'_1}\}$ is frustrated on $|M^2| = \lfloor \frac{m}{2} \rfloor + 1$ proposals. Each voter in $\{3_1, \ldots, 3_{n_2}\} \cup T$ is frustrated on at least $|M^1| + 2 = \lfloor \frac{m-1}{2} \rfloor + 2 > \frac{m}{2}$ proposals. Thus each voter in S is frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X. This is a contradiction since the voting environment is stable.

Case 2: m is odd.

By assumption $|N_3| > 1$. Noting that m is odd, it follows that $\lfloor \frac{m-1}{2} \rfloor + 1 > \frac{m}{2}$. Consider the following vote profile

	X_{1_1}	$\cdots$	$X_{1_{n'_1}}$	X_{2_1}	$\cdots$	$X_{2_{n''_1}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	X_{4_1}	X_{4_2}	X_{4_3}	$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	-1	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1}	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	-1	-1	$\cdots$	-1	$+1$
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	-1	$\cdots$	-1	-1
a^{m_1+2}	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	-1	$\cdots$	-1	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	-1	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	-1	-1	$\cdots$	-1	-1

Let $S = \{1_1, \ldots, 1_{n'_1}\} \cup \{3_1, \ldots, 3_{n_2}\} \cup \{4_1, 4_2\}$, $|S| = \lfloor \frac{n-1}{2} \rfloor + 2$. Each voter in $\{1_1, \ldots, 1_{n'_1}\}$ is frustrated on $|M^2| = \lfloor \frac{m}{2} \rfloor + 1$ proposals. Each voter in $\{3_1, \ldots, 3_{n_2}\} \cup \{4_1, 4_2\}$ is frustrated on $|M^1| + 1 = \lfloor \frac{m-1}{2} \rfloor + 1 > \frac{m}{2}$ proposals. Thus each voter in S is frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X. This is a contradiction since the voting environment is stable.

Sufficiency: Suppose that a voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type II and a nonempty set of independent voters is such that

$$(|N_2| = 1,\ |N_3| = 2,\ |M^2| = 3,\ m \text{ and } n \text{ are even}) \text{ or } (|N_3| = 1).$$

Then $m_1 = \lfloor \frac{m-1}{2} \rfloor$ and $|N_3| \neq \emptyset$. Thus $m_2 = \lfloor \frac{m}{2} \rfloor + 1$.

We show that $(N_1, N_2, N_3, M^1, M^2)$ is stable. On the contrary, suppose that there exist an admissible vote profile X and a majority coalition S such that each voter in S is frustrated on a majority of proposals. Note that:

i) $S \cap N_1 \neq \emptyset$ since $|N_1| > \frac{n}{2}$;

ii) each voter in N_1 is frustrated on no proposal of M^1 since $|N_1| > \frac{n}{2}$. Hence each voter of $S \cap N_1$ is frustrated on all proposals in M^2. Therefore $S \cap (N_2 \cup N_3) \neq \emptyset$ otherwise, all voters in S would be frustrated on the same proposal.

iii) Since m is even and $|M^1| = \left\lfloor \frac{m-1}{2} \right\rfloor$. Then, to be frustrated on a majority of proposals, each voter in $S \cap (N_2 \cup N_3)$ is necessarily frustrated on at least two proposals in M^2.

Consider the following cases:

Case 1: $S \cap N_2 \neq \emptyset$ and $|N_3 \cap S| = \emptyset$.

Let $i \in N_2 \cap S$. Then there exists $a \in M^2$ such that i is frustrated on a. Since X is admissible, then all voters in S are frustrated on a; as well as all voters of $S \cap N_1$. This is a contradiction since S is a majority.

Case 2: $S \cap N_2 = \emptyset$ and $S \cap N_3 \neq \emptyset$.

• Suppose that $|N_3| = 1$. Since $S \subseteq N_1 \cup N_3$ and each voter of $S \cap N_1$ is frustrated on all proposals of M^2, for each proposal in M^2, there exists a voter in $S \cap N_3$ who is not frustrated on that proposal. This means that $|N_3| > 1$. A contradiction occurs.

• Suppose that $|M^2| = 3$ and $|N_3| = 2$. Then $m_1 = 1$ and to be frustrated on a majority of proposals, each voter in $S \cap N_3$ is necessarily frustrated on at least two proposals in M^2. Therefore, there exists a proposal in M^2 on which all voters in $S = (S \cap N_1) \cup (S \cap N_3)$ are frustrated. A contradiction holds since S is a majority.

Case 3: $S \cap N_2 \neq \emptyset$ and $S \cap N_3 \neq \emptyset$.

Let $H \subseteq M^2$ be the set of all proposals on which each voter of $S \cap N_2$ is frustrated on M^2. Each voter in $S \cap (N_1 \cup N_2)$ is also frustrated on all proposals in H. It follows that for each proposal $a \in H$, there exists at least one voter in $S \cap N_3$ who is not frustrated on a. Therefore if $|N_3| = 1$, a contradiction holds. Suppose now that $|S \cap N_3| = 2$. If $|M^2| = 3$ and $|N_1| = 1$, then $2 \leq |H| \leq 3$ and for all proposal $a \in H$ there exists a voter in N_3 who is not frustrated on a. Since $|M^2| = 3$, there exists a proposal $a_0 \in M^2$ such that each voter in N_3 is frustrated on a_0 and $MR(X^{a_0}) = +1$. Note that each voter in $(S \cap N_1) \cup N_3$ is frustrated on a_0. We have $|\{i \in N \ : \ X_i^{a_0} = MR(X^{a_0})\}| \leq |N_1 \backslash S| + |N_2|$. However, $|S \cap N_1| + |N_3| = |S| - |N_2| = |S| - 1 \geq \frac{n}{2}$. Since n is even and each voter in $(S \cap N_1) \cup N_3$ is frustrated on a_0, a contradiction occurs. ■

4.3 Consensual voting environments with an agenda of type III or IV

4.3.1 Agenda of type III or IV with no independent voter

As shown in the result below, a consensual voting environment with an exact majority, an agenda of type III or IV but no independent voter ($N_3 = \emptyset$), is stable provided that the total number of proposals from the opposition party does not exceed a threshold.

Proposition 4.3.1. *A voting environment* (N_1, N_2, M^1, M^2) *with an exact majority, an odd number of voters, an agenda of type III or IV but no independent voter is stable if and only if*

$$|M^2| < \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

Proof.

Necessity part: Consider a voting environment (N_1, N_2, M^1, M^2) with an exact majority, an odd number of voters, an agenda of type III or IV but no independent voter. Then n is odd, $n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $n_2 = \left\lfloor \frac{n-1}{2} \right\rfloor$, $m_1 = \left\lfloor \frac{m-1}{2} \right\rfloor - r$ and $m_2 = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$ with $r \geq 1$.

On the contrary, suppose that

$$|M^2| \geq \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

We have the following cases:

Case 1: n is odd.

Since $r = |M^2| - \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)$ and n odd, we have $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$ and

$$\begin{aligned}
|M^2| \geq \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}{\left\lfloor \frac{n-1}{2} \right\rfloor} &\iff |M^2|\left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 - 1\right) \geq \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \\
&\iff |M^2|\left(\left\lfloor \frac{n}{2} \right\rfloor + 1 - 1\right) \geq \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \\
&\iff |M^2|\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \geq |M^2| \\
&\iff |M^2| - \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \geq \frac{|M^2|}{\left\lfloor \frac{n}{2} \right\rfloor + 1} \\
&\iff r \geq \frac{|M^2|}{n_1}
\end{aligned}$$

Then there exist two integers p and t such that

$$|M^2| = pn_1 + t$$

with ($p < r$ and $t < n_1$) or ($p = r$ and $t = 0$).

Consider a vote profile X such that its restriction on M^2 is defined as follows:

- The matrix of M^2 has p identical blocks that are each defined on n_1 proposals;

	X_{1_1}		$\cdots$		$X_{1_{n_1}}$	X_{2_1}	$\cdots$	$X_{2_{n_2}}$	$MR(X)$
$a^{\lvert M^1\rvert+1}$	$+1$	$\cdots$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
	-1	$\cdots$	$+1$	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\lvert M^1\rvert+n'_1}$	-1	$\cdots$	-1	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\lvert M^1\rvert+ln'_1+1}$	$+1$	$\cdots$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
	-1	$\cdots$	$+1$	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\lvert M^1\rvert+(l+1)n'_1}$	-1	$\cdots$	-1	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$

- The last block refers to the t last proposals $\{a^{pn_1+1}, a^{pn_1+2}, \dots, a^{pn_1+t}\}$ and is defined as follows

	X_{1_1}		$\cdots$		X_{1_t}			$X_{1_{n_1}}$	X_{2_1}	$\cdots$	$X_{2_{n_2}}$	$MR(X)$
$a^{\lvert M^1\rvert+pn_1+1}$	$+1$	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
	-1	$\cdots$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\lvert M^1\rvert+pn_1+t}$	-1	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$

Pose $S = \{1_1, \dots, 1_{n_1}\}$, $|S| = n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Note that each voter of S is frustrated on at least $|M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Therefore voters in S are frustrated on a majority of proposals. Thus the vote profile X exhibits the Anscombe's paradox.

Sufficiency: Consider a voting environment (N_1, N_2, M^1, M^2) with an exact majority, an odd number of voters, an agenda of type III or IV but no independent voter. Suppose that

$$|M^2| < \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

We show that (N_1, N_2, M^1, M^2) is stable. On the contrary, suppose that there exist an admissible vote profile X and a majority coalition S such that each voter in S is frustrated on a majority of proposals.

Pose

$$\delta_m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even} \end{cases}$$

Note that:

i) For each proposal $a \in M^1$, $MR(X) = +1$;

ii) $S \cap N_1 \neq \emptyset$ and each member of $S \cap N_1$ is frustrated on no proposal in M^1 by Remark 4.1.1. Thus each voter in $S \cap N_1$ is frustrated on at least $\lfloor \frac{m}{2} \rfloor + 1$ proposals in M^2;

iii) $\lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor + 1 - \delta_m$;

iv) each member of $S \cap N_2$ is frustrated on at most $\lfloor \frac{m-1}{2} \rfloor - r$ proposals of M^1. Then each such voter, to be frustrated on a majority of proposals, is necessarily frustrated on at least

$$\left\lfloor \frac{m}{2} \right\rfloor + 1 - \left(\left\lfloor \frac{m-1}{2} \right\rfloor - r \right) = r + \delta_m$$

proposals of M^2;

v) recalling that X is admissible, then voters in $S \cap N_2$ are all frustrated on the same subset, say T, of proposals of M^2. By point (iv) above, $|T| = r + \delta_m + u$ with $0 \leq u \leq m_2$; hence $0 \leq u \leq \lfloor \frac{m}{2} \rfloor + 1 - \delta_m = \lfloor \frac{m-1}{2} \rfloor + 1$;

vi) for all proposals $a \in T$, $MR(X^a) = -1$ and $\forall i \in N_2$, $\forall a \in T$, $X_i^a = +1$;

vii) Since voters in $S \cap N_2$ are each frustrated on all proposals in T, then $S \cap N_2$ is not a majority coalition. This implies that

- $|S \cap N_2| \leq \lfloor \frac{n-1}{2} \rfloor$ and $|S \cap N_2| = \lfloor \frac{n-1}{2} \rfloor - v$ where v is an integer such that $0 \leq v \leq \lfloor \frac{n-1}{2} \rfloor$
- $|S \cap N_1| \geq \lfloor \frac{n}{2} \rfloor + 1 - |S \cap N_2| = v + 1$ and $|S \cap N_1| = v + c + 1$ where c is an integer such that $0 \leq c \leq \lfloor \frac{n}{2} \rfloor - v$.
- $|N_1 \backslash S| = |N_1| - |N_1 \cap S| = \lfloor \frac{n}{2} \rfloor + 1 - (v + c + 1) = \lfloor \frac{n-1}{2} \rfloor - (v + c)$.

Consider the following cases

Case 1: $S \cap N_2 \neq \emptyset$

Since X is admissible and $T \subseteq M^2$, then all voters in N_2 vote for the adoption of all proposals in T and each proposal in T is rejected by point (v). Since $|N_1 \backslash S| = \left\lfloor \frac{n-1}{2} \right\rfloor - (v+c)$, each voter from $S \cap N_1$ votes for the rejection of each proposal $a^l \in T$. Therefore each voter of $S \cap N_1$ is frustrated on no proposal in $M^1 \cup T$ and is frustrated on at most $m_2 - |T| = \left\lfloor \frac{m}{2} \right\rfloor + 1 - u - \delta_m \leq \left\lfloor \frac{m}{2} \right\rfloor$. A contradiction holds.

Case 2: $S \cap N_2 = \emptyset$. Since by assumption $N_3 = \emptyset$, it follows that $S \subseteq N_1$. Moreover, N_1 is an exact majority by assumption. Therefore $S = N_1$ and $N \backslash S = N_2$. Progressively, construct a new profile Y from X as follows: given $a \in \mathcal{M}$,

- if $a \in M^1$, pose $Y_i^a = X_i^a$ for all $i \in N$.
- if $a \in M^2$, $MR(X^a) = +1$ and the set $S_a = \{i \in N_1 \ : \ X_i^a = +1\}$ contains at least one voter. Choose a single voter, say i_a, in S_a and set

$$Y_i^a = \begin{cases} -X_i^a & \text{if } i \in S_a \backslash \{i_a\} \\ X_i^a & \text{otherwise.} \end{cases}$$

- if $a \in M^2$ and $MR(X^a) = -1$, then $X_i^a = -1$ for all $i \in N_1$. Choose a single voter, say i_a, in N_1 and set

$$Y_i^a = \begin{cases} -X_i^a & \text{if } i = i_a, \\ X_i^a & \text{otherwise.} \end{cases}$$

By so doing, Y is admissible and the set of all proposals on which each voter $i \in N_1$ is frustrated at X is a subset of the set of all proposals on which i is frustrated at Y. Therefore, the Anscombe's paradox occurs at Y. Moreover, for each proposal $a \in M^2$, there is exactly one voter in N_1 who agrees with the majority decision on a.

Now, given $i \in N_1$, let g_i be the total number of proposals in M^2 on which i agrees with the majority decision. By construction of Y,

$$\sum_{i \in N_1} g_i = m_2.$$

Let j be a voter in N_1 such that $g_j \geq g_i$ for all $i \in N_1$. Then $n_1 g_j \geq m_2$ and therefore $g_j \geq \frac{m_2}{n_1}$. Since $m_2 = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$, and the Anscombe's paradox holds at Y, then

$r \geq g_j \geq \frac{m_2}{n_1}$. We deduce that $m_2 - \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \geq \frac{m_2}{n_1}$. It follows that

$$m_2 - \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \geq \frac{m_2}{n_1} \iff m_2 \left(\frac{n_1 - 1}{n_1}\right) \geq \left\lfloor \frac{m}{2} \right\rfloor + 1 \iff m_2 \geq \frac{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

A contradiction holds. This completes Case 2 and also the proof. ∎

PROPOSITION 4.3.2. *Consider a voting environment (N_1, N_2, M^1, M^2) with an exact majority, an even number of voters and no independent voter. Then (N_1, N_2, M^1, M^2) is not stable if this voting environment has*

- *an agenda of type IV; or*
- *an agenda of type III while (m is odd) or $\left\lfloor \frac{n}{2} \right\rfloor \geq 2\left\lfloor \frac{m}{2} \right\rfloor + 3$.*

Proof.

Consider a voting environment (N_1, N_2, M^1, M^2) with an exact majority, an even number of voters, no independent voter, an agenda of type IV or an agenda of type III while (m is odd) or ($\left\lfloor \frac{n}{2} \right\rfloor \geq 2\left\lfloor \frac{m}{2} \right\rfloor + 3$). Note that n is even, $n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $n_2 = \left\lfloor \frac{n-1}{2} \right\rfloor$. Moreover $m_1 = \left\lfloor \frac{m-1}{2} \right\rfloor - r$ then $m_2 = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$ with $\big(r = 1$ and m odd) or ($r = 1$ and $\left\lfloor \frac{n}{2} \right\rfloor \geq 2\left\lfloor \frac{m}{2} \right\rfloor + 3\big)$ or ($r > 1$). Pose

$$\delta_m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even,} \end{cases}$$

$t = r + \delta_m$, $n = 2p$. We have $n_1 = p + 1$, $n_2 = p - 1$, $\left\lfloor \frac{m-1}{2} \right\rfloor + \delta_m > \frac{m}{2}$.

Case 1: ($r = 1$ if m is odd) or ($r > 1$ otherwise). Consider a vote profile X defined as follows:

	X_{1_1}	$\cdots$	$X_{1_{p-1}}$	X_{1_p}	$X_{1_{p+1}}$	X_{2_1}	$\cdots$	$X_{2_{n_2}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1}	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	$\cdots$	-1	$+1$
	-1	$\cdots$	-1	-1	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	-1	$+1$	$\cdots$	$+1$	-1
a^{m_1+3}	-1	$\cdots$	-1	-1	$+1$	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1+t}	-1	$\cdots$	-1	$+1$	-1	$+1$	$\cdots$	$+1$	-1
	$+1$	$\cdots$	$+1$	-1	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	-1	-1	$+1$	$\cdots$	$+1$	$+1$

Pose $S = \{1_p, 1_{p+1}\} \cup \{2_1 \ldots, 1_{n_2}\}$, $|S| = 2+p-1 = p+1$. Each voter of $\{2_1 \ldots, 1_{n_2}\}$ is frustrated on at least $|M^1| + t = \left\lfloor \frac{m-1}{2} \right\rfloor + \delta_m > \frac{m}{2}$ proposals. Furthermore each voter of $\{1_p, 1_{p+1}\}$ is frustrated on at least $\left\lfloor \frac{t}{2} \right\rfloor + m_2 - t = \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 1 - \delta_m$ proposals.

- If $r = 1$ and $\delta_m = 1$. Thus $\left\lfloor \frac{t}{2} \right\rfloor - \delta_m = 0$ and $\left\lfloor \frac{t}{2} \right\rfloor + m_2 - t = \left\lfloor \frac{m}{2} \right\rfloor + 1$.
- $r > 1$ then $\left\lfloor \frac{t}{2} \right\rfloor - \delta_m \geq 0$. Thus $\left\lfloor \frac{t}{2} \right\rfloor + m_2 - t \geq \frac{m}{2} + 1$.

In both cases, each voter of S is frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X.

Case 2: $r = 1$ and $\left\lfloor \frac{n}{2} \right\rfloor \geq 2 \left\lfloor \frac{m}{2} \right\rfloor + 3$. Then $|N_1| \geq 2|M^2|$. Consider a vote profile X such that its restriction on M^2 is defined as follows:

	X_{1_1}		$\cdots$			$\cdots$		$X_{1_{2m_2}}$		$\cdots$	$X_{1_{n_2}}$	X_{1_1}	$\cdots$	$X_{2_{n_2}}$	$MR(X)$
a^{m_1+1}	$+1$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
	-1	-1	$\cdots$	$+1$	$+1$	$\cdots$	-1	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	-1	-1	$\cdots$	-1	-1	$\cdots$	$+1$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$

Pose $S = N_1$. Each voter of S is frustrated on at least $|M^2| - 1 = \left\lfloor \frac{m}{2} \right\rfloor + 1$. Therefore the Anscombe's paradox holds at X. ■

Proposition 4.3.3. *A voting environment (N_1, N_2, M^1, M^2) with an exact majority, an even number of voters, an agenda of type III, but no independent voter is stable if and only if*

$$m \text{ is even} \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor < 2 \left\lfloor \frac{m}{2} \right\rfloor + 3.$$

Proof.

Necessity: See Proposition 4.3.2.

Sufficiency: Consider a voting environment (N_1, N_2, M^1, M^2) with an exact majority, an even number of voters, an agenda of type III, but no independent voter. Suppose that

$$m \text{ is even} \quad \text{and} \quad \left\lfloor \frac{n}{2} \right\rfloor < 2 \left\lfloor \frac{m}{2} \right\rfloor + 3.$$

We show that (N_1, N_2, M^1, M^2) is stable. On the contrary, suppose that there exist an admissible vote profile X and a majority coalition S such that each voter in S is frustrated on a majority of proposals. Note that:

i) For each proposal $a \in M^1$, $MR(X) = +1$;

ii) $S \cap N_1 \neq \emptyset$ and each member of $S \cap N_1$ is frustrated on no proposal in M^1. Thus each voter in $S \cap N_1$ is frustrated on at least $\lfloor \frac{m}{2} \rfloor + 1$ proposals in M^2;

iii) each member of $S \cap N_2$ is frustrated on at most $\lfloor \frac{m-1}{2} \rfloor - 1$ proposals of M^1. Then each such voter, to be frustrated on a majority of proposals, is necessarily frustrated on at least

$$\left\lfloor \frac{m}{2} \right\rfloor + 1 - \left(\left\lfloor \frac{m-1}{2} \right\rfloor - 1 \right) = 3$$

proposals of M^2 since m is even;

iv) Recalling that X is admissible, then voters in $S \cap N_2$ are all frustrated on the same subset, say T, of proposals of M^2. By point (v) above, $|T| = 3 + u$ with $0 \leq u \leq m_2$; hence $0 \leq u \leq \lfloor \frac{m-1}{2} \rfloor + 1$;

v) for all proposal $a \in T$, $MR(X^a) = -1$ and for all $i \in N_2$, for all $a \in T$, $X_i^a = +1$;

vi) Since voters in $S \cap N_2$ are each frustrated on all proposals in T, then $S \cap N_2$ is not a majority coalition. This implies that

- $|S \cap N_2| \leq \lfloor \frac{n-1}{2} \rfloor$ and $|S \cap N_2| = \lfloor \frac{n-1}{2} \rfloor - v$ where v is an integer such that $0 \leq v \leq \lfloor \frac{n-1}{2} \rfloor$
- $|S \cap N_1| \geq \lfloor \frac{n}{2} \rfloor + 1 - |S \cap N_2| = v + 2$ and $|S \cap N_1| = v + c + 2$ where c is an integer such that $0 \leq c \leq \lfloor \frac{n}{2} \rfloor - v - 1$.
- $|N_1 \backslash S| = |N_1| - |N_1 \cap S| = \lfloor \frac{n}{2} \rfloor + 1 - (v + c + 2) = \lfloor \frac{n-1}{2} \rfloor - (v + c)$.

Consider the following cases

Case 1: $S \cap N_2 \neq \emptyset$

Since X is admissible and $T \subseteq M^2$, then all voters in N_2 vote for the adoption of all proposals in T and each proposal in T is rejected. Since $|N_1 \backslash S| = \lfloor \frac{n-1}{2} \rfloor - (v + c)$ there are at least $v + c + 1$ voters from $S \cap N_1$ who vote for the rejection of each proposal $a \in T$. Recalling that $|S \cap N_1| = v + c + 2$, then there is at most $|S \cap N_1| - (v + c + 1) = 1$ voter of $S \cap N_1$ who is frustrated on a. We have two subcases.

Subcase 1: Suppose that there exists a voter, say i_0 in $S \cap N_1$ who is frustrated on no proposal in T. But $i_0 \in N_1$ is also frustrated on no proposal in M^1. Therefore i_0 is frustrated on at most $|M^2| - |T| = \lfloor \frac{m}{2} \rfloor - 1 - u \leq \lfloor \frac{m}{2} \rfloor$. A contradiction holds.

Subcase 2: Suppose that for each proposal $a^l \in T$, there exists exactly one voter $i_l \in S \cap N_1$ who is frustrated on a^l. This means that $|S \cap N_1| \leq |T|$ otherwise, there exists a voter in $S \cap N_1$ who is frustrated on no proposal in T and a contradiction holds by subcase 1. Now, given $i \in N_1$, let f_i be the total number of proposals in T on which i is frustrated. By assumption,

$$\sum_{i \in S \cap N_1} f_i = |T|.$$

Let j be a voter in $S \cap N_1$ such that $f_j \leq f_i$ for all $i \in S \cap N_1$. Then $|S \cap N_1| f_j \leq |T|$ and therefore $f_j \leq \frac{|T|}{|S \cap N_1|}$. Since $|M^2| - |T| = \left\lfloor \frac{m}{2} \right\rfloor - u - 1$. Therefore, to be frustrated on a majority of proposals, voter j is necessarily frustrated on at least $u+2$ proposals of T. Thus

$$f_j \geq u+2 \iff \frac{|T|}{|S \cap N_1|} \geq u+2 \iff 1+\frac{1}{u+2} = \frac{u+3}{u+2} \geq |S \cap N_1| \iff |S \cap N_1| = 1.$$

A contradiction holds because $|S \cap N_1| = v + c + 2$.

Case 2: $S \cap N_2 = \emptyset$. By ii) $S \subseteq N_1$ and all voters of $S \cap N_1$ are not frustrated on M^1. Thus each of them is frustrated on at least $\left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals in M^2. We have $|S \cap N_1| = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and then $|N \backslash S| = \left\lfloor \frac{n-1}{2} \right\rfloor$. Since n is even, for each proposal $a \in M^2$, there exist at least 2 voters in S who are not frustrated on a. By assumption, $\left\lfloor \frac{n}{2} \right\rfloor < 2\left\lfloor \frac{m}{2} \right\rfloor + 3$. Thus $|S| = |N_1| < 2|M^2|$ and there exists a voter i_0 who is not frustrated on at least 2 proposals. Therefore i_0 is frustrated on no proposal in M^1 and is frustrated on at most $|M^2| - 2 = \left\lfloor \frac{m}{2} \right\rfloor$. A contradiction holds. ■

The following result states that when the half of the number of voters is less than or equal to the half of the proposals, the voting environment with an extra-unit majority, an agenda of type III but no independent voter ($N_3 = \emptyset$), m is even and n odd is stable.

Proposition 4.3.4. *When m is even and n odd, a voting environment (N_1, N_2, M^1, M^2) with an extra-unit majority, an agenda of type III and no independent voter ($N_3 = \emptyset$), is stable if and only if*

$$\left\lfloor \frac{n}{2} \right\rfloor < \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Proof.

Suppose that m is even and n odd.

Necessity part: Consider a voting environment (N_1, N_2, M^1, M^2) with an extra-unit majority, an agenda of type III and no independent voter. Then $n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 2$, $n_2 = \left\lfloor \frac{n-1}{2} \right\rfloor - 1$, $m_1 = \left\lfloor \frac{m-1}{2} \right\rfloor - 1$ and $m_2 = \left\lfloor \frac{m}{2} \right\rfloor + 2$.

To show that $\left\lfloor \frac{n}{2} \right\rfloor < \left\lfloor \frac{m}{2} \right\rfloor + 1$. Suppose on the contrary that

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{m}{2} \right\rfloor + 1.$$

Note that,

$$\left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{m}{2} \right\rfloor + 1 \iff \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq |M^2| \iff n_1 - 1 \geq m_2$$

and $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor$ since n is odd. Consider the vote profile X defined on M^2 as follows:

	X_{1_1}	$\cdots$		$\cdots$	$X_{1_{m_2}}$		$\cdots$	$X_{1_{n_1-1}}$	$X_{1_{n_1}}$	X_{2_1}	$\cdots$	$X_{2_{n_2}}$	$MR(X)$
a^{m_1+1}	$+1$	$\cdots$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	-1	$\cdots$	$+1$	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	-1	$\cdots$	-1	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$+1$	$\cdots$	$+1$	$+1$

Pose $S = \{1_1, \ldots, 1_{n_1-1}\}$, $|S| = n_1 - 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Each voter of S is frustrated on at least $|M^2| - 1 = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Thus each voter of S is frustrated on the majority of proposals. Therefore the Anscombe's paradox occurs at X.

Sufficient part: Suppose that

$$\left\lfloor \frac{n}{2} \right\rfloor < \left\lfloor \frac{m}{2} \right\rfloor + 1$$

We show that (N_1, N_2, M^1, M^2) is stable. On the contrary, suppose that there exist an admissible vote profile X and a majority coalition S such that each voter in S is frustrated on a majority of proposals. Note that

i) For each proposal $a \in M^1$, $MR(X) = +1$;

ii) $S \cap N_1 \neq \emptyset$ and each member of $S \cap N_1$ is frustrated on no proposal in M^1. Thus each voter in $S \cap N_1$ is frustrated on at least $\left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals in M^2;

iii) each member of $S \cap N_2$ is frustrated on at most $\left\lfloor \frac{m-1}{2} \right\rfloor - 1$ proposals of M^1. Then each such voter, to be frustrated on a majority of proposals, is necessarily frustrated on at least

$$\left\lfloor \frac{m}{2} \right\rfloor + 1 - \left(\left\lfloor \frac{m-1}{2} \right\rfloor - 1 \right) = 3$$

proposals of M^2 since m is even;

iv) Recalling that X is admissible, then voters in $S \cap N_2$ are all frustrated on the same subset, say T of proposals of M^2. By point (v) above, $|T| = 3 + u$ with $0 \leq u \leq m_2$; hence $0 \leq u \leq \left\lfloor \frac{m-1}{2} \right\rfloor + 1$;

v) for all proposal $a \in T$, $MR(X^a) = -1$ and for all $i \in N_2$, for all $a \in T$, $X_i^a = +1$;

vi)
- $|S \cap N_2| \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$. Pose $|S \cap N_2| = \left\lfloor \frac{n-1}{2} \right\rfloor - 1 - v$ with v is integer such that $0 \leq v \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$
- $|S \cap N_1| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - |S \cap N_2| = v + 2$. Pose $|S \cap N_1| = v + c + 2$ with $0 \leq c \leq \left\lfloor \frac{n}{2} \right\rfloor - v$.
- $|N_1 \backslash S| = |N_1| - |N_1 \cap S| = \left\lfloor \frac{n}{2} \right\rfloor + 2 - (v + c + 2) = \left\lfloor \frac{n}{2} \right\rfloor - (v + c)$.

Consider the following cases

Case 1: $S \cap N_2 \neq \emptyset$

For each proposal $a^l \in T$, there exist at least $v + c + 1$ voters in $S \cap N_1$ who are not frustrated on a^l. Pose $P_l \subseteq S \cap N_1$ the set of all voters in $S \cap N_1$ who are not frustrated on a^l. Pose $|P_l| = v + c + k_l + 1$ with $0 \leq k_l \leq 1$. We have $|P_l| = v + c + k_l + 1 = |S \cap N_1| + k_l - 1$.

• If $k_l = 1$ for all l then $|P_l| = |S \cap N_1|$. Hence all voters of $S \cap N_1$ are each frustrated on no proposal of T. Therefore each voter of $S \cap N_1$ is frustrated in M^2 on at most $m_2 - |T| = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r - (u + 3) = \left\lfloor \frac{m}{2} \right\rfloor - 1 - u \leq \left\lfloor \frac{m}{2} \right\rfloor$. A contradiction holds since each voter of $S \cap N_1$ is frustrated on a majority of proposals.

• If there exists l_0 such that $k_{l_0} = 0$ then for each proposal $a \in T$, at least $|S \cap N_1| - 1$ voters of $S \cap N_1$ are not frustrated on a. Thus there exists a voter i_0 in $S \cap N_1$ who is frustrated on at most one proposal in T. Therefore i_0 is frustrated in M^2 on at most $m_2 - |T| + 1 \leq \left\lfloor \frac{m}{2} \right\rfloor$. A contradiction holds.

Case 2: $S \cap N_2 = \emptyset$. By ii) $S \subseteq N_1$ and each member of $S \cap N_1$ is frustrated on not proposal in M^1. We have $|S| = |S \cap N_1| = \left\lfloor \frac{n}{2} \right\rfloor + 1 + c$ with $0 \leq c \leq 1$ then

$|N\backslash S| = \lfloor\frac{n-1}{2}\rfloor - c$. Hence for each proposal $a \in M^2$, there exist at least $c+1$ voters in S who are not frustrated on a. We have the following subcases.

subcase 1: Suppose $|S \cap N_1| \geq |M^2|$.

If $|N_1 \cap S| \geq (c+1)|M^2|$ we have,

$|N_1 \cap S| \geq (c+1)|M^2| \iff \lfloor\frac{n}{2}\rfloor + 1 + c \geq (c+1)\left(\lfloor\frac{m}{2}\rfloor + 2\right) \iff \lfloor\frac{n}{2}\rfloor \geq \lfloor\frac{m}{2}\rfloor + 1$. A contradiction holds, since by assumption $\lfloor\frac{n}{2}\rfloor < \lfloor\frac{m}{2}\rfloor + 1$. We have $|M^2| \leq |N_1 \cap S| < (c+1)|M^2|$ them $c = 1$ and for each proposals $a \in M^2$, there exist at least 2 voters in N_1 who are not frustrated on a. Since $|N_1| < 2|M^2|$, there exists a voter i_0 in N_1 who is not frustrated on at least 2 proposals of M^2. As $M^2 = \lfloor\frac{m}{2}\rfloor + 2$, voter i_0 is not frustrated on a majority of proposals; this is a contradiction.

subcase 2: Suppose that $|N_1 \cap S| < |M^2|$.

It is clear that there exists a voter in $N_1 \cap S$ who is not frustrated on at least 2 proposals of M^2. A contradiction holds because $M^2 = \lfloor\frac{m}{2}\rfloor + 2$. ∎

Proposition 4.3.5. *A voting environment (N_1, N_2, M^1, M^2) with (an extra-unit majority and an agenda of type IV) or (a multiple-unit majority, an agenda of type (III or IV)) is not stable.*

Proof.

Consider a voting environment (N_1, N_2, M^1, M^2) with (an extra-unit majority and an agenda of type IV) or (a multiple-unit majority, an agenda of type (III or IV)). Pose $n_1 = \lfloor\frac{n}{2}\rfloor + 1 + d$ with $d \geq 1$. Thus $|N_2| = \lfloor\frac{n-1}{2}\rfloor - d$, $|M^1| = \lfloor\frac{m-1}{2}\rfloor - r$ with $r \geq 1$. Thus $|M^2| = \lfloor\frac{m}{2}\rfloor + 1 + r$. Then ($d = 1$ and $r \geq 2$) or ($d \geq 2$ and $r \geq 1$). $n_1 = n_1' + n_1''$ with $n_1' = \lfloor\frac{n}{2}\rfloor + 1 - \delta_n$ and $n_1'' = d + \delta_n$ with

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad \delta_m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even} \end{cases}$$

$|N_2| = n_2 = \lfloor\frac{n-1}{2}\rfloor - d$. Thus ($d + \delta_n = \delta_n + 1$ and $r + \delta_m \geq \delta_m + 2$) or ($d + \delta_n \geq \delta_n + 2$ and $r + \delta_m \geq \delta_m + 1$). Consider the vote profile X defined as follows:

	X_{1_1}	$\cdots$	$X_{1_{n'_1}}$	X_{2_1}		$\cdots$		$X_{2_{n''_1}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\|M^1\|}$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$
	-1	$\cdots$	-1	-1	$+1$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$+1$	-1	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\|M^1\|+r+\delta_m}$	-1	$\cdots$	-1	-1	-1	-1	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$

If $\delta_n = \delta_m$, $d = 1$ and $r \geq 2$, then $n''_1 = 2$ or 3. In this case, use the following matrix in the vote profile above.

	X_{2_1}	X_{2_2}	X_{2_3}
$a^{\|M^1\|+1}$	-1	$+1$	$+1$
	$+1$	-1	$+1$
	-1	$+1$	$+1$
	$+1$	-1	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a^{\|M^1\|+r+\delta_m}$	-1	$+1$	$+1$

Pose $S = \{2_1, \ldots, 2_{n''_1}\} \cup \{3_1, \ldots, 3_{n_2}\}$, $|S| = n''_1 + n_2 = \left\lfloor \frac{n-1}{2} \right\rfloor + \delta_n$. Each voter in $\{2_1, \ldots, 2n''_1\}$ is frustrated on at least $\delta_m + |M^2| - (r + \delta_m) = |M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Furthermore, each voter in $\{3_1, \ldots, 3_{n_2}\}$ is frustrated on $|M^1| + r + \delta_m = \left\lfloor \frac{m-1}{2} \right\rfloor + \delta_m$ proposals. Thus each voter in S is frustrated on a majority of proposals. Therefore the Anscombe's paradox occurs at X. ■

When the voting environment has an agenda of type III or IV with an extra-unit or a multiple-unit majority, the parity of the total number (n) of voters and the one of the total number (m) of proposals influence the stability of the voting environment.

Proposition 4.3.6. *A voting environment (N_1, N_2, M^1, M^2) with an extra-unit majority, an agenda of type III is not stable if*

$$(m \text{ and } n \text{ have the same parity}) \text{ or } (m \text{ is odd and } n \text{ is even}).$$

Proof.

Consider the vote profile in the proof of Proposition 4.3.5 with $d = r = 1$. ∎

4.3.2 Agenda of type III or IV with a nonempty set of independent voters

When the set of independent voters is nonempty, no matter the type of the prevalent majority, the stability of a consensual voting environment with an agenda of type III depends only on the total number of voters and the parity of the total number of proposals.

Proposition 4.3.7. *A voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type III and a nonempty set of independent voters is stable if*

$$|N| = 6 \text{ and } m \text{ even.}$$

Proof.

Consider a voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type III and a nonempty set of independent voters such that $|N| = 6$ and m even. Then $|N_1| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, $|M^1| = \left\lfloor \frac{m-1}{2} \right\rfloor - 1$, $n = 6$ and m is even. Since $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 2$, $n_2 \neq \emptyset$ and $n_3 \neq \emptyset$, we have $n_2 = n_3 = 1$ and $n_1 = 4$. Pose $m = 2k$ then $|M^1| = k - 2$. Hence $k > 2$ and $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 2 = k + 2$.

Suppose that there exist an admissible vote profile X and a majority coalition S such that each voter in S is frustrated on a majority of proposals.

Note that each voter in $S \cap N_1$ is frustrated on no proposal in M^1. Therefore each of them is frustrated on at least $k + 1$ proposals in M^2. Thus each voter of $S \cap N_1$ is in agreement with the majority decision on at most one proposal in M^2.

Since $|N_1| = 4$ and $|S| \geq 4$, for all proposals $a \in M^2$ there exists at least one voter in $S \cap N_1$ who is in agreement with $MR(X^a)$. It follows from all this that $|M^2| \leq |S \cap N_1|$. Then $k + 2 \leq 4$ and a contradiction holds since $k > 2$. ∎

Proposition 4.3.7 tel us that when the set of independent voters is nonempty, the stability of consensual voting environments impose an agenda of type III, an even number of proposals and six voters. When we are out of these conditions with a nonempty set of independent voters, the environment is not stable. That is the following proposition.

Proposition 4.3.8. *A voting environment $(N_1, N_2, N_3, M^1, M^2)$ with a nonempty set of independent voters is not stable in the following cases:*

a) *the opposition is on type IV; or*

b) *the opposition is on type III and $n \neq 6$; or*

c) *the opposition is on type III and m is odd.*

Proof.

Consider a voting environment $(N_1, N_2, N_3, M^1, M^2)$ with an agenda of type III or IV such that at least one assumption out of a), b) or c) holds. Then $|M^1| \leq \left\lfloor \frac{m-1}{2} \right\rfloor - 2$ or ($|M^1| = \left\lfloor \frac{m-1}{2} \right\rfloor - 1$ and $n \neq 6$) or ($|M^1| = \left\lfloor \frac{m-1}{2} \right\rfloor - 1$ and m is odd). It follows that $|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 3$ or ($|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 2$ and $n \neq 6$) or ($|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 2$ and m is odd). This implies that

$$\begin{cases} |M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 3 \text{ or} \\ \left(|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } n \neq 6\right) \text{ or} \\ \left(|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ odd}\right) \end{cases}$$

which is equivalent to

$$\begin{cases} \left(|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } n \neq 6\right) \text{ or} \\ \left(|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } n = 6 \text{ and } m \text{ odd}\right) \text{ or} \\ \left(|M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 3 \text{ and } n = 6 \text{ and } m \text{ even}\right) \end{cases}$$

Note that $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$ with $r \geq 1$; $|N_1| = n_1$, $|N_2| = n_2$ and $|N_3| = n_3$ and $n_1 + n_2 + n_3 = n$. Note that there exist three integers n_1', n_1'' and m_2' such that $n_1 = n_1' + n_1''$ with $n_1'' = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $m_2' + |M^1| = \left\lfloor \frac{m}{2} \right\rfloor + 1$. Thus $N_1 = \{1_1, \ldots, 1_{n_1'}\} \cup \{2_1, \ldots, 2_{n_1''}\}$, $N_2 = \{3_1, \ldots, 3_{n_2}\}$, $N_3 = \{4_1, \ldots, 4_{n_3}\}$

Case 1: $|M^2| > \left\lfloor \frac{m}{2} \right\rfloor + 1$ and $|N| \neq 6$. We have three subcases

Subcase 1.1: Suppose that ($n_2 \geq 2$ and n even) or ($n_2 \geq 1$ and n odd). Thus $n_1'' + n_2 \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$. Pose $m_1 = |M^1|$ and consider the vote profile X defined as follows

	X_{1_1}	$\cdots$	$X_{1_{n'_1}}$	X_{2_1}	$\cdots$	$X_{2_{n''_1}}$	X_{3_1}	$\cdots$	$X_{3_{n_2}}$	X_{4_1}	$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1}	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	-1
$a^{m_1+m_2'}$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{m_1+m_2'+r-1}$	-1	$\cdots$	-1	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	-1	$\cdots$	-1	$+1$

Let $S = \{1_1, \dots, 1_{n'_1}\} \cup \{3_1, \dots, 3_{n_2}\} \cup \{4_1, \dots, 4_{n_3}\}$ and $|S| = n - n''_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Voters in $\{1_1, \dots, 1_{n'_1}\}$ are each frustrated on $|M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Voters in $\{3_1, \dots, 3_{n_2}\}$ are each frustrated on $|M^1| + m_2' + r - 1 = \left\lfloor \frac{m}{2} \right\rfloor + r$ proposals. Voters in $\{4_1, \dots, 4_{n_3}\}$ are each frustrated on $m - (m_2' - 1)$ proposals and $m - (m_2' - 1) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 - m_2' + 1$. Since $|M^1| + m_2' = \left\lfloor \frac{m}{2} \right\rfloor + 1$ then $m_2' \leq \left\lfloor \frac{m}{2} \right\rfloor \Rightarrow m - (|M^2| - 1) \geq \left\lfloor \frac{m-1}{2} \right\rfloor + 2$. Therefore voters in S are each frustrated on a majority of proposals. Therefore the Anscombe's paradox occurs at X.

Subcase 1.2: Suppose n is even, $n_2 = 1$ and $n_3 \geq 2$. Pose $n'_1 = \left\lfloor \frac{n-1}{2} \right\rfloor$ and note that $|M^2| > 3$ since $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$. Consider the vote profile X defined as follows

	X_{1_1}	$\cdots$	$X_{1_{n'_1}}$	X_{2_1}	$\cdots$			$X_{2_{n''_1}}$	X_{3_1}	X_{4_1}	X_{4_2}	$\cdots$	$X_{4_{n_3}}$	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	$+1$	-1	$\cdots$	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a^{\|m_1\|}$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	$+1$	-1	$\cdots$	-1	$+1$
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	$+1$	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	$+1$	$+1$	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	-1	$+1$	$\cdots$	$+1$	-1
a^{m_1+4}	-1	$\cdots$	-1	-1	$\cdots$	-1	-1	-1	$+1$	-1	$+1$	$\cdots$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^{m_1+r+2}	-1	$\cdots$	-1	-1	$\cdots$	-1	-1	-1	$+1$	-1	$+1$	$\cdots$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	$\cdots$	-1	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
a^m	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	$\cdots$	-1	-1

Pose $S = \{2_1, \dots, 2_{n''_1}\} \cup \{3_1\} \cup \{4_1, \dots, 4_{n_3}\}$. Then $|S| = n - n'_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Voters in $\{2_1, \ldots, 2_{n''_1}\} \cup \{4_1\}$ are each frustrated on at least $|M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Voter 3_1 is frustrated on $|M^1| + |M^2|$ proposals and $|M^1| + |M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$. Voters in $\{4_2, \ldots, 4_{n_3}\}$ are each frustrated on $m - (|M^2| - r - 2)$ proposals and $m - (|M^2| - r - 2) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 - (|M^2| - r - 2) = \left\lfloor \frac{m-1}{2} \right\rfloor + 2$. Therefore voters in S are each frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X.

Subcase 1.3: Suppose that n is even, $n_2 = 1$, $n_3 = 1$ and $n > 6$. Reset $n''_1 = \frac{n}{2} - 1$ then $n''_1 = \frac{n}{2} - 1 \geq 3$. Consider the vote profile X defined as follows:

	X_{1_1}	$\cdots$	$X_{1_{n'_1}}$	X_{2_1}	$\cdots$				$X_{2_{n''_1}}$	X_{3_1}	X_{4_1}	$MR(X)$
a^1	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a^{\lvert m_1\rvert}$	$+1$	$\cdots$	$+1$	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	-1	$+1$	$+1$	$+1$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	-1	$+1$	$+1$	$+1$	-1
a^{m_1+3}	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	-1	$+1$	$+1$	-1
	-1	$\cdots$	-1	-1	$\cdots$	-1	-1	-1	-1	$+1$	$+1$	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{m_1+r+2}	-1	$\cdots$	-1	-1	$\cdots$	-1	-1	-1	-1	$+1$	$+1$	-1
	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^m	-1	$\cdots$	-1	$+1$	$\cdots$	$+1$	$+1$	$+1$	$+1$	$+1$	-1	-1

Pose $S = \{2_1, \ldots, 2_{n''_1}\} \cup \{3_1\} \cup \{4_1\}$. Then $|S| = n''_1 + 2 = \frac{n}{2} + 1$. Voters of $\{2_1, \ldots, 2_{n''_1}\} \cup \{4_1\}$ are each frustrated on at least $|M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Voter 3_1 is frustrated on $|M^1| + |M^2|$ proposals and $|M^1| + |M^2| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$. Voter 4_1 is frustrated on $m - (|M^2| - r - 2)$ proposals and $m - (|M^2| - r - 2) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m-1}{2} \right\rfloor + 1 - (|M^2| - r - 2) = \left\lfloor \frac{m-1}{2} \right\rfloor + 2$ Therefore voters in S are each frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X.

Case 2: Suppose that ($|M^2| > \left\lfloor \frac{m}{2} \right\rfloor + 1$ and $|N| = 6$ and m odd) or ($|M^2| > \left\lfloor \frac{m}{2} \right\rfloor + 2$ and $|N| = 6$ and m even). Thus $|N| = 6$ and $|M^2| = \left\lfloor \frac{m}{2} \right\rfloor + 1 + r$ with ($r \geq 1$ and m is odd) or ($r > 1$ and m is even). It holds that

$$\left\lfloor \frac{m-1}{2} \right\rfloor + r > \frac{m}{2}. \tag{4.1}$$

Since $|N_1| > \frac{n}{2}$ and $|N| = 6$, then $|N_2| = |N_3| = 1$ and $|N_1| = 4$. Note that

$|M^2| > 2r + |M^1|$. Indeed,

$$\begin{aligned} |M^2| - (2r + |M^1|) &= |M^2| - (2r + m - |M^2|) \\ &= \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m-1}{2} \right\rfloor + 1 > 0. \end{aligned}$$

Consider the vote profile X defined as follows

	X_{1_1}	X_{1_2}	X_{1_3}	X_{1_4}	X_{2_1}	X_{3_1}	$MR(X)$
	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a^{\lvert m_1 \rvert}$	$+1$	$+1$	$+1$	$+1$	-1	-1	$+1$
	-1	-1	$+1$	-1	$+1$	$+1$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{m_1+r}	-1	-1	$+1$	-1	$+1$	$+1$	-1
a^{m_1+r+1}	$+1$	$+1$	$+1$	-1	$+1$	-1	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{m_1+r+r}	$+1$	$+1$	$+1$	-1	$+1$	-1	$+1$
	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{m_1+2r}	$+1$	$+1$	-1	-1	$+1$	$+1$	$+1$
	-1	-1	$+1$	$+1$	$+1$	-1	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a^{m}	-1	-1	$+1$	$+1$	$+1$	-1	-1

Let $S = \{1_3, 1_4\} \cup \{2_1\} \cup \{3_1\}$. Voters in $\{1_3, 1_4\}$ are each frustrated on at least $|M^2| - r = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Voter 2_1 is frustrated on $|M^2| - r - |M^1| + |M^1| = \left\lfloor \frac{m}{2} \right\rfloor + 1$ proposals. Voter 3_1 is frustrated on $m - (|M^2| - 1 - r)$ proposals and $|M^1| + 2r = \left\lfloor \frac{m-1}{2} \right\rfloor + r > \frac{m}{2}$ by Equation 4.1. Thus voters in S are each frustrated on a majority of proposals. Therefore the Anscombe's paradox holds at X. ■

Remark 4.3.1. *The following table summarizes the conditions of stability by identifying all combinations of type of opposition and type of leading party.*

M.U.M: *Multiple-unit majority;*

No: *MR never exhibits the Anscombe's paradox over the set of all admissible vote profiles;*

Yes: *MR exhibits the Anscombe's paradox over the set of all admissible vote profiles;*

I, II, III and IV represent the type of opposition.

		Exact Majority	Extra-unit Majority	M.U.M
I	$\lvert N_3\rvert = \emptyset$	No	No	No
II	$\lvert N_3\rvert = \emptyset$	No	No	No
III	$\lvert N_3\rvert = \emptyset$ n is odd	No iff $\lvert M^2\rvert < \frac{(\lfloor \frac{m}{2}\rfloor+1)(\lfloor \frac{n}{2}\rfloor+1)}{\lfloor \frac{n-1}{2}\rfloor}$	No iff (m is even) and $\lfloor \frac{n}{2}\rfloor < \lfloor \frac{m}{2}\rfloor + 1$	Yes
	$\lvert N_3\rvert = \emptyset$ n is even	No iff (m is even) and $\lfloor \frac{n}{2}\rfloor < 2\lfloor \frac{m}{2}\rfloor + 3$	Yes	
IV	$\lvert N_3\rvert = \emptyset$	No iff (m is odd) and $\lvert M^2\rvert < \frac{(\lfloor \frac{m}{2}\rfloor+1)(\lfloor \frac{n}{2}\rfloor+1)}{\lfloor \frac{n-1}{2}\rfloor}$	Yes	Yes

		Exact Majority	Extra-unit Majority	M.U.M
I	$\lvert N_3\rvert \neq \emptyset$	No	No	No
II	$\lvert N_3\rvert \neq \emptyset$	No iff ($\lvert N_3\rvert = 1$) or ($\lvert N_2\rvert = 1,\ \lvert N_3\rvert = 2,\ \lvert M^2\rvert = 3$, m and n even)		
III	$\lvert N_3\rvert \neq \emptyset$	No iff ($\lvert N\rvert = 6$ and m even)		
IV	$\lvert N_3\rvert \neq \emptyset$	Yes	Yes	Yes

CONCLUSION

At the end of the present thesis, it is worth noticing that our aim was to address some open issues on the Anscombe's paradox in order to contribute for a better understanding of the circumstances of its occurrences as well as some conditions to avoid it.

Does the majority rule with a given number n of voters and a given number m of proposals vulnerable to Anscombe's paradox? To this question with no available answer in the literature, we provide a complete landscape of the situation in Theorem 1.2: the majority rule does not exhibit the Anscombe's paradox if and only if ($n \leq 3$) or ($m = 4$ and $n \leq 5$). Moreover, we have shown in Theorem 1.1 that the majority rule is Anscombe's paradox free whenever voters report at most three distinct vote vectors.

Can we escape from observing the Anscombe's paradox by moving from the majority rule to any other binary decision rule? To the best of our knowledge this concern was not yet addressed. To handle this we provide in Chapter 2, generalizations of Anscombe's paradox in two different ways. In the first approach, the rule changes but a majority is any coalition with more than the half of the voters; we then consider any binary voting rule distinct from the majority rule. It appears that all voting rules that are minority sensitive, (for some vote profiles, the decision is supported by a minority coalition) are vulnerable to Anscombe's paradox; see Proposition 2.1.1. Moreover, Proposition 2.1.6 and Proposition 2.1.7 together with Corollary 2.1.1 identify all binary voting rules that are not minority sensitive and do not exhibit Anscombe's paradox for a given number n of voters and a given number m of proposals. In the second approach, the rule is a simple game and an occurrence of the qualified Anscombe's paradox refers to a situation where all members of a winning coalition (a coalition endowed with the power of decision) are frustrated on more than the half of the proposals. Theorem 2.6 is a full characterization of all combinations of a simple game and a set of proposals that do not exhibit the qualified Anscombe's paradox. Interestingly, it appears that simple voting games which are, independently of the number of proposals, immune to the qualified Anscombe's paradox have only singular structures of winning coalitions: the size of a minimal winning coalition for such games is at most 2; and there is necessary a unique vetoer as soon as there are at least four minimal winning coalitions (no matter the size of the set N of all voters); see Proposition 2.2.2 and Corollary 2.2.1.

Is there any preference domain with exponential cardinality that does not exhibit Anscombe's paradox? Any positive response to this question constitutes an improvement of the existing literature on domain restriction for the current paradox. Indeed, this is the case of our findings in chapter 3. Unifying preference domains have being introduced taking into the account the existence of some type of consensus we model with three parameters: a set of unifying proposals, a barometer of consensus and a vector of common standards. The clue of this novelty is the obtention of necessary and sufficient conditions that preclude observing Anscombe's paradox for all admissible vote profiles given a unifying preference domain; see Theorem 2.7. Roughly speaking, the existence of a minimum of consensus on individual preferences may result in ruling out Anscombe's paradox; and we describe a possible way of handling and measuring this "minimum" within a specific framework of consensus. In particular, when all proposals are unifying, the corresponding domain is free of Anscombe's paradox if and only if the total number of proposals on which each voter may deviate from the vector of common standards exceeds the quarter of all proposals by at most one unit; see Corollary 3.2.1.

Is there any intra-profile condition that mirrors the functioning of real life institution that does not exhibit Anscombe's paradox? In chapter 4, we have introduced consensual voting environments attempting to describe vote profiles when one assumes a partition of voters into a leading party (a party with more than the half of representatives), an opposition party and some freethinkers. In our model, we also assume that only parties submit proposals to vote on; the members of a given party all vote for the adoption of the proposals initiated by that party, but may have distinct opinions on other proposals; and independent voters are freethinkers on all proposals. Propositions 4.3.4-4.2.3 provide a complete landscape of the situation when one is interested by necessary and sufficient conditions to avoid all occurrences of Anscombe's paradox. An overview of the coarse conditions we obtain is provided in Remark 4.3.1. For illustrations, the paradox is immediately overcome when the leading party initiates a majority of proposals; or when the opposition party submits an exact majority of proposals in the absence of independent voters.

There are still several other issues on Anscombe's paradox that may desserve further investigations. For example, we have shown that almost all binary voting rules exhibit Anscombe's paradox (qualified or not). For such rules, identifying necessary and sufficient conditions on profiles at which the paradox occurs is clearly left open. Moreover unifying preference domains presented here provide alternatives to single-switchness in avoiding Anscombe's paradox; but how restrictive are those domains is also a question with an unknown answer. One may also consider consensual voting environments with more than two parties and then check necessary and sufficient conditions for their stability. All these concerns are obviously some possible follow up of the present work we will surely address.

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Avoiding Majority Dissatisfaction on a Series of Majority Decisions

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Abstract Applying majority voting on a set of proposals may result in a series of decisions for which there exists a majority of voters who disagree with the collective decision in a majority of cases. This phenomenon is known as Anscombe's paradox. In this paper, we provide new domains of individual opinions free of this paradox. To achieve this, we assume that there are some unifying proposals such that, due to some common values, each voter deviates from a given list of issue-specific standards only on a limited number of unifying proposals. For example, the notion of unifying proposals captures issues such as sovereignty or war against terrorism for which voters, because of patriotism or common-sense, generally unite across the political divide to deal with these crucial issues.

Keywords Majority rule · Unifying proposals · Anscombe's paradox free domains

1 Introduction

For a decision over two alternatives or any "Yes–No" voting, as it is the case in referenda or amendment processes within legislatures of democratic nations, the majority

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